On Two Remarkable Groups of Area-Preserving Homeomorphisms

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We prove that on a symplectic sphere, the group of Hamiltonian homeomorphisms in the sense of Oh and Müller is a proper normal subgroup of the group of finite energy Hamiltonian homeomorphisms. Moreover we detect infinite-dimensional flats inside the quotient of these groups endowed with the natural Hofer pseudo-metric.

Key words: Hamiltonian homeomorphisms, Fathi conjecture, Hofer metric, spectral invariants

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Contents

1. Introduction

Let (M, ω) be a closed and connected symplectic surface. Consider the groups $\mathrm{Ham}(M, \omega) \subset \mathrm{Homeo}(M)$ of Hamiltonian diffeomorphisms of (M, ω) and of all homeomorphisms of M, respectively. The C^0 closure of $\mathrm{Ham}(M, \omega)$ inside $\mathrm{Homeo}(M)$ is denoted by $\overline{\mathrm{Ham}}(M, \omega)$, and it consists exactly of those orientation and area preserving homeomorphisms of (M, ω) that are homotopic to the identity and which belong to the kernel of the mass-flow homomorphism [4]. In the genus 0 case, i.e. when $M = S^2$, the condition of vanishing of the mass-flow is

redundant, and so $\overline{\mathrm{Ham}}(S^2)$ consists of all orientation and area preserving homeomorphisms. The group $\overline{\mathrm{Ham}}(M,\omega)$ can also be defined for a general M, under additional compact support requirements.

The question of Fathi [4] asks whether $\overline{\mathrm{Ham}}(M,\omega)$ is a simple group, and it has been one of the major inspirations for the development of C^0 symplectic geometry. Fathi's question served as an important motivation for the influential work [8] of Oh and Müller which in particular introduced the notion of a continuous Hamiltonian flow on symplectic manifolds. In the 2-dimensional case, the group $\mathrm{Hameo}(M,\omega)$ consisting of time-1 maps of these flows was conjectured [8] to be an example of a proper normal subgroup of $\overline{\mathrm{Ham}}(M,\omega)$.

Recently, a number of breakthrough works addressed the Fathi question, where it was first solved in the case of a two-disc [1], then in the two-sphere case [3,9], and finally for general surfaces of finite type and finite area [2]. Moreover, the works [3,9] have largely contributed to Hofer geometry, in particular solving the Polterovich–Kapovich question. Powerful tools coming from Floer homology, embedded contact homology and periodic Floer homology theories, were central in making that progress possible. The aim of the present article is to give an additional insight on that picture. Our approach is based on application of novel Floer-theoretic invariants from [9] (see also [6]), in combination with a soft approach relying on an idea due to Sikorav [10].

Before discussing our results, we recall some of the relevant definitions. We refer the reader to Section 1.3 for other definitions and notation that we use. Let (M,ω) be a symplectic manifold. We denote by $\mathrm{Homeo}(M)$ the group of all compactly supported homeomorphisms of M. The C^0 convergence of a sequence $\phi_k \in \mathrm{Homeo}(M)$ to some $\phi \in \mathrm{Homeo}(M)$ always assumes that the supports of all ϕ_k lie in some compact subset of M, and is denoted by $\phi = (C^0)\lim_{k\to\infty}\phi_k$. For a continuous compactly supported function $H : M \to \mathbb{R}$, its L^∞ norm is denoted by $\|H\|$. For a continuous compactly supported function $H : M \times [0,1] \to \mathbb{R}$, its $L^{(1,\infty)}$ norm is given by

$$\|H\|_{(1,\infty)} = \int_0^1 \|H_t\| \, dt,$$

where $H_t(\cdot) = H(\cdot,t)$. All Hamiltonian functions are assumed to be compactly supported.

On a closed symplectic manifold (M,ω), for any $\phi \in \mathrm{Ham}(M,\omega)$, its Hofer norm is given by

$$\|\phi\|_{\mathrm{H}} = \inf \|H\|_{(1,\infty)},$$

where H is a smooth and normalized (a Hamiltonian function $H : M \times [0,1] \to \mathbb{R}$ is called normalized if for every $t \in [0,1]$, the function $H_t(x) = H(x,t)$ has zero mean with respect to the Liouville volume form, i.e. $\int_M H_t \omega^n = 0$). Hamiltonian function such that $\phi = \phi_H^1$. We denote by d_{H} the Hofer distance on $\mathrm{Ham}(M,\omega)$, that is, $d_{\mathrm{H}}(\phi,\psi) = \|\phi^{-1}\psi\|_{\mathrm{H}}$.

Now we recall the definitions of the groups $\overline{\mathrm{Ham}}(M,\omega)$ (Hamiltonian homeomorphisms), $\mathrm{Hameo}(M,\omega)$ (strong Hamiltonian homeomorphisms [8]), and $\mathrm{FHomeo}(M,\omega)$ (finite energy Hamiltonian homeomorphisms [1–3]).

Definition 1.1. A homeomorphism $\phi \in \mathrm{Homeo}(M)$ is a *Hamiltonian home-omorphism* if it is a C^0 limit of a sequence of Hamiltonian diffeomorphisms of M. The set of all Hamiltonian homeomorphisms of M is denoted by $\overline{\mathrm{Ham}}(M,\omega)$.

Definition 1.2. A continuous path $(\phi^t)_{t\in[0,1]}$ of homeomorphisms of M is a *continuous Hamiltonian flow*, if there exists a continuous compactly supported function $H : M \times [0,1] \to \mathbb{R}$ and a sequence $H_k : M \times [0,1] \to \mathbb{R}$ of smooth Hamiltonian functions such that:

- The supports of H_k all belong to some compact subset of M.
- $\lim_{k\to\infty} \|H_k - H\|_{(1,\infty)} = 0$.
- We have the C^0 convergence $\phi^t = (C^0) \lim_{k\to\infty} \phi^t_{H_k}$, uniform in $t \in [0,1]$.

In that case the time-1 map ϕ^1 is called a *strong Hamiltonian homeomorphism* of M, and the set of all such homeomorphisms is denoted by $\mathrm{Hameo}(M,\omega)$ (in fact, the original definition of $\mathrm{Hameo}(M,\omega)$ was through the notion of a *topological Hamiltonian path* [8]; however, as it was shown in [7], the two definitions are equivalent).

Definition 1.3. An element $\phi \in \overline{\mathrm{Ham}}(M,\omega)$ is a *finite energy homeomor-phism* if there exists a sequence of smooth Hamiltonians $H_k : M \times [0,1] \to \mathbb{R}$ such that

$$\phi = (C^0) \lim_{k\to\infty} \phi^1_{H_k} \quad \text{and} \quad \|H_k\|_{(1,\infty)} \leqslant C$$

for some constant $C > 0$ which is independent of k. The set of all finite energy Hamiltonian homeomorphisms of M is denoted by $\mathrm{FHomeo}(M,\omega)$.

