Generating Function of Monodromy Symplectomorphism for 2×2 Fuchsian Systems and Its WKB Expansion

Marco Bertola, Dmitry Korotkin, and Fabrizio del Monte

We study the WKB expansion of 2×2 system of linear differential equations with Fuchsian singularities. The main focus is on the generating function of the monodromy symplectomorphism which, according to a recent paper [10], is closely related to the Jimbo–Miwa tau-function. We compute the first three terms of the WKB expansion of the generating function and establish the link to the Bergman tau-function.

Key words: Fuchsian systems, monodromy map, generating function, tau-function, WKB expansion

Mathematical Subject Classification 2020: 34M60, 53D22, 34M56

Contents

1. Introduction

Although a subject of by now venerable age, the Wentzel–Kramers–Brillouin (WKB) approximation, used since early days of quantum mechanics to study the quasi-classical limit of the Schrödinger equation, has enjoyed a surge of interest in the past decades, with a positive feedback of results between the mathematical and physical community. After the method was developed further by many mathematicians in the decades preceding the turn of the millennium (see the relatively recent reviews [18,37]), a new surge of interest in the subject was prompted by the emergence of a connection between the WKB approximation and the geometry of four-dimensional supersymmetric field theories in [22], where the rich geometry arising from the WKB graph and differentials was used to study BPS states of four-dimensional supersymmetric theories. This perspective has been later made more mathematically precise in [2, 12, 14]. The central object in this analysis is the WKB curve, a Riemann surface arising from the leading semiclassical approximation, that coincides with the Seiberg–Witten curve of the associated quantum field theory. Over the curve $\mathcal{C}$ one introduces the graph of horizontal trajectories for the projective connection entering as potential in the Schrödinger equation, defining a triangulation that allows to relate periods of the WKB differential to Fock–Goncharov coordinates [9, 21], defined on the monodromy manifold of the second order ODE.

The monodromy manifold of the Schrödinger equation is the $SL_2(\mathbb{C})$ character variety, that can be parametrized by Fock–Goncharov coordinates (in turn related to WKB periods). Over this space the Goldman Poisson bracket [26] is defined, inverted by the symplectic form Ω_G found in [1] on symplectic leaves $V^{\mathbf{r}}$, which was computed in [9] using complex shear (Fock–Goncharov) coordinates. In the paper [8] a natural set of Darboux coordinates for the Goldman symplectic form Ω_G, called homological shear coordinates, were found. The symplectic leaf $V^{\mathbf{r}}$ of the $SL(2)$ character variety, where $e^{\pm 2\pi i r_j}$ are the monodromy eigenvalues at the punctures, is the image under the monodromy map $\mathcal{F}$ of the moduli space of meromorphic flat connections with fixed residues, $\mathcal{A}^{\mathbf{r}}$. This is also a symplectic manifold, endowed with the Atiyah–Bott (pre-)symplectic form, that reduces to the Kirillov–Konstant symplectic form Ω_{KK} in genus 0. According to the theorem proved in [1, 27, 40], the monodromy map for a Fuchsian differential equation is a symplectomorphism between the two spaces $V^{\mathbf{r}}$ and $\mathcal{A}^{\mathbf{r}}$, i.e.

$$\mathcal{F}^*\Omega_G = 2\pi i \Omega_{KK}. \tag{1.1}$$

Another connection between monodromy of linear ODEs and supersymmetric QFTs comes from the theory of tau functions of isomonodromic systems, first introduced in the '80s by the Japanese School [35]. Starting from [23], the tau function of a large class of isomonodromy problems, including in particular the

sixth Painlevé equation and the Schlesinger system [24,28] as cases relevant to this work, was identified with a Fourier series of non-perturbative four-dimensional gauge theory partition functions. The quantum field theory corresponding to a given isomonodromic problem can be identified by the singularity structure of the linear system, or equivalently by its spectral curve, which coincides with the WKB curve [11]. In this context, the tau function is defined by requiring its logarithmic derivatives to be the isomonodromic Hamiltonian, which makes it determined up to an arbitrary function of the monodromy data.

In [10], after previous results in this direction in the papers [30,31], it was shown that it is possible to extend the definition of the isomonodromic tau function for Fuchsian systems on the Riemann sphere in a way that fixes not only the time dependence, but also the dependence on the monodromy parameters, by defining it to be the generating function for the monodromy symplectomorphism, that is, given symplectic potentials θ_{KK} and θ_G such that $\delta\theta_{KK} = \Omega_{KK}$, $\delta\theta_G = \Omega_G$, the tau function $\mathcal{T}$ is defined as

$$d\log\mathcal{T} := \theta_{KK} - \sum_{k=1}^{n} H_k\, dt_k - \frac{1}{2i\pi}\mathcal{F}^*\theta_G. \tag{1.2}$$

In [17] it was shown that this same expression, together with its generalization for Fuchsian systems over genus one Riemann Surfaces, arises from the Fredholm determinant representation of the tau function [16,24]. The definition (1.2) has the conceptual advantage of fully fixing the functional dependence of $\mathcal{T}$ on all local coordinates of $\mathcal{A}^{\mathbf{r}}$. In particular, it allows to determine the ratio of tau functions expressed in different monodromy coordinates, allowing to determine the so-called connection constant [30,32]. In terms of 2d CFT, the connection constant is interpreted as the ratio between conformal blocks in different channels, known as the fusion kernel [5,25,29], and in terms of the corresponding 4d gauge theory [34,43] it is the ratio of dual partition functions defined in different gauge theory regimes. (A different point of view was taken in [15], where tau functions were characterized as *difference* generating functions. While there are clear similarities between the two definitions, the precise relation is not yet completely clear, and we leave it to future studies.)

In this paper we take the first step towards the WKB analysis of the generating function of monodromy symplectomorphism defined by

$$\delta\log\mathcal{G} := \mathcal{F}^*\theta_G - 2\pi i\theta_{KK} \tag{1.3}$$

for the case of a general Fuchsian system on the Riemann sphere, by computing explicitly its first three contributions. From a WKB standpoint, the main difference with respect to previous works such as [8,22] is that the 2×2 Schlesinger system gives rise to the Schrödinger equation with apparent singularities. Furthermore, we consider variations δ that do not move the position of the (non-apparent) singularities, that we denote by $z_1, \ldots, z_{g+2}$. Even though the apparent singularities introduce technical complications, it is still possible to explicitly integrate the equation for the generating function in the first three orders, resulting in Theorems 3.1–3.3.

To obtain the WKB-expansion of the *isomonodromic* tau function from our computation, one would have to consider also variation of the positions of the poles, and then impose that the apparent singularities evolve according to the isomonodromic deformation equations, as it was done in [13, 33] for the case of Painlevé I. In this case R_j's become $\hbar$ and time-dependent, so to get the true asymptotic expansion in $\hbar$, one would have to further expand the resulting expression imposing the isomonodromic time evolution.

Let us now introduce the necessary definitions and notations (for more details about the notations we refer the reader to the beginning of Section 3). We are going to study the $\hbar$-expansion for the equation

$$\frac{d\Psi}{dz} = R(z)\Psi(z) = \frac{1}{\hbar} \sum_{j=1}^{g+2} \frac{R_j}{z - z_j} \Psi(z). \tag{1.4}$$

Let

$$R_\infty = R_{g+3} = -\sum_{j=1}^{g+2} R_j, \tag{1.5}$$

and assume that R_∞ is diagonal,

$$R_\infty = \begin{pmatrix} r_\infty & 0 \\ 0 & -r_\infty \end{pmatrix}. \tag{1.6}$$

Denote the eigenvalues of the matrices R_j by r_j and $-r_j$ for $j = 1, \ldots, g + 3$. The solution Ψ of (1.4) has monodromies $M_1, \ldots, M_{g+3}$ around the points $z_1, z_2, \ldots, z_{g+2}, \infty$ which satisfy the relation

$$M_1 \cdots M_{g+3} = I.$$

Assume that the matrices R_j are diagonalizable, and let

$$R_j = G_j L_j G_j^{-1}, \tag{1.7}$$

where $L_j = \mathrm{diag}(r_j, -r_j)$. Then, on the space $\mathcal{A}^{\mathbf{r}}$ which is the symplectic leaf $r_j = const$ quotiented over simultaneous transformations of the form $R_j \to GR_jG^{-1}$, $G \in SL(2, \mathbb{C})$, the Kirillov–Kostant symplectic form is defined by

$$\Omega_{KK} = -\frac{1}{\hbar} \mathrm{tr} \sum_{k=1}^{g} L_k G_k^{-1} \delta G_k \wedge G_k^{-1} \delta G_k \tag{1.8}$$

with the symplectic potential θ_{KK} (such that $\delta\theta_{KK} = \omega_{KK}$) given by

$$\theta_{KK} = \frac{1}{\hbar} \mathrm{tr} \sum_{k=1}^{g} L_k G_k^{-1} \delta G_k. \tag{1.9}$$

The construction of the Darboux homological shear coordinates for the Goldman symplectic form [8], parametrizing the monodromy representation of solutions of (1.4), looks as follows: write the coefficient matrix of the linear system (1.4) as

$$R(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}, \tag{1.10}$$

and define

$$Q_0(z) \equiv -\det R(z) = a(z)^2 + b(z)c(z). \tag{1.11}$$

We will assume all zeros x_j, $j = 1, \ldots, 2g+2$ of Q_0 to be simple; then Q_0 takes the form

$$Q_0(z) = C_0 \frac{\prod_{j=1}^{2g+2}(z - x_j)}{\prod_{j=1}^{g+2}(z - z_j)^2}, \tag{1.12}$$

where C_0 is a proportionality constant. Consider the hyperelliptic curve $\mathcal{C}$ of genus g with branch points at $x_1, \ldots x_{2g+2}$ defined by

$$\mu^2 = Q_0(z), \tag{1.13}$$

and introduce the following meromorphic differential of the third kind, with $2g$ poles on $\mathcal{C}$:

$$v = \mu(z)\, z. \tag{1.14}$$

The horizontal trajectories of v generically connect its zeros x_j with its poles z_j; denote the resulting critical graph by Γ. From the graph Γ one can construct two graphs embedded in the Riemann sphere: the graph Σ with vertices at z_j whose faces are triangles, and the tri-valent graph Σ^* dual to Σ with tri-valent vertices at x_j, as in Figure 1.1.
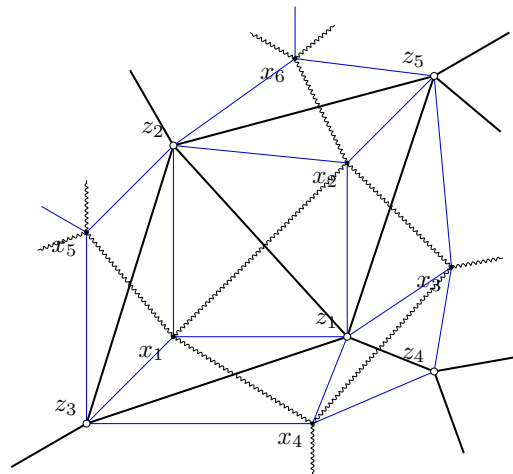

Fig. 1.1: Horizontal critical trajectories (blue lines) connect poles z_j with zeros x_k of Q and form the critical graph Γ. Black edges connecting poles z_j form the graph Σ (the triangulation of $\mathcal{C}$) while the zigzag curves connect zeros x_j and form the dual tri-valent graph Σ^*. All edges of Σ^* can be chosen to be the branch cuts of $\mathcal{C}$.

The (logarithmic) complex shear coordinates on V are assigned to edges of Σ^*; according to [8] these coordinates can be extended by linearity to get homological shear coordinates assigned to elements of $H_1(\mathcal{C})$. Introduce a canonical basis of cycles $(a_j, b_j)_{j=1}^g$ on $\mathcal{C}$ with the intersection index and consider the corresponding set of logarithmic homological shear coordinates (ρ_{a_j}, ρ_{b_j}) on $V^{\mathbf{r}}$ (see App of [8]),

among which there are g relations. In terms of (ρ_{a_j}, ρ_{b_j}), Goldman's symplectic form Ω_G on $V^{\mathbf{r}}$ looks as follows:

$$\Omega_G = \sum_{j=1}^{g} \delta\rho_{a_j} \wedge \delta\rho_{b_j}. \tag{1.15}$$

The corresponding symplectic potential on $V^{\mathbf{r}}$ satisfying $\delta\theta_G = \Omega_G$ will be chosen as follows:

$$\theta_G = \frac{1}{2}\sum_{j=1}^{g}(\rho_{a_j}\delta\rho_{b_j} - \rho_{b_j}\delta\rho_{a_j}). \tag{1.16}$$

Denote the monodromy map by $\mathcal{F}$. The theorem of [1, 27, 40] states that

$$2\pi i \Omega_{KK} = \mathcal{F}^*\Omega_G, \tag{1.17}$$

so that there exists the generating function $\mathcal{G}$ such that

$$\delta\mathcal{G} = \mathcal{F}^*\theta_G - 2\pi i \theta_{KK}. \tag{1.18}$$

In this paper we compute explicitly the first three non-trivial terms in the $\hbar$-expansion of $\mathcal{G}$ for fixed times:

$$\mathcal{G} = \frac{\mathcal{G}_{-2}}{\hbar^2} + \frac{\mathcal{G}_{-1}}{\hbar} + \mathcal{G}_0 + \cdots \tag{1.19}$$

in Theorems 3.1, 3.2 and 3.3. Note that, due to the almost trivial $\hbar$ dependence of the linear system (1.4), the Kirillov–Konstant symplectic potential in (1.18) contributes only at order $\mathcal{O}(\hbar^{-1})$, while the potental θ_G contains terms of every order starting from $\hbar^{-2}$. The existence of the monodromy symplectomorphism $\mathcal{F}$ implies the highly nontrivial consequence that the WKB expansion of Goldman's symplectic potential θ_G gives a closed form at all orders except for $\mathcal{O}(\hbar^{-1})$. The computation of the coefficients in the expansion (1.19) is based on the $\hbar$-expansion of the homological shear coordinates ρ_ℓ for $\ell \in H_1(\mathcal{C}, \mathbb{Z})$ [2, 8]:

$$\rho_\ell = \int_\ell \left(\frac{v}{\hbar} + v_0 + \hbar v_1 + \cdots\right), \tag{1.20}$$

where $v_0, v_1, v_2, \ldots$ are meromorphic differentials on $\mathcal{C}$ arising from the WKB expansion. The integrals in (1.20) are called Voros symbols after [44].

