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Finite-Rank Complex Deformations of Random Band Matrices: Sigma-Model Approximation

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We study the distribution of complex eigenvalues $z_1, \dots, z_N$ of random Hermitian $N \times N$ block band matrices with a complex deformation of a finite rank. Assuming that the width of the band W grows faster than $\sqrt{N}$, we proved that the limiting density of $\Im z_1, \dots, \Im z_N$ in a sigma-model approximation coincides with that for the Gaussian Unitary Ensemble. The method follows the techniques of [16].

Key words: random band matrices, delocalized regime, complex deformation, sigma-model, supersymmetry

Mathematical Subject Classification 2020: 60B20

1. Introduction

The complex eigenvalues of non-Hermitian random matrices have attracted much research interest due to their relevance to several branches of theoretical physics, and in particular to the study of scattering chaotic systems. According to the works [18, 20], universal properties of the poles of the scattering matrix $S(E)$ in the complex plane can be modelled by N complex eigenvalues z_n, $\Im z_n \le 0$ of so-called "effective non-Hermitian Hamiltonian"

$$\mathcal{H}_{eff} = H - i\Gamma, \tag{1.1}$$

where H is a random matrix ensemble with an appropriate symmetry (e.g., Hermitian or real symmetric), and Γ is a positive deformation of a rank $M \ll N$. More details of the approach can be found, e.g., in reviews [9,11,14] and references therein.

One of the most interesting questions about the spectral statistics of H_{eff} is the distribution of $\Im z_i$ (i.e. "resonance widths"). In contrast to the classical non-Hermitian models such as Ginibre ensemble (random matrices with iid entries), if M is fixed and $N \to \infty$, matrices $\mathcal{H}_{eff}$ are weakly non-Hermitian, and so $\Im z_i$ are of order of the typical spacing ω between eigenvalues of H, i.e., $O(1/N)$. It is also expected that the spectral fluctuations on the ω-scale is universal, i.e., independent of the particular form of the distribution of H or the energy dependence of ω.

For the case H taken from Gaussian Unitary Ensemble (GUE) the probability density of the scaled $\Im z_i$ was obtained in [6, 10] for any finite M (for some related models see review [11] and references therein). Let us mention also that the cases of non-Hermitian symmetry, and in particular real symmetric case, are much more involved, and is not well-enough studied even for H taken from Gaussian Orthogonal Ensemble (there are only some partial results for $M = 1$, see physical papers [8, 19] for GOE; let us also mention the paper [13] that gives joint probability distribution of z_i for rank-one perturbation of general β-ensembles).

In this paper we consider H to be a one-dimensional Hermitian block band matrix (block RBM). The 1d block RBM are the special class of Wegner's orbital models (see [21]), i.e., Hermitian $N \times N$ matrices H_N with complex zero-mean random Gaussian entries $H_{jk,\alpha\beta}$, where $j, k = 1, \dots, n$ (they parametrize the lattice sites) and $\alpha, \gamma = 1, \dots, W$ (they parametrize the orbitals on each site), $N = nW$, such that

$$\langle H_{j_1k_1,\alpha_1\gamma_1} H_{j_2k_2,\alpha_2\gamma_2} \rangle = \delta_{j_1k_2}\delta_{j_2k_1}\delta_{\alpha_1\gamma_2}\delta_{\gamma_1\alpha_2} J_{j_1k_1} \tag{1.2}$$

with

$$J = 1/W + \tilde{\beta}\Delta/W, \tag{1.3}$$

where $W \gg 1$ and Δ is the discrete Laplacian on $\{1, 2, .., n\}$. The probability law of H_N can be written in the form

$$P_N(dH_N) = \exp\left\{ -\frac{1}{2} \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{W} \frac{|H_{jk,\alpha\gamma}|^2}{J_{jk}} \right\} dH_N. \tag{1.4}$$

The density of states ρ of a general class of RBM with $W \gg 1$ is given by the well-known Wigner semicircle law (see [2, 15]):

$$\rho(E) = (2\pi)^{-1}\sqrt{4 - E^2}, \quad E \in [-2, 2]. \tag{1.5}$$

The main feature of RBM is that their local spectral statistics is conjectured to exhibit the crossover at $W = \sqrt{N}$: for $W \gg \sqrt{N}$ the eigenvectors are expected to be delocalized and the local spectral statistics is governed by the Wigner-Dyson (GUE/GOE) statistics, and for $W \ll \sqrt{N}$ the eigenvectors are localized and the local spectral statistics is Poisson. The conjecture is supported by the physical derivation due to Fyodorov and Mirlin (see [7]) based on supersymmetric formalism, but is not proved in the full generality yet. For the general RBM the delocalization is proved for $W \gg N^{3/4}$ (see the review [3] and references therein). For the more specific Gaussian model (1.2)–(1.3), the Wigner–Dyson local statistics is proved up to the optimal regime $W \gg \sqrt{N}$ first in the so-called sigma-model approximation [16], and then in the full model [17] by the application of the supersymmetric transfer matrix approach.

The main advantage of the SUSY techniques here is that the main spectral characteristics of the model (1.2)–(1.3) such as a density of states, spectral correlation functions, $\mathbb{E}\{|G_{jk}(E + i\varepsilon)|^2\}$, etc. can be expressed via SUSY as the averages of certain observables in nearest-neighbour statistical mechanics models on a box in $\mathbb{Z}$, which allows to combine the SUSY techniques with a transfer matrix approach. However, the rigorous application of the techniques to the main spectral characteristics of RBM is quite difficult due to the complicated structure and non self-adjointness of the corresponding transfer operator. So it is easier to apply it first to the so-called sigma-model approximation, which is often used by physicists to study complicated statistical mechanics systems. In such approximation spins of the statistical mechanics model take values in some symmetric space (± 1 for Ising model, S^1 for the rotator, S^2 for the classical Heisenberg model, etc.). It is expected that sigma-models have all the qualitative physics of more complicated models with the same symmetry. The sigma-model approximation for RBM was introduced by Fyodorov and Mirlin in [7], where it was demonstrated that the corresponding non-linear sigma-model is equivalent, upon the correct identification of parameters, to one studied in the paper [4] (the spins in this model are 4×4 matrices with both complex and Grassmann entries). The rigorous application of the techniques to the correlation functions of (1.2)–(1.3) was developed in [16].

The aim of the current paper is to derive the sigma-model approximation for the limiting density of the imaginary parts of the eigenvalues of H_{eff} of (1.1) with H of (1.2), and, following the techniques of [16], prove that its limiting behavior in the delocalized regime $W \gg \sqrt{N}$ coincides with that for $H = \mathrm{GUE}$.

Define

$$\mathcal{H} = H_N + i\Gamma_M, \tag{1.6}$$

with H_N of (1.2)–(1.3), where Γ_M is a $N \times N$ matrix

$$\Gamma_M = \begin{pmatrix} \gamma_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \gamma_2 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \gamma_M & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{1.7}$$

with some fixed $\gamma_i > 0$ and fixed M. Notice that for convenience we have changed the sign of Γ_M in order to get positive $\Im z_i$.

In order to access the density $\rho(x, y)$ of complex eigenvalues $z_j = x_j + iy_j$ one can use the formula (see [10] and reference therein)

$$\rho_N(x, y) = -\frac{1}{4\pi N} \lim_{\kappa \to 0} \partial^2 \Phi(x, y, \kappa)$$

with

$$\Phi(x, y, \kappa) = -\frac{1}{N} \log\det\Big((\mathcal{H} - x - iy)(\mathcal{H} - x - iy)^* + \frac{\kappa^2}{N^2}\Big)$$

where ∂^2 stands for the two-dimensional Laplacian and a positive parameter $\kappa$is added to regularize the logarithm.

Introduce the generating function

$$Z_{\beta nW}(\kappa, z_1, z_2) = \mathbb{E}\left[\frac{\det\left\{(\mathcal{H}-z_1)(\mathcal{H}-z_1)^* + \dfrac{\kappa^2}{N^2}\right\}}{\det\left\{(\mathcal{H}-z_2)(\mathcal{H}-z_2)^* + \dfrac{\kappa^2}{N^2}\right\}}\right], \tag{1.8}$$

where z_1 and z_2 are auxiliary spectral parameters in the vicinity of $E + iy/N$:

$$z_l = E_l + \frac{iy_l}{N}, \quad E_l = E + \frac{x_l}{N}, \quad l = 1, 2. \tag{1.9}$$

Given $Z_{\beta nW}$, the density can be obtained using the following identity (see [10] and references therein):

$$\rho_N(E, y/N) = \frac{1}{4\pi}\lim_{\kappa\to 0}\left(\frac{\partial}{\partial y_1}\left(\lim_{y_2\to y_1}\frac{\partial Z_{n,W}}{\partial y_2}\right) + \frac{\partial}{\partial x_1}\left(\lim_{x_2\to x_1}\frac{\partial Z_{n,W}}{\partial x_2}\right)\right)\Bigg|_{\substack{y_1=y\\ x_1=0}}$$

Following [16], to derive sigma-model approximation of $Z_{\beta nW}$ for the model(1.2)–(1.3), we take β in (1.3) of order $1/W$, i.e., put

$$J = 1/W + \beta\Delta/W^2. \tag{1.10}$$

The first main result states that in the model (1.10) with fixed β and n, and with $W \to \infty$, the function $Z_{\beta nW}(\kappa, z_1, z_2)$ of (1.8) converges to the value given by the sigma-model approximation. More precisely, we get

Theorem 1.1. *Given $Z_{\beta nW}(\kappa, z_1, z_2)$ of (1.6)–(1.8), and (1.10), any fixed β, n, $\kappa > 0$, z_1, z_2 of (1.9), and $|E| \le \sqrt{2}$, we have, as $W \to \infty$:*

$$Z_{\beta nW}(\kappa, z_1, z_2) \to Z_{\beta n}(\kappa, z_1, z_2),$$

where

$$Z_{\beta n}(\kappa, z_1, z_2) = e^{E(x_1-x_2)}\int \exp\left\{-\frac{\tilde\beta}{4}\sum \operatorname{Str} Q_j Q_{j-1} + \frac{c_0}{2n}\sum \operatorname{Str} Q_j \Lambda_{\kappa,y_1,y_2}\right\}$$
$$\times \prod_{a=1}^{M}\operatorname{Sdet}^{-1}\Big(Q_1 - \frac{iE}{2\pi\rho(E)} + \frac{i\gamma_a}{\pi\rho(E)}\mathcal{L}\Sigma\Big)dQ, \tag{1.11}$$

$\tilde\beta = (2\pi\rho(E))^2\beta$, $c_0 = 2\pi\rho(E)$ with ρ of (1.5), $U_j \in \mathring{U}(2)$, $S_j \in \mathring{U}(1,1)$ (see notation (1.16) below), and Q_j are 4×4 supermatrices with commuting diagonal and anticommuting off-diagonal 2×2 blocks

$$Q_j = \begin{pmatrix} U_j^* & 0\\ 0 & S_j^{-1}\end{pmatrix}\begin{pmatrix}(I + 2\hat\rho_j\hat\tau_j)L & 2\hat\tau_j\\ 2\hat\rho_j & -(I - 2\hat\rho_j\hat\tau_j)L\end{pmatrix}\begin{pmatrix} U_j & 0\\ 0 & S_j\end{pmatrix},$$
$$dQ = \prod dQ_j, \quad dQ_j = (1 - 2n_{j,1}n_{j,2})\, d\rho_{j,1}d\tau_{j,1}\, d\rho_{j,2}d\tau_{j,2}\, dU_j\, dS_j$$

with

$$n_{j,1} = \rho_{j,1}\tau_{j,1}, \qquad n_{j,2} = \rho_{j,2}\tau_{j,2},$$
$$\hat{\rho}_j = \mathrm{diag}\{\rho_{j1}, \rho_{j2}\}, \qquad \hat{\tau}_j = \mathrm{diag}\{\tau_{j1}, \tau_{j2}\}, \qquad L = \mathrm{diag}\{1, -1\}.$$

Here $\rho_{j,l}$, $\tau_{j,l}$, $l = 1, 2$ *are anticommuting Grassmann variables,*

$$\mathrm{Str}\begin{pmatrix} A & \chi \\ \eta & B \end{pmatrix} = \mathrm{Tr}\, B - \mathrm{Tr}\, A, \quad \mathrm{Sdet}\begin{pmatrix} A & \chi \\ \eta & B \end{pmatrix} = \frac{\det(B - \eta A^{-1}\chi)}{\det A}, \tag{1.12}$$

and

$$\Lambda_{\kappa,y_1,y_2} = \begin{pmatrix} \kappa & -iy_1 & 0 & 0 \\ iy_1 & -\kappa & 0 & 0 \\ 0 & 0 & \kappa & -iy_2 \\ 0 & 0 & iy_2 & -\kappa \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Notice that the conjectured crossover $W \sim \sqrt{N}$ for the RBM is equivalent to $\beta \sim n$ in the sigma-model approximation $Z_{\beta n}$ (see, e.g., [10]). The next theorem gives asymptotic behavior of $Z_{\beta n}$ in the delocalized regime $\beta \gg n$ as $n, \beta \to \infty$:

Theorem 1.2. *Given* $Z_{\beta n}(\kappa, z_1, z_2)$ *of* (1.11)*, we have in the limit* $\beta \to \infty$*,* $n \to \infty$ *with* $\beta > n \log^3 n$*:*

$$Z_{\beta n}(\kappa, z_1, z_2) \to e^{E(x_1 - x_2)} \int \frac{\exp\left\{\pi\rho(E)\, \mathrm{Str}\, Q\Lambda_{\kappa,y_1,y_2}\right\}}{\prod_{a=1}^{M} \mathrm{Sdet}\left(Q - \frac{iE}{2\pi\rho(E)} + \frac{i\gamma_a}{\pi\rho(E)}\mathcal{L}\Sigma\right)} dQ, \tag{1.13}$$

which coincides with $Z(\kappa, z_1, z_2)$ *for the GUE. Therefore, the limiting distribution of the imaginary parts of the eigenvalues of* $\mathcal{H}$ *of* (1.6) *with* H_N *of* (1.2)–(1.3) *in the sigma-model approximation coincides with that for* $H_N = GUE$ *obtained in* [10].

We would like to mention also that the localized regime $\beta \ll n$ was studied in the recent physical paper [12].

The paper is organized as follows. We are going to give a detailed proof for the case $M = 1$ and explain some minor correction that should be done to prove the general case. In Section 2 we obtain the SUSY integral representation of $Z_{\beta n W}$ of (1.8). Section 3 is devoted to the derivation of sigma-model approximation, i.e., to the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2 relying on the similar study in [16].

1.1. Notation. We denote by C, C_1, etc. various n, β, W-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,

- $N = Wn$;

- indices i, j, k vary from 1 to n and correspond to the number of block in H_N, index l is always 1 or 2 (this is the field index), and Greek indices α, γ vary from 1 to W and correspond to the position of the element in the block;
- variables ϕ and Φ with different indices are complex variables or vectors correspondingly; if x_j means some variable (vector or matrix) which corresponds to the site $j = 1, .., n$, then x means vector $\{x_j\}_{j=1}^n$, $dx = \prod dx_j$, and dx_j means the product of the differentials which correspond to functionally independent coefficients of x_j;
- variables ψ, Ψ, ρ, and τ with different indices are Grassmann variables or vectors or matrices correspondingly; if ρ_j corresponds to the site $j = 1, .., n$, then ρ means vector $\{\rho_j\}_{j=1}^n$, $d\rho = \prod d\rho_j$, and $d\rho_j$ means the product of the differentials which correspond the components (for vectors) or entries (for matrices) taken into the lexicographic order;
- $$a_\pm = \frac{-iE \pm \sqrt{4 - E^2}}{2}, \quad c_\pm = 1 + a_\pm^{-2}, \quad c_0 = \sqrt{4 - E^2} = 2\pi\rho(E); \tag{1.14}$$
  $$L = \mathrm{diag}\,\{1, -1\}, \quad L_\pm = \mathrm{diag}\,\{a_+, a_-\}; \tag{1.15}$$
- $$\mathring{U}(2) = U(2)/U(1) \times U(1), \quad \mathring{U}(1,1) = U(1,1)/U(1) \times U(1), \tag{1.16}$$
  where $U(p)$ is a group of $p \times p$ unitary matrices, and $U(1,1)$ is a group of 2×2 hyperbolic matrices S such that $S^* L S = L$;
- $$\mathcal{L}_\pm(E) = \Big\{ r\Big(-iE/2 \pm \sqrt{4 - E^2}/2 \Big) | r \in [0, +\infty) \Big\}; \tag{1.17}$$
- $$\tilde{\beta} = c_0^2 \beta; \tag{1.18}$$
- $$Z_1 = E_1 I + \frac{1}{N} \Lambda_{\kappa, y_1}, \quad Z_2 = E_2 I + \frac{1}{N} \Lambda_{\kappa, y_2}, \tag{1.19}$$
  $$\Lambda_{\kappa, y} = \begin{pmatrix} -i\kappa & -y \\ y & i\kappa \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2. Integral representations

In this section we obtain an integral representation for $Z_{\beta n W}(\kappa, z_1, z_2)$ of (1.8).