It is well known [2,8] that $\mathrm{Hameo}(M,\omega) \subset \mathrm{FHomeo}(M,\omega)$ are normal sub-groups of $\overline{\mathrm{Ham}}(M,\omega)$. For symplectic surfaces (M,ω) of finite type and area, it was shown [1–3,9] that $\mathrm{FHomeo}(M,\omega)$ is in fact a *proper* normal subgroup of $\overline{\mathrm{Ham}}(M,\omega)$. This answers Fathi's question in the negative. This of course also means that $\mathrm{Hameo}(M,\omega)$ is a proper normal subgroup of $\overline{\mathrm{Ham}}(M,\omega)$, confirming the prediction of Oh and Müller. However, a natural question remained whether $\mathrm{Hameo}(M,\omega)$ and $\mathrm{FHomeo}(M,\omega)$ are distinct groups. This question was commu-nicated to us by V. Humilière and S. Seyfaddini. Below we answer this question in the case of S^2 (Corollary 1.5) by showing that these groups are indeed distinct.

Let (M,ω) be a closed symplectic manifold. The group $\mathrm{FHomeo}(M,\omega)$ nat-urally carries a Hofer-like norm and the associated metric, which we denote here by $\|\cdot\|_{\mathcal{H}}$ and $d_{\mathcal{H}}$, respectively [3, Section 5.4]. For any $\phi \in \mathrm{FHomeo}(M,\omega)$, its norm $\|\phi\|_{\mathcal{H}}$ is defined as the minimal possible $\liminf_{k\to\infty} \|\phi_k\|_{\mathrm{H}}$, where (ϕ_k) is a sequence in $\mathrm{Ham}(M,\omega)$ that C^0 converges to ϕ. The fact that $\|\cdot\|_{\mathcal{H}}$ is a norm and is not just a pseudo-norm readily follows from the energy-capacity inequality. It can be easily verified that the norm $\|\cdot\|_{\mathcal{H}}$ is invariant under conjugation by elements of $\overline{\mathrm{Ham}}(M,\omega)$. It is currently completely unknown whether the norm $\|\cdot\|_{\mathcal{H}}$ coincides with $\|\cdot\|_{\mathrm{H}}$ on $\mathrm{Ham}(M,\omega)$ (a question of Le Roux [5]).

Consider the symplectic two-sphere (S^2,ω) of area 1. Our first result is:

Theorem 1.4. *For every $E > 0$, there exists a continuous path $(\varphi^t)_{t\in[0,1]}$ of homeomorphisms of S^2 such that:*

(1) *The flow (φ^t) is the uniform limit of a sequence of smooth Hamiltonian flows $(\phi_{H_k}^t)$, where $H_k \in C^\infty(S^2 \times [0,1])$ is normalized and $\|H_k\|_{(1,\infty)} \leqslant E$ for every k.*

(2) *For every sequence $\psi_k \in \mathrm{Ham}(S^2)$ satisfying $\varphi^1 = (C^0)\lim_{k\to\infty} \psi_k$, and for every $\psi \in \mathrm{Ham}(S^2)$ we have $\liminf_{k\to\infty} d_\mathrm{H}(\psi_k, \psi) \geqslant E$.*

Moreover, the flow (φ^t) can be chosen to be arbitrarily C^0-close to any given smooth Hamiltonian flow in $\mathrm{Ham}(S^2)$ of Hofer length $\leqslant E$.

If (φ^t) is a path of homeomorphisms of S^2 given by Theorem 1.4, then by item (1) of the theorem, the time-1 map $\varphi = \varphi^1$ is an element of $\mathrm{FHomeo}(S^2)$. Both (1) and (2) imply that on the one hand, we have $\|\varphi\|_\mathcal{H} = E$, but on the other hand, the $d_\mathcal{H}$-distance of φ to any element of $\mathrm{Ham}(S^2)$ is greater than or equal to E. The latter property yields $\varphi \notin \mathrm{Hameo}(S^2)$. Indeed, assuming the contrary, that is $\varphi \in \mathrm{Hameo}(S^2)$, we obtain a sequence $H_k' : S^2 \times [0,1] \to \mathbb{R}$ of smooth Hamiltonian functions converging in the $L^{(1,\infty)}$ norm to a continuous function $H' : S^2 \times [0,1] \to \mathbb{R}$ such that in particular we have $\varphi = (C^0)\lim_{k\to\infty} \phi_{H_k'}^1$. But then we come to a contradiction with the item (2) of Theorem 1.4 since we can consider the sequence $\psi_k := \phi_{H_k'}^1$ and then take $\psi := \phi_{H_l'}^1$ for l large enough. Thus we obtain:

Corollary 1.5. $\mathrm{Hameo}(S^2) \neq \mathrm{FHomeo}(S^2)$.

This answers the question mentioned above. Note that $\mathrm{Hameo}(S^2)$ is a normal subgroup of $\mathrm{FHomeo}(S^2)$, and a next possible task is to check how large the quotient group $\mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$ is and to understand its algebraic structure. Moreover, the norm $\|\cdot\|_\mathcal{H}$ naturally descends from $\mathrm{FHomeo}(S^2)$ to the pseudo-norm $\|\cdot\|_\mathcal{H}$ on the quotient $\mathcal{G} = \mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$. It might be interesting to understand that picture better. These questions will be discussed in Section 1.2 below.

In our proof of Theorem 1.4 we use as a tool the novel powerful versions of Lagrangian spectral estimators developed in [9], which are certain functionals defined on the space of time-dependent Hamiltonians (spectral estimators) and on the group of Hamiltonian diffeomorphisms (group estimators). These functionals and their relatives were used in [9] to show several remarkable applications in Hofer's geometry, Lagrangian packing, and C^0 symplectic geometry. Let us briefly describe them and some of their basic properties.

As before, consider the symplectic sphere (S^2, ω), where ω is normalized by $\omega(S^2) = 1$, and think of it as sitting inside $\mathbb{R}^3$ as the sphere of radius $\frac{1}{2}$ centered at the origin, equipped with the standard area form divided by π. Denote by $x_3 : S^2 \to \mathbb{R}$ the x_3-coordinate function, let $k \geqslant 1$ be an integer, and pick a pair of positive rational numbers $0 < C < B$ satisfying $2B + (k-1)C = 1$. For $0 \leqslant j < k$ denote $L_{k,B}^{0,j} = (x_3)^{-1}(-1/2 + B + jC)$. This gives us a finite collection of "horizontal" circles on S^2, which are of course Lagrangian submanifolds. Then there exists a map $c_{k,B}^0 : C^\infty(S^2 \times [0,1]) \to \mathbb{R}$ with the following properties [9] (some properties described in [9] are omitted since we will not use them in the sequel):

1. (*Hofer–Lipschitz*) For each $G, H \in C^\infty(S^2 \times [0,1])$,

$$|c^0_{k,B}(G) - c^0_{k,B}(H)| \leqslant \int_0^1 \max |G_t - H_t| \, dt.$$

2. (*Monotonicity*) If $G, H \in C^\infty(S^2 \times [0,1])$ satisfy $G \leqslant H$ as functions, then

$$c^0_{k,B}(G) \leqslant c^0_{k,B}(H).$$

3. (*Normalization*) For each $H \in C^\infty(S^2 \times [0,1])$ and $b \in C^\infty([0,1])$,

$$c^0_{k,B}(H + b) = c^0_{k,B}(H) + \int_0^1 b(t) \, dt.$$

4. (*Lagrangian control*) For any $H \in C^\infty(S^2 \times [0,1])$ such that $(H_t)|_{L^{0,j}_{k,B}} \equiv c_j(t)$ for all $0 \leqslant j < k$, we have

$$c^0_{k,B}(H) = \frac{1}{k} \sum_{0 \leqslant j < k} \int_0^1 c_j(t) \, dt.$$

5. (*Independence of Hamiltonian*) For a normalized Hamiltonian $H \in C^\infty(S^2 \times [0,1])$, the value

$$c^0_{k,B}(H) = c^0_{k,B}(\phi^1_H)$$

depends only on the time-1 map $\phi^1_H \in \mathrm{Ham}(S^2)$.

