Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 1, pp. 178–190
doi: https://doi.org/10.15407/mag19.01.178

Algebraic Symplectic Reduction and Quantization of Singular Spaces

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The algebraic method of singular reduction is applied for non regular group action on manifolds which provides singular symplectic spaces. The problem of deformation quantization of the singular surfaces is the focus. For some examples of singular Poisson spaces Grönewold–Moyal series is explicitly constructed and convergence is checked. Some examples of deformation quantization of singular Poisson spaces are considered in detail.

Key words: Poisson manifold, constrains, singular symplectic reduction, deformation quantization, Grönewold–Moyal star product, K3 surfaces

Mathematical Subject Classification 2020: 53D55, 53D20

1. Introduction

The problem of quantum systems with constraints goes back to Dirac [6]. The general method of Meyer–Marsden–Weinstein works for reduction of a symplectic manifold with constraints and a free group action. If the group action is not free the constraint locus is singular. The singular points are often the most interesting because they have smaller orbits and larger symmetry. Sniatycki and Weinstein [12] applied a pure algebraic method for symplectic reduction in a modelling case. The problem of singular symplectic reduction of the angular momentum was studied by geometric methods in [1,5]. Batalin–Vilkovisky–Fradkin's method [3,7] was proposed for gauge systems. Stasheff extended this method for a wider class of singular reduced spaces in terms of differential graded free algebraic resolutions. In [4] the BRST method was developed based on the rather complicated homological construction including ghosts fields.

The method of algebraic singular reduction can be applied to any algebraic Poisson manifold (X, q) with an algebraic momentum map and action of an algebraic group G. It ends up on an affine Poisson algebraic variety $(X_{\rm red}, q_{\rm red})$ with an algebra sheaf $\mathcal{O}_{\rm red}$ of G invariant functions restricted to the constraint locus. This variety is singular if the group action is not free. This is the case of the Yang–Mills theory and general relativity where the constraint locus has quadratic singularities and the reduced space $X_{\rm red}$ is singular [2].

We give here explicit constructions of deformation quantization of some singular spaces $X_{\rm red}$. Our method is based on the general Grönewold–Moyal formula.

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We check in the simplest case that the associative product converges for local holomorphic arguments.

The problem of quantization of spaces with singularity was rased by Kontsevich [9]. To my best knowledge there is no examples of deformation quantization of singular spaces so far. See [11] for basics of theory of quantization of singular spaces.

2. Singular reduction

The following construction of singular reduction is close to that of [12]. Let X be a real algebraic variety endowed with a Poisson bracket q defined on the algebra of real rational functions on X. In a more general setting let $(X, \mathcal{O}_X)$ be a real algebraic scheme with a Poisson biderivation $q : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$. Let G be an algebraic group defined on X such that the bracket q is G covariant and $\mathcal{O}_{X/G}$ be the subsheaf of $\mathcal{O}_X$ of G invariant germs. It is a sheaf of algebras defined on the orbit space X/G (which needs not to be an analytic manifold). An invariant Poisson bracket q can be lifted to a Poisson bracket q_G on $\mathcal{O}_{X/G}$.

Let $J : X \to \mathfrak{g}^*$ be an algebraic mapping called momentum map, where $\mathfrak{g}^*$ is the dual space to Lie algebra $\mathfrak{g}$ of G. The set $Y = J^{-1}(0)$ is the subscheme of $\mathcal{O}_X$ (called constraint locus) with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X / (J)$, where (J) denotes the ideal in $\mathcal{O}_X$ generated by elements of J. We suppose that (J) is invariant under action of G and J generates a mapping J_G defined on Y/G making the diagram commutative:

$$\begin{array}{ccccc} Y & \to & X & \overset{J}{\to} & \mathfrak{g}^* \\ \downarrow & & \downarrow & & \| \\ X_{\mathrm{red}} = Y/G & \to & X/G & \overset{J_G}{\to} & \mathfrak{g}^* \end{array} .$$

We assume further that the bracket q is Hamiltonian that is for any $\gamma \in \mathfrak{g}$ and any $a \in \mathcal{O}_X$, we have

$$q(\langle \gamma, J \rangle, a) = \mathrm{d}_G A(\gamma)(a) \tag{2.1}$$

where $A : X \times G \to X$ denotes the group action and $\mathrm{d}_G A : \mathfrak{g} \to T(X)$ is the tangent map.

Proposition 2.1. *The bracket q as above can be lifted to a biderivation q_{red} on $X_{\mathrm{red}} \doteqdot Y/G$. This is a Poisson bracket.*

Proof. Check that inclusion $q(j, b) \in (J)$ holds for any $j \in (J)$ and arbitrary $b \in \mathcal{O}_{X/G}$. Let $j = \langle \gamma, J \rangle a$ for some $a \in \mathcal{O}_X$ and $\gamma \in \mathfrak{g}$. We have

$$q(j, b) = \langle \gamma, J \rangle q(a, b) + aq(\langle \gamma, J \rangle, b)$$

because q is biderivation. The first term belongs to (J) and by (2.1)

$$q(\langle \gamma, J \rangle, b) = \mathrm{d}_G A(\gamma)(b) = 0$$

since b is constant on any orbit and the field $\mathrm{d}_G A(\gamma)$ is tangent to orbits of G. Finally $q(j, b) \in (J)$. □

The Poisson variety $\left(X_{\mathrm{red}}, \mathcal{O}_{Y/G}, q_{\mathrm{red}}\right)$ will be called singular symplectic reduction of (X, q, G, J). This construction is translated to the category of sheaves of smooth functions on X with obvious modifications.