Proposition 2.1. *The determinant ratio $Z_{\beta n W}(\kappa, z_1, z_2)$ of (1.8) can be written as follows:*

$$Z_{\beta n W}(\kappa, z_1, z_2) = C_{n,W} \int \exp\Big\{ -i \sum_{j=1}^n \mathrm{Tr}(L Y_j + \delta_{j1} \sum_{a=1}^M L Q_a) Z_2 \Big\}$$
$$\times \exp\Big\{ -i \sum_{a=1}^M \gamma_a \,\mathrm{Tr}(L Q_a)\sigma - \frac{1}{2} \sum_{j,k=1}^n (J^{-1})_{jk} \,\mathrm{Tr}\, X_j X_k \Big\}$$
$$\times \exp\Big\{ -\frac{1}{2} \sum_{j,k=1}^n J_{jk} \,\mathrm{Tr}(L Y_j + \delta_{j1} \sum_{a=1}^M L Q_a)(L Y_k + \delta_{k1} \sum_{a=1}^M L Q_a) \Big\} \det \widetilde{\mathcal{D}}$$

$$\times \prod_{j=1}^{n} \frac{\det^W (X_j - iZ_1) \det^W Y_j}{\det^2 Y_j} \prod_{a=1}^{M} \frac{\det(X_1 - iZ_1 + i\gamma_a \sigma)}{\det(X_1 - iZ_1) \det Y_1} \, dX \, dY \, dQ, \qquad (2.1)$$

where

$$\widetilde{\mathcal{D}} = J_{jk}^{-1} \mathbf{1}_4 - \delta_{jk} \Big((X_j - iZ_1)^{-1} \otimes (LY_j) + \delta_{j1} \sum_{a=1}^{M} (X_j - iZ_1 + i\gamma_a \sigma)^{-1} \otimes (LQ_a) \Big),$$

$$Q_a = \begin{pmatrix} \bar\phi_{11a}\phi_{11a} & \bar\phi_{11a}\phi_{21a} \\ \bar\phi_{21a}\phi_{11a} & \bar\phi_{21a}\phi_{21a} \end{pmatrix}, \quad dQ = \prod_{a=1}^{M} \prod_{l=1}^{2} \frac{d\Re\phi_{l1a} \, d\Im\phi_{l1a}}{\pi},$$

for complex ϕ_{l1a}. $\{X_j\}_{j=1}^n$ are Hermitian 2×2 matrices with standard dX_j, $\{Y_j\}_{j=1}^n$ are 2×2 positive Hermitian matrices with dY_j of Proposition 5.1, and $Z_{1,2}$ are defined in (1.19), and

$$C_{n,W} = \frac{\det^2 J (-1)^{nW}}{(2\pi^3)^n \big((W-1)!(W-2)!\big)^{n-1} ((W-M-1)!(W-M-2)!)},$$

Proof. To simplify computation, we are going to present the detailed derivation for the case $M = 1$. General case can be obtained similarly with minor modifications.

To obtain SUSY integral representation, it is useful to rewrite $Z_{\beta nW}$ in the more convenient form. Notice that if we set

$$P := P(E, \kappa, y) = \begin{pmatrix} \frac{\kappa}{N} - i(H_N - E) & -i(\Gamma - \frac{y}{N}) \\ -i(\Gamma - \frac{y}{N}) & \frac{\kappa}{N} + i(H_N - E) \end{pmatrix}, \qquad (2.2)$$

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes I_N,$$

then

$$\det(TPT) = \det \left\{ \left(H_N + i\Gamma - E - \frac{iy}{N} \right) \left(H_N - i\Gamma - E + \frac{iy}{N} \right) + \frac{\kappa^2}{N^2} \right\}.$$

Hence

$$Z_{\beta nW}(\kappa, z_1, z_2) = \mathbb{E} \left[\frac{\det \, P_1}{\det \, P_2} \right],$$

where

$$P_1 = P(E_1, \kappa, y_1), \quad P_2 = P(E_2, \kappa, y_2). \qquad (2.3)$$

Such transformation is needed since we want P_1, P_2 to have positive real part.

Introduce complex and Grassmann fields:

$$\Phi_l = (\{\phi_{lj}\}_{j=1}^n)^t, \quad \phi_{lj} = (\phi_{lj1}, \phi_{lj2}, \ldots, \phi_{ljW}), \quad l = 1, 2, \quad \text{are complex},$$

$$\Psi_l = (\{\psi_{lj}\}_{j=1}^n)^t, \quad \psi_{lj} = (\psi_{lj1}, \psi_{lj2}, \ldots, \psi_{ljW}), \quad l = 1, 2, \quad \text{are Grassmann}.$$

Since P_1, P_2 have positive real part, using (5.3)–(5.4) (see Appendix) we can rewrite det P_1 and det P_2 of (2.2)–(2.3) and get

$$
\begin{aligned}
Z_{\beta nW}(\kappa, z_1, z_2) &= \pi^{-2Wn}\mathbf{E}\Big\{\int \exp\Big\{-\Psi_1^+\Big(\frac{\kappa}{N}+iE_1-iH_N\Big)\Psi_1\Big\}\\
&\quad\times\exp\Big\{-\Psi_2^+\Big(\frac{\kappa}{N}-iE_1+iH_N\Big)\Psi_2-\Phi_1^+\Big(\frac{\kappa}{N}+iE_2-iH_N\Big)\Phi_1\Big\}\\
&\quad\times\exp\Big\{-\Phi_2^+\Big(\frac{\kappa}{N}-iE_2+iH_N\Big)\Phi_2+i\Psi_1^+\Big(\Gamma+\frac{y_1}{N}\Big)\Psi_2\Big\}\\
&\quad\times\exp\Big\{i\Psi_2^+\Big(\Gamma+\frac{y_1}{N}\Big)\Psi_1+i\Phi_1^+\Big(\Gamma+\frac{y_2}{N}\Big)\Phi_2+i\Phi_2^+\Big(\Gamma+\frac{y_2}{N}\Big)\Phi_1\Big\}\,d\Phi\,d\Psi\Big\}\\
&=\int\exp\Big\{-\Big(\frac{\kappa}{N}+iE_1\Big)\Psi_1^+\Psi_1-\Big(\frac{\kappa}{N}-iE_1\Big)\Psi_2^+\Psi_2-\Big(\frac{\kappa}{N}+iE_2\Big)\Phi_1^+\Phi_1\Big\}\\
&\quad\times\exp\Big\{-\Big(\frac{\kappa}{N}-iE_1\Big)\Phi_2^+\Phi_2+\frac{iy_1}{N}\big(\Psi_1^+\Psi_2+\Psi_1^+\Psi_2\big)+\frac{iy_2}{N}\big(\Phi_1^+\Phi_2+\Phi_1^+\Phi_2\big)\Big\}\\
&\quad\times\exp\big\{i\gamma\left(\bar\psi_{111}\psi_{211}+\bar\psi_{211}\psi_{111}+\bar\phi_{111}\phi_{211}+\bar\phi_{211}\phi_{111}\right)\big\}\\
&\quad\times\mathbf{E}\Bigg\{\exp\Bigg\{\sum_{j\le k}\sum_{\alpha,\gamma}\Big(i\Re H_{jk,\alpha\gamma}\chi^+_{jk,\alpha\gamma}-\Im H_{jk,\alpha\gamma}\chi^-_{jk,\alpha\gamma}\Big)\Bigg\}\Bigg\}\,d\Phi d\Psi,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^{\pm}_{jk,\alpha\gamma}&=\eta_{jk,\alpha\gamma}\pm\eta_{kj,\gamma\alpha},\\
\eta_{jk,\alpha\gamma}&=\overline{\psi}_{1j\alpha}\psi_{1k\gamma}-\overline{\psi}_{2j\alpha}\psi_{2k\gamma}+\overline{\phi}_{1j\alpha}\phi_{1k\gamma}-\overline{\phi}_{2j\alpha}\phi_{2k\gamma},\\
\eta_{jj,\alpha\alpha}&=(\overline{\psi}_{1j\alpha}\psi_{1j\alpha}-\overline{\psi}_{2j\alpha}\psi_{2j\alpha}+\overline{\phi}_{1j\alpha}\phi_{1j\alpha}-\overline{\phi}_{2j\alpha}\phi_{2j\alpha})/2.
\end{aligned}
$$

Averaging over (1.4), we get

$$
\begin{aligned}
Z_{\beta nW}(\kappa, z_1, z_2) &= \pi^{-2Wn}\int d\Phi\,d\Psi\,\exp\Big\{\frac{iy_1}{N}\Big(\Psi_1^+\Psi_2+\Psi_1^+\Psi_2\Big)\Big\}\\
&\quad\times\exp\Big\{\frac{iy_2}{N}\Big(\Phi_1^+\Phi_2+\Phi_1^+\Phi_2\Big)-\Big(\frac{\kappa}{N}+iE_1\Big)\Psi_1^+\Psi_1\Big\}\\
&\quad\times\exp\Big\{-\Big(\frac{\kappa}{N}-iE_1\Big)\Psi_2^+\Psi_2-\Big(\frac{\kappa}{N}+iE_2\Big)\Phi_1^+\Phi_1-\Big(\frac{\kappa}{N}-iE_1\Big)\Phi_2^+\Phi_2\Big\}\\
&\quad\times\exp\big\{i\gamma(\bar\psi_{111}\psi_{211}+\bar\psi_{211}\psi_{111}+\bar\phi_{111}\phi_{211}+\bar\phi_{211}\phi_{111})\big\}\\
&\quad\times\exp\Bigg\{-\sum_{j<k}\sum_{\alpha,\gamma}J_{jk}\,\eta_{jk,\alpha\gamma}\eta_{kj,\gamma\alpha}-\frac12\sum_{j,\alpha}J_{jj}\,\eta^2_{jj,\alpha\alpha}\Bigg\}.
\end{aligned}
$$

Define

$$
Q=\begin{pmatrix}\bar\phi_{111}\phi_{111} & \bar\phi_{111}\phi_{211}\\ \bar\phi_{211}\phi_{111} & \bar\phi_{211}\phi_{211}\end{pmatrix}
$$

and set

$$
\tilde Y_j=\begin{pmatrix}\phi_{1j}^+\phi_{1j} & \phi_{1j}^+\phi_{2j}\\ \phi_{2j}^+\phi_{1j} & \phi_{2j}^+\phi_{2j}\end{pmatrix},\quad j\neq 1,\qquad \tilde Y_1=\begin{pmatrix}\sum\limits_{\alpha=2}^{W}\bar\phi_{11\alpha}\phi_{11\alpha} & \sum\limits_{\alpha=2}^{W}\bar\phi_{11\alpha}\phi_{21\alpha}\\ \sum\limits_{\alpha=2}^{W}\bar\phi_{21\alpha}\phi_{11\alpha} & \sum\limits_{\alpha=2}^{W}\bar\phi_{21\alpha}\phi_{21\alpha}\end{pmatrix}
$$

$$\tilde{X}_j = \begin{pmatrix} \psi_{1j}^{+}\psi_{1j} & \psi_{1j}^{+}\psi_{2j} \\ \psi_{2j}^{+}\psi_{1j} & \psi_{2j}^{+}\psi_{2j} \end{pmatrix}.$$

Thus,

$$\begin{aligned} Z_{\beta nW}(\kappa, z_1, z_2) = \pi^{-2Wn} \int d\Phi\, d\Psi\, \exp\left\{ i\gamma(\bar{\psi}_{111}\psi_{211} + \bar{\psi}_{211}\psi_{111}) \right\} \\ \times \exp\left\{ -i\sum_{j=1}^{n} \operatorname{Tr} \tilde{X}_j L Z_1 - i\sum_{j=1}^{n} \operatorname{Tr}\left(L\tilde{Y}_j + \delta_{j1}LQ\right) Z_2 \right\} \\ \times \exp\left\{ -i\gamma \operatorname{Tr}(LQ)\sigma + \frac{1}{2}\sum_{j,k=1}^{n} J_{jk} \operatorname{Tr}\left(\tilde{X}_j L\right)\left(\tilde{X}_k L\right) \right\} \\ \times \exp\left\{ -\frac{1}{2}\sum_{j,k=1}^{n} J_{jk} \operatorname{Tr}\left(L\tilde{Y}_j + \delta_{j1}LQ\right)\left(L\tilde{Y}_k + \delta_{k1}LQ\right) \right\} \\ \times \exp\left\{ -\sum_{j,k=1}^{n} J_{jk}\Big(\overline{\psi}_{1j}\psi_{1k}\left(\overline{\phi}_{1k}\phi_{1j} - \overline{\phi}_{2k}\phi_{2j}\right) \right. \\ \left. + \overline{\psi}_{2j}\psi_{2k}\left(\overline{\phi}_{2k}\phi_{2j} - \overline{\phi}_{1k}\phi_{1j}\right)\Big) \right\}, \quad (2.4) \end{aligned}$$

where L, $Z_{1,2}$, σ are defined in (1.15) and (1.19).

Using the standard Hubbard–Stratonovich transformation, we obtain

$$\begin{aligned} \left(2\pi^2\right)^n \det^2 J \exp\left\{ \frac{1}{2}\sum_{j,k=1}^{n} J_{jk} \operatorname{Tr}(\tilde{X}_j L)(\tilde{X}_k L) \right\} \\ = \int \exp\left\{ -\frac{1}{2}\sum_{j,k=1}^{n} \left(J^{-1}\right)_{jk} \operatorname{Tr} X_j X_k + \sum_{j=1}^{n} \operatorname{Tr} X_j \left(\tilde{X}_j L\right) \right\} dX, \quad (2.5) \end{aligned}$$

where X_j are 2×2 Hermitian matrices with the standard measure dX_j.

Substituting (2.5) to (2.4) and integrating over $d\Psi$ (see (5.4)), we get

$$\begin{aligned} Z(\kappa, z_1, z_2) = \frac{\det^{-2} J}{\left(2\pi^{2(1+W)}\right)^n} \int \exp\left\{ -\frac{1}{2}\sum_{j,k=1}^{n} \left(J^{-1}\right)_{jk} \operatorname{Tr} X_j X_k - i\gamma \operatorname{Tr}(LQ)\sigma \right\} \\ \times \exp\left\{ -i\sum_{j=1}^{n} \operatorname{Tr}\left(L\tilde{Y}_j + \delta_{j1}LQ\right) Z_2 \right\} \det M \\ \times \exp\left\{ -\frac{1}{2}\sum_{j,k=1}^{n} J_{jk} \operatorname{Tr}\left(L\tilde{Y}_j + \delta_{j1}LQ\right)\left(L\tilde{Y}_k + \delta_{k1}LQ\right) \right\} d\Phi\, dX \quad (2.6) \end{aligned}$$

with $M = M^{(1)} - M^{(2)}$. Here $M^{(1)}$ and $M^{(2)}$ are $2Wn \times 2Wn$ matrices with entries

$$M^{(1)}_{lj\alpha,l'k\gamma} = \delta_{jk}\delta_{\alpha\gamma}(C_{j\alpha})_{ll'}L_{ll}, \quad j,k = 1,\dots,n,\ \alpha,\gamma = 1,\dots,W,\ l,l' = 1,2,$$
$$M^{(2)}_{lj\alpha,l'k\gamma} = J_{jk}\delta_{ll'}L_{ll}\sum_{\nu=1}^{2}\phi_{\nu j\alpha}\overline{\phi}_{\nu k\gamma}L_{\nu\nu} \tag{2.7}$$

with

$$C_{j\alpha} = \begin{cases} X_1 - iZ_1 + i\gamma\sigma, & j = \alpha = 1 \\ X_j - iZ_1, & \text{otherwise} \end{cases}. \tag{2.8}$$

We can rewrite

$$\det M = \det M^{(1)} \cdot \det\Big(1 - \big(M^{(1)}\big)^{-1}M^{(2)}\Big) =: \det M^{(1)} \det\big(1 - \mathcal{M}\big)$$

with

$$\mathcal{M}_{lj\alpha,l'k\gamma} = J_{jk}(C_{j\alpha})^{-1}_{ll'}\sum_{\nu=1}^{2}\phi_{\nu j\alpha}\overline{\phi}_{\nu k\gamma}L_{\nu\nu}. \tag{2.9}$$

Note that $\mathcal{M} = AB$, where

$$A_{lj\alpha,l'k\sigma} = J_{jk}(C_{j\alpha})^{-1}_{ll'}\,\phi_{\sigma j\alpha}, \quad j,k \in \Lambda,\ \alpha,\gamma = 1,\dots,W,\ l,l',\sigma = 1,2,$$
$$B_{lj\sigma,l'k\alpha} = \delta_{jk}\delta_{ll'}L_{\sigma\sigma}\,\overline{\phi}_{\sigma k\alpha}. \tag{2.10}$$

Therefore, using that $\det(1 - AB) = \det(1 - BA)$, (2.9), and (2.10), we get

$$\det\big(1 - \mathcal{M}\big) = \det\big(1 - BA\big) =: \det\big(1 - \tilde{\mathcal{M}}\big), \tag{2.11}$$

where

$$\tilde{\mathcal{M}}_{lj\sigma,l'k\sigma'} = \sum_{p,\alpha,\nu} B_{lj\sigma,\nu p\alpha}A_{\nu p\alpha,l'k\sigma'} = J_{jk}\sum_{\alpha=1}^{W}(C_{j\alpha})^{-1}_{ll'}\overline{\phi}_{\sigma j\alpha}\phi_{\sigma' j\alpha}L_{\sigma\sigma}$$
$$= \begin{cases} J_{jk}\big(X_j - iZ_1\big)^{-1}_{ll'}\big(L\tilde{Y}_j\big)_{\sigma\sigma'}, & j > 1 \\ J_{1k}\big(X_1 - iZ_1\big)^{-1}_{ll'}\big(L\tilde{Y}_1\big)_{\sigma\sigma'} + J_{1k}\big(X_1 - iZ_1 + i\gamma\sigma\big)^{-1}_{ll'}\big(LQ\big)_{\sigma\sigma'}, & j = 1 \end{cases}.$$

Here we substituted (2.8).