6. (*Subadditivity*) For all $\phi, \psi \in \mathrm{Ham}(S^2)$,

$$c^0_{k,B}(\phi\psi) \leqslant c^0_{k,B}(\phi) + c^0_{k,B}(\psi).$$

7. (C^0-*continuity*) The map

$$\tau_{k,k',B,B'} : \mathrm{Ham}(S^2) \to \mathbb{R}, \quad \tau_{k,k',B,B'} = c^0_{k,B} - c^0_{k',B'}$$

is 2-Lipschitz in Hofer's metric, it is C^0-continuous, and it extends to $\overline{\mathrm{Ham}}(S^2)$ by continuity.

Recall that the above Lagrangian control property of the functionals $c^0_{k,B}$ is associated with the collection $\{L^{0,j}_{k,B}\}_{0 \leqslant j < k}$ of horizontal circles on S^2. The work [9] by Polterovich and Shelukhin has inspired a later work [2] by Cristofaro-Gardiner, Humilière, Mak, Seyfaddini, and Smith, which introduced new invariants sharing properties similar to these of $c^0_{k,B}$. These invariants from [2] can be defined on general closed symplectic surfaces (M, ω), and moreover the corresponding Lagrangian control property for them holds for quite general collections of circles on M. The invariants are denoted by $c_{\underline{L}}$, where $\underline{L}$ is a suitable given collection of circles on M. In particular, one may choose a collection $\underline{L}$ so that most of the circles in it (all except for a restricted number of them) are small circles bounding discs of the same area which are "spread uniformly" over the surface and capture almost all of its area. That particular property of $\underline{L}$ was crucial for deriving the so-called "Calabi property" for the $c_{\underline{L}}$'s [2, p. 3, Theorem 1.1], which in turn has remarkable applications such as the Fathi conjecture [2]. We wish to remark that by using the invariants $c_{\underline{L}}$ from [2] instead of $c^0_{k,B}$ from [9], one can extend Theorem 1.4 to all closed symplectic surfaces.

1.1. The Calabi property. The mentioned above Calabi property for the invariants c_L states the following [2]:

Theorem. *Let $\underline{L}^m$ be a sequence of equidistributed Lagrangian links in a closed symplectic surface (M, ω). Then, for any $H \in C^\infty(M \times [0, 1])$ we have*

$$\lim_{m \to \infty} c_{\underline{L}^m}(H) = (\omega(M))^{-1} \int_0^1 \int_M H_t \omega \, dt.$$

Here a Lagrangian link means a collection of pairwise disjoint smoothly embedded circles, and for the precise definition of a sequence of equidistributed Lagrangian links we refer the reader to [2]. We claim, however, that in principle one can relax the assumptions and still have the Calabi property, so that in particular, relying on the "equidistribution property" in unnecessary. In the next proposition, $\widetilde{\mathrm{Ham}}(M, \omega)$ stands for the universal cover of the Hamiltonian group $\mathrm{Ham}(M, \omega)$.

Proposition 1.6. *Let (M, ω) be a closed and connected symplectic surface, and let $c_m : C^\infty(M \times [0, 1]) \to \mathbb{R}$ be a sequence of functionals, satisfying:*

1. *(Hofer–Lipschitz) For each $G, H \in C^\infty(M \times [0, 1])$,*

$$|c_m(G) - c_m(H)| \leqslant \int_0^1 \max |G_t - H_t| \, dt.$$

2. *(Monotonicity) If $G, H \in C^\infty(M \times [0, 1])$ satisfy $G \leqslant H$ as functions, then*

$$c_m(G) \leqslant c_m(H).$$

3. *(Normalization) For each $H \in C^\infty(M \times [0, 1])$ and $b \in C^\infty([0, 1])$,*

$$c_m(H + b) = c_m(H) + \int_0^1 b(t) \, dt.$$

   Moreover, $c_m(\mathbf{0}) = 0$ where $\mathbf{0} : M \times [0, 1] \to \mathbb{R}$ is the zero function.

4. *(Independence of Hamiltonian) For a normalized Hamiltonian $H \in C^\infty(S^2 \times [0, 1])$, the value*

$$c_m(H) = c_m(\phi_H)$$

   depends only on the class $\phi_H = [(\phi_H^t)] \in \widetilde{\mathrm{Ham}}(M, \omega)$.

5. *(Subadditivity) For any $\phi, \psi \in \widetilde{\mathrm{Ham}}(M, \omega)$,*

$$c_m(\phi\psi) \leqslant c_m(\phi) + c_m(\psi).$$

6. *(Locality) For each m, there exists a smooth function $h_m : M \to [0, 1]$ and an open topological disc $\mathcal{D}_m \subset M$ such that:*

   a) *$\mathrm{supp}(h_m) \subset \mathcal{D}_m$,*

   b) *The diameter of $\mathcal{D}_m$ converges to 0 when $m \to \infty$.*

   c) *$\lim_{m \to \infty} (\omega(M)/\omega(\mathcal{D}_m)) \, c_m(-h_m) = -1$.*

Then for every $H \in C^\infty(M \times [0, 1])$ we have

$$\lim_{m \to \infty} c_m(H) = (\omega(M))^{-1} \int_0^1 \int_M H_t \omega \, dt. \tag{1.1}$$

1.1.1. Some remarks. Note that the Normalization, Independence of Hamiltonian, and Subadditivity properties for c_m in Proposition 1.6 are stated differently from the ones which were previously stated for the invariants $c^0_{k,B}$. Also, the Locality property combined with the normalization condition $c_m(\mathbf{0}) = 0$ in some sense replace the previously stated Lagrangian control property for $c^0_{k,B}$. The main reason to relax the Independence of Hamiltonian is that other than $c^0_{k,B}$ relevant functionals introduced in [9] and [2] do not necessarily satisfy the original version of the Independence of Hamiltonian property.