3. The Poisson bracket of Hamiltonian fields

Let $\mathcal{A}$ be a unitary commutative algebra over a field $\mathbb{K}$ of zero characteristic.

Proposition 3.1. *Let q be a Poisson bracket on $\mathcal{A}$. If $q(q(a,b),\cdot) = 0$ for some $a, b \in \mathcal{A}$, then the Hamiltonian fields $\mathrm{A}(\cdot) = q(\cdot, a)$ and $\mathrm{B}(\cdot) = q(b, \cdot)$ commute.*

This follows from the Jacobi identity.

For arbitrary derivations A, B on $\mathcal{A}$, we define the biderivation $(\mathrm{A} \wedge \mathrm{B})(a,b) = \mathrm{A}(a)\,\mathrm{B}(b) - \mathrm{B}(a)\,\mathrm{A}(b)$, $a, b \in \mathcal{A}$. For a biderivation q, we denote

$$\mathrm{Jac}[q](a,b,c) \equiv q(q(a,b),c) + q(q(b,c),a) + q(q(c,a),b)$$

and have $\mathrm{Jac}[q] = 0$ if q is a Poisson bracket.

Proposition 3.2. *If fields A_i, B_i, $i = 1, \dots, n$, are defined on $\mathcal{A}$ and satisfy $[\mathrm{A}_i, \mathrm{B}_j] = 0$ for any i and j then bracket $q = \sum \mathrm{A}_i \wedge \mathrm{B}_i$ fulfils the Jacobi identity.*

Proof. For $n = 1$ this identity can be checked by a direct computation. In the general case we set $U = \sum t^i \mathrm{A}_i$, $V = \sum t^{n-i} \mathrm{B}_i$ where t is a real parameter. The fields U and V commute, hence $\mathrm{Jac}[U \wedge V] = 0$. The left hand side is a polynomial in t which vanishes identically. In particular the term with t^n is equal zero which implies the statement. □

We say that a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is dense, if any derivation δ in $\mathcal{A}$ that vanishes on $\mathcal{B}$ vanishes also on $\mathcal{A}$.

Proposition 3.3. *Let q be a Poisson bracket on $\mathcal{A}$. If there exist elements $\mathrm{a}_i, \mathrm{b}_i \in \mathcal{A}$, $i = 1, \dots, n$, such that*

$$q\left(\mathrm{a}_i, \mathrm{a}_j\right) = q\left(\mathrm{b}_i, \mathrm{b}_j\right) = 0, \quad q\left(\mathrm{a}_i, \mathrm{b}_j\right) = \delta_{ij}, \quad i, j = 1, \dots, n, \tag{3.1}$$

and $\mathrm{a}_i, \mathrm{b}_i$ generate the dense subalgebra $\mathcal{B}$ of $\mathcal{A}$ then

$$q(\cdot,\cdot) = \sum_1^n q(\cdot, \mathrm{b}_k) \wedge q(\mathrm{a}_k, \cdot). \tag{3.2}$$

Proof. Proposition 3.1 implies commutativity of any pair of the fields $q(\cdot, \mathrm{b}_k)$, $q(\mathrm{a}_k, \cdot)$, $i, j = 1, 2, \dots, n$. By (3.1) the biderivation

$$[\cdot,\cdot] \div \sum_{\div 1}^n q(\cdot, \mathrm{b}_k) \wedge q(\mathrm{a}_k, \cdot) \div \sum_1^n q(\cdot, \mathrm{b}_k)\, q(\mathrm{a}_k, \cdot) - \sum_1^n q(\cdot, \mathrm{a}_k)\, q(\mathrm{b}_k, \cdot)$$

fulfils

$$[\mathrm{a}_i, \mathrm{b}_j] = \sum_{k=1}^n q(\mathrm{a}_i, \mathrm{b}_k)\, q(\mathrm{a}_k, \mathrm{b}_j) - \sum_{k=1}^n q(\mathrm{b}_j, \mathrm{b}_k)\, q(\mathrm{a}_k, \mathrm{a}_i) = \delta_{ij}$$

for any i, j that is $[\mathrm{a}_i, \mathrm{b}_j] = q\left(\mathrm{a}_i, \mathrm{b}_j\right)$. Therefore $[\mathrm{p}, \mathrm{q}] = q(\mathrm{p}, \mathrm{q})$ for arbitrary $\mathrm{p}, \mathrm{q} \in \mathcal{B}$.This implies that the brackets coincide on $\mathcal{A}$ since $\mathcal{B}$ is dense in $\mathcal{A}$. □

4. The Grönewold–Moyal star product

The idea of quantization of a physical space-time supplied with a Poisson structure goes back to Weyl [15]. It was developed by Grönewold [8] and Moyal [10]). The idea was used later in the form of *deformation quantization* by Kontsevich [9] and formalized by the author for the category of singular analytic spaces [11].