This yields

$$\det\big(1 - \tilde{\mathcal{M}}\big) = \det\big\{\delta_{jk} - J_{jk}D_j\big\} = \det{}^4 J \cdot \det\big\{J^{-1}_{jk}\mathbf{1}_4 - \delta_{jk}D_j\big\}$$

with

$$D_j = (X_j - iZ_1)^{-1} \otimes (L\tilde{Y}_j) + \delta_{j1}\,(X_1 - iZ_1 + i\gamma\sigma)^{-1} \otimes (LQ).$$

Besides,

$$\det\, M^{(1)} = (-1)^{nW}\frac{\det(X_1 - iZ_1 + i\gamma\sigma)}{\det(X_1 - iZ_1)}\prod_{j=1}^{n}\det{}^W(X_j - iZ_1). \tag{2.12}$$

Now substituting (2.7)–(2.9) and (2.11)–(2.12) to (2.6) and applying the bosonization formula (see Proposition 5.1), we obtain (2.1) which finishes the proof for the case $M = 1$. □

3. Derivation of the sigma-model approximation

3.1. Proof of Theorem 1.1. Again we are going to concentrate on the case $M = 1$.

Let β and n be fixed, and $W \to \infty$. Defining $n \times n$ matrix R as

$$J^{-1} = W\left(1 - \frac{\beta}{W}\triangle + \frac{\beta^2}{W^2}\triangle^2 - \cdots\right) =: W\left(1 - \frac{\beta}{W}\triangle + \frac{1}{W^2}R\right),$$

putting $B_j = W^{-1}LY_j$, and shifting $X_j - iZ_1 \to X_j$, we can rewrite (2.1) of Proposition 2.1 as

$$\begin{aligned}
Z_{\beta n W}(\kappa, z_1, z_2) &= C^{(1)}_{W,n} \int dX\, dB\, dQ\, \det D \prod_{j=1}^{n} \frac{\det^W X_j \det^W B_j}{\det^2 B_j} \\
&\times \exp\left\{ -\operatorname{Tr}(LQ)(iZ_2 + i\gamma\sigma) - \frac{W}{2}\sum_{j=1}^{n}\Big(\operatorname{Tr}(B_j + iZ_2)^2 + \operatorname{Tr}(X_j + iZ_1)^2\Big)\right\} \\
&\times \exp\left\{ -\frac{1}{2W}\operatorname{Tr}(LQ)^2 - \operatorname{Tr} B_1(LQ) + \frac{\beta}{W}\operatorname{Tr}(B_1 - B_2)(LQ)\right\} \\
&\times \exp\left\{ \frac{\beta}{2}\sum_{j=1}^{n-1}\Big(\operatorname{Tr}(B_j - B_{j+1})^2 - \operatorname{Tr}(X_j - X_{j+1})^2\Big)\right\} \frac{\det(X_1 + i\gamma\sigma)}{\det X_1 \det B_1} \\
&\times \exp\left\{ \frac{1}{2W}\sum_{j,k} R_{jk} \operatorname{Tr}(X_j + iZ_1)(X_k + iZ_1) + \frac{\beta}{2W^2}\operatorname{Tr}(LQ)^2\right\}, \qquad (3.1)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D} = \Bigg\{ &\left(1 - \frac{\beta}{W}\triangle + \frac{1}{W^2}R\right)_{jk} \mathbf{1}_4 \\
&- \delta_{jk}\left(X_j^{-1} \otimes B_j + \frac{\delta_{j1}}{W}(X_1 + i\gamma\sigma)^{-1} \otimes (LQ)\right)\Bigg\}_{j,k=1}^{n}
\end{aligned}$$

and

$$\begin{aligned}
C^{(1)}_{W,n} &= \frac{\det^2 J\, W^{8n}\, W^{2(W-2)(n-1)}\, W^{2(W-3)}\, e^{-Wn \operatorname{Tr} Z_2^2/2}}{(2\pi^3)^n\big((W-1)!(W-2)!\big)^{n-1}((W-2)!(W-3)!)} \\
&= \frac{W^{4n}\, e^{2nW - Wn \operatorname{Tr} Z_2^2/2}}{(2\pi^2)^{2n}}\Big(1 + O\big(W^{-1}\big)\Big).
\end{aligned}$$

Change the variables to

$$\begin{aligned}
&X_j = U_j^* \hat{X}_j U_j, \quad \hat{X}_j = \operatorname{diag}\{x_{j,1}, x_{j,2}\}, \quad U_j \in \mathring{U}(2), \qquad x_{j,1}, x_{j,2} \in \mathbb{R}, \\
&B_j = S_j^{-1} \hat{B}_j S_j, \quad \hat{B}_j = \operatorname{diag}\{b_{j,1}, b_{j,2}\}, \quad S_j \in \mathring{U}(1,1), \quad b_{j,1} \in \mathbb{R}^+,\ b_{j,2} \in \mathbb{R}^-.
\end{aligned}$$

The Jacobian of such a change is

$$2^n(\pi/2)^{2n}\prod_{j=1}^{n}(x_{j,1}-x_{j,2})^2\prod_{j=1}^{n}(b_{j,1}-b_{j,2})^2.$$

This and (3.1) yield

$$Z_{\beta nW}(\kappa,z_1,z_2)=C^{(2)}_{W,n}\int dS\,dU\,dQ\int dx\int_{\mathbb{R}^n_+\times\mathbb{R}^n_-}db\,\det\mathcal{D} \tag{3.2}$$

$$\times\prod_{j=1}^{n}\frac{(x_{j,1}-x_{j,2})^2(b_{j,1}-b_{j,2})^2}{b_{j,1}^2b_{j,2}^2}\,\frac{\det(\hat{X}_1+i\gamma U_1\sigma U_1^*)}{x_{1,1}x_{1,2}\cdot b_{1,1}b_{1,2}}$$

$$\times\exp\Big\{-W\sum_{j=1}^{n}\sum_{l=1}^{2}\big(f(x_{j,l})+f(b_{j,l})\big)\Big\}$$

$$\times\exp\Big\{-\operatorname{Tr}(LQ)(iZ_2+i\gamma\sigma)-\operatorname{Tr}S_1^{-1}\hat{B}_1S_1(LQ)+\frac{\beta}{2W^2}\operatorname{Tr}(LQ)^2\Big\} \tag{3.3}$$

$$\times\exp\Big\{-\frac{1}{2W}\operatorname{Tr}(LQ)^2+\frac{\beta}{W}\operatorname{Tr}\Big(S_1^{-1}\hat{B}_1S_1-S_2^{-1}\hat{B}_2S_2\Big)(LQ)\Big\}$$

$$\times\exp\Big\{\frac{\beta}{2}\sum_{j=2}^{n}\operatorname{Tr}\Big(S_j^{-1}\hat{B}_jS_j-S_{j-1}^{-1}\hat{B}_{j-1}S_{j-1}\Big)^2\Big\} \tag{3.4}$$

$$\times\exp\Big\{-\frac{\beta}{2}\sum_{j=2}^{n}\operatorname{Tr}\Big(U_j^*\hat{X}_jU_j-U_{j-1}^*\hat{X}_{j-1}U_{j-1})^2\Big)\Big\}$$

$$\times\exp\Big\{\frac{1}{2W}\sum_{j,k}R_{jk}\,\operatorname{Tr}\Big(U_j^*\hat{X}_jU_j+iZ_1\Big)\Big(U_k^*\hat{X}_kU_k+iZ_1\Big)\Big\}$$

$$\times\exp\Big\{-\frac{1}{n}\sum_{j=1}^{n}\Big(\operatorname{Tr}U_j^*\hat{X}_jU_j\Lambda_1+\operatorname{Tr}S_j^{-1}\hat{B}_jS_j\Lambda_2\Big)\Big\}, \tag{3.5}$$

where

$$\det\mathcal{D}=\det\Big\{\delta_{jk}\Big(\mathbf{1}_4-\hat{X}_j^{-1}\otimes\hat{B}_j\Big)-\frac{\delta_{jk}\delta_{j1}}{W}\Big(\hat{X}_1+i\gamma U_1\sigma U_1^*\Big)^{-1}\otimes\big(S_1(LQ)S_1^{-1}\big)$$

$$+\frac{1}{W}\Big(-\beta\triangle+\frac{1}{W}R\Big)_{jk}U_jU_k^*\otimes S_jS_k^{-1}\Big\}_{j,k=1}^{n}, \tag{3.6}$$

$$\Lambda_l=\begin{pmatrix}\kappa+ix_l & -iy_l\\ iy_l & -\kappa+ix_l\end{pmatrix},\quad l=1,2,$$

and

$$C^{(2)}_{W,n}=2^n(\pi/2)^{2n}\,e^{Wn(\operatorname{Tr}Z_1^2+\operatorname{Tr}Z_2^2)/2-Wn(2+E^2)}\,C^{(1)}_{W,n}$$
$$=\frac{W^{4n}e^{2Ex_1}}{2^{3n}\pi^{2n}}\Big(1+O(W^{-1})\Big),$$

$$f(x) = x^2/2 + iEx - \log x - (2 + E^2)/4.$$

The constant in $f(x)$ is chosen in such a way that $\Re f(a_\pm) = 0$. Measures dU_j, dS_j in (3.2) are the Haar measures over $\mathring{U}(2)$ and $\mathring{U}(1,1)$ correspondingly.

Also it is easy to see that for $|E| \le \sqrt{2}$ we can deform the contours of integration as

- for $x_{j,1}$, $x_{j,2}$ to $-iE/2 + \mathbb{R}$;
- for $b_{j,1}$ to $\mathcal{L}_+(E)$ of (1.17);
- for $b_{j,2}$ to $\mathcal{L}_-(E)$ of (1.17).

To prove Theorem 1.1, we are going to integrate (3.2) over the "fast" variables: $\{x_{j,l}\}, \{b_{j,l}\}$, $l = 1, 2$, $j = 1, .., n$. The first step is the following lemma:

Lemma 3.1. *The integral* (3.2) *over* $\{x_{j,l}\}, \{b_{j,l}\}$, $l = 1, 2$, $j = 1, \dots, n$ *can be restricted to the integral over the* $W^{-(1-\kappa)/2}$*-neighbourhoods* (*with a small* $\kappa >$ 0) *of the points*

I. $x_{j,1} = a_+$, $x_{j,2} = a_-$ *or* $x_{j,1} = a_-$, $x_{j,2} = a_+$, $b_{j,1} = a_+$, $b_{j,2} = a_-$ *for any* $j = 1, \dots, n$;

II. $x_{j,1} = x_{j,2} = a_+$, $b_{j,1} = a_+$, $b_{j,2} = a_-$ *for any* $j = 1, \dots, n$;

III. $x_{j,1} = x_{j,2} = a_-$, $b_{j,1} = a_+$, $b_{j,2} = a_-$ *for any* $j = 1, \dots, n$.

Moreover, the contributions of the points II *and* III *are* $o(1)$, *as* $W \to \infty$.

Proof. The proof of the first part of the lemma is straightforward and based on the fact that $\Re f(z)$ for $z = x - iE/2$, $x \in \mathbb{R}$ has two global minimums at $z = a_\pm$, and for $z \in \mathcal{L}_\pm(E)$ has one global minimum at $z = a_\pm$.

To prove the second part of the lemma, consider the neighbourhood of the point II (the point III can be treated in a similar way). Change the variables as

$$x_{j,1} = a_+ + \tilde{x}_{j,1}/\sqrt{W}, \qquad x_{j,2} = a_+ + \tilde{x}_{j,2}/\sqrt{W},$$
$$b_{j,1} = a_+\big(1 + \tilde{b}_{j,1}/\sqrt{W}\big), \qquad b_{j,2} = a_-\big(1 + \tilde{b}_{j,2}/\sqrt{W}\big).$$

This gives the Jacobian $(-1)^n W^{-2n}$ and also the additional W^{-n} since

$$x_{j,1} - x_{j,2} = (\tilde{x}_{j,1} - \tilde{x}_{j,2})/\sqrt{W}.$$

Together with $C^{(2)}_{W,n}$ this gives W^n in front of the integral (3.2). In addition, expanding f into the series, we get

$$f(x_{j,l}) = f(a_+) + \frac{c_+}{2}\frac{\tilde{x}^2_{j,l}}{W} - \frac{1}{2a_+^3}\frac{\tilde{x}^3_{j,l}}{W^{3/2}} + O\left(\frac{\tilde{x}^4_{j,l}}{W^2}\right), \qquad l = 1, 2, \tag{3.7}$$
$$f(b_{j,1}) = f(a_+) + \frac{a_+^2 c_+}{2}\frac{\tilde{b}^2_{j,1}}{W} - \frac{1}{2}\frac{\tilde{b}^3_{j,1}}{W^{3/2}} + O\left(\frac{\tilde{b}^4_{j,1}}{W^2}\right),$$
$$f(b_{j,2}) = f(a_-) + \frac{a_-^2 c_-}{2}\frac{\tilde{b}^2_{j,2}}{W} - \frac{1}{2}\cdot\frac{\tilde{b}^3_{j,2}}{W^{3/2}} + O\left(\frac{\tilde{b}^4_{j,2}}{W^2}\right),$$

where

$$c_\pm = 1 + a_\pm^{-2}, \quad f(a_+) = -f(a_-) \in i\mathbb{R}. \tag{3.8}$$

We are going to compute the leading order of the integral over $\{\tilde{x}_{j,l}\}$, $\{\tilde{b}_{j,l}\}$, $l = 1, 2$, $j = 1, \dots, n$. To this end, we leave the quadratic part of f (see (3.7)) in the exponent, expand everything else into the series of $\tilde{x}_{j,l}/\sqrt{W}$, $\tilde{b}_{j,l}/\sqrt{W}$ around the saddle-point $\tilde{x}_{j,l} = \tilde{b}_{j,l} = 0$, and compute the Gaussian integral of each term of this expansion. We are going to prove that all this terms are $o(1)$.