As a corollary of Proposition 1.6 we conclude that the functionals $c^0_{k,B}$ satisfy the Calabi property:

Theorem 1.7. *For a sequence* $B_k \in (\frac{1}{k+1}, \frac{1}{2})$ *that converges to* 0, *for any* $H \in C^\infty(S^2 \times [0,1])$ *we have*

$$\lim_{k\to\infty} c^0_{k,B_k}(H) = \int_0^1 \int_{S^2} H_t \omega \, dt.$$

Proof. For deriving the Calabi property from Proposition 1.6, it is enough to verify the Locality property for the sequence c^0_{k,B_k} of functionals. Choose a sequence $B'_k \in (B_k, \frac{1}{2})$ that converges to 0 and such that $\lim_{k\to\infty} B'_k/B_k = \infty$. Then for the sequence $\mathcal{D}_m = z^{-1}([-1/2, -1/2 + B'_m))$ of spherical discs, the Lagrangian control property for $c_m := c^0_{m,B_m}$ implies the locality property (as in the statement of Proposition 1.6) for c_m with h_m being an approximation the characteristic function of $\mathcal{D}_m$. Indeed, the corresponding collection $(L^{0,j}_{m,B_m})_{0 \leqslant j < m}$ of circles divides the disc $\mathcal{D}_m$ into a collection of annuli of equal area C_m (except for possibly one annulus of area less than C_m) and a disc of area B_m which is negligible with respect to $B'_m = \omega(\mathcal{D}_m)$. Therefore the number of circles that lie in $\mathcal{D}_m$ is asymptotically approximated by $\omega(\mathcal{D}_m)/C_m$ and hence by $m \, \omega(\mathcal{D}_m)/\omega(M)$. Because of that, the Lagrangian control property for c_m and the function $-h_m$ (where h_m is an approximation the characteristic function of $\mathcal{D}_m$) implies the locality property for c_m. $\qquad\square$

1.2. The quotient $\mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$**.** Recall that for any $\phi \in \mathrm{FHomeo}(S^2)$, its norm $\|\phi\|_{\mathcal{H}}$ is defined as the minimal possible $\liminf_{k\to\infty} \|\phi_k\|$, where (ϕ_k) is a sequence in $\mathrm{Ham}(S^2)$ that C^0 converges to ϕ. The quotient $\mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$ is naturally endowed with the induced pseudo-norm $\|\cdot\|_{\mathcal{H}}$.

Consider the normed Abelian additive group $(l^\infty, \|\cdot\|_\infty)$ that consists of infinite bounded sequences $s = (s_1, s_2, \ldots)$ of real numbers such that $\|s\|_\infty = \sup|s_k|$, where the group structure is standard. Moreover, consider the subgroup $c_0 \subset l^\infty$ that consists of all sequences that converge to 0. The norm $\|\cdot\|_\infty$ on l^∞ naturally descends to a norm on the quotient l^∞/c_0. We denote that norm on l^∞/c_0 also by $\|\cdot\|_\infty$. Note that for every $s \in l^\infty$ and the corresponding element $[s] \in l^\infty/c_0$, we have

$$\|[s]\|_\infty = \limsup_{k\to\infty} |s_k|.$$

Using properties of the functionals $c_{k,B}^0$ (in particular, the Calabi property stated in Theorem 1.7), combined with a soft approach, we show (cf. [9, Theorem A]):

Theorem 1.8. *The normed group $(l^\infty/c_0, \|\cdot\|_\infty)$ embeds isometrically into the group $\mathcal{G} = \mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$ endowed with the Hofer pseudo-norm $\|\cdot\|_\mathcal{H}$.*

Corollary 1.9. *One can isometrically embed into $\mathcal{G}$ the normed group $(l^\infty, \|\cdot\|_\infty)$ and also the normed group $(C(X), \|\cdot\|_\infty)$ when X is a separable topological space (e.g. when $X = \mathbb{R}$).*

The corollary readily follows from the theorem and from the fact that one can isometrically embed the normed group $(l^\infty, \|\cdot\|_\infty)$ into the normed group $(l^\infty/c_0, \|\cdot\|_\infty)$ and moreover isometrically embed $(C(X), \|\cdot\|_\infty)$ into $(l^\infty, \|\cdot\|_\infty)$ (for a separable topological space X). We also remark that Theorem 1.8 is stronger than Corollary 1.9 in the sense that there is no isometric group embedding of $(l^\infty/c_0, \|\cdot\|_\infty)$ into $(l^\infty, \|\cdot\|_\infty)$. See Section 2.6.2 for more details.

It is known that the group $\mathcal{G}$ is abelian [2, Proposition 2.2]. Theorem 1.8 implies that the torsion-free rank of $\mathcal{G}$ is continuum (since the cardinality of $\mathcal{G}$ is continuum itself). Our approach, however, does not seem to help understanding the torsion part of $\mathcal{G}$. Also, it is would be interesting to verify whether the Hofer pseudo-norm on $\mathcal{G}$ is non-degenerate (that is, a genuine norm). In addition, the following questions remain unanswered. For a smooth function $h : (-1/2, 1/2) \to \mathbb{R}$, consider the time-1 map ϕ of the Hamiltonian flow of the autonomous function $H(x_1, x_2, x_3) = h(x_3)$ defined on the sphere without the north and south poles, and extend ϕ to a homeomorphism of S^2. For which h do we have $\phi \in \mathrm{FHomeo}(S^2)$? In case when $\phi \in \mathrm{FHomeo}(S^2)$, how to compute the norm $\|\phi\|_\mathcal{H}$ (if not precisely, then up to an "almost equivalence", i.e. up to an additive and multiplicative constant)? For which h do we have $\phi \in \mathrm{Hameo}(S^2)$? Our proof of Theorem 1.8 shows only a very partial answer to these questions.

To conclude, we remark that the proofs of Theorems 1.4, 1.8 and Proposition 1.6 are quite close. Morally speaking, Theorem 1.8 is more general than Theorem 1.4. We still preferred to keep Theorem 1.4 for the convenience of the reader since it serves a good motivation for the latter and since its proof is simpler than that of Theorem 1.8 and does not rely on the Calabi property for the functionals $c_{k,B}^0$.

1.3. Notation and preliminary remarks. Let (M, ω) be a symplectic surface. For a subset $A \subset M$, its ω-area is denoted by $\omega(A)$. For a Hamiltonian function $H \in C^\infty(M \times [0,1])$, $(\phi_H^t)_{t \in [0,1]}$ denotes the Hamiltonian flow of H. For given Hamiltonian functions $H, K \in C^\infty(M \times [0,1])$, we denote $H \sharp K(x, t) = H(x, t) + K((\phi_H^t)^{-1}(x))$ (the Hamiltonian flow of this Hamiltonian function is the composition of flows $(\phi_H^t \circ \phi_K^t)$). Moreover by $\overline{H}(x, t) = -H(\phi_H^t(x), t)$ we denote Hamiltonian function that generates the inverse flow $((\phi_H^t)^{-1})$. The L^∞ norm of a Hamiltonian $H \in C^\infty(M \times [0,1])$ is denoted by $\|H\| = \max |H|$. We say that a Hamiltonian $H \in C^\infty(M \times [0,1])$ is normalized if $\int_M H_t \omega = 0$ for all $t \in [0,1]$.