Our main result is the integration of definition 1.3, resulting in the explicit determination of $\mathcal{G}_{-2}, \mathcal{G}_{-1}, \mathcal{G}_0$ in terms of contour integrals of WKB differentials v, v_0, v_1 on the WKB curve (1.13). The leading contribution to $\mathcal{G}$ is

$$\mathcal{G}_{-2} = -\pi i \left(r_\infty \text{reg} \int_{\infty^{(2)}}^{\infty^{(1)}} v + \sum_{j=1}^{g} r_j \text{reg} \int_{z_j^{(2)}}^{z_j^{(1)}} v\right), \tag{1.21}$$

where $p^{(j)}$ denotes the image of the point $p \in \mathbb{P}^1$ on the j-th sheet of $\mathcal{C}$, and the regularized integrals are defined by

$$\text{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v := \lim_{\epsilon \to 0}\left(\int_{z_j^{(2)}+\epsilon}^{z_j^{(1)}+\epsilon} v - 2r_j \log\epsilon\right) \tag{1.22}$$

and

$$\operatorname{reg} \int_{\infty^{(2)}}^{\infty^{(1)}} v := \lim_{R \to \infty} \left(\int_{R^{(2)}}^{R^{(1)}} v + 2r_\infty \log R \right). \tag{1.23}$$

The subleading term $\mathcal{G}_{-1}$ is given by the formula

$$\mathcal{G}_{-1} = \frac{1}{2} \langle v, v_0 \rangle - 2\pi i \sum_{j=1}^{g} q_j A_j - \pi i \sum_{k=1}^{g} B_k - \pi i \sum_{j=1}^{g} j A_j, \tag{1.24}$$

where A_j, B_j are the A- and B-periods of v, and $\langle \cdot, \cdot \rangle$ is the antisymmetric pairing defined by Riemann's bilinear relations

$$\langle w, \tilde{w} \rangle := \oint_{\partial \tilde{\mathcal{C}}} \left(\int^x w \right) \tilde{w}, \tag{1.25}$$

where $\tilde{\mathcal{C}}$ is the fundamental polygon of $\mathcal{C}$. Finally, the constant term $\mathcal{G}_0$ in the WKB expansion of the generating function is

$$\mathcal{G}_0 = -12\pi i \ln \tau_B(CP^1, Q_0) + F - \frac{1}{2} \langle v_1, v \rangle, \tag{1.26}$$

$$\frac{F}{i\pi} = \frac{1}{2} \sum_{j=1}^{g} \int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 + \frac{1}{2} \int_{\infty^{(2)}}^{\infty^{(1)}} v_0 + \ln \frac{\prod_{a,k}(\lambda_a - z_k)}{\prod_j \mu_j \prod_{a<b}(\lambda_a - \lambda_b)}$$

$$- \frac{1}{4r_\infty} \int_{\infty^{(2)}}^{\infty^{(1)}} v - \sum_{k=1}^{g+2} \frac{1}{4r_k} \int_{z_k^{(2)}}^{z_k^{(1)}} v. \tag{1.27}$$

Here λ_j's are the location of the zeros of the $(2,1)$ entry of the matrix $R(z)$ (1.4), τ_B is Bergman's tau function (see Appendix B) and the regularization in the integrals is defined by

$$\int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 := \lim_{\substack{p \to \lambda_j^{(1)} \\ q \to \lambda_j^{(2)}}} \int_q^p v_0 - \frac{1}{2} \ln(z(p) - \lambda_j) - \frac{1}{2} \ln(z(q) - \lambda_j), \tag{1.28}$$

where $z : \mathcal{C} \to \mathbb{C}$ is the projection on the z-coordinate.

2. Second order equation and its WKB expansion

The starting point of our discussion is the linear system (1.4), with coefficient matrix (1.10). Let us denote the zeros of the component $c(z) = R_{21}(z)$ by $\lambda_1, \ldots, \lambda_g$. Then, since R_∞ is diagonal, we have $c(z) \sim C/z^2$ as $z \to \infty$ and we can write:

$$c(z) = C \frac{\prod_{j=1}^{g}(z - \lambda_j)}{\prod_{k=1}^{g+2}(z - z_k)} \tag{2.1}$$

for some constant C. If we also denote by

$$\mu_j = a(\lambda_j), \quad j = 1, \ldots, g, \tag{2.2}$$

we can write $a(z)$ as follows:

$$a(z) = \sum_{j=1}^{g} \mu_j \prod_{k=1}^{n} \frac{\lambda_j - z_k}{z - z_k} \prod_{l \neq k}^{g} \frac{z - \lambda_l}{\lambda_j - \lambda_l}. \tag{2.3}$$

Proposition 2.1. *Let $\psi(z; \hbar)$ be the second component of the vector-valued solution of* (1.4). *Then the function $f = \sqrt{\frac{\hbar}{c(z)}} \psi(z; \hbar)$ satisfies the following second order ODE:*

$$\frac{d^2 f}{dz^2} - Q(z; \hbar) f = 0, \tag{2.4}$$

where the potential $Q(z; \hbar)$ *is given by*

$$Q(z; \hbar) = \frac{Q_0}{\hbar^2} + \frac{Q_1}{\hbar} + Q_2. \tag{2.5}$$

Here,

$$Q_0 = -\det R = a^2 + bc, \tag{2.6}$$

$$Q_1 = a \frac{d}{dz} \ln \left(\frac{c}{a} \right), \tag{2.7}$$

$$Q_2 = \frac{1}{4} \left(\frac{c'}{c} \right)^2 - \frac{1}{2} \left(\frac{c'}{c} \right)' = -\frac{1}{2} \mathcal{S} \left(\int^z c(x) \, dx, z \right), \tag{2.8}$$

where $\mathcal{S}(f, z)$ is the Schwarzian derivative

$$\mathcal{S}(f, z) \equiv \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2. \tag{2.9}$$

Proof. Let us apply to the linear system (1.10) the following (singular) gauge transformation

$$\Psi(z) = c^{-\frac{\sigma_3}{2}} \begin{pmatrix} 1 & \frac{\hbar}{2} \frac{c'}{c} + a \\ 0 & 1 \end{pmatrix} \hbar^{\frac{\sigma_3}{2}} F(z) \equiv g(z) F(z), \tag{2.10}$$

which maps the matrix $R(z)$ to the matrix

$$g^{-1} R g - \hbar g^{-1} \frac{dg}{dz} = \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix}, \tag{2.11}$$

where

$$Q = \frac{a^2 + bc}{\hbar^2} + \frac{a \log \left(\frac{c}{a} \right)'}{\hbar} + \left[\frac{3}{4} \left(\frac{c'}{c} \right)^2 - \frac{1}{2} \frac{c''}{c} \right]. \tag{2.12}$$

Then, the elements F_{11}, F_{12} of the matrix F are the two independent solutions of (2.4). $\square$

2.1. Properties of Q_0, Q_1, and Q_2. Let us discuss the properties of the meromorphic functions Q_0, Q_1, and Q_2:

- The function Q_0 can be written as follows:

$$Q_0(z) = C_0 \frac{P(z)}{\prod_{j=1}^{g+2}(z-z_j)^2} = \sum_{j=1}^{g+2}\left(\frac{r_j^2}{(z-z_j)^2} + \frac{H_j}{z-z_j}\right), \qquad (2.13)$$

where $P(z) = \prod_{j=1}^{2g+2}(x-x_j)$ is a polynomial of degree $2g+2$ and

$$C_0 = \sum_{j=1}^{g+2} r_j^2. \qquad (2.14)$$

- Function Q_1:
  Notice that $a \sim -r_\infty/z + \cdots$ as $z \to \infty$ while c behaves as C/z^2. Therefore, Q_1 behaves as $r_\infty z^{-2}$ as $z \to \infty$. If we write

$$Q_1(z) = \sum_{j=1}^{g}\frac{\mu_j}{z-\lambda_j} + \sum_{j=1}^{g+2}\frac{\gamma_j}{z-z_j} \qquad (2.15)$$

for some parameters $\gamma_j \in \mathbb{C}$, we get the following condition on the parameters entering in Q_1:

$$\sum_{j=1}^{g}\mu_j + \sum_{j=1}^{g+2}\gamma_j = 0. \qquad (2.16)$$

- Function Q_2:
  The function Q_2 in (2.8) can be written as follows:

$$Q_2 = \frac{1}{4}\left(\sum_{j=1}^{g}\frac{1}{(z-\lambda_j)} - \sum_{k=1}^{g+2}\frac{1}{z-z_k}\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{g}\frac{1}{(z-\lambda_j)^2} - \sum_{k=1}^{g+2}\frac{1}{(z-z_k)^2}\right), \qquad (2.17)$$

and its Laurent expansion near λ_ℓ looks as follows:

$$Q_2(z) = \frac{3}{4(z-\lambda_\ell)^2} + \frac{E_\ell}{z-\lambda_\ell} + F_\ell + \cdots \qquad (2.18)$$

with

$$E_\ell = \frac{1}{2}\left(\sum_{\substack{i \\ i\neq\ell}}\frac{1}{\lambda_\ell-\lambda_i} - \sum_{k=1}^{n}\frac{1}{\lambda_\ell-z_k}\right), \qquad (2.19)$$

$$F_\ell = \frac{1}{4}\left(\sum_{\substack{i \\ i\neq\ell}}\frac{1}{\lambda_\ell-\lambda_i} - \sum_{k=1}^{g+2}\frac{1}{\lambda_\ell-z_k}\right)^2. \qquad (2.20)$$

Near $z = \infty$ we have

$$Q_2(z) = \frac{(\sum_j \lambda_j - \sum_k z_k)^2 + 2\sum_j \lambda_j^2 - 2\sum_k z_k^2}{4z^4} + O(z^{-5}). \qquad (2.21)$$

The potential (2.5) of the resulting Schrödinger equation has second order poles at the points $z_1, \ldots, z_{g+2}$ with biresidues $\hbar^{-2} r_1^2 - \frac{1}{4}, \ldots, \hbar^{-2} r_{g+2}^2 - \frac{1}{4}$:

$$Q = \left(\frac{r_j^2}{\hbar^2} - \frac{1}{4} \right) \frac{1}{(z - z_j)^2} + O((z - z_j)^{-1}), \quad z \to z_j, \tag{2.22}$$

it has the following behaviour at $z \sim \infty$

$$Q = \left(\frac{r_\infty^2}{\hbar^2} + \frac{r_\infty}{\hbar} \right) \frac{1}{z^2} + \ldots,$$

and second order poles at the points $\lambda_1, \ldots, \lambda_{n-3}$ with biresidues $3/4$:

$$Q = \frac{3/4}{(z - \lambda_j)^2} + \left(\frac{\mu_j}{\hbar} + E_j \right) \frac{1}{z - \lambda_j} + \mathcal{O}(1). \tag{2.23}$$

The singularities at λ_j are *apparent*, i.e. the monodromy of the fundamental matrix of equation (2.4) around λ_j is $-\mathbf{1}$ (the terminology here is the one accepted in the specific literature but it is a misnomer. According to the classical use of the term, an "apparent" singularity in an ODE is a point of singularity of the coefficients such that all the solutions are *analytic* in a neighbourhood thereof. Here, on the other hand, both solutions have a branchpoint with exponents $\pm \frac{1}{2}$. In general this could be a resonance, but the fact that there are no logarithms in the solution is the property that is termed improperly "apparent".

This can be seen by inspection of the gauge transformation (2.10) and is a consequence of the following (for a proof, see [13]):

Lemma 2.2. *The ODE*

$$f''(z) = \left(\frac{3}{4z^2} + \frac{A}{z} + B + \mathcal{O}(z) \right) f(z) \tag{2.24}$$

has two linearly independent solutions with Frobenius exponents $\pm \frac{1}{2}$ at $z = 0$ if and only if $A^2 = B$.

The triviality (in $\mathbb{P}SL_2$) of monodromies at the points λ_j translates to the following Bethe equations for the quantities γ_k appearing in Q_1:

Lemma 2.3. *The monodromy of the ODE (2.4) around the apparent singularities λ_j is $-\mathbf{1}$ if and only if the following set of equations are satisfied for coefficients γ_j, $j = 1, \ldots, g + 2$ from (2.15):*

$$\sum_{k=1}^{g+2} \frac{\gamma_k}{\lambda_\ell - z_k} = \sum_{\substack{i \\ i \neq \ell}} \frac{\mu_\ell - \mu_i}{\lambda_\ell - \lambda_i} - \mu_\ell \sum_k \frac{1}{\lambda_\ell - z_k} \tag{2.25}$$

for $\ell = 1, \ldots, g$.