Indeed, consider the expansion of the diagonal elements of $\mathcal{D}$ of (3.6):

$$\begin{aligned} d_{j,l1} &= 1 - x_{j,l}^{-1} b_{j,1} = (\tilde{x}_{j,l}/a_+ - \tilde{b}_{j,1})/\sqrt{W} + O\big(W^{-1+2\kappa}\big), \\ d_{j,l2} &= 1 - x_{j,l}^{-1} b_{j,2} \\ &= c_- - (\tilde{x}_{j,l}/a_+ - \tilde{b}_{j,2})/a_-^2\sqrt{W} + O\big(W^{-1+2\kappa}\big), \quad l = 1, 2. \end{aligned} \tag{3.9}$$

If we rewrite the determinant of $\mathcal{D}$ in a standard way, then each summand has strictly one element from each row and column. Because of (3.9), each element in the rows $(j, 11)$ and $(j, 21)$ has at least $W^{-1/2}$, and so the expansion of det $\mathcal{D}$ starts from W^{-n}. Moreover, to obtain W^{-n} (i.e., non-zero contribution) we must consider the summands of the determinant expansion that have only diagonal elements $d_{j,ls}$ (since non-diagonal elements of $\mathcal{D}$ are $O(W^{-1})$ or less), and furthermore only the first terms in the expansions (3.9) and all other function in (3.2). Thus we get

$$C \left\langle \prod_{j=1}^{n} \frac{\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,1}}{\sqrt{W}} \, \frac{\tilde{x}_{j,2}/a_+ - \tilde{b}_{j,1}}{\sqrt{W}} \, (\tilde{x}_{j,1} - \tilde{x}_{j,2})^2 \right\rangle_{++} + o(1), \tag{3.10}$$

where

$$\begin{aligned} &\langle \cdot \rangle_{++} \\ &\quad = \int (\,\cdot\,) \exp\Big\{ -\frac{1}{2} \sum_{j=1,\dots,n} \Big(c_+(\tilde{x}_{j,1}^2 + \tilde{x}_{j,2}^2) + a_+^2 c_+ \tilde{b}_{j,1}^2 + a_-^2 c_- \tilde{b}_{j,2}^2 \Big) \Big\} \, d\tilde{x}\, d\tilde{b}. \end{aligned}$$

But it is easy to see that the Gaussian integral in (3.10) is zero, which completes the proof of the lemma. □

According to Lemma 3.1 the main contribution to (3.2) is given by the neighbourhoods of the saddle points $x_{j,1} = a_+$, $x_{j,2} = a_-$ or $x_{j,1} = a_-$, $x_{j,2} = a_+$. All such points can be obtained from each other by rotations of U_j, so we can consider only $x_{j,1} = a_+$, $x_{j,2} = a_-$ for all $j = 1, \dots, n$. Similarly to the proof of Lemma 3.1, change variables as

$$\begin{aligned} x_{j,1} &= a_+ + \tilde{x}_{j,1}/\sqrt{W}, & x_{j,2} &= a_- + \tilde{x}_{j,2}/\sqrt{W}, \\ b_{j,1} &= a_+\big(1 + \tilde{b}_{j,1}/\sqrt{W}\big), & b_{j,2} &= a_-\big(1 + \tilde{b}_{j,2}/\sqrt{W}\big). \end{aligned} \tag{3.11}$$

That slightly change the expansions (3.7) and (3.9). We get

$$f(x_{j,2}) = f(a_-) + \frac{c_-}{2}\frac{\tilde{x}_{j,2}^2}{W} - \frac{1}{2a_-^3}\frac{\tilde{x}_{j,2}^3}{W^{3/2}} + O\left(\frac{\tilde{x}_{j,2}^4}{W^2}\right), \tag{3.12}$$

and

$$\begin{aligned}
d_{j,11} &= 1 - x_{j,1}^{-1}b_{j,1} \\
&= \frac{\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,1}}{\sqrt{W}} + \frac{a_+\tilde{x}_{j,1}\tilde{b}_{j,1} - \tilde{x}_{j,1}^2}{a_+^2 W} + \frac{\delta_{j1}}{W}T_{11,11} + O\left(W^{-3(1-\kappa)/2}\right), \\
d_{j,22} &= 1 - x_{j,2}^{-1}b_{j,2} \\
&= \frac{\tilde{x}_{j,2}/a_- - \tilde{b}_{j,2}}{\sqrt{W}} + \frac{a_-\tilde{x}_{j,2}\tilde{b}_{j,2} - \tilde{x}_{j,2}^2}{a_-^2 W} + \frac{\delta_{j1}}{W}T_{22,22} + O\left(W^{-3(1-\kappa)/2}\right), \\
d_{j,12} &= 1 - x_{j,1}^{-1}b_{j,2} \\
&= c_+ - \frac{\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,2}}{a_+^2\sqrt{W}} - \frac{a_+\tilde{x}_{j,1}\tilde{b}_{j,2} - \tilde{x}_{j,1}^2}{a_+^4 W} + \frac{\delta_{j1}}{W}T_{11,22} + O\left(W^{-3(1-\kappa)/2}\right), \\
d_{j,21} &= 1 - x_{j,2}^{-1}b_{j,1} \\
&= c_- - \frac{\tilde{x}_{j,2}/a_- - \tilde{b}_{j,1}}{a_-^2\sqrt{W}} - \frac{a_-\tilde{x}_{j,2}\tilde{b}_{j,1} - \tilde{x}_{j,2}^2}{a_-^4 W} + \frac{\delta_{j1}}{W}T_{22,11} + O\left(W^{-3(1-\kappa)/2}\right),
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
T &= \left(\hat{X}_1 + i\gamma U_1\sigma U_1^*\right)^{-1} \otimes \left(S_1(LQ)S_1^{-1}\right) \\
&= \left(A^{-1} - \frac{1}{\sqrt{W}}A^{-1}\begin{pmatrix}\tilde{x}_{1,1} & 0 \\ 0 & \tilde{x}_{1,2}\end{pmatrix}A^{-1}\right) \otimes (S_1(LQ)S_1^{-1}) \\
&\qquad + O\left(W^{-1+2\kappa}\right)
\end{aligned} \tag{3.14}$$

with

$$A = \left(\hat{X}_1 + i\gamma U_1\sigma U_1^*\right)\Big|_{\tilde{x}_{1,1}=\tilde{x}_{1,2}=0} = -\frac{iE}{2} + \frac{c_0}{2}L + i\gamma U_1\sigma U_1^*. \tag{3.15}$$

The change (3.11) gives the Jacobian W^{-2n}, which together with $C_{W,n}^{(2)}$ gives W^{2n} in front of the integral (3.2). Similarly to the proof of Lemma 3.1 we are going to compute the leading order of the integral (3.2) over $\{\tilde{x}_{j,l}\}$, $\{\tilde{b}_{j,l}\}$, $l = 1, 2$, $j = 1, \ldots, n$, and so we leave the quadratic part of f (see (3.7) and (3.12)) in the exponent, expand everything else into the series of $\tilde{x}_{j,l}/\sqrt{W}$, $\tilde{b}_{j,l}/\sqrt{W}$ around the saddle-point $\tilde{x}_{j,l} = \tilde{b}_{j,l} = 0$, and compute the Gaussian integral of each term of this expansion. We are going to prove, that the non-zero contribution is given by the terms having at least W^{-2n}.

Lemma 3.2. *Formula* (3.2) *can be rewritten as*

$$
\begin{aligned}
Z_{\beta nW}(\kappa,z_1,z_2) &= (c_0/2\pi)^{2n}\, e^{E(x_1-x_2)}\int dz\, d\tilde\rho\, d\tilde\tau\, dU\, dS\, dQ\\
&\times\exp\Big\{-\frac{1}{2}(Mz,z)+W^{1/2}(z,h^0)+W^{-1/2}(z,h+\zeta/n)\Big\}\\
&\times\exp\Big\{-\operatorname{Tr}(LQ)(iE/2+i\gamma\sigma)-\frac{c_0}{2}\operatorname{Tr}S_1^{-1}LS_1(LQ)\Big\}\\
&\times\exp\big\{-\operatorname{Tr}A^{-1}\tilde\rho_1 S_1(LQ)S_1^{-1}\tilde\tau_1\big\}\det A\\
&\times\exp\Big\{\beta\sum\operatorname{Tr}\Big(U_j^*\tilde\rho_jS_j-U_{j-1}^*\tilde\rho_{j-1}S_{j-1}\Big)\Big(S_j^{-1}\tilde\tau_jU_j-S_{j-1}^{-1}\tilde\tau_{j-1}U_{j-1}\Big)\Big\}\\
&\times\exp\Big\{\sum\big(c_+n_{j,12}+c_-n_{j,21}-n_{j,1}/c_0a_++n_{j,2}/c_0a_-\big)-\beta c_0^2\sum\big(v_j^2+t_j^2\big)\Big\}\\
&\times\exp\left\{-\frac{c_0}{2n}\sum_{j=1}^n\left(\operatorname{Tr}U_j^*LU_j\begin{pmatrix}\kappa & -iy_1\\ iy_1 & -\kappa\end{pmatrix}+\operatorname{Tr}S_j^{-1}LS_j\begin{pmatrix}\kappa & -iy_2\\ iy_2 & -\kappa\end{pmatrix}\right)\right\}\\
&\qquad\qquad + o(1), \qquad (3.16)
\end{aligned}
$$

where A is defined in (3.15),

$$
\tilde\rho_j=\begin{pmatrix}\rho_{j,11} & \rho_{j,12}/\sqrt{W}\\ \rho_{j,21}/\sqrt{W} & \rho_{j,22}\end{pmatrix},\qquad \tilde\tau_j=\begin{pmatrix}\tau_{j,11} & \tau_{j,12}/\sqrt{W}\\ \tau_{j,21}/\sqrt{W} & \tau_{j,22}\end{pmatrix}, \tag{3.17}
$$

$$
n_{j,12}=\rho_{j,12}\tau_{j,12},\quad n_{j,21}=\rho_{j,21}\tau_{j,21},\quad n_{j,1}=\rho_{j,11}\tau_{j,11},\quad n_{j,2}=\rho_{j,22}\tau_{j,22},
$$
$$
z=(z_{j,11},z_{j,22},z_{j,12},z_{j,21})=(\tilde x_{j,1},\tilde x_{j,2},\tilde b_{j,1},\tilde b_{j,1}),
$$

and

$$
M=M_0+W^{-1}\tilde M \tag{3.18}
$$

$$
(M_0z,z)=\sum_{j=1,..,n}\Big(c_+\tilde x_{j,1}^2+c_-\tilde x_{j,2}^2+a_+^2c_+\tilde b_{j,1}^2+a_-^2c_-\tilde b_{j,2}^2\Big) \tag{3.19}
$$

$$
\begin{aligned}
(\tilde Mz,z) &= -2\beta\sum\Big(\tilde x_{j,1}\tilde x_{j-1,1}+\tilde x_{j,2}\tilde x_{j-1,2}-a_+^2\tilde b_{j,1}\tilde b_{j-1,1}-a_-^2\tilde b_{j,2}\tilde b_{j-1,2}\Big)\\
&\quad+2\beta\sum v_j^2\,(\tilde x_{j,1}-\tilde x_{j,2})\,(\tilde x_{j-1,1}-\tilde x_{j-1,2})\\
&\quad+2\beta\sum t_j^2\Big(a_+\tilde b_{j,1}-a_-\tilde b_{j,2}\Big)\Big(a_+\tilde b_{j-1,1}-a_-\tilde b_{j-1,2}\Big)\\
&\quad-\sum\Big(\frac{4}{c_0^2}(\tilde x_{j,1}\tilde x_{j,2}-\tilde b_{j,1}\tilde b_{j,2})-2(a_+^{-3}n_{j,12}\tilde x_{j,1}\tilde b_{j,2}+a_-^{-3}n_{j,21}\tilde x_{j,2}\tilde b_{j,1})\Big)\\
&\quad+\operatorname{Tr}A^{-1}\begin{pmatrix}\tilde x_{1,1} & 0\\ 0 & \tilde x_{1,2}\end{pmatrix}A^{-1}\begin{pmatrix}\tilde x_{1,1} & 0\\ 0 & \tilde x_{1,2}\end{pmatrix}.
\end{aligned} \tag{3.20}
$$

Here $\zeta=\{\zeta_j\}_{j=1,\dots,n}$, $\zeta_j=(\zeta_{j,11},\zeta_{j,22},a_+\zeta_{j,12},a_-\zeta_{j,21})$ with

$$
\zeta_{j,11}=-\left(U_j\begin{pmatrix}\kappa & -iy_1\\ iy_1 & -\kappa\end{pmatrix}U_j^*\right)_{11},\qquad \zeta_{j,22}=-\left(U_j\begin{pmatrix}\kappa & -iy_1\\ iy_1 & -\kappa\end{pmatrix}U_j^*\right)_{22},
$$
$$
\zeta_{j,12}=-\left(S_j\begin{pmatrix}\kappa & -iy_2\\ iy_2 & -\kappa\end{pmatrix}S_j^{-1}\right)_{11},\qquad \zeta_{j,21}=-\left(S_j\begin{pmatrix}\kappa & -iy_2\\ iy_2 & -\kappa\end{pmatrix}S_j^{-1}\right)_{22}.
$$

We also denoted $h=\{h_{j,ls}+h^q_{j,ls}\}_{j=1,..,n,l,s=1,2}$, $h^0=\{h^0_{j,ls}\}_{j=1,..,n,l,s=1,2}$ *with*

$$\begin{aligned}
&h_{j,11}=2/c_0-\beta c_0v_j^2-\beta c_0v_{j+1}^2+a_-n_{j,12}/a_+^2,\\
&h_{j,22}=-2/c_0+\beta c_0v_j^2+\beta c_0v_{j+1}^2+a_+n_{j,21}/a_-^2,\\
&h_{j,12}=2a_+/c_0-2-\beta c_0a_+t_j^2-\beta c_0a_+t_{j+1}^2-n_{j,21}a_+/a_-,\\
&h_{j,21}=-2a_-/c_0-2+\beta c_0a_-t_j^2+\beta c_0a_-t_{j+1}^2-n_{j,12}a_-/a_+,\\
&h^0_{j,11}=n_{j,1}/a_+,\quad h^0_{j,22}=n_{j,2}/a_-,\quad h^0_{j12}=-n_{j,1},\quad h^0_{j,21}=-n_{j,2}
\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}
&h^q_{j,ls}=0,\quad j\ne 1,\\
&h^q_{1,11}=-\frac{1}{a_+}+(A^{-1})_{11}+\big(A^{-1}\tilde\rho_1S_1(LQ)S_1^{-1}\tilde\tau_1A^{-1}\big)_{11},\\
&h^q_{1,12}=-\frac{1}{a_-}+(A^{-1})_{22}+\big(A^{-1}\tilde\rho_1S_1(LQ)S_1^{-1}\tilde\tau_1A^{-1}\big)_{22},\\
&h^q_{1,21}=-1-a_+\big(S_1(LQ)S_1^{-1}\big)_{11},\\
&h^q_{1,22}=-1-a_-\big(S_1(LQ)S_1^{-1}\big)_{22}.
\end{aligned}$$

We also set

$$v_j=|(U_jU^*_{j-1})_{12}|\quad t_j=|(S_jS^{-1}_{j-1})_{12}|.$$

Proof. Rewriting the determinant in (3.6) in a standard way, we obtain

$$\det\mathcal{D}=\sum_{\bar\sigma}(-1)^{|\sigma|}\prod_{j=1}^nP_{j,\bar\sigma_j}(\tilde x_{j,1},\tilde x_{2j},\tilde b_{j,1},\tilde b_{j,1}),\tag{3.22}$$

where $\bar\sigma$ is a permutation of $\{(j,ls)\}$, $l,s=1,2$, $j=1,\dots,n$, $\bar\sigma_j$ is its restriction on $\{(j,ls)\}^2_{l,s=1}$, $(-1)^{|\sigma|}$ is a sign of σ and $P_{j,\bar\sigma_j}$ is an expansion in $\tilde x_{j,1}$, $\tilde x_{2j}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$ of the product of four elements from the rows $\{(j,ls)\}^2_{l,s=1}$ taken with respect to $\bar\sigma_j$.

Let us prove that for each $j=1,\dots,n$ and any $\bar\sigma$ each term of $P_{j,\bar\sigma_j}(\tilde x_{j,1},\tilde x_{2j},\tilde b_{j,1},\tilde b_{j,1})$ of (3.22) belongs to one of the three following groups:

(i) has a coefficient W^{-2} or lower;

(ii) has a coefficient $W^{-3/2}$ and at least one of variables $\tilde x_{j,1}$, $\tilde x_{2j}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$ of the odd degree;

(iii) has a coefficient W^{-1} and at least two variables of $\tilde x_{j,1}$, $\tilde x_{2j}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$ of the odd degree;

Note that each element in the expansion of the coefficients of the rows $(j,11)$ and $(j,22)$ has a coefficient $W^{-1/2}$ or lower, and so $P_{j,\bar\sigma_j}(\tilde x_{j,1},\tilde x_{2j},\tilde b_{j,1},\tilde b_{j,1})$ has a coefficient W^{-1} or lower. In addition, if $P_{j,\bar\sigma_j}(\tilde x_{j,1},\tilde x_{j,2},\tilde b_{j,1},\tilde b_{j,1})$ contains any terms with R_{jk} (see (3.6)), or at least one off-diagonal elements in $(j,12)$ and $(j,21)$, we get a coefficient W^{-2} or lower (and so obtain the group (i)).