Theorem 4.1. *For a commutative* $\mathbb{R}$*-algebra* $\mathcal{A}$ *with Poisson bracket* q *and elements* $\mathrm{a}_i, \mathrm{b}_j$ *fulfilling* (3.1)*, the generalized Grönewold–Moyal (GM) product is a bilinear associative operation*

$$(u * v)(t) \doteqdot uv + \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathrm{Q}_k(u, v), \;\; u, v \in \mathcal{A}, \tag{4.1}$$

where for any $k = 1, 2, 3, \ldots$

$$\mathrm{Q}_k(u, v) \doteqdot \sum_{j=0}^{k} \frac{(-1)^j k!}{j!(k-j)!} \sum_{i_l=1}^{n} \mathrm{A}_{i_1} \cdots \mathrm{A}_{i_j} \mathrm{B}_{i_{j+1}} \cdots \mathrm{B}_{i_k}(u)\, \mathrm{B}_{i_1} \ldots \mathrm{B}_{i_j} \mathrm{A}_{i_{j+1}} \cdots \mathrm{A}_{i_k}(v)$$

and

$$\mathrm{A}_k = q(\cdot, \mathrm{b}_k), \;\; \mathrm{B}_k = q(\mathrm{a}_k, \cdot), \;\; k = 1, \ldots, n.$$

In particular $Q_1(u, v) = q(u, v)$.

Proof. For the phase space $T^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$ the classical Poisson bracket

$$q(a, b) = \sum \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial \xi_i} - \frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial x^i} \tag{4.2}$$

is a particular case of formula (3.2) written for coordinate functions $\mathrm{a}_i = x^i$, $\mathrm{b}_i = \xi_i$. Operation (4.1) has the same form as the classical Grönewold–Moyal star series. Only Jacobi identity is necessary for the proof of this property [8,10]. This implies that (4.1) is an associative operation in the general case. □

5. Invariant quantization of a flat phase space

The action of the orthogonal group $O(n)$ on $\mathbb{R}^n \times \mathbb{R}^n : (x, \xi) \mapsto (Ux, U\xi)$ preserves the Poisson bracket and momentum map

$$J : \mathbb{R}^n \times \mathbb{R}^n \to \wedge^2 \mathbb{R}^n, \;\; J(x, \xi) = x \wedge \xi.$$

The constraint locus $Y = J^{-1}(0)$ consists of pairs (x, ξ) of proportional vectors x and ξ. For elements $e_{jk} = y^j \partial/\partial y^k - y^k \partial/\partial y^j$, $j \neq k = 1, \ldots, n$ of Lie algebra of the group $O(n)$, we have $\langle e_{jk}, J \rangle = x^j \xi_k - x^k \xi_j$ and equation

$$q(\langle e_{jk}, J \rangle, a) = \xi_k \frac{\partial a}{\partial \xi_j} - \xi_j \frac{\partial a}{\partial \xi_k} + x^j \frac{\partial a}{\partial x^k} - x^k \frac{\partial a}{\partial x^j} = \mathrm{d}_G A(e_{jk})(a)$$

implies (2.1). By Proposition 2.1 the bracket q is lifted to the Poisson bracket q_{red} in Y/G.

Let $\mathcal{A}$ be the algebra of real polynomials on $X = \mathbb{R}^n \times \mathbb{R}^n$. The algebra $\mathcal{A}_{Y/G}$ of invariant polynomials on X with respect to the action of $O(n)$ is generated by

$$s_1 = |x|^2, \quad s_2 = |\xi|^2, \quad s_3 = \langle x, \xi \rangle .$$

The generators fulfils one equation $f(s) \doteqdot s_3^2 - s_1 s_2 = 0$, hence $\mathcal{A}_{Y/G} \cong \mathbb{R}[s_1, s_2, s_3] / (f)$. The bracket is defined by

$$q_{\mathrm{red}}(s_1, s_2) = 4s_3, \quad q(s_1, s_3) = 2s_1, \quad q(s_2, s_3) = -2s_2$$

or equivalently

$$q_{\mathrm{red}} = 4s_3 \partial_1 \wedge \partial_2 - 2s_2 \partial_2 \wedge \partial_3 - 2s_1 \partial_3 \wedge \partial_1. \tag{5.1}$$

The elements $\mathrm{a}_1 = \sqrt{s_1}$, $\mathrm{b}_1 = \sqrt{s_2}$ belong to the quadratic extension $\mathcal{A}^*$ of the algebra $\mathcal{A}_{Y/G}$ and

$$q_{\mathrm{red}}(\mathrm{a}_1, \mathrm{b}_1) = \frac{4s_3}{2\sqrt{s_1} 2\sqrt{s_2}} = 1.$$

This bracket fulfils conditions of Proposition 3.3 for $n = 1$ and the algebra $\mathcal{B}$ of polynomials of a_1 and b_1 is dense in $\mathcal{A}^*$. It follows that the bracket q_{red} admits the quantization of GM type on the algebra $\mathcal{A}^*$.