Proof. Use the behaviour (2.23) of Q at λ_j, and apply Lemma 2.2 to the Schrödinger equation (2.4):

$$\left(\frac{\mu_j}{\hbar} + \operatorname*{res}_{\lambda_j} Q_2\right)^2 = \frac{Q_0(\lambda_j)}{\hbar^2} + \frac{1}{\hbar}Q_1^{\mathrm{reg}}(\lambda_j) + Q_2^{\mathrm{reg}}(\lambda_j), \qquad (2.26)$$

where $\operatorname{res}_{\lambda_j} Q_2$ denotes the coefficient of $\frac{1}{z-\lambda_j}$ in the expansion of Q_2 around λ_j, as in equation (2.24), and Q_i^{reg} the regular part of the expansion at λ_j. Using the explicit form of Q_0, Q_1, Q_2 in equations (2.13), (2.15), (2.18), we find

$$\left(\frac{\mu_\ell}{\hbar} + \frac{1}{2}\sum_{\substack{i \\ i\neq\ell}} \frac{1}{\lambda_\ell - \lambda_i} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{\lambda_\ell - z_k}\right)^2$$

$$= \frac{Q_0(\lambda_\ell)}{\hbar^2} + \frac{1}{\hbar}\sum_{k=1}^{n} \frac{\gamma_k}{\lambda_\ell - z_k} + F_\ell + \frac{1}{\hbar}\sum_{\substack{i \\ i\neq\ell}} \frac{\mu_i}{\lambda_\ell - \lambda_i}.$$

On account that $\mu_\ell^2 = Q_0(\lambda_\ell)$ and equation (2.20) we are left with the Bethe equations for the quantities γ_k. $\qquad\square$

2.2. Canonical cover and WKB differentials.

We now introduce the WKB approximation of equation (2.4):

$$f_{zz} - \left(Q_2 + \frac{Q_1}{\hbar} + \frac{Q_0}{\hbar^2}\right) f = 0. \qquad (2.27)$$

Introduce the cover $\mathcal{C}$ given by

$$v^2 = Q_0(z)dz^2. \qquad (2.28)$$

The curve $\mathcal{C}$ has $2g + 2$ branch points which we denote by $x_1, \ldots, x_{2g+2}$. We denote the projection of $\mathcal{C}$ to the z-plane by f, and the hyperelliptic involution on $\mathcal{C}$ by ν. The homology group of the curve $\mathcal{C}$, punctured at $2g + 4$ points $f^{-1}(z_j)$, can be represented as a direct sum of even and odd components under the involution ν:

$$H_1\left(\mathcal{C} \setminus \{f^{-1}(z_j)\}_{j=1}^{g+2}\right) = H_+ \oplus H_-, \quad \dim H_+ = g+1, \quad \dim H_- = 3g+1. \quad (2.29)$$

Denote the points projecting to the poles z_j by $z_j^{(1,2)}$:

$$f^{-1}(z_j) = \left\{z_j^{(1)}, z_j^{(2)}\right\}.$$

The enumeration is chosen such that

$$\operatorname*{res}_{z_j^{(1)}} v = r_j, \quad \operatorname*{res}_{z_j^{(2)}} v = -r_j.$$

Let small positively-oriented loops around points $\{z_j^{(1)}\}_{j=1}^{g+1}$ be denoted by $\{t_j\}_{j=1}^{g+1}$. Then generators of H_+ can be chosen to be $t_j^+ = t_j + \nu_* t_j$, $j = 1, \ldots, g+1$. The generators of H_- can be chosen to be

$$\{a_j, b_j\}_{j=1}^g, \quad \{t_j^-\}_{j=1}^{g+1}, \tag{2.30}$$

where

$$t_j^- = \frac{1}{2}(t_j - \nu_* t_j).$$

Introduce the divisor of degree g given by

$$D = \lambda_1^{(1)} + \cdots + \lambda_g^{(1)}, \tag{2.31}$$

where $\lambda_j^{(1)} = (\lambda_j, \mu_j)$ with λ_j being the zeros of $c(z)$ and $\mu_j = a(\lambda_j)$. Then $\lambda_j^{(2)}$ is the point having the same projection on z-plane but lying on another sheet of $\mathcal{C}$, i.e. $\lambda_j^{(2)} = (\lambda_j, -\mu_j)$.

To study the limit $\hbar \to 0$ of equation (2.27) we introduce the asymptotic series $s = \sum_{k=-1}^{\infty} \hbar^k s_k$ and write the asymptotic series for the solution f in the form

$$f = v^{-1/2} \exp\left\{ \int_{x_0}^x (\hbar^{-1} s_{-1} + s_0 + \hbar s_1 + \ldots) v \right\}, \tag{2.32}$$

where s_k are meromorphic functions on $\mathcal{C}$ and x_0 is a basepoint. We introduce also the meromorphic differentials

$$v_k = \frac{1}{2}(s_k + \nu^* s_k)v. \tag{2.33}$$

The differential v_k satisfies

$$\nu^* v_k = -v_k. \tag{2.34}$$

As a corollary of (2.27) the function s satisfies the Riccati equation which in coordinate-independent form can be written as follows:

$$ds + vs^2 = -qv + \frac{Q_1 v}{\hbar} + \frac{v}{\hbar^2}, \tag{2.35}$$

where $q = -Q_2 - \frac{\mathcal{S}_v}{2}$ and $\mathcal{S}_v = \mathcal{S}(\int^z v, \cdot)$. Equivalently, since $Q_2 = -\frac{1}{2}\mathcal{S}\left(\int^z c(x)\,dx, \cdot\right)$, we can represent the meromorphic function q on $\mathcal{C}$ in the coordinate-independent form:

$$q = \frac{1}{2}\left(\mathcal{S}\left(\int^z c(x)\,dx, \cdot\right) - \mathcal{S}\left(\int^z v, \cdot\right)\right) = \frac{1}{2}\mathcal{S}\left(\int^z c(x)\,dx, \int^z v\right). \tag{2.36}$$

Equivalently, (2.35) can be written as

$$d\left(\sum_{k=-1}^{\infty} \hbar^k s_k\right) + v\left(\sum_{k=-1}^{\infty} \hbar^k s_k\right)^2 = -qv + \frac{Q_1 v}{\hbar} + \frac{v}{\hbar^2}.$$

The coefficients of $\hbar^{-2}$, $\hbar^{-1}$ and $\hbar^0$ give

$$s_{-1} = \pm 1, \quad s_0 = \frac{Q_1}{2Q_0}, \quad s_1 = -\frac{s_0^2}{2} - \frac{q}{2} - \frac{\mathrm{d}s_0}{2v} \tag{2.37}$$

so that $v_{-1} = \pm v$. We shall choose the "+" sign.

The higher functions s_k, $k \geq 1$ can be found recursively from relations

$$\mathrm{d}s_k + v \sum_{\substack{j+l=k \\ j,l \geq -1}} s_j s_l = 0, \quad k \geq 1, \tag{2.38}$$

which imply

$$s_{k+1} = -\frac{1}{2s_{-1}} \left(\frac{\mathrm{d}s_k}{v} + \sum_{\substack{j+l=k \\ j,l \geq 0}} s_j s_l \right), \quad k \geq 1.$$

The first three differentials in the WKB expansion are obtained from (2.37) taking into account that the only term in (2.37) which is not skew-symmetric under ν^* is $-\mathrm{d}s_0/2v$:

$$v^2 = Q_0(z)(\mathrm{d}z)^2, \quad v_0 = \frac{1}{2v} Q_1(z)(\mathrm{d}z)^2, \tag{2.39}$$

$$v_1 = -\frac{v_0^2}{2v} + \frac{1}{2v} Q_2(z)(\mathrm{d}z)^2 + \frac{1}{4v} \mathcal{S}\left(\int^z v, z \right) (\mathrm{d}z)^2. \tag{2.40}$$

2.3. Properties of WKB differentials. Here are the properties of the WKB differentials (2.39), (2.40):

- The differential of the third kind on $\mathcal{C}$ given by

$$v = \mu \, dz \tag{2.41}$$

has first order poles at points $z_j^{(1,2)}$ with the residues $\pm r_j$, and at $\infty^{(1,2)}$ with residues $\pm r_\infty$. The latter statement follows from the fact that near $\infty^{(1,2)}$ we have

$$v \sim a(z) \, \mathrm{d}z = \mp r_\infty \frac{dz}{z}, \tag{2.42}$$

$$\operatorname*{res}_{\infty^{(1,2)}} v = \pm r_\infty. \tag{2.43}$$

- The differential of the third kind

$$v_0 = \frac{Q_1}{2v} (dz)^2 \tag{2.44}$$

is holomorphic at branch points of $\mathcal{C}$ (at the branch points, v has second order zero, while the differential $\mathrm{d}z$ has a first order zero). Let us now discuss the

behaviour of v_0 near $\lambda_j^{(1,2)}$. Near $\lambda_j^{(1)}$ we have $v \sim a(\lambda_j)dz$ and $Q_1 \sim \frac{\mu_j}{z-\lambda_j}$. Therefore, near $\lambda_j^{(1,2)}$ one has the asymptotics

$$v_0 \sim \pm\frac{\mathrm{d}z}{2(\lambda - \lambda_j)}.$$

Moreover, near $\infty^{(1,2)}$ we have $Q_1 \sim \frac{r_\infty}{z}$; thus, using (2.42), v_0 has simple poles at $\infty^{(1,2)}$ with residues $\pm 1/2$, respectively.

- We can write the differential v_1 (which turns out to be of second kind) as

$$v_1 = -\frac{v_0^2}{2v} + \frac{Q_2}{2v}(\mathrm{d}z)^2 + \frac{1}{4v}\mathcal{S}\left(\int v, z\right)(\mathrm{d}z)^2, \tag{2.45}$$

or more more symmetrically as

$$v_1 = -\frac{v_0^2}{2v} + \frac{(\mathrm{d}z)^2}{4v}\left(\mathcal{S}\left(\int v, z\right) - \mathcal{S}(c, z)\right). \tag{2.46}$$

Using the composition property of the Schwarzian derivative, it can also be written as follows:

$$v_1 = -\frac{v_0^2}{2v} - \frac{v}{4}\mathcal{S}(c, \xi), \tag{2.47}$$

where $\xi = \int^z v$ is the flat coordinate defined by differential v.

To prove (2.47) we used the composition rule for $c(\xi(z))$:

$$\mathcal{S}(c, z)(\mathrm{d}z)^2 = \mathcal{S}(c, \xi)(\mathrm{d}\xi)^2 + \mathcal{S}(\xi, z)(\mathrm{d}z)^2. \tag{2.48}$$

The differential v_1 has second order poles at branch points and second order poles at $\infty^{(1,2)}$ with residues

$$\operatorname*{res}_{\infty^{(1,2)}} v_1 = \mp\frac{1}{4r_\infty}. \tag{2.49}$$

Lemma 2.4. *The conditions* (2.25) *of triviality of monodromies around* λ_j *are equivalent to equations*

$$\operatorname{res}\big|_{\lambda_j^{(1,2)}} v_1 = 0. \tag{2.50}$$

Proof. We use the expression

$$v_1 = \frac{1}{2v}\left[\frac{1}{2}\mathcal{S}\left(\int v, z\right) + Q_2(\mathrm{d}z)^2 - v_0^2\right]. \tag{2.51}$$

The term involving the Schwarzian derivative can be ignored since it is regular at λ_j. Using the expansions at $z = \lambda_j$

$$Q_0(z) = \mu_j^2 + (z - \lambda_j)Q_0'(\lambda_j) + \mathcal{O}(z - \lambda_j)^2,$$

$$Q_1(z)^2 = \frac{\mu_j^2}{(z - \lambda_j)^2} + \frac{2\mu_j}{z - \lambda_j}Q_1^{reg}(\lambda_j) + \mathcal{O}(1),$$

$$Q_2(z) = \frac{3}{4}\frac{1}{(z-\lambda_j)^2} + \frac{E_j}{z-\lambda_j} + \mathcal{O}(1)$$

and expressing v and v_0 in terms of Q_0, Q_1 we have

$$\frac{1}{2v}\left[Q_2(dz)^2 - v_0^2\right] = \pm\frac{1}{4\mu_j}\left[\frac{1}{(z-\lambda_j)^2} + \frac{1}{(z-\lambda_j)}(2\mu_j E_j - Q_1^{\text{reg}}(\lambda_j)) + \mathcal{O}(1)\right].$$

The vanishing of the residue is equivalent to the $O(\hbar^{-1})$ equation of (2.26). $\quad\square$

Let us summarize in a table all the properties of the WKB differential discussed in this section:

Differential	Pole	Order of the pole	Residue
$v = \mu dz$	$z_j^{(1,2)}$	1	$\pm r_j$
	$\infty^{(1,2)}$	1	$\pm r_\infty$
$v_0 = \frac{Q_1}{2v}(dz)^2$	$z_j^{(1,2)}$	1	$\pm\frac{1}{2}$
	$\lambda_j^{(1,2)}$	1	$\pm\frac{1}{2}$
$v_1 = -\frac{v_0^2}{2v} - \frac{v}{4}\mathcal{S}(c,\xi)$	$z_j^{(1,2)}$	2	$\mp\frac{1}{4r_\infty}$
	$\infty^{(1,2)}$	2	$\mp\frac{1}{4r_k}$
	$\lambda_j^{(1,2)}$	2	0
	x_j	2	

2.4. WKB expansion of complex shear coordinates. Let us assume that Q_0 is a Gaiotto–Moore–Nietzke (GMN) differential (i.e. it has no horizontal trajectories connecting two zeros [22]). Then the cover (2.28) (also called the "WKB curve") admits a canonical triangulation Σ, constructed as follows. Due to the genericity assumption on Q_0, the horizontal trajectories always start at a zero x_j and end at a pole z_k: let us call by Γ the critical graph whose edges are such trajectories. The WKB triangulation Σ is defined as the graph having the poles z_k's as vertices, with every face containing exactly one zero of Q. The dual graph to Σ, whose vertices are instead the zeros x_j, will be denoted by Σ^*. This construction is shown in Figure 1.1.

To each edge e of the graph Σ one assigns a coordinate $\rho_e \in \mathbb{C}$ whose exponential is a complex shear coordinate, the simplest example of a Fock–Goncharov coordinate (see Appendix A.2 of [8] for more details).