We are left to consider terms with $d_{j,12}d_{j,21}$. Consider first $j>1$. If $P_{j,\bar\sigma_j}(\tilde x_{j,1}, \tilde x_{j,2}, \tilde b_{j,1}, \tilde b_{j,1})$ contains two off-diagonal elements in rows $(j,11)$ and $(j,22)$, we get group (i). One off-diagonal element and $d_{j,11}$ (or $d_{j,22}$) gives group (ii) or group (i) (since off-diagonal elements do not depend on $\tilde x_{j,1}$, $\tilde x_{j,2}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$), and it is easy to see from (3.13) that all the terms in expansion of $d_{j,11}d_{j,22}d_{j,12}d_{j,21}$ belongs to groups (i)–(iii). For $j=1$ everything will be similar since the zero order term of T of (3.14) (which gives contribution to the W^{-1} order of elements) does not depend on $\tilde x_{j,1}$, $\tilde x_{j,2}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$, and the next orders contribute to the orders $W^{-3/2}$ or smaller.

To get a non-zero contribution, we have to complete the expression $P_{j,\bar\sigma_j}(\tilde x_{j,1}, \tilde x_{j,2}, \tilde b_{j,1}, \tilde b_{j,1})$ by some other terms of the expansion of the exponent of (3.2) in order to get an even degree of each variable $\tilde x_{j,1}$, $\tilde x_{j,2}$, $\tilde b_{j,1}$, $\tilde b_{j,1}$. But all such a terms have the coefficient $W^{-1/2}$ or lower, and therefore Lemma 3.2 yields that the coefficient near each j in terms that gives a non-zero contribution must be W^{-2} or lower. Since we have a coefficient W^{2n} in (3.2) after the change (3.11), this means that to get a non-zero contribution each coefficient must be exactly W^{-2}. Note that the terms of $P_{j,\bar\sigma_j}(\tilde x_{j,1}, \tilde x_{j,2}, \tilde b_{j,1}, \tilde b_{j,1})$ that can be completed to the monomial with all even degrees and with a coefficients W^{-2} does not contain any terms with R_{jk}, any terms of (3.14) higher than linear in $\tilde x$'s, and any terms of the expansion $d_{j,ls}$, $l,s=1,2$ of order $W^{-3/2}$ or lower (except those that comes from T). They also cannot be completed to the monomial with all even degrees and with a coefficients W^{-2} by any terms of the exponent of (3.2) that has a coefficient lower then $W^{-1/2}$ for some j. Thus we need to consider the terms up to the third order in the expansions (3.7) and (3.12), the linear terms of the functions in the exponents (3.3)–(3.5), the linear terms coming from

$$b_{j,1}^{-2}b_{j,2}^{-2} = \exp\left\{-\frac{2\tilde b_{j,1}}{\sqrt W} - \frac{2\tilde b_{j,2}}{\sqrt W} + O\left(W^{-1}\right)\right\}, \tag{3.23}$$

$$(x_{1,1}x_{1,2}b_{1,1}b_{1,2})^{-1} = \exp\left\{-\frac{\tilde x_{1,1}}{a_+\sqrt W} - \frac{\tilde x_{1,2}}{a_-\sqrt W} - \frac{\tilde b_{1,1}}{\sqrt W} - \frac{\tilde b_{1,2}}{\sqrt W} + O\left(W^{-1}\right)\right\},$$

and no more than quadratic terms in

$$\begin{aligned}&\det\left(\hat X_1 + i\gamma U_1\sigma U_1^*\right)\\ &\quad= \det A\,\exp\left\{\frac{1}{\sqrt W}\operatorname{Tr} A^{-1}\widetilde X_1 - \frac{1}{2W}\operatorname{Tr} A^{-1}\widetilde X_1 A^{-1}\widetilde X_1 + O\left(W^{-3/2}\right)\right\}\end{aligned} \tag{3.24}$$

with

$$\widetilde X_1 = \begin{pmatrix}\tilde x_{1,1} & 0\\ 0 & \tilde x_{1,2}\end{pmatrix}.$$

Note that the terms containing $\tilde x_{j,1}\tilde b_{j,1}/W$ in $d_{j,11}$ (see (3.13)) cannot contribute to the limit, since if we complete them to the monomial with even degrees of $\tilde x_{j,1}, \tilde b_{j,1}$, then it will contain W^{-2} and an additional W^{-1} should come from the line containing $d_{j,22}$. Moreover, the terms containing $\tilde x_{j,1}^2$ in $d_{j,11}$ can give a non-zero contribution only if the resulting monomial contains only $\tilde{x}_{j,1}^2$, since otherwise, taking into account the contribution of the line containing $d_{j,22}$, we again obtain at least W^{-3}. Thus we can replace $\tilde{x}_{j,1}^2$ by its average via Gaussian measure $(2\pi/c_+)^{-1/2}e^{-c_+\tilde{x}_{j,1}^2/2}$, i.e., by c_+^{-1}. The same is true for $\tilde{x}_{j,2}\tilde{b}_{j,2}/W$ and for $\tilde{x}_{j,2}^2$ which could be replaced by c_-^{-1}. Similar argument yields that the contribution of the terms with $\tilde{x}_{j,1}^2$ in the line containing $d_{j,12}$ and $\tilde{x}_{j,2}^2$ in the line containing $d_{j,21}$ disappear in the limit $W \to \infty$. Thus the term corresponding to $W^{2n}\det\mathcal{D}$ in (3.2) can be replaced by the term

$$\begin{aligned}\int d\rho\, d\tau \exp\Big\{ & \sum \Big(c_+n_{j,12} + c_-n_{j,21} - n_{j,1}/c_0a_+ + n_{j,2}/c_0a_-\Big) \\ & + \beta \sum \mathrm{Tr}\Big(U_j^*\tilde{\rho}_jS_j - U_{j-1}^*\tilde{\rho}_{j-1}S_{j-1}\Big)\Big(S_j^{-1}\tilde{\tau}_jU_j - S_{j-1}^{-1}\tilde{\tau}_{j-1}U_{j-1}\Big) \\ & + W^{1/2}\sum\Big((\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,1})n_{j,1} + (\tilde{x}_{j,2}/a_- - \tilde{b}_{j,2})n_{j,2}\Big) \\ & - W^{-1/2}\sum\Big(a_+^{-2}(\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,2})n_{j,12} + a_-^{-2}(\tilde{x}_{j,2}/a_- - \tilde{b}_{j,1})n_{j,21}\Big)\Big\} \\ & \times \exp\Big\{-\mathrm{Tr}\Big(A^{-1} - \tfrac{1}{\sqrt{W}}A^{-1}\begin{pmatrix}\tilde{x}_{1,1} & 0 \\ 0 & \tilde{x}_{1,2}\end{pmatrix}A^{-1}\Big)\tilde{\rho}_1\big(S_1(LQ)S_1^{-1}\big)\tilde{\tau}_1\Big\} \\ & + o(1), \quad (3.25)\end{aligned}$$

where $\tilde{\rho}_j$, $\tilde{\tau}_j$, $n_{j,12}$, $n_{j,21}$, $n_{j,1}$, $n_{j,2}$ are defined in (3.17). Here we have used Grassmann variables $\{\rho_{j,ls}\}$, $\{\tau_{j,ls}\}$, $j = 1,\dots,n$, $l,s = 1,2$ to rewrite the determinant (3.6) with respect to (5.4), have substituted (3.13) and left only terms that give the contribution (according to arguments above), and then have changed $\rho_{j,11} \to \sqrt{W}\rho_{j,11}$, $\tau_{j,11} \to \sqrt{W}\rho_{j,11}$. Note also

$$c_+a_+^2 = c_0a_+, \quad c_-a_-^2 = -c_0a_-. \qquad (3.26)$$

Now let us prove that the contribution of the third order in the expansions (3.7) and (3.12) is small. Indeed, the terms $P_{j,\bar{\sigma}_j}(\tilde{x}_{j,1}, \tilde{x}_{j,2}, \tilde{b}_{j,1}, \tilde{b}_{j,1})$ that can be completed to the monomial with all even degrees and with a coefficients W^{-2} by these cubic terms cannot come from the contribution of T of (3.14) and can be one of two types

(1) terms $(\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,1})x\,c_+c_-$, where c_+, c_- come from the zero terms of $d_{j,12}$, $d_{j,21}$ (see (3.13)) and x is an element of the row $(j,22)$ and so does not depend on $\tilde{x}_{j,1}$, $\tilde{b}_{j,1}$ (or similar terms with $(\tilde{x}_{j,2}/a_- - \tilde{b}_{j,2})$);

(2) terms of $(\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,1})(\tilde{x}_{j,2}/a_- - \tilde{b}_{j,2})(\tilde{x}_{j,1}/a_+ - \tilde{b}_{j,2})c_-$ with $\tilde{x}_{j,1}^2$ or $\tilde{b}_{j,2}^2$ (or similar terms with c_+ coming from $d_{j,12}$);

But it is easy to see that

$$\int (\tilde{x}_{j,1}^4/(3a_+^4) - \tilde{b}_{j,1}^4/3)e^{-\frac{c_+\tilde{x}_{j,1}^2}{2} - \frac{a_+^2c_+\tilde{b}_{j,1}^2}{2}}\, d\tilde{x}_{j,1}\, d\tilde{b}_{j,1} = \frac{2\pi}{a_+c_+}\left(\frac{1}{a_+^4c_+^2} - \frac{1}{a_+^4c_+^2}\right) = 0,$$

and so the contribution of (1) is zero. Similarly the contribution (2) is zero.

Therefore, the contribution of the third order in the expansions (3.7) is small, and using (3.25), (3.23)–(3.24), and also

$$\exp\Big\{-\frac{1}{n}\sum_{j=1}^{n}\Big(\mathrm{Tr}\,U_j^*L_\pm U_j\Lambda_1+\mathrm{Tr}\,S_j^{-1}L_\pm S_j\Lambda_2\Big)\Big\}=\exp\{-E(x_1+x_2)\}$$
$$\times\exp\Big\{-\frac{c_0}{2n}\sum_{j=1}^{n}\Big(\mathrm{Tr}\,U_j^*LU_j\begin{pmatrix}\kappa & -iy_1\\ iy_1 & -\kappa\end{pmatrix}+\mathrm{Tr}\,S_j^{-1}LS_j\begin{pmatrix}\kappa & -iy_2\\ iy_2 & -\kappa\end{pmatrix}\Big)\Big\}$$

for $L_\pm$, L defined in (1.15), we get (3.16). □

Denoting the exponent in the second line of (3.16) by $\mathcal{E}(z)$ and taking the Gaussian integral over dz with z of (3.17), we get

$$\int_{\mathbb{R}^{4n}}\mathcal{E}(z)dz=(2\pi)^{2n}\det{}^{-1/2}M$$
$$\times\exp\Big\{\frac{1}{2}(M^{-1}(W^{1/2}h^0+W^{-1/2}(h+\zeta/n)),W^{1/2}h^0+W^{-1/2}(h+\zeta/n))\Big\}. \tag{3.27}$$

It is easy to see from (3.18)–(3.20) that

$$\begin{aligned}\det M&=\det M_0\left(1+O\left(W^{-1}\right)\right)\\&=(c_+^2c_-^2a_+^2a_-^2)^n\left(1+O\left(W^{-1}\right)\right)=c_0^{4n}\left(1+O\left(W^{-1}\right)\right)\end{aligned}$$

with $c_\pm$ of (3.8). Note now that

$$M^{-1}=\Big(M_0+\frac{1}{W}\tilde M\Big)^{-1}=M_0^{-1}-\frac{1}{W}M_0^{-1}\tilde MM_0^{-1}+O\left(W^{-2}\right).$$

Since M_0 is diagonal and $h^0_{j,ls}$ is proportional to $n_{j,1}$ or $n_{j,2}$ and $n_{j,l}^2=0$, we have

$$\left(M_0^{-1}h^0,h^0\right)=0.$$

Hence, the exponent in the right-hand side of (3.27) takes the form

$$\frac{1}{2}\Big(\left(M_0^{-1}h^0,h+\zeta/n\right)+\left(M_0^{-1}(h+\zeta/n),h^0\right)-\Big(M_0^{-1}\tilde MM_0^{-1}h^0,h^0\Big)\Big)+o(1)=I_1+I_2-I_3+o(1).$$

Then we can rewrite (recall (3.21) and (3.26))

$$I_1+I_2=\sum\Big(\frac{(h_{j,11}+\zeta_{j,11}/n)n_{j,1}}{a_+c_+}+\frac{(h_{j,22}+\zeta_{j,22}/n)n_{j,2}}{a_-c_-}-\frac{(h_{j,12}+a_+\zeta_{j,12}/n)n_{j,1}}{a_+^2c_+}-\frac{(h_{j,21}+a_-\zeta_{j,21}/n)n_{j,2}}{a_-^2c_-}\Big)$$

$$
\begin{aligned}
&+\Big(\frac{h^q_{1,11}n_{j11}}{a_+c_+}+\frac{h_{1,22}n_{1,2}}{a_-c_-}-\frac{h_{1,12}n_{1,1}}{a_+^2c_+}-\frac{h_{1,21}n_{1,2}}{a_-^2c_-}\Big)\\
&=\sum\Big[n_{j,1}\Big(\frac{2}{a_+c_0}+\beta\big(t_j^2+t_{j+1}^2-v_j^2-v_{j+1}^2\big)\\
&\quad+\frac{a_-n_{j,12}}{a_+^2c_0}+\frac{n_{j,21}}{a_-c_0}+\frac{\zeta_{j,11}-\zeta_{j,12}}{c_0n}\Big)\\
&\quad+n_{j,2}\Big(-\frac{2}{a_-c_0}+\beta\big(t_j^2+t_{j+1}^2-v_j^2-v_{j+1}^2\big)\\
&\quad-\frac{a_+n_{j,21}}{a_-^2c_0}-\frac{n_{j,12}}{a_+c_0}-\frac{\zeta_{j,22}-\zeta_{j,21}}{c_0n}\Big)\Big]\\
&\quad+n_{1,1}n_{1,2}\Big(\frac{(A^{-1})_{12}(S_1LQS_1^{-1})_{22}(A^{-1})_{21}}{c_+a_+}\\
&\quad+\frac{(A^{-1})_{21}(S_1LQS_1^{-1})_{11}(A^{-1})_{12}}{c_-a_-}\Big)\\
&\quad+n_{1,1}\frac{(A^{-1})_{11}-(S_1(LQ)S_1^{-1})_{11}}{c_+a_+}\\
&\quad+n_{1,2}\frac{(A^{-1})_{22}-(S_1(LQ)S_1^{-1})_{22}}{c_-a_-}+O(\tfrac{1}{\sqrt{W}});
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
I_3&=\frac{4}{c_0^4}\sum n_{j,1}n_{j,2}-\frac{1}{a_+^2c_0^2}\sum n_{j,12}n_{j,1}n_{j,2}-\frac{1}{a_-^2c_0^2}\sum n_{j,21}n_{j,1}n_{j,2}\\
&\quad+\sum\frac{\beta(v_j^2+t_j^2)}{c_0^2}\big(n_{j,1}n_{j+1,1}+n_{j,1}n_{j+1,2}+n_{j,2}n_{j+1,1}+n_{j,2}n_{j+1,2}\big)\\
&\quad-\frac{1}{c_0^2}(A^{-1})_{12}(A^{-1})_{21}n_{1,1}n_{1,2}+O(W^{-1}).
\end{aligned}
\tag{3.29}
$$

Moreover,

$$
\begin{aligned}
&e^{\beta\sum\mathrm{Tr}\,\big(U_j^*\tilde\rho_jS_j-U_{j-1}^*\tilde\rho_{j-1}S_{j-1}\big)\big(S_j^{-1}\tilde\tau_jU_j-S_{j-1}^{-1}\tilde\tau_{j-1}U_{j-1}\big)}\\
&\quad=e^{\frac{\beta}{W}\sum\mathrm{Tr}\,\big(U_j^*\hat\rho_jS_j-U_{j-1}^*\hat\rho_{j-1}S_{j-1}\big)\big(S_j^{-1}\hat\tau_jU_j-S_{j-1}^{-1}\hat\tau_{j-1}U_{j-1}\big)}+O(W^{-1/2}),
\end{aligned}
\tag{3.30}
$$

where

$$
\hat\rho_j=\mathrm{diag}\{\rho_{j,11},\rho_{j,22}\},\qquad \hat\tau_j=\mathrm{diag}\{\tau_{j,11},\tau_{j,22}\}.
\tag{3.31}
$$