6. Convergence of the Grönewold–Moyal series

Theorem 6.1. *The terms Q_m of the GM quantization of bracket (5.1) are bidifferential operators with polynomial coefficients of degree $\leq m$ in each argument.*

Proof. The fields

$$\begin{aligned} \mathrm{A} &= q(\cdot, \mathrm{b}_1) = q(\cdot, \sqrt{s_2}) = 2\sqrt{s_1}\partial_1 + \sqrt{s_2}\partial_3, \\ \mathrm{B} &= q(\mathrm{a}_1, \cdot) = q(\cdot, \sqrt{s_1}) = 2\sqrt{s_2}\partial_2 + \sqrt{s_1}\partial_3 \end{aligned}$$

commute, vanish on f and satisfy $\mathrm{A} \wedge \mathrm{B} = q$. By Theorem 4.1 for an arbitrary even k, we have

$$\begin{aligned} \mathrm{Q}_k(a, b) = &\sum_{i+j=k/2} \frac{k!}{2i!2j!} \mathrm{A}^{2i}\mathrm{B}^{2j}(a)\, \mathrm{B}^{2i}\mathrm{A}^{2j}(b) \\ &- \sum_{i+j+1=k/2} \frac{k!}{(2i+1)!\,(2j+1)!} \mathrm{ABA}^{2i}\mathrm{B}^{2j}(a) \cdot \mathrm{ABA}^{2j}\mathrm{B}^{2i}(b) . \end{aligned} \tag{6.1}$$

For any odd k,

$$\mathrm{Q}_k(a, b) = \sum_{i+j=k-1} (-1)^j \frac{(k-1)!}{i!j!} q\left(\mathrm{A}^i\mathrm{B}^j(a), \mathrm{A}^j\mathrm{B}^i(b)\right).$$

where second order differential operators

$$\mathrm{A}^2 = 4s_1\partial_1^2 + 2\partial_1 + s_2\partial_3^2,\ \mathrm{B}^2 = 4s_2\partial_2^2 + 2\partial_2 + s_1\partial_3,$$
$$\mathrm{BA} = \mathrm{AB} = 4s_3\partial_1\partial_2 + 2s_1\partial_1\partial_3 + 2s_2\partial_2\partial_3 + s_3\partial_3^2$$

have linear coefficients. □

Theorem 6.2. *For arbitrary holomorphic functions a, b defined on the ball $\left\{s \in \mathbb{C}^3,\ |s| \le r\right\}$, the GM series (4.1) for the Poisson bracket (5.1) converges if $\{|s| < r/4\}$ and $|t| < r^{1/2}/18$.*

The proof is given in the last section.

7. Commuting matrices

Let M_2 be the space of 2×2 matrices with complex entries. The manifold $X = M_2 \times M_2$ is endowed with Poisson bracket

$$q = \sum_{k=1}^{4} \frac{\partial}{\partial a_k} \wedge \frac{\partial}{\partial b_k}, \tag{7.1}$$

where

$$\mathsf{A} = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} b_1 & b_3 \\ b_4 & b_2 \end{pmatrix}$$

are coordinates in X. The group $\mathbf{Sl}(2, \mathbb{C})$ acts diagonally by

$$g : (\mathsf{A}, \mathsf{B}) \mapsto \left(g\mathsf{A}g^{-1}, g\mathsf{B}g^{-1}\right).$$

Let $J : (\mathsf{A}, \mathsf{B}) \mapsto [\mathsf{A}, \mathsf{B}]$ be the momentum map on X; the constraint locus is the cone

$$\begin{aligned} Y = \{(\mathsf{A}, \mathsf{B}) : & b_3(a_1 - a_2) - a_3(b_1 - b_2) = 0, \\ & b_4(a_1 - a_2) - a_4(b_1 - b_2) = 0\}. \end{aligned} \tag{7.2}$$

Condition (2.1) is easy to check. The polynomials

$$\alpha_1 = \mathrm{tr}\mathsf{A}, \quad \alpha_2 = \det\mathsf{A}, \quad \beta_1 = \mathrm{tr}\mathsf{B}, \quad \beta_2 = \det\mathsf{B}, \quad \gamma = \mathrm{tr}\mathsf{AB}$$

generate the algebra $\mathcal{A}_{X/G}$ of invariant polynomials on X. The reduced Poisson bracket equals

$$\begin{aligned} q_{\mathrm{red}} = {} & 2\frac{\partial}{\partial\alpha_1} \wedge \frac{\partial}{\partial\beta_1} + \beta_1\frac{\partial}{\partial\alpha_1} \wedge \frac{\partial}{\partial\beta_2} + \alpha_1\frac{\partial}{\partial\alpha_2} \wedge \frac{\partial}{\partial\beta_1} + \gamma\frac{\partial}{\partial\alpha_2} \wedge \frac{\partial}{\partial\beta_2} \\ & + \left(\alpha_1\frac{\partial}{\partial\alpha_1} - \beta_1\frac{\partial}{\partial\beta_1} + 2\alpha_2\frac{\partial}{\partial\alpha_2} - 2\beta_2\frac{\partial}{\partial\beta_2}\right) \wedge \frac{\partial}{\partial\gamma}. \end{aligned} \tag{7.3}$$