Now let (M, ω) be a closed symplectic surface. The Hofer distance on $\mathrm{Ham}(M, \omega)$ is denoted by d_H, and for every $\phi \in \mathrm{Ham}(M, \omega)$, $\|\phi\|_\mathrm{H}$ stands for the Hofer norm of ϕ. We also denote by d_H and $\| \cdot \|_\mathrm{H}$ the Hofer distance and norm on $\widetilde{\mathrm{Ham}}(M, \omega)$ (the universal cover of $\mathrm{Ham}(M, \omega)$). In order to speak about C^0 convergence, we equip M with an auxiliary Riemannian metric which defines the distance function $d : M \times M \to \mathbb{R}$. For homeomorphisms $\phi, \psi : M \to M$, we define $d_{C^0}(\phi, \psi) = \max_{x \in M} d(\phi(x), \psi(x))$ and $\bar{d}_{C^0}(\phi, \psi) = \max(d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1}))$. The reason for considering the metric $\bar{d}_{C^0}$ is related to the following important property: $\mathrm{Homeo}(M)$ is complete with respect to $\bar{d}_{C^0}$, that is, if we have a Cauchy sequence in $\mathrm{Homeo}(M)$ with respect to $\bar{d}_{C^0}$ then it necessarily converges (with respect to $\bar{d}_{C^0}$) to a homeomorphism of M. For a sequence of homeomorphisms of M, in order to conclude its uniform convergence to some homeomorphism, the Cauchy property with respect to d_{C^0} is generally not enough. Note however that if a sequence in $\mathrm{Homeo}(M)$ is known to converge uniformly (i.e. with respect to d_{C^0}) a homeomorphism, then it in fact converges to that homeomorphism with respect to $\bar{d}_{C^0}$. For $\phi \in \mathrm{Homeo}(M)$ and a sequence $\phi_k \in \mathrm{Homeo}(M)$, we will write $\phi = (C^0) \lim_{k \to \infty} \phi_k$ if $\lim_{k \to \infty} d_{C^0}(\phi, \phi_k) = 0$ (which is equivalent to $\lim_{k \to \infty} \bar{d}_{C^0}(\phi, \phi_k) = 0$).

Assume that we have a smooth manifold M, two open subsets $U, V \subset M$, and a diffeomorphism $\phi : U \to V$. For a diffeomorphism $f : M \to M$ compactly supported in V by $\phi^* f$, we denote the diffeomorphism $\phi^* f : M \to M$ which is given by $\phi^* f = \phi^{-1} f \phi$ on U and for which $\phi^* f = \mathbb{1}$ on $M \setminus U$. Similarly, for a diffeomorphism $h : M \to M$ compactly supported in U by $\phi_* f$, we denote the diffeomorphism $\phi_* f : M \to M$ which is given by $\phi_* f = \phi h \phi^{-1}$ on V and for which $\phi_* f = \mathbb{1}$ on $M \setminus V$.

2. Proofs

The central lemma which is used in the proofs is essentially due to Sikorav [10, Section 8.4]:

Lemma 2.1. *Let (M, ω) be a closed and connected symplectic surface. Let $\varepsilon > 0$, let m be a positive integer, and let $\mathcal{D}_0, \ldots, \mathcal{D}_m \subset M$ be topological open discs of area ε each such that for every $1 \leqslant j \leqslant m$ we are given a symplectic diffeomorphism $\phi_j : \mathcal{D}_0 \to \mathcal{D}_j$. Moreover, let $f_0, f_1, \ldots, f_m \in \mathrm{Ham}(M, \omega)$ with $\mathrm{supp}(f_j) \subset \mathcal{D}_j$. Define $\Phi, \Phi' \in \mathrm{Ham}(M, \omega)$ by*

$$\Phi = f_0 f_1 \cdots f_m$$

and

$$\Phi' = f_0 \Pi_{j=1}^m \phi_j^* f_j,$$

where $\phi_j^* f_j$ is given by $\phi_j^* f_j = (\phi_j)^{-1} f_j \phi_j$ on $\mathcal{D}_0$ and $\phi_j^* f_j = \mathbb{1}$ on $M \setminus \mathcal{D}_0$. Then

$$d_\mathrm{H}(\Phi, \Phi') < 3\varepsilon. \tag{2.1}$$

Its proof will be given in Section 2.5 below.

Remark 2.2. As it can be seen from the proof of Lemma 2.1, if one removes from the statement of the lemma the assumption that M is closed, one still gets a similar conclusion that $\Phi^{-1}\Phi'$ is generated by a normalized Hamiltonian function H of $L^{(1,\infty)}$ norm less than 3ε. Moreover, by a reparametrization we can have $\|H\| < 3\varepsilon$.

Proofs of Theorems 1.4 and 1.8 use the notation that we now introduce. For each integer $m \geqslant 3$ make a choice of rational numbers $\frac{1}{m+1} < B_m < B'_m < \frac{1}{m}$ and denote $C_m = (1 - 2B_m)/(m-1)$ and $C'_m = (1 - 2B'_m)/(m-1)$. Then denote $\sigma_m := c^0_{m,B_m}$ and $\sigma'_m := c^0_{m,B'_m}$, and let $\tau_m := \sigma_m - \sigma'_m$.

The main ingredient in the proof of Theorem 1.4 is the following lemma:

Lemma 2.3. *Let* $H : S^2 \times [0,1] \to \mathbb{R}$ *be a smooth normalized Hamiltonian function. Then for every* $E > \|H\|$, $\varepsilon > 0$ *and every smooth Hamiltonian function* $F : S^2 \times [0,1] \to \mathbb{R}$, *one can find an integer* $m \geqslant 3$ *and a smooth normalized Hamiltonian function* $H' : S^2 \times [0,1] \to \mathbb{R}$ *such that:*

1. $\overline{d}_{C^0}(\phi^t_H, \phi^t_{H'}) < \varepsilon$ *for every* $t \in [0,1]$.
2. $\|H'\| < E$.
3. $|\tau_m(H' \sharp F)| > 2E - \varepsilon$.

Its proof relies on Lemma 2.1, see Section 2.2.

2.1. Proof of Theorem 1.4. Let $E > 0$ and let $H : S^2 \times [0,1] \to \mathbb{R}$ be a smooth normalized Hamiltonian function with $\|H\|_{(1,\infty)} \leqslant E$. We need to show that for every $\varepsilon > 0$ we can find a continuous path $(\varphi^t)_{t\in[0,1]}$ in $\mathrm{Homeo}(S^2)$ such that $d_{C^0}(\varphi^t, \phi^t_H) < \varepsilon$ for all $t \in [0,1]$, and which satisfies the properties (1) and (2) stated in the theorem. By choosing some $c \in (0,1)$ which is very close to 1, and replacing $H(x,t)$ by $cH(x,ct)$ (defined on $M \times [0,1]$), we may assume that $\|H\|_{(1,\infty)} < E$. Now, by further time-reparametization (i.e. replacing H by $c'(t)H(x,c(t))$ for some smooth bijection $c : [0,1] \to [0,1]$ such that $c'(t) > 0$ for all $t \in [0,1]$), we may without loss of generality assume that $\|H\| < E$.

The space of smooth normalized functions $S^2 \times [0,1] \to \mathbb{R}$ is separable when endowed with the $L^{(1,\infty)}$ norm. Choose a corresponding dense sequence of normalized Hamiltonian functions $F_j : S^2 \times [0,1] \to \mathbb{R}$, $(j = 1, 2, \ldots)$, and then the sequence of time-1 maps $\Phi_j := \phi^1_{F_j}$ is dense in $\mathrm{Ham}(S^2)$ with respect to the Hofer metric. We now inductively construct a sequence $H_0, H_1, \ldots$ of time dependent normalized Hamiltonian functions on S^2. Set $H_0 = H$. For each $k \geqslant 1$, Lemma 2.3 provides us a smooth Hamiltonian function $H_k : S^2 \times [0,1] \to \mathbb{R}$ such that $\overline{d}_{C^0}(\phi^t_{H_{k-1}}, \phi^t_{H_k}) < \varepsilon/2^k$ for every $t \in [0,1]$, $\|H_k\| < E$, and $|\tau_{m_k}(H_k \sharp \overline{F_k})| > 2E - 1/k$ for some $m_k \geqslant 3$. Moreover by the C^0-continuity of τ_{m_j}'s as functionals on $\mathrm{Ham}(S^2)$, we may inductively assume that we also have $|\tau_{m_j}(H_k \sharp \overline{F_j})| > 2E - 1/j$ for every $1 \leqslant j < k$ (indeed, on the step $k \geqslant 1$ when we obtain H_k, we get the bound $|\tau_{m_j}(H_{k-1} \sharp \overline{F_j})| > 2E - 1/j$ from the previous step, and then the inequality $|\tau_{m_j}(H_k \sharp \overline{F_j})| > 2E - 1/j$ follows from the C^0 continuity of τ_{m_j} provided that $d_{C^0}(\phi^1_{H_k}, \phi^1_{H_{k-1}})$ is small enough. And by Lemma 2.3 we are indeed able to