Combining (3.28)–(3.30) we can integrate the main term of (3.27) with respect to $\rho_{j,12}$, $\tau_{j,12}$, $\rho_{j,21}$, $\tau_{j,21}$ according to (5.4). This integration gives

$$
\begin{aligned}
&\prod_{j=1}^n\Big(c_++\frac{a_-n_{j,1}}{a_+^2c_0}-\frac{n_{j,2}}{a_+c_0}+\frac{n_{j,1}n_{j,2}}{a_+^2c_0^2}\Big)\Big(c_-+\frac{n_{j,1}}{a_-c_0}-\frac{a_+n_{j,2}}{a_-^2c_0}+\frac{n_{j,1}n_{j,2}}{a_-^2c_0^2}\Big)\\
&\qquad=c_0^2+\frac{c_0n_{j,2}}{a_-}-\frac{c_0n_{j,1}}{a_+}+\big(1+2/c_0^2\big)n_{j,1}n_{j,2}\\
&\qquad=c_0^2\exp\Big\{-\frac{n_{j,1}}{a_+c_0}+\frac{n_{j,2}}{a_-c_0}\Big\}\Big(1+\frac{2}{c_0^4}n_{j,1}n_{j,2}\Big),
\end{aligned}
$$

which, together with (3.28)–(3.30), yields

$$Z_{\beta nW}(\kappa, z_1, z_2) = c_0^{2n}\, e^{E(x_1-x_2)} \int d\hat\rho\, d\hat\tau\, dU\, dS \prod_{j=1}^{n}\Big(1 - \frac{2}{c_0^4} n_{j,1}n_{j,2}\Big)$$
$$\times \exp\Big\{\beta \sum \mathrm{Tr}\Big(U_j^*\hat\rho_j S_j - U_{j-1}^*\hat\rho_{j-1}S_{j-1}\Big)\Big(S_j^{-1}\hat\tau_j U_j - S_{j-1}^{-1}\hat\tau_{j-1}U_{j-1}\Big)\Big\}$$
$$\times \exp\Big\{\sum n_{j,1}\Big(\beta\big(t_j^2 + t_{j+1}^2 - v_j^2 - v_{j+1}^2\big) + \frac{\zeta_{j,11}-\zeta_{j,12}}{c_0 n}\Big)\Big\}$$
$$\times \exp\Big\{\sum n_{j,2}\Big(\beta\big(t_j^2 + t_{j+1}^2 - v_j^2 - v_{j+1}^2\big) - \frac{\zeta_{j,22}-\zeta_{j,21}}{c_0 n}\Big)\Big\}$$
$$\times \exp\Big\{-\beta c_0^2 \sum (v_j^2 + t_j^2)\Big\} \int F(A, Q, \hat\rho_1, \hat\tau_1, S_1)\, dQ$$
$$\times \exp\Bigg\{-\frac{c_0}{2n}\sum_{j=1}^{n}\Bigg(\mathrm{Tr}\, U_j^* L U_j \begin{pmatrix}\kappa & -iy_1\\ iy_1 & -\kappa\end{pmatrix} + \mathrm{Tr}\, S_j^{-1} L S_j \begin{pmatrix}\kappa & -iy_2\\ iy_2 & -\kappa\end{pmatrix}\Bigg)\Bigg\}$$
$$+ o(1)$$

where we have used $a_+c_+ = c_0$, $a_-c_- = -c_0$, and

$$\big(1 + 2n_{j,1}n_{j,2}/c_0^4\big)\, e^{-4n_{j,1}n_{j,2}/c_0^4} = 1 - 2n_{j,1}n_{j,2}/c_0^4.$$

Here

$$F(A, Q, \hat\rho_1, \hat\tau_1, S_1) = \exp\Big\{ - \mathrm{Tr}(iE/2 + i\gamma\sigma)(LQ) - \frac{c_0}{2}\,\mathrm{Tr}\, S_1^{-1} L S_1 (LQ)\Big\}$$
$$\times \exp\Big\{ - \mathrm{Tr}\, A^{-1}\hat\rho_1 S_1 (LQ) S_1^{-1}\hat\tau_1 + \frac{1}{c_0^2}(A^{-1})_{12}(A^{-1})_{21} n_{1,1}n_{1,2}\Big\}\,\det A$$
$$\times\, e^{\big(n_{1,1}(A^{-1}-S_1(LQ)S_1^{-1})_{11} - n_{1,2}(A^{-1}-S_1(LQ)S_1^{-1})_{22})\big)/c_0}$$
$$\times\, e^{n_{1,1}n_{1,2}\big((A^{-1})_{12}(S_1LQS_1^{-1})_{22}(A^{-1})_{21} - (A^{-1})_{21}(S_1LQS_1^{-1})_{11}(A^{-1})_{12}\big)/c_0}.$$

Notice

$$\exp\Big\{\frac{1}{c_0}\Big(\big(A^{-1}\big)_{11}n_{1,1} - \big(A^{-1}\big)_{22}n_{1,2}\Big) + \frac{1}{c_0^2}\big(A^{-1}\big)_{12}\big(A^{-1}\big)_{21}n_{1,1}n_{1,2}\Big\}\,\det A$$
$$= \det\big(A + \frac{1}{c_0}L\hat\rho_1\hat\tau_1\big),$$

where $\hat\rho_1, \hat\tau_1$ is defined in (3.31). In addition,

$$e^{-\mathrm{Tr}\, A^{-1}\hat\rho_1 S_1(LQ)S_1^{-1}\hat\tau_1 - (S_1(LQ)S_1^{-1})_{11}n_{1,1}/c_0 + (S_1(LQ)S_1^{-1})_{22}n_{1,2}/c_0}$$
$$\times\, e^{n_{1,1}n_{1,2}\big((A^{-1})_{12}(S_1LQS_1^{-1})_{22}(A^{-1})_{21} - (A^{-1})_{21}(S_1LQS_1^{-1})_{11}(A^{-1})_{12}\big)/c_0}$$
$$= \exp\Big\{\mathrm{Tr}\, S_1^{-1}\hat\tau_1\Big(A + \frac{1}{c_0}L\hat\rho_1\hat\tau_1\Big)^{-1}\hat\rho_1 S_1 LQ - \frac{1}{c_0}\,\mathrm{Tr}\, S_1^{-1}L\hat\rho_1\hat\tau_1 S_1(LQ)\Big\}$$

hence we can perform the integration with respect to Q to get

$$\int F(A, Q, \hat\rho_1, \hat\tau_1, S_1)\, dQ$$
$$= \frac{\det\Big(A + \dfrac{1}{c_0} L\hat\rho_1\hat\tau_1\Big)}{\det\Big(iE/2 + i\gamma\sigma + \dfrac{c_0}{2} S_1^{-1} L\Big(1 - \dfrac{2}{c_0^2}\hat\rho_1\hat\tau_1\Big)S_1 - S_1^{-1}\hat\tau_1\Big(A + \dfrac{1}{c_0}L\hat\rho_1\hat\tau_1\Big)^{-1}\hat\rho_1 S_1\Big)}.$$

Using

$$U_1^*\Big(A + \frac{1}{c_0}L\hat\rho_1\hat\tau_1\Big)U_1 = -\frac{iE}{2} + \frac{c_0}{2}U_1^* L\Big(1 + \frac{2}{c_0^2}\hat\rho_1\hat\tau_1\Big)U_1 + i\gamma\sigma,$$

we get finally

$$\int F(A, Q, \hat\rho_1, \hat\tau_1, S_1)\, dQ$$
$$= \mathrm{sdet}^{-1}\begin{pmatrix} U_1^* L(1 + \frac{2}{c_0^2}\hat\rho_1\hat\tau_1)U_1 - \frac{iE - 2i\gamma\sigma}{c_0} & \dfrac{2}{c_0} S_1^{-1}\hat\tau_1 U_1 \\ \dfrac{2}{c_0}U_1^*\hat\rho_1 S_1 & -S_1^{-1}L(1 - \frac{2}{c_0^2}\hat\rho_1\hat\tau_1)S_1 - \frac{iE + 2i\gamma\sigma}{c_0}\end{pmatrix}.$$

Now changing

$$\rho_{j,11} \to c_0\rho_{j,1}, \quad \tau_{j,11} \to c_0\tau_{j,1}, \quad \rho_{j,22} \to c_0\rho_{j,2}, \quad \tau_{j,22} \to c_0\rho_{j,2}$$

with an appropriate change in $n_{j,1}$, $n_{j,2}$, $\hat\rho_j$, $\hat\tau_j$, and recalling (1.18), we get (1.11) which finishes the proof of Theorem 1.1 for $M = 1$. The general case can be obtain very similar: since M is finite, the additional terms (2.1) do not affect the saddle-points and the main terms in representation (3.16), they just add some additional terms to $\tilde M$ of (3.20), $h^{(q)}_{1,ls}$ of (3.21) and to (3.23) – (3.24) which can be handled in the same way. □

4. Proof of Theorem 1.2

To simplify formulas below we handle again the case $M = 1$. We explain the difference with the case $M > 1$ at the end of the section.

It is easy to see that (1.11) implies that $Z_{\beta n}(\kappa, z_1, z_2)$ can be written in the form

$$Z_{\beta n}(\kappa, z_1, z_2) = e^{E(x_1 - x_2)} \int D(Q)\tilde{\mathcal F}(Q)\tilde{\mathcal M}^{n-1}(Q, Q')\tilde{\mathcal F}(Q')dQdQ',$$

where

$$\tilde{\mathcal F}(Q) := \exp\{\frac{c_0}{4n}\,\mathrm{Str}\, Q\Lambda_{\kappa, y_1, y_2}\},$$
$$\tilde{\mathcal M}(Q, Q') = \tilde{\mathcal F}(Q)\exp\big\{-\frac{\tilde\beta}{4}\,\mathrm{Str}\, QQ'\big\}\tilde{\mathcal F}(Q'),$$

and

$$D(Q) := \operatorname{Sdet}^{-1}\Big(Q - \frac{iE}{2\pi\rho(E)} + \frac{i\gamma}{\pi\rho(E)}\mathcal{L}\Sigma\Big) = D(U, S, \hat\rho, \hat\tau).$$

But for the proof of Theorem 1.2, it is convenient to change variables $\{U_i\}_{i=1}^n$ and $\{S_i\}_{i=1}^n$ in order to obtain a little bit different representation.

Proposition 4.1. *We have*

$$Z_{\beta n}(\kappa, z_1, z_2) = e^{E(x_1-x_2)} \int D_1(Q)\mathcal{F}(Q)\mathcal{M}^{n-1}(Q, Q')\mathcal{F}(Q')dQdQ', \qquad (4.1)$$

where

$$\mathcal{F}(Q) := \exp\Big\{\frac{c_0}{4n}\operatorname{Str} Q\Lambda_1\Big\}, \quad \Lambda_1 = \begin{pmatrix} L\kappa_1 & 0 \\ 0 & L\kappa_2 \end{pmatrix}, \quad \kappa_{1,2} = (\kappa^2 + y_{1,2}^2)^{1/2},$$

$$\mathcal{M}(Q, Q') = \mathcal{F}(Q)\exp\big\{-\frac{\tilde\beta}{4}\operatorname{Str} QQ'\big\}\mathcal{F}(Q'),$$

$$D_1(Q) = c_1 + c_2 n_1 + c_3 n_2 + c_4 n_1 n_2 + d_1\rho_1\tau_2 + d_2\rho_2\tau_1, \qquad (4.2)$$

$$c_\nu = \sum_{k=1}^{3} c_\nu^{(k)}(\tau - i\sinh t\,\cos\alpha_2\,\cos\theta + \cosh t\,\sin\alpha_2)^{-k}, \quad \nu = 1,2,3,4,$$

$$d_\nu = d_\nu^{(1)}(\tau - i\sinh t\,\cos\alpha_2\,\cos\theta + \cosh t\,\sin\alpha_2)^{-1}, \quad \nu = 1,2,$$

$$\tau = (\gamma + \gamma^{-1})/c_0 > 0. \qquad (4.3)$$

Here $n_1 = \rho_1\tau_1$, $n_2 = \rho_2\tau_2$ *and* α_1, α_2 *are defined as*

$$\sin\alpha_\sigma = y_\sigma(\kappa^2 + y_\sigma^2)^{-1/2}, \quad 0 < \alpha_\sigma < \pi/2, \quad \sigma = 1, 2.$$

In addition, $c_\nu^{(k)}$ *and* $d_\nu^{(1)}$ *are polynomials with respect to entries of* U*, whose coefficients are independent of* S *in the case of* $c_\nu^{(k)}$*, and are bounded functions of* S *in the case of* $d_\nu^{(1)}$*. Parameters* t*,* θ *here correspond to the following parametrizations of* $U \in \mathring{U}(2)$ *and* $S \in \mathring{U}(1,1)$*:*

$$U = \begin{pmatrix} e^{i\psi/2}\cos\frac{\varphi}{2} & e^{-i\psi/2}\sin\frac{\varphi}{2} \\ -e^{i\psi/2}\sin\frac{\varphi}{2} & e^{-i\psi/2}\cos\frac{\varphi}{2} \end{pmatrix}, \quad S = \begin{pmatrix} e^{i\theta/2}\cosh\frac{t}{2} & e^{-i\theta/2}\sinh\frac{t}{2} \\ e^{i\theta/2}\sinh\frac{t}{2} & e^{-i\theta/2}\cosh\frac{t}{2} \end{pmatrix}. \qquad (4.4)$$

Proof. Let us introduce unitary matrices

$$V_\sigma = \begin{pmatrix} \cos(\alpha_\sigma/2) & -i\sin(\alpha_\sigma/2) \\ -i\sin(\alpha_\sigma/2) & \cos(\alpha_\sigma/2) \end{pmatrix}, \quad \sigma = 1, 2,$$

where $\alpha_{1,2}$ are defined in (4.1). It is straightforward to check that

$$V_\sigma\Lambda_{\kappa,y_\sigma}V_\sigma^* = \kappa_\sigma L, \quad \sigma = 1, 2.$$

For the unitary group we can just change the variables $U_i \to U_i V_1$, and since the Haar measure is invariant with respect to this change of variables, we obtain the desired transformation for the "unitary" part of $\mathcal{M}$. Unfortunately, similar transformation for the hyperbolic group does not work directly, since the matrix $\tilde{S}_i = S_i V_2$ is not hyperbolic. But if we use another parametrization of the Hyperbolic group

$$S(t,s) = \begin{pmatrix} \cosh(t/2) + ise^{t/2}/2 & -\sinh(t/2) - ise^{t/2}/2 \\ -\sinh(t/2) + ise^{t/2}/2 & \cosh(t/2) - ise^{t/2}/2 \end{pmatrix},$$

then it is straightforward to check that

$$S(t,s)V_2 = S(t + i\alpha_2, s).$$

On the other hand, $\mathcal{M}(S_1, S_2)$ depends only on $S_1 S_2^{-1}$ and the entries of $S_1 S_2^{-1}$ depend only on $t_1 - t_2$

$$(S(t_1,s_1)S^{-1}(t_2,s_2)))_{11} = \cosh((t_1-t_2)/2) + (is_1 e^{(t_1-t_2)/2} - is_2 e^{-(t_1-t_2)/2})/2,$$
$$(S(t_1,s_1)S^{-1}(t_2,s_2)))_{12} = -\sinh((t_1-t_2)/2) - (is_1 e^{(t_1-t_2)/2} - is_2 e^{-(t_1-t_2)/2})/2.$$

Hence, if we change the integration contour with respect to all $t_j + i\alpha_2 \to t_j$, then

$$\tilde{\mathcal{F}} \to \mathcal{F}, \quad \tilde{\mathcal{M}} \to \mathcal{M}, \quad D(U,S) \to D(UV_1^*, SV_2^*).$$

Thus we are left to study $D_1 = D(UV_1^*, SV_2^*)$. Denote

$$\tilde{U} = UV_1^*, \quad \tilde{S} = SV_2^*.$$

Using formulas (1.11) and (1.12), we conclude that

$$\begin{aligned} D_1 &:= \frac{\det \tilde{A}}{\det \tilde{B}} \frac{\det(1 + 2L\hat{n}\tilde{A}^{-1})}{\det(1 - 2L\hat{n}\tilde{B}^{-1} + 4\hat{\rho}(\tilde{A} + 2L\hat{n})^{-1}\hat{\tau})} \\ \tilde{A} &= -\frac{iE}{c_0} + \frac{2i\gamma}{c_0}\tilde{U}\hat{\sigma}\tilde{U}^{-1} + L, \quad \tilde{B} = \frac{iE}{c_0} + \frac{2i\gamma}{c_0}\tilde{S}\hat{\sigma}\tilde{S}^{-1} + L, \\ \hat{n} &= \mathrm{diag}\{n_1, n_2\} \end{aligned} \tag{4.5}$$

It is easy to see that

$$\begin{aligned} \det \tilde{A} &= \det\Big(-\frac{iE}{c_0} + \frac{2i\gamma}{c_0}\tilde{U}\hat{\sigma}\tilde{U}^{-1} + L\Big) \\ &= -\frac{E^2}{c_0^2} - 1 - \frac{4\gamma^2}{c_0^2} - \frac{4i\gamma}{c_0}\Big(\tilde{U}_{11}\bar{\tilde{U}}_{12} - \tilde{U}_{12}\bar{\tilde{U}}_{11}\Big) = \\ &= -\frac{4\gamma}{c_0}(\tau - \sin\varphi \cdot \cos\alpha_1 \sin\psi + \sin\alpha_1 \cos\varphi) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \det \tilde{B} &= \det\Big(\frac{iE}{c_0} + \frac{2i\gamma}{c_0}\tilde{S}\hat{\sigma}\tilde{S}^{-1} + L\Big) \\ &= -\frac{E^2}{c_0^2} - 1 - \frac{4\gamma^2}{c_0^2} - \frac{4i\gamma}{c_0}\Big(\tilde{S}_{11}(\tilde{S}^{-1})_{21} - \tilde{S}_{12}(\tilde{S}^{-1})_{11}\Big) = \end{aligned}$$

$$= -\frac{4\gamma}{c_0}(\tau - i\sinh t \cdot \cos\alpha_2 \cos\theta + \cosh t \cdot \sin\alpha_2), \tag{4.7}$$

where τ is defined in (4.3) and we used parametrizations (4.4) for U and S. Here we used also that

$$\tilde{S}^{-1} = \begin{pmatrix} \tilde{S}_{22} & -\tilde{S}_{12} \\ -\tilde{S}_{21} & \tilde{S}_{11} \end{pmatrix},$$

and so

$$\begin{aligned} -\tilde{S}_{11}(\tilde{S}^{-1})_{21} + \tilde{S}_{12}(\tilde{S}^{-1})_{11} &= \tilde{S}_{11}\tilde{S}_{21} + \tilde{S}_{12}\tilde{S}_{22} \\ &= (SV_2^*)_{11}(SV_2^*)_{21} + (SV_2^*)_{12}(SV_2^*)_{22} \\ &= \cos\alpha_2(S_{11}S_{21} + S_{12}S_{22}) + i\sin\alpha_2(S_{11}S_{22} + S_{12}S_{21}) \\ &= \sinh t\, \cos\alpha_2\, \cos\theta + i\cosh t\, t \sin\alpha_2. \end{aligned} \tag{4.8}$$

Similar formulas can be obtained for $\tilde{U}_{11}\bar{\tilde{U}}_{12} - \tilde{U}_{12}\bar{\tilde{U}}_{11}$.