To every edge e of the triangulation it is possible to associate a cycle $\ell_e \in H_-$, defined to be the loop that goes clockwise around the edge e^* of Σ^*; to the cycle e one assigns the Fock-Goncharov coordinate ρ_e

$$\{\rho_e, \rho_{e'}\} = e \circ e'.$$

Because the Poisson bracket is constant, it is possible to choose linear combinations of the ρ_e's that are Darboux conjugate. These coordinates extend by linearity to the a_j and b_j cycles, that we denote by ρ_{a_j}, ρ_{b_j}, and their Goldman bracket is

$$\{\rho_{a_j}, \rho_{b_k}\}_G = \frac{\delta_{jk}}{2}. \tag{2.52}$$

The following proposition is an analog of Proposition 5.2 in [8]. It shows that the homological shear coordinates ρ_ℓ (for $\ell \in H^-(\mathcal{C})$) admit an asymptotic expansion in terms of periods of S_{odd}.

Proposition 2.5. *The formal asymptotic expansion of the homological shear coordinate ρ_ℓ for each $\ell \in H_-$ looks as follows:*

$$\rho_\ell \sim \frac{1}{\hbar} \int_\ell v + \int_\ell v_0 + \hbar \int_\ell v_1 + \cdots, \tag{2.53}$$

where v_k are given by (2.33). The relation (2.53) is understood in $PSL(2)$ sense, i.e. up to an addition of $\pi i k$ for $k \in \mathbb{Z}$.

Proof. The proof is parallel to the proof of Proposition 5.2 of [8]. The difference is the presence of the term Q_1 in this paper which was absent in [8]; moreover, Q_1 has additional singularities at λ_j which are apparent singularities of the equation (2.4) were absent in [8] and also in [3]. In the frameworks of [8] and [3] the differential V contains only odd powers of $\hbar$ (since in these papers it is assumed that $Q_1 = 0$) while in our present context we also have all even powers.

The presence of apparent singularities at λ_j does not modify the asymptotics (2.53) by the following reason: the issue of apparent singularities at the poles of Q_1 is completely analogous to the case of the Lax pair for the sixth Painlevé equation (i.e. our case with $g = 1$); we refer to [37], Theorem 4.4. The reason why there are no different Stokes' regions in the WKB analysis near a λ_j is that, up to a (Borel resummable $\hbar$–dependent) conformal change of coordinate $z \to \xi(z, \hbar)$, the local model of the equation in a neighbourhood of $z = \lambda_j$ is

$$-\frac{d^2}{d\xi^2} f + \left(\frac{4\xi^2}{\hbar^2} - \frac{3}{4\xi^2} \right) f = 0. \tag{2.54}$$

This equation, while formally displaying a double turning point at $\xi = 0$ (corresponding to $z = \lambda_j$), does not, in fact, exhibit any Stokes' phenomenon since its general solution is explicitly written as

$$f(\xi) = \frac{A}{\sqrt{\xi}} e^{\frac{1}{\hbar}\xi^2} + \frac{B}{\sqrt{\xi}} e^{-\frac{1}{\hbar}\xi^2}. \tag{2.55}$$

This implies that the analysis of [8] goes through without further modifications. In the case of the opers relevant for all the Painlevé equations I–VI this was discussed in detail in Chapter 4 of [37] (and references therein). $\square$

3. WKB expansion of the generating function

The generating function $\mathcal{G}$ defined by (1.18) has the following formal expansion in powers of $\hbar$:

$$\mathcal{G} = \frac{\mathcal{G}_{-2}}{\hbar^2} + \frac{\mathcal{G}_{-1}}{\hbar} + \mathcal{G}_0 + \ldots. \tag{3.1}$$

In this section we compute the first three coefficients, $\mathcal{G}_{-2}$, $\mathcal{G}_{-1}$ and $\mathcal{G}_0$. Let us introduce the following notation: for any two 1-forms w and $\tilde{w}$ we consider their periods (A_j, B_j) and $(\tilde{A}_j, \tilde{B}_j)$ and introduce the pairing

$$\langle w, \tilde{w} \rangle = \sum_{j=1}^{g} A_j \tilde{B}_j - B_j \tilde{A}_j. \tag{3.2}$$

The expansion of the symplectic potential θ_G from equation (1.16) can be written as follows using (2.53) and the pairing (3.2):

$$\theta_G = \frac{\theta_G^{(-2)}}{\hbar^2} + \frac{\theta_G^{(-1)}}{\hbar} + \theta_G^{(0)} + O(\hbar), \tag{3.3}$$

where

$$\theta_G^{(-2)} = \frac{1}{2} \langle v, \delta v \rangle, \tag{3.4}$$

$$\theta_G^{(-1)} = \langle v_0, \delta v \rangle - \frac{1}{2} \delta \langle v_0, v \rangle, \tag{3.5}$$

$$\theta_G^{(0)} = \frac{1}{2} \langle v_0, \delta v_0 \rangle + \langle v_1, \delta v \rangle - \frac{1}{2} \delta \langle v_1, v \rangle. \tag{3.6}$$

3.1. Formula for $\mathcal{G}_{-2}$. Chose a set of generators of $H_1(\mathcal{C})$ which we denote by $\{a_j, b_j\}_{j=1}^{g}$, $\{t_j\}_{j=1}^{g-1}$. Introduce the a and b-periods of the differential v:

$$A_j = \int_{a_j} v, \quad B_j = \int_{b_j} v.$$

Theorem 3.1. *The equation* (3.4) *for* $\mathcal{G}_{-2}$ *can be written as*

$$\delta \mathcal{G}_{-2} = \frac{1}{2} \sum_{j=1}^{g} (A_k \delta B_k - B_k \delta A_k) \tag{3.7}$$

and its solution is given by

$$\mathcal{G}_{-2} = -\pi i \left(r_\infty \mathrm{reg} \int_{\infty^{(2)}}^{\infty^{(1)}} v + \sum_{j=1}^{g} r_j \mathrm{reg} \int_{z_j^{(2)}}^{z_j^{(1)}} v \right), \tag{3.8}$$

where

$$\mathrm{reg} \int_{z_j^{(2)}}^{z_j^{(1)}} v = \lim_{\epsilon \to 0} \left(\int_{z_j^{(2)} + \epsilon}^{z_j^{(1)} + \epsilon} v - 2 r_j \log \epsilon \right)$$

and

$$\mathrm{reg} \int_{\infty^{(2)}}^{\infty^{(1)}} v = \lim_{R \to \infty} \left(\int_{R^{(2)}}^{R^{(1)}} v + 2 r_\infty \log R \right).$$

Proof. In the coordinate system (A_j, λ_j), the form in the r.h.s. of (3.7) has only δA_j-contributions since it depends only on the moduli of $\mathcal{C}$, and not on the point of Jacobian. One can write v as follows, using the properties of v that we listed in Section 2.3:

$$v = r_\infty w_{\infty^{(2)}, \infty^{(1)}} + \sum_{j=1}^{g} r_j w_{z_j^{(2)}, z_j^{(1)}} + \sum_{j=1}^{g} A_j u_j, \tag{3.9}$$

where $w_{x,y}$ is the differential of third kind on $\mathcal{C}$ with residues -1 and $+1$ at x and y, respectively, normalized by the condition of vanishing a-periods, and u_j is the holomorphic differential normalized via $\int_{a_k} u_j = \delta_{jk}$. The b-period of $w_{x,y}$ is given by

$$\oint_{b_j} w_{x,y} = 2\pi i \int_x^y u_j, \tag{3.10}$$

so that

$$B_k = \int_{b_k} v = 2\pi i \left(r_\infty \int_{\infty^{(2)}}^{\infty^{(1)}} u_k + \sum_{j=1}^{g} r_j \int_{z_j^{(2)}}^{z_j^{(1)}} u_k \right) + \sum_{j=1}^{g} A_j \Omega_{jk}. \tag{3.11}$$

Moreover, since z_j and r_j are independent of the periods A_j, from (3.9) and (3.11) we have

$$\frac{\delta v}{\delta A_j} = u_j, \quad \frac{\delta B_k}{\delta A_j} = \Omega_{jk} \tag{3.12}$$

and, therefore,

$$\sum_{k=1}^{g} (A_k \delta B_k - B_k \delta A_k) = -2\pi i \sum_{k=1}^{g} \left(r_\infty \int_{\infty^{(2)}}^{\infty^{(1)}} u_k + \sum_{j=1}^{g} r_j \int_{z_j^{(2)}}^{z_j^{(1)}} u_k \right) \delta A_k, \tag{3.13}$$

which, due to (3.12), equals to

$$-2\pi i \delta \left(r_\infty \text{reg} \int_{\infty^{(2)}}^{\infty^{(1)}} v + \sum_{j=1}^{g} r_j \text{reg} \int_{z_j^{(2)}}^{z_j^{(1)}} v \right)$$

leading to (3.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. Formula for $\mathcal{G}_{-1}$. We shall use the version of Riemann bilinear relations (see [19], equation (2.5.6)) given in (1.25).

The first theorem we need is the following: let $\mu_j = a(\lambda_j)$ and

$$\lambda_j^{(1)} = (\lambda_j, \mu_j) \tag{3.14}$$

and consider the divisor $D = \lambda_1^{(1)} + \cdots + \lambda_g^{(1)}$.

According to Th.A.5 the symplectic potential for Kirillov–Kostant symplectic form can be expressed as follows in terms a-periods of v and the divisor D as follows:

$$\sum_{j=1}^{g} \operatorname{tr} L_j G_j^{-1} \delta G_j = \sum_{j=1}^{g} A_j \delta q_j = - \left(\sum_{j=1}^{g} q_j \delta A_j \right) + \delta \left(\sum_{j=1}^{g} q_j A_j \right), \qquad (3.15)$$

where

$$q = \mathcal{A}_{\infty^{(2)}}(D) - K^{\infty^{(2)}}, \qquad (3.16)$$

$K^{\infty^{(2)}}$ is the vector of Riemann constants at the point $\infty^{(2)}$.

Using this fact we shall prove the following formula for $\mathcal{G}_{-1}$:

Theorem 3.2. *The equations for $\mathcal{G}_{-1}$ look as follows*

$$\delta \mathcal{G}_{-1} = \langle v_0, \delta v \rangle - 2\pi i \sum_{j=1}^{g} \operatorname{tr} L_j G_j^{-1} \delta G_j + \frac{1}{2} \delta \langle v, v_0 \rangle \qquad (3.17)$$

and the solution is given by the formula

$$\mathcal{G}_{-1} = \frac{1}{2} \langle v, v_0 \rangle - 2\pi i \sum_{j=1}^{g} q_j A_j - \pi i \sum_{k=1}^{g} B_k - \pi i \sum_{j=1}^{g} j A_j. \qquad (3.18)$$

Proof. Equation (3.17) follows from $\delta \mathcal{G} = \theta_G - 2\pi i \theta_{KK}$, together with equations (3.5) and the (1.9). Let us work in the coordinate system (A_j, λ_j). Then $\delta_{\lambda_j} v = 0$. On the other hand, since $\partial_{A_j} v = u_j$ we compute

$$\langle v_0, \partial_{A_j} v \rangle = \langle v_0, u_j \rangle = - \oint_{\partial \tilde{\mathcal{C}}} \left(\int^x u_j \right) v_0 = -2\pi i \sum \operatorname{res} \left(\int^x u_j \right) v_0$$

$$= - \left(\pi i \sum_{k=1}^{g} \int_{\lambda_k^{(2)}}^{\lambda_k^{(1)}} u_j + \pi i \int_{\infty^{(2)}}^{\infty^{(1)}} u_j \right), \qquad (3.19)$$

where we used that v_0 has residues $\pm \frac{1}{2}$ at $\lambda_j^{(1,2)}$ and $\infty^{(1,2)}$, respectively. Therefore,

$$\mathcal{G}_{-1} = \frac{1}{2} \langle v, v_0 \rangle - 2\pi i \sum_{j=1}^{g} q_j A_j + f, \qquad (3.20)$$

where

$$\delta f = \langle v_0, \delta v \rangle + 2\pi i \sum_{j=1}^{g} q_j \delta A_j.$$

Therefore, $\delta_{\lambda_j} f = 0$ and

$$\partial_{A_j} f = \langle v_0, u_j \rangle + 2\pi i q_j.$$

Using (3.19) and (3.16) we get

$$\partial_{A_j} f = -\pi i \left(\sum_{k=1}^{g} \int_{\lambda_k^{(2)}}^{\lambda_k^{(1)}} u_j + \int_{\infty^{(2)}}^{\infty^{(1)}} u_j \right) + 2\pi i \left(\sum_{k=1}^{g} \int_{\infty^{(2)}}^{\lambda_k^{(1)}} u_j - K_j^{\infty^{(2)}} \right).$$

Choose a branch point x_1 as the corner of the fundamental polygon. Then this equation can be written as

$$\frac{1}{2\pi i}\partial_{A_j} f = -\left(\sum_{k=1}^{g} \int_{x_1}^{\lambda_k^{(1)}} u_j + \int_{x_1}^{\infty^{(1)}} u_j \right) + \sum_{k=1}^{g} \int_{x_1}^{\lambda_k^{(1)}} u_j + g \int_{\infty^{(2)}}^{x_1} u_j - K_j^{\infty^{(2)}}$$

or

$$\frac{1}{2\pi i}\partial_{A_j} f = -(g-1) \int_{\infty^{(2)}}^{x_1} u_j - K_j^{\infty^{(2)}}.$$