Proposition 7.1. *The algebra $\mathcal{A}_{Y/G}$ of invariant polynomials of the algebra $\mathcal{A}_{X/G}$ restricted to Y is isomorphic to $\mathcal{B}/(\rho)$, where $\mathcal{B} = \mathbb{R}[\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma]$ and*

$$\rho = \gamma^2 - \alpha_1\beta_1\gamma + \alpha_2\beta_1^2 + \beta_2\alpha_1^2 - 4\alpha_2\beta_2 = \left(\gamma - \frac{1}{2}\alpha_1\beta_1\right)^2 - \frac{1}{4}\pi,$$
$$\pi = \left(4\alpha_2 - \alpha_1^2\right)\left(4\beta_2 - \beta_1^2\right) \tag{7.4}$$

Proof. Check that $\rho = 0$ on Y. For any pair $(\mathsf{A}, \mathsf{B}) \in Y$, there exists $g \in \mathbf{Sl}(2, C)$ such that both matrices $g\mathsf{A}g^{-1}$ and $g\mathsf{B}g^{-1}$ have Jordan form. This is easy to prove by means of (7.2). Let (a_1, a_2) and (b_1, b_2) be its diagonal elements, respectively. Then

$$\alpha_1 = a_1 + a_2,\ \alpha_2 = -(a_1 - a_2)^2,\ \beta_1 = b_1 + b_2,\ \beta_2 = -(b_1 - b_2)^2,\ \gamma = a_1 b_1 + a_2 b_2$$

and (7.4) can be checked directly. It is easy to show that this equation generates all algebraic relations. □

It follows that the spectrum of the algebra $\mathcal{A}_{Y/G}$ is a two-fold covering of $\mathbb{C}^4$ ramified over the discriminant set $\{\pi = 0\}$.

Conclusion 7.2. *The singular symplectic reduction of the variety $(X, \mathbf{O}(2), q)$ is singular hypersurface $X_{\mathrm{red}} = \{\rho = 0\}$ with coordinate functions α_1, α_2, β_1, β_2, γ defined by (7.4) with the Poisson bracket q_{red} as in (7.3).*

Let $\mathcal{A}^*$ be the extension of the algebra $\mathcal{A}_{Y/G}$ by means of the element $(2\gamma - \alpha_1\beta_1)^{-1/2} = \pi^{-1/4}$.

Proposition 7.3. *Elements*

$$\mathrm{a}_1 = \frac{1}{\sqrt{2}}\alpha_1, \qquad \mathrm{b}_1 = \frac{1}{\sqrt{2}}\beta_1,$$
$$\mathrm{a}_2 = \frac{1}{2}\frac{\tilde{\alpha}_2^{3/4}}{\tilde{\beta}_2^{1/4}}, \qquad \mathrm{b}_2 = \frac{1}{2}\frac{\tilde{\beta}_2^{3/4}}{\tilde{\alpha}_2^{1/4}} \tag{7.5}$$

of algebra $\mathcal{A}^*$ fulfil (3.1) with $n = 2$ where $\tilde{\alpha}_2 = 4\alpha_2 - \alpha_1^2$, $\tilde{\beta}_2 = 4\beta_2 - \beta_1^2$.

Proof. It is easy to check that

$$q(\mathrm{a}_1, \mathrm{b}_1) = 1, \quad q(\mathrm{a}_1, \mathrm{a}_2) = q(\mathrm{a}_1, \mathrm{b}_2) = q(\mathrm{a}_2, \mathrm{b}_1) = q(\mathrm{b}_1, \mathrm{b}_2) = 0.$$

By (7.3)

$$q\left(\tilde{\alpha}_2, \tilde{\beta}_2\right) = 16\gamma - 8\alpha_1\beta_1 = 8\pi^{1/2} \tag{7.6}$$

on $\mathcal{A}^*$ hence by (7.6)

$$q(\mathrm{a}_2, \mathrm{b}_2) = \frac{1}{4}\left[\tilde{\alpha}_2^{-1/4}\tilde{\beta}_2^{-1/4} q\left(\tilde{\alpha}_2^{3/4},\ \tilde{\beta}_2^{3/4}\right) - \tilde{\alpha}_2^{3/4}\tilde{\beta}_2^{3/4} q\left(\tilde{\alpha}_2^{1/4},\ \tilde{\beta}_2^{1/4}\right)\right] = 1. \quad \square$$

Corollary 7.4. *The Poisson bracket q_{red} admits a quantization by means of GM series with the Hamiltonian fields $\mathrm{A}_k = q(\cdot, \mathrm{b}_k)$, $\mathrm{B}_k = q(\mathrm{a}_k, \cdot)$, $k = 1, 2$.*