pick the Hamiltonian H_k such that (in addition to the conditions $\|H_k\| < E$ and $|\tau_{m_k}(H_k \sharp \overline{F_k})| > 2E - 1/k$) the C^0 distance $\max_{t \in [0,1]} \overline{d}_{C^0}(\phi^t_{H_{k-1}}, \phi^t_{H_k})$ between Hamiltonian flows is arbitrarily small, in particular $d_{C^0}(\phi^1_{H_k}, \phi^1_{H_{k-1}})$ is arbitrarily small.

To summarise, we now have a Hofer dense in $\mathrm{Ham}(S^2)$ sequence $\Phi_1, \Phi_2, \ldots,$ where $\Phi_j = \phi^1_{F_j}$, and we have a sequence of smooth normalized Hamiltonian functions (H_k) and a sequence of indices (m_k) such that:

(a) $H_0 = H$.
(b) $\overline{d}_{C^0}(\phi^t_{H_{k-1}}, \phi^t_{H_k}) < \varepsilon/2^k$ for all $t \in [0,1]$.
(c) $\|H_k\| < E$.
(d) $|\tau_{m_j}(H_k \sharp \overline{F_j})| > 2E - 1/j$ for $1 \leqslant j \leqslant k$.

The property (b) implies that the sequence of flows $(\phi^t_{H_k})$ uniformly converges to a continuous flow φ^t of homeomorphisms of S^2. Then by (c) we readily get the property (1) from the statement of the theorem. Also note that (b) yields

$$\overline{d}_{C^0}(\phi^t_H, \varphi^t) = \overline{d}_{C^0}(\phi^t_{H_0}, \varphi^t) < \varepsilon, \quad t \in [0,1].$$

To see the property (2) we use the fact that τ_{m_j} are Hofer–Lipschitz with constant 2. As in (2), assume that we have a sequence $\psi_k \in \mathrm{Ham}(S^2)$ such that $\varphi^1 = (C^0) \lim_{k \to \infty} \psi_k$, and assume that $\psi \in \mathrm{Ham}(S^2)$. Given any $\delta > 0$ there exist infinitely many indices j for which $d_{\mathrm{H}}(\psi, \Phi_j) < \delta$. Then for such j and for every k have

$$2d_{\mathrm{H}}(\psi_k, \psi) \geqslant 2d_{\mathrm{H}}(\psi_k, \Phi_j) - 2\delta \geqslant |\tau_{m_j}(\psi_k \circ \Phi_j^{-1})| - 2\delta. \qquad (2.2)$$

Since τ_{m_j} is C^0 continuous on $\mathrm{Ham}(S^2)$ and moreover extends by continuity to $\overline{\mathrm{Ham}}(S^2)$, we get

$$\lim_{k \to \infty} \tau_{m_j}(\psi_k \circ \Phi_j^{-1}) = \tau_{m_j}(\varphi^1 \circ \Phi_j^{-1}) = \lim_{k \to \infty} \tau_{m_j}(\phi^1_{H_k} \circ \Phi_j^{-1}). \qquad (2.3)$$

But then by (2.2), (2.3) and the property (d) we get

$$2 \liminf_{k \to \infty} d_{\mathrm{H}}(\psi_k, \psi) \geqslant \lim_{k \to \infty} |\tau_{m_j}(\psi_k \circ \Phi_j^{-1})| - 2\delta = \lim_{k \to \infty} |\tau_{m_j}(\phi^1_{H_k} \circ \Phi_j^{-1})| - 2\delta$$
$$= \lim_{k \to \infty} |\tau_{m_j}(H_k \sharp \overline{F_j})| - 2\delta \geqslant 2E - 1/j - 2\delta.$$

We conclude that the inequality

$$\liminf_{k \to \infty} d_{\mathrm{H}}(\psi_k, \psi) \geqslant E - 1/j - \delta$$

holds for every $\delta > 0$ and for infinitely many integer values of j. The property (2) now follows.

2.2. Proof of Lemma 2.3. Recall (see the beginning of Section 2, right after the formulation of Lemma 2.1) that for each integer $m \geqslant 3$ we have chosen rational numbers $\frac{1}{m+1} < B_m < B'_m < \frac{1}{m}$, and denoted $C_m = (1 - 2B_m)/(m - 1)$ and $C'_m = (1 - 2B'_m)/(m - 1)$. Also, we denoted $\sigma_m = c^0_{m,B_m}$, $\sigma'_m = c^0_{m,B'_m}$, and $\tau_m = \sigma_m - \sigma'_m$.

Let $m > 6/\varepsilon$ be large enough. One can cover S^2 by open topological discs $\mathcal{D}_j$ $(0 \leqslant j < m)$ of equal area $A_m \in (B'_m, \frac{1}{m})$ such that the their diameters are bounded from above by $o(1)$ when $m \to \infty$ (the discs can in fact be chosen such that the diameters are bounded by $m^{-1/2}$ up to a constant). We may without loss of generality assume that $\mathcal{D}_0 = (x_3)^{-1}([-1/2, -1/2 + A_m))$. Denote $L = (x_3)^{-1}(-1/2 + B_m)$ and $L' = (x_3)^{-1}(-1/2 + B'_m)$, where $x_3 : S^2 \to \mathbb{R}$ is the coordinate function as before. We have $L, L' \subset \mathcal{D}_0 \subset S^2$. Choose area-preserving diffeomorphisms $\phi_j : \mathcal{D}_0 \to \mathcal{D}_j$, for $0 \leqslant j < m$ (where ϕ_0 is taken to be the identity map).

Define $\hat{a}_j(t) = \inf_{x \in \mathcal{D}_j} H(x, t)$ and $\hat{b}_j(t) = \sup_{x \in \mathcal{D}_j} H(x, t)$ for $t \in [0, 1]$. These functions are continuous on $[0, 1]$ and attain their values in $(-E, E)$. Moreover, if we chose m to be sufficiently large, we get $0 \leqslant \hat{b}_j(t) - \hat{a}_j(t) < \varepsilon/2$ for every j and t. Then we can approximate $\hat{a}_j, \hat{b}_j$ by smooth functions $a_j, b_j : [0, 1] \to (-E, E)$ satisfying $\hat{a}_j > a_j$ and $\hat{b}_j < b_j$ on $[0, 1]$ for which we still have $b_j(t) - a_j(t) < \varepsilon/2$.