Therefore, we get

$$\frac{\delta f}{\delta A_j} = -2\pi i K_j^{x_1}.$$

Using the representation

$$K_j^{x_1} = \frac{j}{2} + \frac{1}{2}\sum_k \Omega_{jk}$$

and relation

$$\Omega_{jk} = \frac{\delta B_k}{\delta A_j},$$

we have

$$f = -\pi i \sum_{j=1}^{g} j A_j - \pi i \sum_{k=1}^{g} B_k. \qquad \square$$

3.3. Formula for $\mathcal{G}_0$.

Theorem 3.3. *The equations for $\mathcal{G}_0$ looks as follows*

$$\delta \mathcal{G}_0 = \theta_G^{(0)} = \frac{1}{2}\langle v_0, \delta v_0 \rangle + \langle v_1, \delta v \rangle - \frac{1}{2}\delta\langle v_1, v \rangle \qquad (3.21)$$

and the solution is given by the formula

$$\mathcal{G}_0 = -12\pi i \ln \tau_B(CP^1, Q_0) + F - \frac{1}{2}\langle v_1, v \rangle,$$

where

$$\frac{F}{i\pi} = \frac{1}{2}\sum_{j=1}^{g} \oint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 + \frac{1}{2}\oint_{\infty^{(2)}}^{\infty^{(1)}} v_0 + \ln \frac{\prod_{a,k}(\lambda_a - z_k)}{\prod_j \mu_j \prod_{a<b}(\lambda_a - \lambda_b)}$$

$$- \frac{1}{4r_\infty}\oint_{\infty^{(2)}}^{\infty^{(1)}} v - \sum_{k=1}^{g+2}\frac{1}{4r_k}\oint_{z_k^{(2)}}^{z_k^{(1)}} v,$$

and $\tau_B(CP^1, Q_0)$ is the Bergman tau-function defined by (3.28). The regularization is the "z"–regularization, where we subtract the singular part computed in the z–coordinate. Specifically,

$$\int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 = \lim_{\substack{p \to \lambda_j^{(1)} \\ q \to \lambda_j^{(2)}}} \int_q^p v_0 - \frac{1}{2}\ln(z(p) - \lambda_j) - \frac{1}{2}\ln(z(q) - \lambda_j). \tag{3.22}$$

Proof. Let us write the potential $\theta_G^{(0)}$ as follows:

$$\theta_G^{(0)} = \Theta^{(0)} + \langle \widehat{v}_1, \delta v \rangle + \delta \langle v, v_1 \rangle, \tag{3.23}$$

where

$$\Theta^{(0)} = \frac{1}{2} \langle v_0, \delta v_0 \rangle + \langle \widetilde{v}_1, \delta v \rangle. \tag{3.24}$$

Here we have set

$$\widehat{v}_1 = \frac{(dz)^2}{2v} \mathcal{S}\left(\int v, z\right), \quad \widetilde{v}_1 = -\frac{v_0^2}{2v} - \frac{(dz)^2}{2v}\mathcal{S}(c, z), \tag{3.25}$$

so that the differential v_1 is the sum

$$v_1 = \widetilde{v}_1 + \widehat{v}_1. \tag{3.26}$$

The integration of the last term in (3.23) is trivial. The second term can be integrated in terms of the Bergman tau-function described in the Proposition 3.4 The integration of $\Theta^{(0)}$ is provided in Lemma 3.5. $\square$

Proposition 3.4 (Proposition B.1)**.** *The solution to the equation*

$$-12\pi i\, \delta \ln \tau_B(\mathcal{C}, \mathrm{d}z) = \langle \widehat{v}_1, \delta v \rangle \tag{3.27}$$

is given by the Bergman tau-function:

$$\tau_B = \exp\left\{-\frac{1}{6}\sum_{k=1}^{g+2}\frac{1}{r_k}\mathrm{reg}\int_{x_1}^{z_k} v\right\}\prod_{j<k}(x_j - x_k)^{5/144}\prod_{j<k}(z_j - z_k)^{1/6}\prod_{j,k}(x_j - z_k)^{-7/72}, \tag{3.28}$$

where the regularized integrals are defined by (B.4).

The integration of the term $\Theta^{(0)}$ is provided by the following lemma:

Lemma 3.5. *The solution to equation*

$$\delta F = \Theta^{(0)} \tag{3.29}$$

is given by

$$\frac{F}{i\pi} = \frac{1}{2}\sum_{j=1}^{g}\int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 + \frac{1}{2}\int_{\infty^{(2)}}^{\infty^{(1)}} v_0 + \ln\frac{\prod_{a,k}(\lambda_a - z_k)}{\prod_j \mu_j \prod_{a<b}(\lambda_a - \lambda_b)}$$
$$- \frac{1}{4r_\infty}\int_{\infty^{(2)}}^{\infty^{(1)}} v - \sum_{k=1}^{g+2}\frac{1}{4r_k}\int_{z_k^{(2)}}^{z_k^{(1)}} v. \tag{3.30}$$

Proof. The computation can be performed in any set of coordinates; we choose to use coordinates $\lambda'_j s$ and periods $\{A_j\}$ (alternatively one could use any any moduli of the spectral curve). Then the spectral curve is λ_j–independent. By application of the Riemann bilinear identities we obtain

$$\frac{1}{i\pi}\Theta^{(0)} = \sum \operatorname{res}\left(v_0(x)\int^x \delta v_0 + 2\widetilde{v}_1 \int^x \delta v\right),\qquad(3.31)$$

where the sum extends over all poles of the expression in the residue bracket. These poles are located at: the points $z_j^{(1,2)}$, the branch points x_j of the curve $\mathcal{C}$, the points $\lambda_j^{(1,2)}$, the points $\infty^{(1,2)}$.

Branch points.   We are going to show that the residues at the branchpoints of the two terms in (3.31) cancel each other. To this end we observe that from

$$v_0 = \frac{Q_1 \mathrm{d}z^2}{2v},\qquad(3.32)$$

we have

$$\delta v_0 = \frac{\delta Q_1 (\mathrm{d}z)^2}{2v} - \frac{Q_1 (\mathrm{d}z)^2}{2v^2}\delta v = \frac{\delta Q_1 (\mathrm{d}z)^2}{2v} - \frac{v_0}{v}\delta v.\qquad(3.33)$$

The first term does not have poles at the branch points because $\mathrm{d}z^2$ and v both have double zeros. The second term is present only for differential in the moduli of the curve (leaving λ_j's constant). Secondly, in the expression for $\widetilde{v}_1$ the only term with poles at the branch point are

$$\widetilde{v}_1 = -\frac{v_0^2}{v} + \mathcal{O}(1).\qquad(3.34)$$

Thus, applying (3.33) we get

$$\sum_{\substack{b.pts}} \operatorname*{res}_{x_k}\left(v_0(p)\int^p \partial_{A_j}v_0 + 2\widetilde{v}_1 \int^p \partial_{A_j}v\right)$$

$$= \sum_{\substack{b.pts}} \operatorname*{res}_{x_k}\left(-v_0(p)\int^p \frac{v_0}{v}\partial_{A_j}v + 2\widetilde{v}_1 \int^p \partial_{A_j}v\right)$$

$$= \sum_{\substack{b.pts}} \operatorname*{res}_{x_k}\left(-v_0(p)\int^p \frac{v_0}{v}\partial_{A_j}v - \frac{v_0^2}{v}\int^p \partial_{A_j}v\right).\qquad(3.35)$$

The computation of this residue is easier if done in the local coordinate ζ given by $z = x_k + \zeta^2$: in this coordinate each of the differentials (being all odd under the hyperelliptic involution) are expressed as functions of ζ^2. We denote

$$v_0 = f_0(\zeta^2)\mathrm{d}\zeta,\quad v = \zeta^2 h(\zeta^2)\mathrm{d}\zeta,\quad \partial_{A_j}v = g(\zeta^2)\mathrm{d}\zeta,\qquad(3.36)$$

where we have used that v has a double zero at $\zeta = 0$ (and $h(0) \neq 0$). We also observe that in the computation of the residues the base point of integration of the integrals is irrelevant because it adds a constant and this yields no

residue. Then we can represent the indefinite integral $\int^p \frac{v_0}{v} \partial_{A_j} v$ as a locally defined meromorphic *odd* function with a simple pole at $\zeta = 0$. In explicit terms we have

$$\sum_{\substack{b.pts}} \operatorname*{res}_{x_k} \left(v_0(p) \int^p \partial_{A_j} v_0 + 2\widetilde{v}_1 \int^p \partial_{A_j} v \right)$$

$$= -\operatorname*{res}_{\zeta=0} \left(-f_0(\zeta^2) \frac{f_0(0)g(0)}{\zeta h(0)} + \frac{f_0(0)^2}{\zeta^2 h(\zeta^2)} \int_0^\zeta g(\xi^2) \mathrm{d}\xi + \mathcal{O}(1) \right) \mathrm{d}\zeta = 0. \quad (3.37)$$

Contribution of the other residues. Consider first one of the moduli, A_j, of the curve that does not modify the z–projection of the divisor D.

Recall that $\operatorname{res}_{\lambda_j^{(1,2)}} v_0 = \operatorname{res}_{\infty^{(1,2)}} v_0 = \pm\frac{1}{2}$ and thus $\partial_{A_j} v_0$ is locally analytic at the points λ_j and ∞; the same applies to $\partial_{A_j} v$. Viceversa, from (3.34) it follows that $\widetilde{v}_1$ has a double pole at λ_j with coefficient

$$\widetilde{v}_1 = \pm\frac{1}{4\mu_j} \frac{1}{(z - \lambda_j)^2} + \mathcal{O}(1) \quad \text{near } \lambda_j^{(1,2)}, \quad (3.38)$$

where we emphasize the absence of residue. Finally we need the residues of $\widetilde{v}_1$ at the points $z_k^{(1,2)}$'s and $\infty^{(1,2)}$; a short computation using (3.25) yields

$$\operatorname*{res}_{z_k^{(1,2)}} \widetilde{v}_1 = \mp\frac{1}{8r_k}, \quad \operatorname*{res}_{\infty^{(1,2)}} \widetilde{v}_1 = \mp\frac{1}{8r_\infty}. \quad (3.39)$$

Keeping this in mind, the result is then

$$\sum_{\lambda_j^{(1,2)}, \infty^{(1,2)}, z_k^{(1,2)}} \operatorname{res} \left(v_0(p) \int^p \partial_{A_j} v_0 + 2\widetilde{v}_1 \int^p \partial_{A_j} v \right)$$

$$= \frac{1}{2} \sum_j \int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \partial_{A_j} v_0 + \frac{1}{2} \int_{\infty_-}^{\infty+} \partial_{A_j} v_0 - \sum_j \frac{1}{2\mu_j} \frac{\partial_{A_j} v}{\mathrm{d}z} \Big|_{\lambda_j^{(2)}}^{\lambda_j^{(1)}}$$

$$- \frac{1}{4r_\infty} \int_{\infty^{(2)}}^{\infty^{(1)}} \partial_{A_j} v - \sum_{k=1}^{n+2} \frac{1}{4r_k} \int_{z_k^{(2)}}^{z_k^{(1)}} \partial_{A_j} v$$

$$= \frac{1}{2} \sum_j \int_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \partial_{A_j} v_0 + \frac{1}{2} \int_{\infty_-}^{\infty+} \partial_{A_j} v_0$$

$$- \partial_{A_j} \ln \prod_{k=1}^{g+2} \mu_k - \frac{1}{4r_\infty} \int_{\infty^{(2)}}^{\infty^{(1)}} \partial_{A_j} - \sum_{k=1}^{g+2} \frac{1}{4r_k} \int_{z_k^{(2)}}^{z_k^{(1)}} \partial_{A_j} v,$$

where we have used that $\frac{\partial_{A_j} v}{\mathrm{d}z} \Big|_{\lambda_j^{(1,2)}} = \pm\partial_{A_j}\mu_j$. We now observe that the derivative is made at z–value fixed under the integral sign; therefore we can pull the

derivative outside provided we interpret the integration as a z–regularized integral:

$$\sum_j \operatorname*{res}_{\lambda_j^{\pm}} \left(v_0(p) \int^p \partial_{A_j} v_0 + 2v_1 \int^p \partial_{A_j} v \right)$$

$$= \partial_{A_j} \left(\frac{1}{2} \sum_j \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 + \frac{1}{2} \fint_{\infty_-}^{\infty_+} v_0 - \ln \prod_{k=1}^{g+2} \mu_k - \frac{1}{4r_\infty} \fint_{\infty^{(2)}}^{\infty^{(1)}} v - \sum_{k=1}^{n+2} \frac{1}{4r_k} \fint_{z_k^{(2)}}^{z_k^{(1)} v} \right),$$

where, by definition, the regularization is made by subtraction of the singular part in the z–coordinate of the antiderivative.

Variations of divisor D. We now consider a derivative ∂_{λ_j}. Since v and the spectral curve are independent of λ_j's, only the first term in (3.31) gives a nonzero contribution.