This follows from Proposition 3.3. These fields are well defined on $\mathcal{A}^*$ since they vanish on ρ. Explicitly,

$$\begin{aligned}
\mathrm{A}_2 &= q\left(\cdot, \mathrm{b}_2\right) \\
&= \mathrm{b}_2 \frac{\partial}{\partial \gamma} + \frac{3}{4}\pi^{-1/4}\left(4\gamma - 2\alpha_1\beta_1\right)\frac{\partial}{\partial \alpha_2} - \frac{1}{4}\pi^{-1/4}\left(4\gamma - 2\alpha_1\beta_1\right)\frac{\tilde{\beta}_2}{\tilde{\alpha}_2}\frac{\partial}{\partial \beta_2} \\
\sqrt{2}\mathrm{A}_1 &= 2\frac{\partial}{\partial \alpha_1} + \alpha_1\frac{\partial}{\partial \alpha_2} + \beta_1\frac{\partial}{\partial \gamma}, \\
\sqrt{2}\mathrm{B}_1 &= 2\frac{\partial}{\partial \beta_1} + \beta_1\frac{\partial}{\partial \beta_2} + \alpha_1\frac{\partial}{\partial \gamma}, \\
2\mathrm{A}_2 &= \frac{3}{2}\pi^{1/4}\frac{\partial}{\partial \alpha_2} - \frac{1}{2}\pi^{1/4}\frac{\tilde{\beta}_2}{\tilde{\alpha}_2}\frac{\partial}{\partial \beta_2} + \mathrm{b}_2\frac{\partial}{\partial \gamma} = q\left(\cdot, \mathrm{b}_2\right), \\
2\mathrm{B}_2 &= \frac{3}{2}\pi^{1/4}\frac{\partial}{\partial \beta_2} - \frac{1}{2}\pi^{1/4}\frac{\tilde{\alpha}_2}{\tilde{\beta}_2}\frac{\partial}{\partial \alpha_2} + \mathrm{a}_2\frac{\partial}{\partial \gamma} = q\left(\mathrm{a}_2, \cdot\right)
\end{aligned}$$

since

$$\pi^{1/4}\frac{\tilde{\beta}_2}{\tilde{\alpha}_2} = \frac{\tilde{\beta}_2^{5/4}}{\tilde{\alpha}_2^{3/4}}.$$

Conjecture 7.5. *The Grönewold–Moyal quantization of $\mathcal{A}^*$ generated by elements (7.5) can be lifted to $\mathcal{A}$.*

This conjectured is fulfilled at least for the second term $\mathrm{Q}_2\left(a, b\right)$.

Other groups. The above method works for the conjugate action of the orthogonal group $\mathbf{O}\left(2\right)$ on the space of pairs of real symmetric 2×2 matrices as well for action of the unitary group $\mathbf{SU}\left(2\right)$ on the space of pairs of Hermitian 2×2 matrices. The algebra of invariants is generated by the same five symmetric polynomials. This bracket can be quantized in a similar way.

8. K3 surfaces

K3 surfaces are topologically trivial Calabi–Yau 2-manifolds. Any nonsingular variety X_f given in $\mathbb{CP}^3$ by an equation $f = 0$ of degree 4 is a K3 surface. The Poisson bracket q_f on $\mathcal{O}\left(\mathbb{CP}^3\right)/\left(f\right)$ is equal to const $x_0^{-1}q_0$ on the chart $X_0 = \{x_0 \neq 0\}$, where

$$q_0\left(a, b\right) = \det\begin{pmatrix} \partial_1 a & \partial_2 a & \partial_3 a \\ \partial_1 b & \partial_2 b & \partial_3 b \\ \partial_1 f & \partial_2 f & \partial_3 f \end{pmatrix}, \quad \partial_i = \partial/\partial x_i, \tag{8.1}$$

and x_0, x_1, x_2, x_3 are arbitrary homogeneous coordinates on $\mathbb{CP}^3$. Theorem 4.1 can be applied to the nonsinular K3 variety X_f where

$$f = \frac{1}{4}\left(-x_0^4 + x_1^4 + x_2^4 + x_3^4\right).$$

The canonical Poisson bracket is given by

$$q_f = x_3^3 \partial_1 \wedge \partial_2 + x_1^3 \partial_2 \wedge \partial_3 + x_2^3 \partial_3 \wedge \partial_1$$

on the chart $X_0 = \{x_0 = 1\}$. We set $\mathrm{a} = \varphi(x_3) x_1$, $\mathrm{b} = \varphi(x_3) x_2$ for an unknown function φ and solve equation

$$q_f(\mathrm{a}, \mathrm{b}) = q_f(\varphi(x_3) x_1, \varphi(x_3) x_2) = 1. \tag{8.2}$$