Denoting $\delta = \frac{1}{4E}(E - \|H\|) > 0$, pick smooth functions $h, h' : S^2 \to [-\delta, 1]$, which are compactly supported in $\mathcal{D}_0$, have disjoint supports, satisfy the relation

$$\int_{S^2} h\omega = \int_{S^2} h'\omega = 0,$$

and the relations $h|_L \equiv 1$, $h|_{L'} \equiv 0$, $h'|_L \equiv 0$, $h'|_{L'} \equiv 1$. For each j, define smooth functions $h_j, h'_j : S^2 \to [-\delta, 1]$, compactly supported in $\mathcal{D}_j$, as push-forwards $h_j = (\phi_j)_* h$ and $h'_j = (\phi_j)_* h'$.

Now define normalized Hamiltonian functions $K, K' : S^2 \times [0, 1] \to \mathbb{R}$ by

$$K(x, t) = \sum_{j=0}^{m-1} (E - b_j(t))h_j(x) + (-E - a_j(t))h'_j(x),$$

$$K'(x, t) = \sum_{j=0}^{m-1} (-E - a_j(t))h_j(x) + (E - b_j(t))h'_j(x).$$

Clearly, K and K' have commuting Hamiltonian flows, and are both compactly supported in the disjoint union $\cup_{j=0}^{m-1} \mathcal{D}_j$. Define Hamiltonian functions $H_1, H_2 : S^2 \times [0, 1] \to \mathbb{R}$ by $H_1 = K \sharp H$ and $H_2 = K' \sharp H$. Let us show that we can choose the desired Hamiltonian H' to be either H_1 or H_2.

Introduce new Hamiltonian functions

$$K_0(x, t) = \sum_{j=0}^{m-1} (E - b_j(t))h(x) + (-E - a_j(t))h'(x),$$

$$K_0'(x,t) = \sum_{j=0}^{m-1}(-E - a_j(t))h(x) + (E - b_j(t))h'(x),$$

which are compactly supported in $\mathcal{D}_0$. Lemma 2.1 ensures that

$$d_{\mathrm{H}}(\phi_{K'}^1(\phi_K^1)^{-1}, \phi_{K_0'}^1(\phi_{K_0}^1)^{-1}) < 3A_m < 3/m < \varepsilon/2. \tag{2.4}$$

Recall our notation $\sigma_m = c_{m,B_m}^0$ and $\sigma_m' = c_{m,B_m'}^0$, and $\tau_m = \sigma_m - \sigma_m'$. The inequality (2.4), and the Hofer Lipschitz and Independence of Hamiltonian properties of σ_m yield

$$\sigma_m(\phi_{K'}^1 \circ (\phi_K^1)^{-1}) \leqslant \sigma_m(\phi_{K_0'}^1 \circ (\phi_{K_0}^1)^{-1}) + d_{\mathrm{H}}(\phi_{K'}^1(\phi_K^1)^{-1}, \phi_{K_0'}^1(\phi_{K_0}^1)^{-1})$$
$$< \sigma_m(\phi_{K_0'}^1 \circ (\phi_{K_0}^1)^{-1}) + \varepsilon/2.$$

Now because $H_1 = K \sharp H$ and $H_2 = K' \sharp H$, we therefore get

$$\sigma_m(H_2 \sharp \overline{H_1}) = \sigma_m(K' \sharp H \sharp \overline{H} \sharp \overline{K}) = \sigma_m(K' \sharp \overline{K}) = \sigma_m(\phi_{K'}^1 \circ (\phi_K^1)^{-1})$$
$$< \sigma_m(\phi_{K_0'}^1 \circ (\phi_{K_0}^1)^{-1}) + \varepsilon/2 = \sigma_m(K_0' \sharp \overline{K_0}) + \varepsilon/2. \tag{2.5}$$

The value $\sigma_m(K_0' \sharp \overline{K_0})$ can be computed explicitly. Indeed, notice that

$$K_0' \sharp \overline{K_0}(x,t) = K_0'(x,t) - K_0(x,t) = \left(-2mE + \sum_{j=0}^{m-1}(b_j(t) - a_j(t))\right)h(x)$$
$$+ \left(2mE + \sum_{j=0}^{m-1}(a_j(t) - b_j(t))\right)h'(x).$$

Hence by our choice of the functions h and h', we see that for each $t \in [0,1]$ we have $K_0' \sharp \overline{K_0}(x,t) = -2mE + \sum_{j=0}^{m-1}(b_j(t) - a_j(t))$ when $x \in L = z^{-1}(-1/2 + B_m)$. Moreover we have $K_0' \sharp \overline{K_0}(x,t) = 0$ when $x \in z^{-1}(-1/2 + B_m + iC_m)$ for $i = 1, \ldots, m-1$. To see the this, note that

$$B_m + C_m = B_m + (1 - 2B_m)/(m-1) = ((m-3)B_m + 1)/(m-1) > 1/m > A_m,$$

and so the circles $z^{-1}(-1/2 + B_m + iC_m)$ lie in the complement of the supports of h and h' $(i = 1, \ldots, m-1)$. Therefore by the Lagrangian control property we get

$$\sigma_m(K_0' \sharp \overline{K_0}) = \frac{1}{m}\int_0^1 \left(-2mE + \sum_{j=0}^{m-1}(b_j(t) - a_j(t))\right) dt \leqslant -2E + \varepsilon/2. \tag{2.6}$$

Combining this estimate with (2.5), we conclude

$$\sigma_m(H_2 \sharp \overline{H_1}) < \sigma_m(K_0' \sharp \overline{K_0}) + \varepsilon/2 \leqslant -2E + \varepsilon. \tag{2.7}$$

In an analogous way we obtain the upper bound

$$\sigma'_m(H_1 \sharp \overline{H_2}) < -2E + \varepsilon. \tag{2.8}$$

Indeed, similarly as in (2.5) and (2.6), we have estimates

$$\sigma'_m(H_1 \sharp \overline{H_2}) < \sigma'_m(K_0 \sharp \overline{K'_0}) + \varepsilon/2,$$

and

$$\sigma'_m(K_0 \sharp \overline{K'_0}) = \frac{1}{m} \int_0^1 \left(-2mE + \sum_{j=0}^{m-1} (b_j(t) - a_j(t)) \right) dt \leqslant -2E + \varepsilon/2,$$

and the inequality (2.8) follows.