Using the Riemann bilinear relations we find

$$\Theta(\partial_{\lambda_j}) = \left(\left(\operatorname*{res}_{\infty^{(1)}} + \operatorname*{res}_{\infty^{(2)}} \right) + \sum_\ell \left(\operatorname*{res}_{\lambda_\ell^{(1)}} + \operatorname*{res}_{\lambda_\ell^{(2)}} \right) \right) v_0 \int \frac{\partial v_0}{\partial \lambda_j}. \tag{3.40}$$

Now observe that for $\ell \neq j$ and for the residues at infinity the integrand is locally analytic and hence the differential $v_0 \int \partial_{\lambda_j} v_0$ has a simple pole; we can pull the derivative outside of the integration because the regularization depends on λ_ℓ but not on λ_j. Thus we have

$$\Theta(\partial_{\lambda_j}) = \frac{1}{2} \left(\int_{\infty^{(2)}}^{\infty^{(1)}} + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{g} \int_{\lambda_\ell^{(2)}}^{\lambda_\ell^{(1)}} \right) \frac{\partial v_0}{\partial \lambda_j} + \left(\operatorname*{res}_{\lambda_j^{(1)}} + \operatorname*{res}_{\lambda_j^{(2)}} \right) v_0 \int \frac{\partial v_0}{\partial \lambda_j}. \tag{3.41}$$

We are left with the contribution of $\ell = j$:

$$\operatorname*{res}_{(\lambda_j, \pm\mu_j)} v_0 \int \frac{\partial v_0}{\partial \lambda_j}. \tag{3.42}$$

The local behaviour of the indefinite integral is (near $\lambda_j^{(1)}$):

$$\int^p \partial_{\lambda_j} v_0 = \int^p \frac{\partial}{\partial \lambda_j} \left(\frac{1}{2(z - \lambda_j)} + \mathcal{O}(1) \right) \delta z$$

$$= \int^p \left(\frac{1}{2(z - \lambda_j)^2} + \mathcal{O}(1) \right) \delta z = \frac{-1}{2(z - \lambda_j)} + \mathcal{O}(1).$$

Therefore, the local behaviour of the function we are taking the residue of in (3.42) is

$$v_0 \int \frac{\partial v_0}{\partial \lambda_j} = \left(\frac{\pm 1}{2(z - \lambda_j)} + A_j^{(1,2)} + \cdots \right) \left(\frac{\mp 1}{2(z - \lambda_j)} + C_j^{(1,2)} + \cdots \right), \tag{3.43}$$

where $A_j^{(1,2)} = v_0^{\text{reg}}(\lambda_j^{(1,2)}) = \pm v_0^{\text{reg}}(\lambda_j^{(1)})$. This means that the result is $\frac{A_j^{(2)} - A_j^{(1)} + C_j^{(1)} - C_j^{(2)}}{2}$. Now, by definition of regularization:

$$\frac{C_j^{(1)} - C_j^{(2)}}{2} = \frac{1}{2} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \frac{\partial v_0}{\partial \lambda_j}. \tag{3.44}$$

We then observe that

$$\frac{1}{2} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \frac{\partial v_0}{\partial \lambda_j} = \frac{1}{2} \frac{\partial}{\partial \lambda_j} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 - \frac{1}{2} v_0^{reg} \Big|_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} = \frac{1}{2} \frac{\partial}{\partial \lambda_j} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 + \frac{A_j^{(2)} - A_j^{(1)}}{2}. \tag{3.45}$$

Therefore we have

$$\operatorname*{res}_{\lambda_j^{(1,2)}} \left(v_0 \int \partial_{\lambda_j} v_0 \right) = \frac{A_j^{(2)} - A_j^{(1)} + C_j^{(1)} - C_j^{(2)}}{2} = \frac{1}{2} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \frac{\partial}{\partial \lambda_j} v_0 + \frac{A_j^{(2)} - A_j^{(1)}}{2}$$

$$= \frac{1}{2} \frac{\partial}{\partial \lambda_j} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0 - 2 A_j^{(1)}. \tag{3.46}$$

It remains to compute $A_j^{(1)}$: from the definition of v_0 (3.32) it follows promptly

$$A_j^{(1)} = \frac{Q_1^{reg}(\lambda_j)}{2\mu_j} + \frac{1}{2} \partial_{\lambda_j} \ln \mu_j$$

$$= \frac{1}{2\mu_j} \left(\sum_{\ell \neq j} \frac{\mu_\ell}{\lambda_j - \lambda_\ell} + \sum_k \frac{\gamma_k}{\lambda_j - z_k} \right) + \frac{1}{2} \partial_{\lambda_j} \ln \mu_j. \tag{3.47}$$

Using the Bethe equations (2.25) one finds then

$$A_j^{(1)} = \frac{1}{2} \left(\sum_{\ell \neq j} \frac{1}{\lambda_j - \lambda_\ell} - \sum_k \frac{1}{\lambda_j - z_k} \right) + \frac{1}{2} \partial_{\lambda_j} \ln \mu_j. \tag{3.48}$$

Inserting this expression into (3.46) we then obtain:

$$\operatorname*{res}_{\lambda_j^{(1,2)}} v_0 \int \partial_{\lambda_j} v_0 = \frac{1}{2} \frac{\partial}{\partial \lambda_j} \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} v_0$$

$$- \left(\sum_{\ell \neq j} \frac{1}{\lambda_j - \lambda_\ell} - \sum_k \frac{1}{\lambda_j - z_k} \right) - \partial_{\lambda_j} \ln \mu_j, \tag{3.49}$$

where, again, the regularization so far is made in the z-coordinate.

Thus

$$\sum_{j=1}^g \left\langle v_0, \frac{\partial v_0}{\partial \lambda_j} \right\rangle \delta \lambda_j = \frac{1}{2} \delta_\lambda \left(\fint_{\infty^{(2)}}^{\infty^{(1)}} + \sum_{j=1}^g \fint_{\lambda_j^{(2)}}^{\lambda_j^{(1)}} \right) v_0$$

$$+ \delta_\lambda \ln \frac{\prod_{a,k} (\lambda_a - z_k)}{\prod \mu_j \prod_{a<b} (\lambda_a - \lambda_b)}. \tag{3.50}$$

This concludes the proof. $\qquad\qquad\square$

A. Kirillov–Kostant symplectic potential

A.1. Szegö kernel and its variations. Here we list the necessary information about Szegö kernel and its variations. For a Riemann surface of genus g denote the Abel map by $\mathcal{A}(x)$, introduce holomorphic differentials u_j normalized via $\int_{a_j} u_k = \delta_{jk}$ and the prime-form $E(x,y)$. Let $q \in \mathbb{C}^g$. The Szegö kernel S_q is then given by

$$S_q(x,y) = \frac{\Theta(\mathcal{A}(x) - \mathcal{A}(y) + q)}{\Theta(q)E(x,y)}. \tag{A.1}$$

The Szegö kernel has the following properties. First, it has simple pole on the diagonal of the form:

$$S_q(x,y) = \left(\frac{1}{\xi(x) - \xi(y)} + \mathcal{O}(1) \right) \sqrt{d\xi(x)}\sqrt{d\xi(y)}, \tag{A.2}$$

where ξ is a local coordinate near the diagonal. Second, it has the following periodicity properties: $S_q(x,y)$ remains invariant (up to a sign) when x or y are analytically continued along any a-cycle a_j; under analytical continuation along b_j one has

$$S_q(x + b_j, y) = e^{-2\pi i q_j} S_q(x,y), \quad S_q(x, y + b_j) = e^{2\pi i q_j} S_q(x,y).$$

The Szegö kernel satisfies the following identity due to Fay [20]:

$$S_q(x,y)S_q(y,x) = -B(x,y) - \sum_{\alpha,\beta=1}^{g} \partial_\alpha \partial_\beta \log \Theta_q(0) u_\alpha(x) u_\beta(y), \tag{A.3}$$

where $B(x,y) = d_x d_y \log E(x,y)$ is the canonical bimeromorphic differential.

The Szegö kernel depends on the moduli of the Riemann surface $\mathcal{C}$ (we consider here the moduli space of hyperelliptic curves of genus g defined by (1.13)) and on the vector q, which defines a point of the Jacobian of $\mathcal{C}$. Variational formula for Szegö kernel on the space $\mathcal{A}^{\mathbf{r}}$ can be conveniently written in terms of coordinate system (A_j, q_j) where $A_j = \int_{a_j} v$ and q_j are components of vector q. The moduli of the curve $\mathcal{C}$ (for fixed $\mathbf{r}$ and z_j) depend (locally) only on the periods A_j.

The variational formulas are given in the next proposition.

Proposition A.1. *The following variational formulas hold:*

$$\frac{\partial}{\partial A_j} S_q(x,y) = -\frac{2\pi i}{4} \sum_{i=1}^{2g+2} \frac{u_j}{d\ln(v/dz)}(x_i) \operatorname*{res}_{x_i} \frac{W_t[S_q(x,t),\, S_q(t,y)]}{v(t)}, \tag{A.4}$$

$$\frac{\partial}{\partial q_\gamma} S_q(x,y) = -\int_{t \in a_\gamma} S_q(x,t)S_q(t,y), \tag{A.5}$$

where $W_t(f,g)$ denotes Wronskian of two functions $f(t)$ and $g(t)$.

Proof. The formula (A.5) was proved in [36] (Proposition 1).

The formula (A.4) follows from two results. The first is Theorem 2 of [36] where the variational formulas for S_q on moduli spaces of meromorphic abelian differentials were derived. Then (A.4) can be obtained from the formula of [36] via chain rule, following verbatim the proof of formula (3.10) from [6] where the variational formulas for Abelian differentials were derived. $\square$

We shall need the following lemma which is valid for any n-sheeted cover of $\mathbb{P}^1$; this statement is equivalent to relations (4.12), (4.13) from [39].

Lemma A.2. *Let $\mathcal{C}$ be an n-sheeted cover of $\mathbb{P}^1$ with projection $f : \mathcal{C} \to \mathbb{P}^1$. Then the following identity holds:*

$$\sum_{i=1}^{n} S_q(x, t^{(i)}) S_q(t^{(i)}, y) = S_q(x, y) \left(\frac{1}{f(x) - t} - \frac{1}{f(y) - t} \right) dt. \qquad (A.6)$$

Proof. For completeness here we give a short independent proof of this fact. The left-hand side of (A.6) is a 1-form in t depending only on the point of the base. It has simple poles at $z = f(x)$ and $z = f(y)$. The coefficient depends on x and y and must coincide with $S_q(x, y)$ due to the holonomy properties of S_q. $\square$

We shall also use the following notations. First, introduce the system of distinguished local coordinates on $\mathcal{C}$. Near a branch point x_j it is given by $\zeta_j(z) = \sqrt{z - x_j}$. Near $\infty^{(1,2)}$ the distinguished coordinate is chosen to be $\xi(z) = 1/z$. Finally, near any other point with projection z_0 on z-plane the distinguished coordinate is $z - z_0$. Now we define the following:

$$S_q(x, \infty^{(j)}) = \left. \frac{S_q(x, y)}{\sqrt{\mathrm{d}(z^{-1}(y))}} \right|_{y = \infty^{(j)}}, \qquad j = 1, 2, \qquad (A.7)$$

$$S_q(x, x_k) = \left. \frac{S_q(x, y)}{\sqrt{\mathrm{d}\sqrt{f(y) - x_j}}} \right|_{y = x_k}, \qquad k = 1, \ldots, 2g + 2. \qquad (A.8)$$

Using these notations we get from (A.6) in the limit $t \to \infty$:

$$\sum_{i=1}^{n} S_q(x, \infty^{(i)}) S_q(\infty^{(i)}, y) = S_q(x, y)(f(x) - f(y)). \qquad (A.9)$$

A.2. Eigenvectors of rational matrix functions via Szegö kernel on spectral curve. The Szegö kernel can be conveniently used to construct eigenvectors of the rational matrix-valued function $R(z)$ (1.10). The construction is parallel to the one used in [39] to construct solutions of matrix Riemann-Hilbert problems.

Proposition A.3. *Consider a pair $(\mathcal{C}, q)$ where $q \in \mathbb{C}^g$ and $\mathcal{C}$ is the spectral curve given by equation*

$$\mu^2 = C \frac{\prod_{j=1}^{2g+2}(z - x_j)}{\prod_{j=1}^{g+2}(z - z_j)^2} \tag{A.10}$$

such that for the differential $v = \mu \mathrm{d}z$ we have

$$\operatorname*{res}_{z_j^{(1,2)}} v = \pm r_j, \qquad \operatorname*{res}_{\infty^{(1,2)}} v = \pm r_\infty, \tag{A.11}$$

where $r_1, \ldots, r_{2g+2}, r_\infty$ are some constants. Consider the divisor $D = \lambda_1^{(1)} + \cdots + \lambda_g^{(1)}$ defined by

$$\mathcal{A}_{\infty^{(2)}}(D) = -q + K^{\infty^{(2)}}. \tag{A.12}$$

Consider the canonical polygon $\tilde{\mathcal{C}}$ invariant under the hyperelliptic involution ν. Define the following column-vector for $x \in \tilde{\mathcal{C}}$:

$$\psi(x) = \frac{1}{\sqrt{\mathrm{d}f(x)}} \begin{pmatrix} S_q(x, \infty^{(1)}) \\ S_q(x, \infty^{(2)}) \end{pmatrix} \tag{A.13}$$

and the 2×2 matrix on $\tilde{\mathcal{C}}$

$$\Psi(x) = (\psi(x), \psi(x^*)) = \frac{1}{\sqrt{\mathrm{d}f(x)}} \begin{pmatrix} S_q(x, \infty^{(1)}) & S_q(x^*, \infty^{(1)}) \\ S_q(x, \infty^{(2)}) & S_q(x^*, \infty^{(2)}) \end{pmatrix}. \tag{A.14}$$

Then the matrix R defined by

$$R(x)\,\mathrm{d}z(x) = \Psi(x) \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \Psi^{-1}(x) \tag{A.15}$$

is a rational matrix invariant under the transformation $x \to x^\nu$, i.e. it depends only on z, and

1. *$R(z)$ has simple poles only at z_j:*

$$R(z) = \sum_{j=1}^{g+2} \frac{R_j}{z - z_j}. \tag{A.16}$$

2. *The eigenvalues of R_j are equal to $\pm r_j$ and*

$$R_\infty := -\sum_{j=1}^{g+2} R_j = \begin{pmatrix} r_\infty & 0 \\ 0 & -r_\infty \end{pmatrix}.$$

3. *$\mathcal{C}$ coincides with the spectral curve $\det(R(z) - \mu I) = 0$.*

4. *The matrix entry $R_{21}(z)$ has on $\mathcal{C}$ exactly g zeros situated at $\lambda_j \equiv f(\lambda_j^{(1)})$.*

5. *The points $\lambda_j^{(1)} = (\lambda_j, \mu_j)$ are such that $\mu_j = R_{11}(\lambda_j)$, i.e. the divisor D defined by (A.12) is given by $D = \sum_{j=1}^g (\lambda_j, R_{11}(\lambda_j))$. Another characterization of the divisor is that it consists of the points of the spectral curve above the finite part of the plane where the eigenvector is proportional to the vector $(1, 0)$.*

Proof. To prove that the matrix R is invariant under the involution $x \to x^\nu$ we notice that $\Psi(x^\nu) = \Psi(x)\sigma_1$ and $v(x^\nu) = -v(x)$. Therefore, $R(x^\nu) = R(x)$.