Since

$$\hat{\rho}(\tilde{A} + 2L\hat{n})^{-1}\hat{\tau} = \hat{\rho}\tilde{A}^{-1}(1 - 2L\hat{n}\tilde{A}^{-1})\hat{\tau},$$
$$\hat{\rho}\tilde{A}^{-1}L\hat{n}\tilde{A}^{-1}\hat{\tau}\tilde{B}^{-1} = -n_1n_2\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{21}L\tilde{B}^{-1},$$

we have

$$\begin{aligned} &\det\Big(1 - 2L\hat{n}\tilde{B}^{-1} + 4\hat{\rho}\left(\tilde{A} + 2L\hat{n}\right)^{-1}\hat{\tau}\tilde{B}^{-1}\Big) \\ &= \det\left(1 - 2L\hat{n}\tilde{B}^{-1} + 4\hat{\rho}\tilde{A}^{-1}\left(1 - 2L\hat{n}\tilde{A}^{-1}\right)\hat{\tau}\tilde{B}^{-1}\right) \\ &= \det\left(1 - 2L\hat{n}\tilde{B}^{-1} + 4\hat{\rho}\tilde{A}^{-1}\hat{\tau}\tilde{B}^{-1}\right)\det\left(1 + 8n_1n_2\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{21}L\tilde{B}^{-1}\right) \\ &= \Big(1 + n_1(4\tilde{A}^{-1}_{11} - 2)\tilde{B}^{-1}_{11} + n_2\left(4\tilde{A}^{-1}_{22} + 2\right)\tilde{B}^{-1}_{22} \\ &\qquad + 4\tilde{A}^{-1}_{12}\tilde{B}^{-1}_{21}\rho_1\tau_2 + 4\tilde{A}^{-1}_{21}\tilde{B}^{-1}_{12}\rho_2\tau_1\Big) \\ &\times\left(1 + n_1n_2\left(\frac{(4\tilde{A}^{-1}_{11} - 2)(4\tilde{A}^{-1}_{22} + 2) + 16\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{21}}{\det\tilde{B}} + 8\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{12}\operatorname{Tr}\tilde{B}^{-1}L\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\det{}^{-1}\Big(1 - 2L\hat{n}\tilde{B}^{-1} + 4\hat{\rho}(\tilde{A} + 2L\hat{n})^{-1}\hat{\tau}\tilde{B}^{-1}\Big) \\ &\quad = (1 - k_bn_1n_2)\Big(1 - n_1\left(4\tilde{A}^{-1}_{11} - 2\right)\tilde{B}^{-1}_{11} - n_2\left(4\tilde{A}^{-1}_{22} + 2\right)\tilde{B}^{-1}_{22} \\ &\qquad - 4\tilde{A}^{-1}_{12}\tilde{B}^{-1}_{21}\rho_1\tau_2 - 4\tilde{A}^{-1}_{21}\tilde{B}^{-1}_{12}\rho_2\tau_1\Big), \end{aligned}$$

where

$$\begin{aligned} k_b = &\frac{\left(4\tilde{A}^{-1}_{11} - 2\right)\left(4\tilde{A}^{-1}_{22} + 2\right) + 16\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{21}}{\det\tilde{B}} + 8\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{12}\operatorname{Tr}\tilde{B}^{-1}L \\ &- 2\left(4\tilde{A}^{-1}_{11} - 2\right)\left(4\tilde{A}^{-1}_{22} + 2\right)\tilde{B}^{-1}_{11}\tilde{B}^{-1}_{22} + 32\tilde{A}^{-1}_{12}\tilde{A}^{-1}_{21}\tilde{B}^{-1}_{12}\tilde{B}^{-1}_{21} \end{aligned}$$

$$= -\big(16\det{}^{-1}\tilde{A} + 8\,\mathrm{Tr}\,\tilde{A}^{-1}L - 4\big)\big(\tilde{B}_{11}^{-1}\tilde{B}_{22}^{-1} + \tilde{B}_{12}^{-1}\tilde{B}_{21}^{-1}\big) + 8\tilde{A}_{12}^{-1}\tilde{A}_{21}^{-1}\,\mathrm{Tr}\,\tilde{B}^{-1}L.$$

Similarly,

$$\det(1 + 2L\hat{n}\tilde{A}^{-1}) = (1 + 2n_1\tilde{A}_{11}^{-1} - 2n_2\tilde{A}_{22}^{-1})(1 - 4n_1n_2(\det\tilde{A})^{-1}).$$

Then finally D_1 of (4.5) can be rewritten as

$$\begin{aligned} D_1 &= \Big(\det\tilde{B}\Big)^{-1}\Big[\det\tilde{A} - n_1\Big(4\tilde{A}_{22}\tilde{B}_{11}^{-1} - 2\tilde{A}_{22} - 2\tilde{B}_{11}^{-1}\det\tilde{A}\Big) \\ &\quad - n_2\Big(4\tilde{B}_{22}^{-1}\tilde{A}_{11} + 2\tilde{A}_{11} + 2\tilde{B}_{22}^{-1}\det\tilde{A}\Big) \\ &\quad + 4\tilde{A}_{12}\tilde{B}_{21}^{-1}\rho_1\tau_2 + 4\tilde{A}_{21}\tilde{B}_{12}^{-1}\rho_2\tau_1 - kn_1n_2\Big], \\ k &= -\Big(16 - 8\,\mathrm{Tr}\,\tilde{A}L - 4\det\tilde{A}\Big)\Big(\tilde{B}_{11}^{-1}\tilde{B}_{22}^{-1} + \tilde{B}_{12}^{-1}\tilde{B}_{21}^{-1}\Big) \\ &\quad - 8\ \mathrm{Tr}\,\tilde{B}^{-1}L + 4\tilde{B}_{11}^{-1}\tilde{A}_{11} + 4\tilde{B}_{22}^{-1}\tilde{A}_{22} + 4, \end{aligned} \tag{4.9}$$

where we used

$$\tilde{A}^{-1} = \begin{pmatrix} \dfrac{\tilde{A}_{22}}{\det\tilde{A}} & -\dfrac{\tilde{A}_{12}}{\det\tilde{A}} \\ -\dfrac{\tilde{A}_{21}}{\det\tilde{A}} & \dfrac{\tilde{A}_{11}}{\det\tilde{A}} \end{pmatrix}. \tag{4.10}$$

Using (4.10) for $\tilde{B}$, and taking into account that (see (4.8))

$$\begin{aligned} \tilde{B}_{jj} &= \frac{iE}{c_0} - (-1)^j + \frac{2i\gamma}{c_0}(\tilde{S}\hat{\sigma}\tilde{S}^{-1})_{jj} \\ &= \frac{iE}{c_0} - (-1)^j + (-1)^j\frac{2i\gamma}{c_0}(\tilde{S}_{12}\tilde{S}_{22} + \tilde{S}_{11}\tilde{S}_{21}) \\ &= \frac{iE}{c_0} - (-1)^j + (-1)^j\frac{2i\gamma}{c_0}(\sinh t\,\cos\alpha_2\,\cos\theta + i\cosh t\,\sin\alpha_2), \quad j = 1, 2, \\ \tilde{B}_{11}^{-1}\tilde{B}_{22}^{-1} &+ \tilde{B}_{12}^{-1}\tilde{B}_{21}^{-1} = -(\det\tilde{B})^{-1} + 2(\det\tilde{B})^{-2}\tilde{B}_{11}\tilde{B}_{22}, \end{aligned}$$

we obtain (4.2). □

For the next step we will use the following notations:

$$\begin{aligned} F(U,S) &= \exp\Big\{-\frac{c_0}{n}\Big(\kappa_1\Big(\frac{1}{2} - |U_{12}|^2\Big) + \kappa_2\Big(\frac{1}{2} + |S_{12}|^2\Big)\Big)\Big\}, \quad (4.11) \\ F_1(U,S) &= -\frac{c_0}{n}\Big(\kappa_1\Big(\frac{1}{2} - |U_{12}|^2\Big) - \kappa_2\Big(\frac{1}{2} + |S_{12}|^2\Big)\Big) \end{aligned}$$

with $\kappa_{1,2}$ defined in Proposition 4.1.

Proposition 4.2. *We have*

$$Z_{\beta n}(\kappa, z_1, z_2) = -\frac{e^{E(x_1 - x_2)}}{2\pi i}\oint_{\omega_A} z^{n-1}(\widehat{G}(z)\widehat{f}, \widehat{g})dz, \tag{4.12}$$

where $\omega_A = \{z : |z| = 1 + A/n\}$,

$$\widehat{G}(z) = \left(\widehat{M} - z\right)^{-1}, \quad \widehat{M} = \widehat{F}\widehat{K}\widehat{F}, \quad \widehat{K} = \widehat{K}_0 + O\left(\beta^{-1}\right), \tag{4.13}$$

where operators $\widehat{K}_0$, $\widehat{F}$ *and the vectors* $\widehat{f}$, $\widehat{g}$ *have the form*

$$\widehat{K}_0 = \begin{pmatrix} K_{US} & \widetilde{K}_1 & \widetilde{K}_2 & \widetilde{K}_3 \\ 0 & K_{US} & 0 & \widetilde{K}_2 \\ 0 & 0 & K_{US} & \widetilde{K}_1 \\ 0 & 0 & 0 & K_{US} \end{pmatrix}, \quad \widehat{F} = F \begin{pmatrix} 1 & F_1 & F_1 & F_1^2 \\ 0 & 1 & 0 & F_1 \\ 0 & 0 & 1 & F_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.14}$$

$$\hat{f} = \widehat{F}(e_4 - e_1), \quad \hat{g} = \widehat{F}(c_1 e_0 + c_2 e_2 + c_3 e_3 + (c_4 - c_1) e_4) + O\big(\tilde{\beta}^{-1}\big) \tag{4.15}$$

with F *and* F_1 *being the operator of multiplication by the functions* F *and* F_1 *defined in* (4.11) *on* $L_2(U) \otimes L_2(S)$, $K_{US} = K_U \otimes K_S$ *and* K_U *and* K_S *being the integral operators in* $L_2(U)$ *and* $L_2(S)$ *with a "difference" kernels*

$$K_U(U, U') = K_U(U(U')^*) = \tilde{\beta} e^{-\tilde{\beta}|(U(U')^*)_{12}|^2},$$
$$K_S(S, S') = K_S(S(S')^{-1}) = \tilde{\beta} e^{-\tilde{\beta}|(S(S')^{-1})_{12}|^2}.$$

and c_δ *having the form* (4.2)*. Here* $\widetilde{K}_p$, $p = 1, 2, 3$ *are normal operators on* $L_2(U) \otimes L_2(S)$*, they commute with* K_{US} *and with the Laplace operators* $\widetilde{\Delta}_U, \widetilde{\Delta}_S$ *on the corresponding groups and satisfy the bounds*

$$|\widetilde{K}_p| \le C(1 - K_{US}) \le -C(\widetilde{\Delta}_U + \widetilde{\Delta}_S)/\beta, \tag{4.16}$$

where the Laplace operators $\widetilde{\Delta}_U, \widetilde{\Delta}_S$ *for the functions depending only on* $|S_{12}|^2$ *and* $|U_{12}|^2$ *have the form*

$$\widetilde{\Delta}_S(\varphi) = -\frac{d}{dx} x(x+1) \frac{d\varphi}{dx} \quad (x = |S_{12}|^2),$$
$$\widetilde{\Delta}_U(\varphi) = -\frac{d}{dx} x(1-x) \frac{d\varphi}{dx} \quad (x = |U_{12}|^2).$$

The proposition is basically identical to the Proposition 5.1 of [16]. The only change is the different form of $\widehat{g}$ coming from the presence of the factor D_1 in (4.1). The from of $\widehat{g}$ in (4.14) follows from (1.11) and Proposition 4.1. Indeed, consider the basis $e_1 = 1$, $e_2 = n_1$, $e_3 = n_2$, $e_4 = n_1 n_2$, $e_5 = \rho_1 \tau_2$, $e_6 = \rho_2 \tau_1$, and let $\mathcal{L}_1 = \mathrm{span}\{e_1, e_2, e_3, e_4\}$. Write the transfer operator matrix H as a block matrix with the first block corresponding to $\mathcal{L}_1$ (see the proof of Proposition 5.1 in [16]):

$$H = \begin{pmatrix} H^{(11)} & H^{(12)} \\ H^{(21)} & H^{(22)} \end{pmatrix}, \qquad H^{(22)} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$H^{(21)} = \begin{pmatrix} 2x_d & x & x & 0 \\ -2\overline{x}_d & -\overline{x} & -\overline{x} & 0 \end{pmatrix}, \qquad H^{(12)} = \begin{pmatrix} 0 & 0 \\ y & -\overline{y} \\ y & -\overline{y} \\ 2y_d & -2\overline{y_d} \end{pmatrix}. \tag{4.17}$$

Here h_{ij}, x, y, x_d, y_d are "difference" operators whose kernels are defined with the functions

$$\begin{aligned} h_{ij} &= h_{ijU}h_{ijS}, \quad h_{ijU} = U_{ij}^2 K_U, \qquad h_{ijS} = \bar{S}_{ij}^2 K_S \\ x &= x_U x_S, \qquad x_U = U_{11}U_{12}K_U, \qquad x_S = \bar{S}_{11}\bar{S}_{12}K_S, \quad x_d = x\, d, \\ y &= y_U y_S, \qquad y_U = U_{11}\bar{U}_{12}K_U \qquad y_S = \bar{S}_{11}S_{12}K_S, \quad y_d = y\, d, \end{aligned} \tag{4.18}$$

and $\bar{x}, \bar{y}, \bar{x}_d, \bar{y}_d$ mean the complex conjugate kernels. We recall that we are saying that the operator in $L_2(\mathring{U}_2)$ is a "difference" one with a kernel f, if its kernel $k(U_1, U_2)$ has the form $k(U_1, U_2) = f(U_1 U_2^*)$. The operator on $L_2(\mathring{U}(1,1))$ is a"difference" one with a kernel f, if $k(S_1, S_2) = f(S_1 S_2^{-1})$. Let us recall also that K (and consequently its resolvent) was obtained from H by the transformation

$$K = \hat{T}H\hat{T}, \quad \hat{T} = \mathrm{diag}\{T, I\}, \quad T = \begin{pmatrix} 0 & 0 & 0 & \tilde{\beta} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \tilde{\beta}^{-1} & 0 & 0 & 0 \end{pmatrix}$$

Hence the entries of the off-diagonal blocks of the resolvent of K are obtained from those of the off-diagonal blocks of the resolvent of H by multiplication by β, 1, or β^{-1}. Thus, to obtain the bound $O(\beta^{-1})$ in (4.15), it is sufficient to get the bound $O(\beta^{-2})$ for the corresponding entries of the resolvent of H.