This equation reads

$$\det \begin{pmatrix} \varphi & 0 & x_1 \varphi' \\ 0 & \varphi & x_2 \varphi' \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix} = x_3^3 \varphi^2 - \left(x_1^4 + x_2^4\right) \varphi \varphi' = 1$$

where $\varphi' = \partial \varphi / \partial x_3$. For $\kappa = \varphi^2$, we get

$$\kappa' = \frac{2x_3^3}{x_1^4 + x_2^4} \kappa - \frac{2}{x_1^4 + x_2^4} = \frac{2x_3^3}{1 - x_3^4} \kappa - \frac{2}{1 - x_3^4}$$

Calculate a solution of this equation

$$\exp\left(2 \int_1^{x_3} \frac{y^3}{1 - y^4} \mathrm{d}y\right) = \exp\left(\frac{1}{2} \int_1^{x_3^4} \frac{\mathrm{d}z}{1 - z}\right) = \left(1 - x_3^4\right)^{-1/2},$$
$$\kappa = -\left(1 - x_3^4\right)^{-1/2} \int_1^{x_3} \frac{\mathrm{d}y}{\left(1 - y^4\right)^{1/2}}$$

This yields

$$\varphi(x_3) = \kappa^{1/2}(z) = (1 - z)^{-1/4} \lambda^{1/2}, \ \lambda(z) \doteqdot \int_z^1 \left(1 - y^4\right)^{-1/2} \mathrm{d}y.$$

The products

$$\mathrm{a}(x) = \frac{x_1}{(1 - z)^{1/4}} \lambda^{1/2}\left(x_3^4\right), \ \mathrm{b}(x) = \frac{x_2}{(1 - z)^{1/4}} \lambda^{1/2}\left(x_3^4\right)$$

belong to extension of the algebra $\mathbb{C}[x_1, x_2, x_3]$ by means of $\kappa^{1/2}$ and fulfil (8.2). The corresponding GM series is generated by the fields

$$\mathrm{A} = \det \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ 0 & \partial_2 \mathrm{b} & \varphi' \mathrm{b} \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix} = \kappa^{1/2}\left(\left(x_3^3 - \varphi' x_2^4\right) \partial_1 + \varphi' x_1^3 x_2 \partial_2 - x_1^3 \partial_3\right)$$
$$\mathrm{B} = \det \begin{pmatrix} \partial_1 \mathrm{a} & 0 & \varphi' \mathrm{a} \\ \partial_1 & \partial_2 & \partial_3 \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix} = \kappa^{1/2}\left(x_1 x_2^3 \varphi' \partial_1 + \left(x_3^3 - x_1^4 \varphi'\right) \partial_2 - x_2^3 \partial_3\right)$$

such that $\mathrm{A} \wedge \mathrm{B} = q_f$.

9. Singular surfaces in $\mathbb{CP}^3$ of degree 4

Few more examples of quantization of singular surfaces of degree 4 are given below.

I. The singular hypersurface of degree $4x_0x_3^3 - x_1^2x_2^2 = 0$ has singularity at four points where both terms $x_0x_3^3$, $x_1^2x_2^2$ vanish. The bracket

$$q = 3x_0x_3^2\partial_1 \wedge \partial_2 - 2x_1x_2^2\partial_2 \wedge \partial_3 - 2x_1^2x_2\partial_3 \wedge \partial_1$$

is quantized on X_0 by the functions

$$\mathrm{a} = \frac{x_2}{x_3\sqrt{x_0}}, \qquad \mathrm{b} = \frac{x_1}{x_3\sqrt{x_0}}.$$

Equation $q_f(\mathrm{a},\mathrm{b}) = 1$ implies that the Hamiltonian fields

$$\mathrm{B} = q_f(\mathrm{a},\cdot) = -\frac{1}{\sqrt{x_0}}\left(x_0x_3\partial_1 + 2x_1x_2^3x_3^{-2}\partial_2 + 2x_1x_2^2x_3^{-1}\partial_3\right),$$

$$\mathrm{A} = q_f(\cdot,\mathrm{b}) = -\frac{1}{\sqrt{x_0}}\left(2x_1^3x_2x_3^{-2}\partial_1 + x_0x_3\partial_2 + 2x_1^2x_2x_3^{-1}\partial_3\right)$$

generate a quantization of GM type.

II. If $f = x_0^2x_3^2 - x_1^2x_2^2$ then

$$q_f = 2x_0^2x_3\partial_1 \wedge \partial_2 + 2x_1x_2^2\partial_2 \wedge \partial_3 + 2x_1^2x_2\partial_3 \wedge \partial_1$$

and have $q_f(\mathrm{a}_1,\mathrm{b}_1) = 1$ if we take

$$\mathrm{a} = \frac{x_1}{2x_0\sqrt{x_3}}, \qquad \mathrm{b} = \frac{x_2}{2x_0\sqrt{x_3}}.$$

III. For $f = x_3^4 - x_1^2x_2^2$ we have $q_f(\mathrm{a},\mathrm{b}) = 1$ for the elements

$$\mathrm{a} = \frac{x_1}{x_3\sqrt{x_3}}, \qquad \mathrm{b} = -\frac{x_2}{x_3\sqrt{x_3}}.$$