The eigenvalues of the matrices R_j coincide with residues of v at the points $z_j^{(1,2)}$ which are equal to $\pm r_j$ due to (A.11).

According to [39] (see p.350, and formula (4.14)), Fay's identities imply that $\det \Psi = \pm 1$ and

$$\Psi^{-1}(x) = \frac{1}{\sqrt{\mathrm{d}f(x)}} \begin{pmatrix} S_q(\infty^{(1)}, x) & S_q(\infty^{(2)}, x) \\ S_q(\infty^{(1)}, x^\nu) & S_q(\infty^{(2)}, x^\nu) \end{pmatrix}. \tag{A.17}$$

The matrix element R_{21} is a rational function of $z = f(x)$ given by

$$R_{21}(z) = 2 \frac{v(x)}{(\mathrm{d}f(x))^2} S_q(x, \infty^{(2)}) S_q(x^\nu, \infty^{(2)}),$$

which vanishes at the points of divisor $D + D^\nu$ due to (A.12). Equivalently, it means that R_{21}, considered as function of z, vanishes at the points of $\pi(D)$.

Finally, the eigenvalues of R_j from (A.16) are equal to $\pm$ residues of v at $z_j^{(1,2)}$, i.e. $\pm r_j$. □

The expression (A.14) can be alternatively written as follows:

$$\Psi_{\alpha\beta}(z) = \frac{1}{\sqrt{\mathrm{d}f(x)}} S_q(z^{(\beta)}, \infty^{(\alpha)}) \tag{A.18}$$

and (A.17) as

$$(\Psi^{-1})_{\alpha\beta}(z) = \frac{1}{\sqrt{\mathrm{d}f(x)}} S_q(\infty^{(\beta)}, z^{(\alpha)}). \tag{A.19}$$

Corollary A.4. *The matrices R_j can be diagonalized as follows:*

$$R_j = G_j \begin{pmatrix} r_j & 0 \\ 0 & -r_j \end{pmatrix} G_j^{-1}, \tag{A.20}$$

where the formulas for G_j are obtained for (A.14), (A.18):

$$(G_j)_{\alpha\beta} = S_q(z_j^{(\beta)}, \infty^{(\alpha)}) \tag{A.21}$$

such that

$$(G_j^{-1})_{\alpha\beta} = S_q(\infty^{(\beta)}, z_j^{(\alpha)}). \tag{A.22}$$

A.3. Kirillov–Kostant potential. The Kirillov–Kostant symplectic form looks as follows in terms of eigenvector matrices G_j [4]:

$$\omega_{KK} = -\sum_{j=1}^n \operatorname{tr} L_j G_j^{-1} \delta G_j \wedge G_j^{-1} \delta G_j. \tag{A.23}$$

The natural choice of symplectic potential θ_{KK} such that $\mathrm{d}\theta_{KK} = \omega_{KK}$ is

$$\theta_{KK} = \sum_{j=1}^{n} \mathrm{tr}\, L_j G_j^{-1} \delta G_j \qquad (A.24)$$

or, alternatively,

$$\theta_{KK} = \sum_{j=1}^{g+2} \operatorname*{res}_{z_j} \mathrm{tr}\left(\widehat{v}\Psi^{-1}\delta\Psi\right) = \sum_{j=1}^{g+2} \operatorname*{res}_{z_j} \mathrm{tr}\left(\widehat{\mu}\Psi^{-1}\delta\Psi\right)\mathrm{d}z, \qquad (A.25)$$

where

$$\widehat{v} = \widehat{\mu}\mathrm{d}z = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}.$$

Theorem A.5. *The form θ_{KK} in (A_j, q_j)-coordinates looks as follows:*

$$\theta_{KK} = \sum_{j=1}^{g} A_j \delta q_j, \qquad (A.26)$$

where $A_j = \int_{a_j} v$ are a-period of v (defined up to an integer linear combination of Casimirs r_j).

Proof. Consider first the contribution of δq_j in (A.24). First, using (A.18), (A.19) and variational formula (A.5), we get

$$\left(\Psi^{-1}(z)\partial_{q_j}\Psi(z)\right)_{\alpha\beta}\mathrm{d}z$$

$$= -\sum_{\gamma=1}^{2} \oint_{x\in a_j} S_q\left(z^{(\beta)}, x\right) S_q\left(x, \infty^{(\gamma)}\right) S_q\left(\infty^{(\gamma)}, z^{(\alpha)}\right) \quad (A.27)$$

or, using Lemma A.2,

$$\left(\Psi^{-1}(z)\partial_{q_j}\Psi(z)\right)_{\alpha\beta}\mathrm{d}z = -\oint_{x\in a_j} S_q\left(z^{(\beta)}, x\right) S_q\left(x, z^{(\alpha)}\right)(f(x) - z). \qquad (A.28)$$

Recall that $\mu(z)$ is a meromorphic function with simple poles at the points $p \to z_j^{(1,2)}$ with singular parts $\mu = \frac{\pm r_j}{z - z_j}$.

The contribution to θ_{KK} of δq_j is therefore given by:

$$\theta_{KK}(\partial_{q_j}) = -\sum_{k=1}^{g+2}\sum_{\alpha=1}^{2}(L_k)_{\alpha\alpha} \oint_{x\in a_j} \frac{S_q(p,x)S_q(x,p)}{\mathrm{d}f(p)}\bigg|_{p=z_k^{(\alpha)}} (f(x) - f(p)), \quad (A.29)$$

which, using (A.25), gives

$$-\sum_{k=1}^{g+2}\sum_{\alpha=1}^{2} \operatorname*{res}_{p=z_k^{(\alpha)}} \mu(p) \oint_{x\in a_j} S_q(p,x)S_q(x,p)(f(x) - f(p)). \qquad (A.30)$$

The integration contours a_j in the x–variable can be chosen so as not to intersect the integration contours for the residues in the p–variable and hence the integrand is regular. Thus we can interchange the order of integrations:

$$\theta_{KK}(\partial_{q_j}) = -\oint_{x \in a_j} \sum_{k=1}^{g+2} \sum_{\alpha=1}^{2} \operatorname*{res}_{p=z_k^{(\alpha)}} \mu(p) S_q(p,x) S_q(x,p)(f(x) - f(p)). \qquad (A.31)$$

The sum over the residues is the sum over all poles above the z_j's of the differential (in the p variable)

$$\mu(p) S_q(p,x) S_q(x,p)(f(x) - f(p)). \qquad (A.32)$$

This differential does not have a pole at $p = \infty^{(1,2)}$ because the eigenvalue, $\mu(p)$ of R has a simple zero, which cancels the pole of $f(p)$. Moreover it has an additional simple pole at $p = x$, with residue

$$\operatorname*{res}_{p=x} \mu(p) S_q(p,x) S_q(x,p)(f(x) - f(p)) = \mu(x)\,df(x). \qquad (A.33)$$

Thus we can use residue theorem and rewrite (A.31) as follows:

$$\theta_{KK}(\partial_{q_j}) = \oint_{x \in a_j} \operatorname*{res}_{p=x} \mu(p) S_q(p,x) S_q(x,p)(f(x) - f(p)). \qquad (A.34)$$

Furthermore, using Fay's identity (A.3), and using the fact that $B(p,x)$ behaves on the diagonal as $(f(p) - f(x))^{-2} df(p) df(x)$, we get

$$\operatorname*{res}_{p=x} \mu(p) S_q(p,x) S_q(x,p)(f(x) - f(p)) = \mu(x)\,df(x) \qquad (A.35)$$

and, therefore,

$$\theta_{KK}(\partial_{q_j}) = \oint_{a_j} v = A_j. \qquad (A.36)$$

Consider now the contribution of δA_j to θ_{KK}. Remind that $\Psi(\lambda_j) = G_j$ while $\operatorname{res}|_{z_j} \hat{v} = L_j$. Now, using variational formulas (A.4) we have

$$\left(\Psi^{-1}\partial_{A_j}\Psi\right)_{\alpha\alpha}$$
$$= \sum_{\gamma=1}^{2} \frac{S_q(\infty^{(\gamma)}, p)}{df(p)} \left(\frac{\pi i}{2} \sum_{x_i} u_j(x_i) \operatorname{res}|_{x=x_i} \frac{W_x[S_q(p,x), S_q(x, \infty^{(\gamma)})]}{v(x)} \right)\Bigg|_{p=z_j^{(\alpha)}},$$

where

$$u_j(x_i) = \frac{u_j}{d\ln(v/\delta\xi)}(x_i). \qquad (A.37)$$

Therefore, using (A.9) we have

$$\theta_{KK}(\partial_{A_j}) = \sum_{k=1}^{g+2} \sum_{\alpha=1}^{2} (L_k)_{\alpha\alpha} \left(\Psi^{-1}\partial_{A_j}\Psi\right)_{\alpha\alpha} = -\frac{\pi i}{2} \sum_{\ell=1}^{g+2} \sum_{\alpha=1}^{2} \operatorname{res}\Big|_{p=z_\ell^{(\alpha)}}$$

$$\times \left(\sum_{x_i \in b.pts} u_j(x_i) \operatorname*{res}_{t=x_i} \frac{W_x\big[S_q(p,x),\, S_q(x,p)(f(x) - f(p))\big]}{\mu(x)\, \mathrm{d}f(x)} \mu(p) \right) \tag{A.38}$$

or, since the branch points are simple,

$$\theta_{KK}(\partial_{A_j}) = -\frac{\pi i}{2} \sum_{\ell=1}^{g+2} \sum_{\alpha=1}^{2} \operatorname*{res}_{p=z_\ell^{(\alpha)}} \left(\sum_{x_i \in b.pts} \operatorname*{res}_{t=x_i} \frac{u_j(x)}{\mathrm{d}\ln\mu(x)} \right.$$

$$\left. \times \frac{W_x\big[S_q(p,x),\, S_q(x,p)(f(x) - f(p))\big]}{\mu(x)\, \mathrm{d}f(x)} \mu(p) \right). \tag{A.39}$$

Once again we can swap the order of residues because the branch-points are away from the points $z_j^{(\alpha)}$. One can verify that the differential of p in the inmost residue has poles only above the z_j's and at $p = t$, with no pole at ∞, for the same reason as in equation (A.32). To compute the residue at $p = x$, we use (A.2) that specifies the behavior of $S_q(p,x)$ for $p \sim x$:

$$\operatorname*{res}_{p=x} \frac{W_x\big(S_q(p,x),\, S_q(x,p)[f(x) - f(p)]\big)}{\mu(x)\mathrm{d}f(x)} \mu(p)$$

$$= \operatorname*{res}_{p=x} \frac{\mu(p)}{\mu(x)\mathrm{d}f(x)} \Big(\partial_x S_q(p,x) S_q(x,p)(f(x) - f(p))$$

$$- S_q(p,x)\partial_x S_q(x,p)(f(x) - f(p)) - S_q(p,x)S_q(x,p) \Big)$$

$$= \operatorname*{res}_{p=x} \frac{1}{(f(p) - f(x))^2} \frac{\mu(p)}{\mu(x)} = d\log\mu(x).$$

Therefore,

$$\theta_{KK}(\partial_{A_j}) = -\frac{\pi i}{2} \sum_{x_i} \operatorname*{res}_{x=x_i} \frac{u_j(x)}{\mathrm{d}\ln\mu(x)} \mathrm{d}\ln\mu(x) = 0. \qquad \square$$

B. Bergman tau-function

Here we summarize the key facts from the theory of Bergman tau-function (see [36, 38, 39] and the review [41]).

First, let us write the function (2.13) as follows:

$$Q_0(z) = C_0 \frac{P(z)}{R^2(z)}, \tag{B.1}$$

where

$$P(z) = \prod_{j=1}^{2g+2} (x - x_j), \quad R(z) = \prod_{j=1}^{g+2} (z - z_j),$$

and

$$C_0 = \sum_{j=1}^{g+2} r_j^2.$$

B.1. Definition of $\tau_B(Q_0)$ and its properties. Here we remind the definition of the Bergman tau-function on the space of quadratic differentials (see [7], where the tau-function needed here is denoted by τ_+, and the review [41]). The constructions of [7] are adjusted to the case of genus zero base curve (then in the notations of [7] we have $g_- = g$).

In the genus zero case the Bergman tau-function τ_B is defined by the following equations with respect to the periods of v along canonical cycles on $\mathcal{C}$ (see [7,36]):

$$\frac{\delta \log \tau_B}{\delta A_i} = -\frac{1}{24\pi i} \int_{b_i} \frac{\mathcal{S}\left(\int^x v, z(x)\right)(dz)^2}{v},$$

$$\frac{\delta \log \tau_B}{\delta B_i} = \frac{1}{24\pi i} \int_{a_i} \frac{\mathcal{S}\left(\int^x v, z(x)\right)(dz)^2}{v} \tag{B.2}$$

for $i = 1, \ldots, g$. Here z is the global coordinate on $CP1$ and $\mathcal{S}(\cdot, \cdot)$ is the Schwarzian derivative (notice that in z-coordinate the Bergman projective connection S_B on the Riemann sphere is identically zero).