According to the Schur formula, the resolvent $G(z) = (H - z)^{-1}$ has the form

$$\begin{aligned} G(z) &:= \begin{pmatrix} G^{(11)} & -G^{(11)}H^{(12)}G_2 \\ -G_2 H^{(21)}G^{(11)} & G_2 + G_2 H^{(21)}G^{(11)}H^{(12)}G_2 \end{pmatrix}, \\ G^{(11)} &= M_1^{-1}, \quad M_1 = H^{(11)} - z - H^{(12)}G_2 H^{(21)}, \quad G_2(z) = (H^{(22)} - z)^{-1}. \end{aligned}$$

Since $\hat{f}_{5,6} = 0$, we can write

$$\big(G(z)\hat{f}, \hat{g}\big) = \big(G^{(11)}f^{(1)}, g^{(1)} - (H^{(21)})^* G_2^* g^{(2)}\big),$$

where $f^{(1)}, g^{(1)}$ are the projection of $\hat{f}, \hat{g}$ on $\mathcal{L}_1$ and $g^{(2)}$ is a projection of $\hat{g}$ on $\mathrm{span}\{e_5, e_6\}$. Let us consider $H^{(22)} = \hat{h} + \tilde{h}$, where $\hat{h}$ is a diagonal part and $\tilde{h}$-off diagonal part of $H^{(22)}$, and let $G_{2d} = (\hat{h} - z)^{-1}$. By the resolvent identity we can write

$$G_2 = G_{2d} - G_{2d}\tilde{h}G_2 \tag{4.19}$$

Moreover, it was proven in [16] (see the proof of Lemma 6.2) that

$$\begin{aligned} \|G_2(z)\| \le Cn, \quad \|G_{2d}(z)\| \le Cn, \quad \|\tilde{h}\| \le C\beta^{-2}, \quad \|H^{(21)}\| \le C\beta^{-2} \\ \Rightarrow \|(H^{(21)})^*(G_{2d}\tilde{h}G_2)^*\| \le C\beta^{-2} \end{aligned}$$

For the first terms of the right-hand side of (4.19) we use the expansion

$$G_{2d}(z) = -z^{-1}\sum_{s=0}^{\infty} z^{-s}\big(\hat{h}\big)^s \tag{4.20}$$

It is easy to see that, due to the form $g^{(2)}$ (see (4.9)), $\hat{h}$ and $H^{(21)}$ (see (4.17) and (4.18)), after the integration with respect to U only the term corresponding to $s = 1$ in the above expansion will give non zero contribution. Hence, using that

$$\big\|\big(H^{(21)}\big)^*\hat{h}^*g^{(2)}\big\| \le C\beta^{-2},$$

after the multiplication by β we get (4.15). □

Now let us derive (1.13) from Proposition 4.2. To this end, set

$$\widehat{M}_0 = \widehat{F}^2, \quad \widehat{G}_0 = \Big(\widehat{M}_0 - z\Big)^{-1},$$

and consider

$$\Delta G := \widehat{G} - \widehat{G}_0 = -\widehat{G}_0\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}_0 - \widehat{G}_0\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}_0.$$

We apply the following lemma:

Lemma 4.3. *For any $z \in \omega_A$ (see (4.12)) we have the bounds*

$$\Big\|\widehat{G}\Big\| \le C\log^2 n/|z-1|, \qquad \Big\|\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}_0\widehat{f}\Big\|^2 \le C\big(n/\tilde{\beta}\big)^2, \tag{4.21}$$

$$\Big\|\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}_0\widehat{g}\Big\|^2 \le C\big(n/\tilde{\beta}\big)^2, \quad \Big|\Big(\widehat{G}_0\Big(\widehat{M} - \widehat{M}_0\Big)\widehat{G}_0\widehat{f}, \widehat{g}\Big)\Big| \le \frac{n\log n}{\tilde{\beta}|z-1|}. \tag{4.22}$$

Inequalities (4.21) were proven in [16] (see Lemma 5.1). Hence we need to prove only inequalities (4.22). We postpone the proof to the end of the section, and continue with the proof of (1.13) using Lemma 4.3.

Let us write

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\omega_A} z^{n-1}(\Delta G\widehat{f}, \widehat{g})dz\right| &\le C\oint_{\omega_A}\Big|(\widehat{G}_0(\widehat{M} - \widehat{M}_0)\widehat{G}_0\widehat{f}, \widehat{g})\Big|\,|dz| \\ &\quad + C\oint_{\omega_A}\Big\|\widehat{G}(z)\Big\|\,\Big\|(\widehat{M} - \widehat{M}_0)\widehat{G}_0(z)\widehat{f}\Big\|\,\Big\|(\widehat{M} - \widehat{M}_0)\widehat{G}_0(\bar{z})\widehat{g}\Big\|\,|dz| \\ &\le C\big(n\log n/\tilde{\beta}\big)\oint_{\omega_A}\frac{|dz|}{|z-1|} \le Cn\log^2 n/\tilde{\beta} \to 0,\end{aligned}$$

where we used $n\log^2 n \ll \tilde{\beta}$ and

$$\oint_{\omega_A}\frac{|dz|}{|z-1|} \le C\log n.$$

Thus we have proved that (recall (4.14))

$$\begin{aligned}Z_{\beta n}(\kappa, z_1, z_2) &= -\frac{e^{E(x_1 - x_2)}}{2\pi i}\oint_{\omega_A} z^{n-1}\Big(\widehat{G}_0(z)\widehat{f}, \widehat{g}\Big)\,dz + o(1) \\ &= e^{E(x_1 - x_2)}\Big(\widehat{F}^{2n-2}\widehat{f}, \widehat{g}\Big) + o(1).\end{aligned}$$

Proof of inequalities (4.22). Using inequalities (4.16) it is easy to conclude the it is sufficient to prove that

$$\|(\Delta_S+\Delta_U)\widehat{G}_0\widehat{g}\|^2\le Cn^2,\quad |(\widehat{G}_0(\Delta_S+\Delta_U)\widehat{G}_0\widehat{f},\widehat{g})|\le n\log n/|z-1|. \tag{4.23}$$

Notice that since $\widehat{G}_0=(\widehat{F}^2-z)^{-1}$, we can write

$$\widehat{G}_0=\begin{pmatrix} G_0 & G_0F_1FG_0 & G_0F_1FG_0 & G_0F_1FG_0F_1FG_0+G_0F_1^2FG_0\\ 0 & G_0 & 0 & G_0F_1FG_0\\ 0 & 0 & G_0 & G_0F_1FG_0\\ 0 & 0 & 0 & G_0\end{pmatrix}$$
$$G_0=(F^2-z)^{-1},\quad F(x)=e^{-(2c_2x-2c_1u+c_1-c_2)/2n} \tag{4.24}$$

with $x=|S_{12}|^2$, $u=|U_{12}|^2$. Observe that coefficients of $\widehat{G}_0$ do not depend on θ of (4.4). Hence we can integrate over θ in expression for $\hat{g}$ of (4.14) and (4.2). Using that

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{d\theta}{(\tau-i\cos\alpha_2\sinh t\cos\theta+\sin\alpha_2\cosh t)^\delta}$$
$$=C_\delta\frac{\partial^{\delta-1}}{\partial\tau^{\delta-1}}\Big((\tau+\sin\alpha_2\cosh t)^2+\sinh^2 t\cos^2\alpha_2\Big)^{-1/2},\quad \delta=1,2,3,$$

one can conclude that (4.23) will follow from the bounds

$$\Big\|(\Delta_S+\Delta_U)\tilde{f}_\nu\tilde{g}_\nu\Big\|^2\le Cn^2,\quad \nu=1,\dots,4, \tag{4.25}$$

where for $x=\sinh^2(t/2)$

$$\begin{aligned}\tilde{f}_\nu(x)&=a_\nu G_0F+b_\nu F_1(FG_0)^2+d_\nu F_1^2(FG_0)^3\\ \tilde{g}_\nu(x)&=a_\nu^{(1)}\Big((\tau+\sin\alpha_2\cosh t)^2+\sinh^2 t\cos^2\alpha_2\Big)^{-1/2}\\ &\quad+b_\nu^{(1)}\frac{\partial}{\partial\tau}\Big((\tau+\sin\alpha_2\cosh t)^2+\sinh^2 t\cos^2\alpha_2\Big)^{-1/2}\\ &\quad+d_\nu^{(1)}\frac{\partial^2}{\partial\tau^2}\Big((\tau+\sin\alpha_2\cosh t)^2+\sinh^2 t\cos^2\alpha_2\Big)^{-1/2},\end{aligned} \tag{4.26}$$

where a_ν, b_ν, d_ν and $a_\nu^{(1)}, b_\nu^{(1)}, d_\nu^{(1)}$ are bounded functions depending only on u.

It is straightforward to check that

$$\begin{aligned}&\big|\tilde{g}_\nu(x)\big|\le C(x^2+1)^{-1/2},\quad |(x+1)\tilde{g}_\nu'(x)|\le C(x^2+1)^{-1/2},\\ &x(x+1)\big|\tilde{g}_\nu''(x)\big|\le C(x^2+1)^{-1/2},\end{aligned} \tag{4.27}$$

and

$$\big|\tilde{f}_\nu''(x)\big|\le Cn,\quad \big|(x+1)f_\nu'(x)\big|\le Cn,\quad \big|f_\nu(x)\big|\le Cn.$$

Then, since

$$\Delta_S\big(\tilde{f}_\nu\tilde{g}_\nu\big) = x(x+1)\big(\tilde{f}_\nu\tilde{g}_\nu\big)'' + (2x+1)\big(\tilde{f}_\nu\tilde{g}_\nu\big)' + \tilde{f}_\nu\tilde{g}_\nu,$$

we conclude that

$$\big|\Delta_S(\tilde{f}_\nu\tilde{g}_\nu)\big| \le Cn(x^2+1)^{-1/2} \quad \Rightarrow \quad \big\|\Delta_S(\tilde{f}_\nu\tilde{g}_\nu)\big\|^2 \le Cn^2.$$

In addition, one can obtain by the same way that

$$\big\|\Delta_U\big(\tilde{f}_\nu\tilde{g}_\nu\big)\big\|^2 \le Cn^2.$$

Thus, we obtain (4.25).

To prove the second inequality in (4.23), we observe that for any f_ν of the same type as in (4.26) we have

$$|G_0\Delta_S f_\nu| \le \frac{Ce^{-cx/n}}{|z-1|^2} \le \frac{Cne^{-cx/n}}{|z-1|}.$$

Hence

$$\begin{aligned}\Big|\int \Delta_S(f_\nu)g_\nu dx\Big| &\le \frac{Cn}{|z-1|}\int_0^\infty \frac{e^{-cx/n}dx}{(x^2+1)^{1/2}}\\ &= \frac{Cn}{|z-1|}\int_0^\infty \frac{e^{-c\tilde{x}}dx}{(\tilde{x}^2+n^{-2})^{1/2}} \le \frac{Cn\log n}{|z-1|}.\end{aligned}$$

Here we changed the variable $x \to n\tilde{x}$. Repeating the argument for Δ_U we obtain the second inequality in (4.25). □

The case $M > 1$ is very similar, since in this case the transfer matrix $\mathcal{M}$ and $\mathcal{F}$ in (4.1) remain the same and only D_1 is replaced by the product of D_α of the same form but with different γ_α (see (1.11)). Hence, in (4.12) the resolvent $\widehat{G}$ and the function $\widehat{f}$ are the same and only the function $\widehat{g}$ will be different. But one can see from the argument given after (4.27) that for our proof we need only bounds (4.27), and the fact that $\widehat{g}$ depends polynomially on entries of U (recall that we used polynomial dependence on U in (4.19)-(4.20) to prove that only a finite number of terms in (4.20) are non zero). But for $M > 1$ D_1 should be replaced by the product of D_α and each of them has the form (4.2) with τ replaced by τ_α defined by (4.3) with $\gamma = \gamma_\alpha$. Hence it is evident that that resulting $\widehat{g}$ will satisfy (4.27) and will depend on entries of U polynomially.

5. Appendix

5.1. Grassmann integration. Let us consider two sets of formal variables $\{\psi_j\}_{j=1}^n, \{\overline{\psi}_j\}_{j=1}^n$, which satisfy the anticommutation conditions

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0, \quad j,k = 1,\dots,n. \tag{5.1}$$

Note that this definition implies $\psi_j^2 = \overline{\psi}_j^2 = 0$. These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. Taking into account that $\psi_j^2 = 0$, we have that all elements of $\mathfrak{A}$ are polynomials of $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ of degree at most one in each variable. We can also define functions of the Grassmann variables. Let χ be an element of $\mathfrak{A}$, i.e.,

$$\chi = a + \sum_{j=1}^{n}(a_j\psi_j + b_j\overline{\psi}_j) + \sum_{j\neq k}(a_{j,k}\psi_j\psi_k + b_{j,k}\psi_j\overline{\psi}_k + c_{j,k}\overline{\psi}_j\overline{\psi}_k) + \cdots . \tag{5.2}$$

For any sufficiently smooth function f we define by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of f at the point a. Since χ is a polynomial of $\{\psi_j\}_{j=1}^n$, $\{\overline{\psi}_j\}_{j=1}^n$ of the form (5.2), according to (5.1) there exists such l that $(\chi - a)^l = 0$, and hence the series terminates after a finite number of terms and so $f(\chi) \in \mathfrak{A}$.

Following Berezin [1], we define the operation of integration with respect to the anticommuting variables in a formal way:

$$\int d\,\psi_j = \int d\,\overline{\psi}_j = 0, \quad \int \psi_j d\,\psi_j = \int \overline{\psi}_j d\,\overline{\psi}_j = 1,$$

and then extend the definition to the general element of $\mathfrak{A}$ by the linearity. A multiple integral is defined to be a repeated integral. Assume also that the "differentials" $d\,\psi_j$ and $d\,\overline{\psi}_k$ anticommute with each other and with the variables ψ_j and $\overline{\psi}_k$. Thus, according to the definition, if

$$f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1}\psi_{j_1} + \sum_{j_1<j_2} p_{j_1,j_2}\psi_{j_1}\psi_{j_2} + \cdots + p_{1,2,\cdots,k}\psi_1 \ldots \psi_k,$$

then

$$\int f(\psi_1, \ldots, \psi_k) d\,\psi_k \cdots d\,\psi_1 = p_{1,2,\ldots,k}.$$

Let A be an ordinary Hermitian matrix with a positive real part. The following Gaussian integral is well-known

$$\int \exp\left\{-\sum_{j,k=1}^{n} A_{jk} z_j \overline{z}_k\right\} \prod_{j=1}^{n} \frac{d\,\Re z_j d\,\Im z_j}{\pi} = \frac{1}{\det A}. \tag{5.3}$$

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see [1]):

$$\int \exp\left\{-\sum_{j,k=1}^{n} A_{jk}\overline{\psi}_j\psi_k\right\} \prod_{j=1}^{n} d\,\overline{\psi}_j d\,\psi_j = \det A, \tag{5.4}$$

where A now is any $n \times n$ matrix.

We will also need the following bosonization formula

Proposition 5.1 (see, e.g., [5]). *Let $F : \mathbb{R} \to \mathbb{C}$ be some function that depends only on combinations*

$$\bar{\phi}\phi := \left\{ \sum_{\alpha=1}^{p} \bar{\phi}_{l\alpha}\phi_{s\alpha} \right\}_{l,s=1}^{2},$$

and set

$$d\Phi = \prod_{l=1}^{2} \prod_{\alpha=1}^{p} d\Re\phi_{l\alpha}\, d\Im\phi_{l\alpha}.$$

Assume also that $p \geq 2$. Then

$$\int F\left(\bar{\phi}\phi\right) d\Phi = \frac{\pi^{2p-1}}{(p-1)!(p-2)!} \int F(B) \det{}^{p-2} B\, dB,$$

where B is a 2×2 positive Hermitian matrix, and

$$dB = \mathbf{1}_{B>0}\, dB_{11}\, dB_{22}\, d\Re B_{12}\, d\Im B_{12}.$$

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Комплексна деформація скінченного рангу для випадкових стрічкових матриць: апрокимація сігма-моделі

Mariya Shcherbina and Tatyana Shcherbina

Ми вивчаємо розподіл комплексних власних значень $z_1, \ldots, z_N$ випадкової ермітової блокової стрічкової матриці розміру $N \times N$ з комплексною деформацією скінченного рангу. У режимі, коли розмір блоков W зростає швидше за N, ми доводимо, що гранична щільність $\Im z_1, \ldots, \Im z_N$ у сігма-модельній апроксимації збігається з відповідною щільністю для Ґаусівського унітарного ансамблю. Для цього ми використовуємо метод, розроблений в [16].

Ключові слова: випадкові стрічкові матриці, делокалізований режим, комплексна деформація, сігма-модель, суперсиметрія