10. Convergence of GM series

Proof of Theorem 6.2. Denote

$$\|a\| = \max_{\max|s_i|\le 1} |a(s)|$$

for any polynomial a on $\mathbb{C}^3$. It is easy to check that $\|s_ia\| \le \|a\|$, $i = 1,2,3$, and $\|\partial_i a_m\| \le m\,\|a_m\|$ for any polynomial a_m of degree m. For any operator

$$p(s,D) = \sum p_{ijk}s_i\partial_j\partial_k, \quad p_{ijk} \in \mathbb{C},$$

degree of the polynomial $p(s,D)\,a_m$ is $\le m-1$ and

$$\|p(s,D)\,a_m\| \le \frac{m!}{(m-2)!}\,\|p\|\,\|a_m\|, \quad \|p\| = \sum |p_{ijk}|.$$

For arbitrary i, j, $\mathrm{A}^{2i}\mathrm{B}^{2j}(a_m)$ is a polynomial of degree $m-i-j$ and

$$\|\mathrm{AB}(a_m)\| \le 9^2 \frac{m!\,(m-1)!}{(m-2)!\,(m-3)!}\|a_m\| \le 9^2\left(\frac{m!}{(m-2)!}\right)^2\|a_m\|$$

$$\begin{aligned}\left\|\mathrm{A}^i\mathrm{B}^j(a_m)\right\| &\le 9^{i+j}\frac{m!\,(m-1)!}{(m-i-j)!\,(m-i-j-1)!}\|a_m\| \\ &\le 9^{i+j}\left(\frac{m!}{(m-i-j)!}\right)^2\|a_m\|\end{aligned}$$

since

$$\max\left(\|\mathrm{AB}\|, \left\|\mathrm{A}^2\right\|, \left\|\mathrm{B}^2\right\|\right) \le 9.$$

It follows that for an arbitrary homogeneous polynomial b_n of degree n, and any even k,

$$\begin{aligned}\|\mathrm{Q}_k(a_m, b_n)\| &\le 9^k\left(\sum_{i+j=k/2}\frac{k!}{2i!2j!} + \sum_{i+j=k-1}\frac{(k-1)!}{i!j!}\right) \\ &\quad\times\left(\frac{m!}{(m-i)!}\frac{n!}{(n-j)!}\right)^2\|a_m\|\,\|b_n\| \\ &\le (36)^k\left(\frac{k}{2}!\right)^2 4^{m+n}\|a_m\|\,\|b_n\| \le C\,(18)^k\,k!4^{m+n}\|a_m\|\,\|b_n\|\end{aligned}$$

for $m+n > k/2$. Otherwise $\mathrm{Q}_k(a_m, b_n) = 0$. Similar estimate holds for any odd k, since $\|q\| \le 9$. Let

$$a = \sum a_m, \quad b = \sum b_m \tag{10.1}$$

be series of homogeneous polynomials a_m, b_m. We assume that both series converge on the ball of radius r which implies

$$\max\{\|a_m\|, \|b_m\|\} \le C_\varepsilon \varepsilon^m \tag{10.2}$$

for arbitrary $\varepsilon > 1/r$ and some constant C_ε that does not depend on m. For any k, by (10.2)

$$|\mathrm{Q}_k(a, b; s)| \le \|\mathrm{Q}_k(a_m, b_n)\|\,|s|^{m+n+k/2}$$

since $\mathrm{Q}_k(a, b; s)$ is a homogeneous polynomial in s. Finally obtain the inequality

$$\begin{aligned}\sum_k \frac{t^k}{k!}|\mathrm{Q}_k(a, b; s)| &\le C'_\varepsilon \sum_k (18\,|t|)^k\,|s|^{-k/2}\sum_{m+n\ge k/2}(4\varepsilon\,|s|)^{m+n} \\ &\le \frac{C'_\varepsilon}{1-4\varepsilon\,|s|}\sum_{k\ge 0}\left(18\,|t|\,|s|^{-1/2}\right)^k(4\varepsilon\,|s|)^{k/2} \\ &= \frac{C'_\varepsilon}{1-(4\varepsilon\,|s|)\left(1-18\varepsilon^{1/2}\,|t|\right)},\end{aligned}$$

where $|s| < 1/4\varepsilon$ and for $|t| < \varepsilon^{-1/2}/18$. The estimate implies that the series converges for any s and t such that $|s| < r/4$ and $|t| < r^{1/2}/18$. □

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Received March 6, 2022, revised October 10, 2022.

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Алгебраїчна симплектична редукція і квантизація сингулярних просторів

Victor Palamodov

Алгебраїчний метод сингулярної редукції застосовано для нерегулярних груп дій на многовидах, які забезпечують сингулярні симплектичні простори. У фокусі проблема квантизації деформації сингулярних просторів. Для деяких прикладів сингулярних просторів Пуассона побудовано ряди Ґроневолда–Мойала та перевірено їх збіжність. Детально розглянуто деякі приклади квантизації деформацій сингулярних просторів Пуассона.

Ключові слова: многовид Пуассона, обмеження, сингулярна симплектична редукція, квантизация деформації, добуток з зірочкою Ґроневолда–Мойала, K3 поверхні