Therefore, the differential of $\log \tau_B$ on the symplectic leaf $\mathcal{Q}_{g,n}[\mathbf{r}]$ is given by the following expression:

$$\delta \log \tau_B = \frac{1}{24\pi i} \sum_{j=1}^{g} \left[\left(\int_{a_i} \frac{\mathcal{S}\left(\int^x v, z(x)\right)(dz)^2}{v} \right) \delta B_j \right.$$

$$\left. - \left(\int_{b_i} \frac{\mathcal{S}\left(\int^x v, z(x)\right)(dz)^2}{v} \right) \delta A_j \right]. \tag{B.3}$$

B.2. Explicit formula for $\tau(Q_0)$. Let us introduce the following regularized integrals on $\mathcal{C}$ of $v = \sqrt{Q_0}dz$:

$$\text{reg} \int_{x_1}^{z_k} v := \lim_{x \to z_k} \left\{ \int_{x_1}^{x} v - r_k \ln(z(x) - z_k) \right\}. \tag{B.4}$$

The explicit formula for $\tau_B(Q_0)$ is then given by the following proposition:

Proposition B.1. *Choose the contours l_j connecting x_1 with z_j such that they lie entirely inside of the fundamental polygon $\mathcal{C}$. Then*

$$\tau_B = \exp\left\{ -\frac{1}{6} \sum_{k=1}^{g+2} \frac{1}{r_k} \text{reg} \int_{x_1}^{z_k} v \right\} \prod_{j<k}(x_j - x_k)^{5/144} \prod_{j<k}(z_j - z_k)^{1/6} \prod_{j,k}(x_j - z_k)^{-7/72}. \tag{B.5}$$

Proof. Denote by

$$E(z, w) = \frac{z - w}{\sqrt{dz}\sqrt{dw}} \tag{B.6}$$

the prime-form on $\mathbb{P}^1$. In terms of the prime-form the Bergman tau-function is given by the following expression valid for the base curve of genus zero in terms

of the divisor $(Q_0) = \sum d_j p_j$ (see [7, 36, 38, 42]):

$$\tau_B = \left(\frac{Q_0(x)}{\prod_j E^{d_i}(x, p_j)} \right)^{-1/6} \prod_{i<j} E(p_i, p_j)^{\frac{d_i d_j}{24}} . \tag{B.7}$$

The prime-forms in (B.7) are evaluated at the points p_i as follows:

$$E(x, q_i) = \lim_{y \to p_i} E(x, y) \sqrt{d\zeta_i(y)}, \tag{B.8}$$

$$E(p_i, p_j) = \lim_{x \to p_j, y \to p_i} E(x, y) \sqrt{d\zeta_i(y)} \sqrt{d\zeta_j(x)}, \tag{B.9}$$

where ζ are distinguished local coordinates near points p_j.

Let us apply (B.7) to our case, when

$$(Q_0) = \sum_{k=1}^{2g+2} x_j - 2 \sum_{k=1}^{g+2} z_k$$

and

$$\tau_B = \left(\frac{Q_0(x) \prod_j E^2(x, z_k)}{\prod_j E(x, x_j)} \right)^{-1/6} \left(\frac{\prod_{i<j} E(x_i, x_j) \prod_{k<l} E^4(z_k, z_l)}{\prod_{k,l} E^2(x_k, z_l)} \right)^{1/24} . \tag{B.10}$$

Locally near $z_k^{(1)}$ we have

$$v \sim \frac{r_k}{z - z_k} dz + \cdots \tag{B.11}$$

and the distinguished local coordinate ζ_k near z_k is given by

$$\zeta_k(x) = \exp \left\{ \frac{1}{r_k} \int_{x_1}^{z} v \right\}, \tag{B.12}$$

such that

$$\frac{d\zeta_k}{dz} \bigg|_{z_k} = \exp \left\{ \frac{1}{r_k} \mathrm{reg} \int_{x_1}^{z_k} v \right\}, \tag{B.13}$$

where the regularized integral is given by (B.4).

The total power of $d\zeta_j/dz(z_j)$ in (B.7) is

$$-\frac{1}{6} + \frac{1}{12}(g+1) - \frac{1}{24}(2g+2) = -\frac{1}{6} \tag{B.14}$$

which gives the second term in (B.5).

The distinguished local coordinates ξ_j near x_j are given by:

$$\xi_j(x) = \left(\int_{x_j}^{z} v \right)^{2/3} . \tag{B.15}$$

Since locally, near the branch point x_j, we have

$$v \sim a_j(z - x_j)^{1/2}, \tag{B.16}$$

we get

$$\xi_j(z) \sim a_j^{2/3}(z - x_j) \tag{B.17}$$

and

$$\frac{d\xi_j(z)}{dz}(x_j) = a_j^{2/3}, \tag{B.18}$$

where

$$a_j = C_0^{1/2} \frac{\prod_{k \neq j}(x_k - x_j)^{1/2}}{\prod_k(x_j - z_k)}.$$

The total power of $d\xi_j/dz(x_j)$ in (B.10) is

$$\frac{1}{12} + \frac{1}{2}\frac{1}{24}(2g + 1) - \frac{1}{24}(g + 2) = \frac{1}{48}. \tag{B.19}$$

Therefore, the total power of a_j is $\frac{1}{72}$.

The total power of C_0 comes from Q and all a_j which gives

$$-\frac{1}{6} + \frac{1}{2}\frac{1}{72}(2g + 2) = \frac{g - 11}{72}.$$

However, we don't include the C_0 multiplier into τ_B since the latter is defined up to a constant which might depend on residues.

Let us now compute the powers of $x_j - x_k$, $z_j - x_k$ and $z_j - z_k$.

The product of $z_j - z_k$ comes only from $E(z_j, z_k)$, thus equals to $1/6$.

The product of $x_j - z_k$ comes from $E(x_j, z_k)$ (gives $-1/12$) and from product of a_j (gives $-1/72$), and in total we get

$$-\frac{1}{12} - \frac{1}{72} = -\frac{7}{72}.$$

Finally, the product of $x_j - x_k$ comes from $E(x_j, x_k)$ (gives $1/48$) and product of a_j (gives $1/72$). In total we get $5/144$. $\qquad\square$

References

[1] A.Y. Alekseev and A.Z. Malkin, *The hyperbolic moduli space of flat connections and the isomorphism of symplectic multiplicity spaces*, Duke Math. J. **93** (1998), 575–595.

[2] D. Allegretti and T. Bridgeland, *The monodromy of meromorphic projective structures*, Trans. Amer. Math. Soc. **373** (2020), 6321–6367.

[3] D. Allegretti, *Voros symbols as cluster coordinates*, J. Topol. **12** (2019), 1031–1068.

[4] O. Babelon, D. Bernard, and M. Talon, *Introduction to Classical Integrable Systems*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2003.

[5] M. Bershtein, P. Gavrylenko, and A. Grassi, *Quantum spectral problems and isomonodromic deformations*, Comm. Math. Phys. **393** (2022), 347–418.

[6] M. Bertola and D. Korotkin, *Spaces of Abelian differentials and Hitchin's spectral covers*, Int. Math. Res. Not. IMRN **2021** (2021), 11246–11269.

[7] M. Bertola and D. Korotkin, *Hodge and Prym tau functions, Strebel differentials and combinatorial model of* $\mathcal{M}_{g,n}$, Commun. Math. Phys. **378** (2020), 1279–1341.

[8] M. Bertola and D. A. Korotkin, *WKB expansion for a Yang–Yang generating function and the Bergman tau function*, Theor. Math. Phys. **206** (2021), 258–295.

[9] M. Bertola and D. Korotkin, *Extended Goldman symplectic structure in fock-goncharov coordinates*, J. Diff. Geom., to appear (2021), https://arxiv.org/abs/1910.06744.

[10] M. Bertola and D. Korotkin, *Tau-functions and monodromy symplectomorphisms*, Commun. Math. Phys. **388** (2021), 245–290.

[11] G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa, and A. Tanzini, *On Painlevé/gauge theory correspondence*, Lett. Math. Phys. **107** (2017), 2359–2417.

[12] T. Bridgeland, *Riemann-Hilbert problems from Donaldson–Thomas theory*, Invent. Math. **216** (2019), 69–124.

[13] T. Bridgeland and D. Masoero, *On the monodromy of the deformed cubic oscillator*, Math. Annalen, **385** (2022), 193–258.

[14] T. Bridgeland and I. Smith, *Quadratic differentials as stability conditions*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 155–278.

[15] I. Coman, P. Longhi, and J. Teschner, *From quantum curves to topological string partition functions II*, Comm. Math. Phys. **399** (2023), 1501–1548.

[16] F. Del Monte, H. Desiraju, and P. Gavrylenko, *Isomonodromic tau functions on a torus as Fredholm determinants, and charged partitions*, Comm. Math. Phys. **398** (2023), 1–56.

[17] F. Del Monte, H. Desiraju and P. Gavrylenko, *Monodromy dependence and symplectic geometry of isomonodromic tau functions on the torus*, preprint, https://arxiv.org/abs/2211.01139.

[18] H. Dillinger, E. Delabaere, and F. Pham, *Résurgence de Voros et périodes des courbes hyperelliptiques*, Ann. Inst. Fourier (Grenoble) **43** (1993), 163–199.

[19] B. Dubrovin, *Integrable systems and Riemann surfaces*, Lecture Notes, 2009. Available from: https://people.sissa.it/~dubrovin/rsnleq_web.pdf.

[20] J.D. Fay, *Theta functions on Riemann surfaces*, **352**, Springer, 2006.

[21] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.

[22] D. Gaiotto, G.W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math. **234** (2013), 239–403.

[23] O. Gamayun, N. Iorgov, and O. Lisovyy, *Conformal field theory of Painlevé VI*, J. High Energ. Phys. **2012** (2012), 38.

[24] P. Gavrylenko and O. Lisovyy, *Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions*, Commun. Math. Phys. **363** (2018), 1-58.

[25] P. Gavrylenko, A. Marshakov, and A. Stoyan, *Irregular conformal blocks, Painlevé III and the blow-up equations*, J. High Energ. Phys. **2020** (2020), 125.

[26] W.M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.

[27] N. Hitchin, *Frobenius manifolds*, Gauge theory and symplectic geometry, (Eds. J. Hurtubise, F. Lalonde, and G. Sabidussi), Nato Science Series C, **488**, Springer, 1997, 69–112.

[28] N. Iorgov, O. Lisovyy, and J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*, Commun. Math. Phys. **336** (2015), 671–694.

[29] N. Iorgov, O. Lisovyy, and Y. Tykhyy, *Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks*, J. High Energ. Phys. **2013** (2013), 029.

[30] A.R. Its, O. Lisovyy, and A. Prokhorov, *Monodromy dependence and connection formulae for isomonodromic tau functions*, Duke Math. J. **167** (2018), 1347–1432.

[31] A. Its, O. Lisovyy, and Y. Tykhyy, *Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks*, Int. Math. Res. Not. IMRN **2015** (2015), 8903–8924.

[32] A. Its and A. Prokhorov, *Connection problem for the tau-function of the sine-Gordon reduction of Painlevé-iii equation via the Riemann–Hilbert approach*, Int. Math. Res. Not. IMRN **2016** (2016), 6856–6883.

[33] K. Iwaki, *2-Parameter τ-function for the first Painlevé equation: Topological recursion and direct monodromy problem via exact WKB analysis*, Commun. Math. Phys. **377** (2020), 1047–1098.

[34] S. Jeong and N. Nekrasov, *Riemann–Hilbert correspondence and blown up surface defects*, J. High Energ. Phys. **2020** (2020), 6.

[35] M. Jimbo, T. Miwa, and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function*, Physica D: Nonlinear Phenomena **2** (1981), 306–352.

[36] C. Kalla and D. Korotkin, *Baker–Akhiezer spinor kernel and tau-functions on moduli spaces of meromorphic differentials*, Comm. Math. Physics **331** (2014), 1191–1235.

[37] T. Kawai and Y. Takei, *Algebraic Analysis of Singular Perturbation Theory*, **227**, Amer. Math. Soc., Providence, RI, 2005.

[38] A. Kokotov and D. Korotkin, *Tau-functions on spaces of Abelian differentials and higher genus generalizations of Ray–Singer formula*, J. Differential Geom. **82** (2009), 35–100.

[39] D. Korotkin, *Solution of matrix Riemann–Hilbert problems with quasi-permutation monodromy matrices*, Math. Ann. **329** (2004), 335–364.

[40] D. Korotkin and H. Samtleben, *Quantization of coset space sigma models coupled to two-dimensional gravity*, Commun. Math. Phys. **190** (1997), 411–457.

[41] D. Korotkin, *Bergman tau-function: From Einstein equations and Dubrovin–Frobenius manifolds to geometry of moduli spaces*, Integrable Systems and Algebraic Geometry, **2**, (Eds. R. Donagi and T. Shaska), LMS Lecture Notes, **459**, 2020, 215–287.

[42] D. Korotkin and P. Zograf, *Tau function and the Prym class*, Algebraic and Geometric Aspects of Integrable Systems and Random Matrices (Eds. A. Dzhamay, K. Maruno, and V. Pierce), Contemp. Math., **593**, 2013, 241–262.

[43] N. Nekrasov, *Blowups in BPS/CFT correspondence, and Painlevé VI*, preprint, https://arxiv.org/abs/2007.03646.

[44] A. Voros, *The return of the quadratic oscillator. The complex WKB method*, Ann. IHP Phys. Théor. **39** (1983), 211–338.

Marco Bertola,

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8,
E-mail: Marco.Bertola@concordia.ca

Dmitry Korotkin,

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8,
E-mail: Dmitry.Korotkin@concordia.ca

Fabrizio del Monte,

Centre de recherches mathématiques, Université de Montréal, C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7,
E-mail: Fabrizio.Delmonte@concordia.ca

Генерувальна функція монодромного симплектоморфізму для 2×2 системи Фукса та її ВКБ розвинення

Marco Bertola, Dmitry Korotkin, and Fabrizio del Monte

Ми вивчаємо ВКБ розвинення 2×2 системи лінійних диференціальних рівнянь з фуксовими сингулярностями. Основна увага сфокусована на генерувальній функції монодромного симплектоморфізму, яка, відповідно до недавньої роботи [10], є тісно пов'язаною з тау-функцією Джимбо–Міви. Ми обчислюємо перші три члени ВКБ розвинення генерувальної функції та встановлюємо її зв'язок з тау-функцією Бергмана.

Ключові слова: системи Фукса, відображення монодромії, генерувальна функція, тау-функція, ВКБ розвинення