By the Subadditivity property, (2.7) and (2.8) we have

$$
\begin{aligned}
\tau_m(H_2 \sharp F) &= \sigma_m(H_2 \sharp F) - \sigma'_m(H_2 \sharp F) \\
&\leqslant (\sigma_m(H_2 \sharp \overline{H_1}) + \sigma_m(H_1 \sharp F)) + (-\sigma'_m(H_1 \sharp F) + \sigma'_m(H_1 \sharp \overline{H_2})) \\
&= \tau_m(H_1 \sharp F) + \sigma_m(H_2 \sharp \overline{H_1}) + \sigma'_m(H_1 \sharp \overline{H_2}) < \tau_m(H_1 \sharp F) - 4E + 2\varepsilon.
\end{aligned}
$$

This means that we have either $\tau_m(H_1 \sharp F) > 2E - \varepsilon$ or $\tau_m(H_2 \sharp F) < -2E + \varepsilon$. In the first case we put $H' = H_1$, and in the second $H' = H_2$. Then H' satisfies the property (3) from the statement of the lemma. To see the property (2), it is enough to check that $\|H_1\| < E$ and $\|H_2\| < E$. We have $H_1(x, t) = K \sharp H(x, t) = K(x, t) + H((\Phi_K^t)^{-1}(x), t)$. Therefore on each $\mathcal{D}_j$, since the Hamiltonian flow of K preserves it, we have

$$H_1(x, t) \leqslant \max(E - b_j(t), 2E\delta) + \sup_{\mathcal{D}_j} H(\cdot, t) < E$$

and

$$-H_1(x, t) \leqslant \max(E + a_j(t), 2E\delta) - \inf_{\mathcal{D}_j} H(\cdot, t) < E$$

(recall that $a_j(t), b_j(t) \in (-E, E)$, $a_j(t) < \hat{a}_j(t) = \inf_{\mathcal{D}_j} H(\cdot, t)$, that $b_j(t) > \hat{b}_j(t) = \sup_{\mathcal{D}_j} H(\cdot, t)$, and $\delta = \frac{1}{4E}(E - \|H\|)$). Moreover, on the complement of the union $\cup_j \mathcal{D}_j$ we have $H_1(x, t) = H(x, t)$, and so we obtain $|H_1(x, t)| < E$ as well. We conclude that $\|H_1\| < E$. The inequality $\|H_2\| < E$ follows similarly. Finally, the property (1) holds if m is sufficiently large. Indeed, the Hamiltonian flows of K and K' are compactly supported in the disjoint union $\cup_j \mathcal{D}_j$, and the diameters of all the $\mathcal{D}_j$'s are bounded by $o(1)$ when $m \to \infty$. Hence $\max_{t \in [0,1]} \overline{d}_{C^0}(\mathbb{1}, \phi_K^t)$ and $\max_{t \in [0,1]} \overline{d}_{C^0}(\mathbb{1}, \phi_{K'}^t)$ become as small as we wish when m is large enough, and consequently, for large enough m we get $\overline{d}_{C^0}(\phi_H^t, \phi_{H_i}^t) < \varepsilon$ for every $t \in [0, 1]$ and $i = 1, 2$, because $\phi_{H_1}^t = \phi_K^t \circ \phi_H^t$ and $\phi_{H_2}^t = \phi_{K'}^t \circ \phi_H^t$ for all $t \in [0, 1]$.

2.3. Proof of Proposition 1.6. Note that the Normalization and Independence of Hamiltonian properties of c_m imply that the Subadditivity property can be reformulated into the statement that for $G, H \in C^\infty(M \times [0,1])$ we have

$$c_m(G \sharp H) \leqslant c_m(G) + c_m(H). \tag{2.9}$$

Then the Monotonicity and Normalization properties imply that for every smooth $h : M \to (-\infty, 0]$ and $t \geqslant 0$ we have

$$c_m(th) \leqslant (t-1)c_m(h). \tag{2.10}$$

Indeed, from the Monotonicity and Normalization properties we get $c_m(h) \leqslant c(\mathbf{0}) = 0$, and then denoting $n = \lfloor t \rfloor$, from (2.9) and Monotonicity we conclude $c_m(th) \leqslant c_m(nh) \leqslant nc_m(h) \leqslant (t-1)c_m(h)$.

Denote $N_m = \lceil \omega(M)/\omega(\mathcal{D}_m) \rceil - 2$ and note that

$$\lim_{m \to \infty} N_m = +\infty,$$

$$\lim_{m \to \infty} N_m c_m(-h_m) = -1. \tag{2.11}$$

We can without loss of generality assume that the open disc $\mathcal{D}_m$ has a smooth boundary. Pack M by open topological discs $\mathcal{D}_{m,j}$ ($0 \leqslant j \leqslant N_m$) of equal areas that have smooth boundaries and mutually disjoint closures such that $\mathcal{D}_{m,0} = \mathcal{D}_m$ and the diameters of $\mathcal{D}_{m,0}, \ldots, \mathcal{D}_{m,N_m}$ are bounded from above by $o(1)$ when $m \to \infty$. Choose area-preserving diffeomorphisms $\phi_{m,j} : \mathcal{D}_{m,0} \to \mathcal{D}_{m,j}$, for $0 \leqslant j \leqslant N_m$ (where $\phi_{m,0}$ is taken to be the identity map). For each $0 \leqslant j \leqslant N_m$, define smooth functions $h_{m,j} : S^2 \to [0,1]$, compactly supported in $\mathcal{D}_{m,j}$, as push-forwards $h_{m,j} = (\phi_{m,j})_* h_m$.

Now let $H \in C^\infty(M \times [0,1])$, and let $\varepsilon > 0$ be arbitrary. Choose some $E > \|H\|$. Then, choose m to be large enough such that in particular we have $3\omega(\mathcal{D}_m) < \varepsilon$ and $\omega(\mathcal{D}_m)\|H\| < \varepsilon$. Define

$$\hat{a}_j(t) = \inf_{\mathcal{D}_{m,j}} H(\cdot, t), \quad \hat{b}_j(t) = \sup_{\mathcal{D}_{m,j}} H(\cdot, t), \qquad t \in [0,1].$$

These functions are continuous on $[0,1]$ and attain their values in $(-E, E)$. Moreover if we choose m to be sufficiently large, we get $0 \leqslant \hat{b}_j(t) - \hat{a}_j(t) < \varepsilon$ for every j and t. Then approximate $\hat{a}_j, \hat{b}_j$ by smooth functions $a_j, b_j : [0,1] \to (-E, E)$ such that we have $\hat{a}_j > a_j$ and $\hat{b}_j < b_j$ on $[0,1]$ and such that we still have $b_j(t) - a_j(t) < \varepsilon$. Denote

$$I = \int_0^1 \int_M H(x, t) \omega \, dt.$$

Then I is well–approximated by $\omega(\mathcal{D}_m) \left(\sum_{j=0}^{N_m} \int_0^1 b_j(t) \, dt \right)$. Indeed, note that the complement $K := M \setminus \cup_{j=0}^{N_m} \mathcal{D}_{m,j}$ of the union of the discs, has a small area:

$$\omega(K) = \omega(M) - (N_m + 1)\omega(\mathcal{D}_m) \leqslant \omega(\mathcal{D}_m).$$

Moreover,

$$I = \int_0^1 \int_M H(x,t)\omega\,dt = \left(\sum_{j=0}^{N_m} \int_0^1 \int_{\mathcal{D}_{m,j}} H(x,t)\omega\,dt\right) + \int_0^1 \int_K H(x,t)\omega\,dt.$$

We have

$$\left|\int_0^1 \int_K H(x,t)\omega\,dt\right| \leqslant \omega(K)\|H\| \leqslant \omega(\mathcal{D}_m)\|H\| < \varepsilon.$$

Also, by

$$a_j(t) < \inf_{\mathcal{D}_{m,j}} H(\cdot,t) \leqslant \sup_{\mathcal{D}_{m,j}} H(\cdot,t) < b_j(t)$$

and by $b_j(t) - a_j(t) < \varepsilon$, we have

$$|H(x,t) - b_j(t)| < \varepsilon$$

on $\mathcal{D}_{m,j}$, and therefore

$$\left|\int_0^1 \int_{\mathcal{D}_{m,j}} H(x,t)\omega\,dt - \omega(\mathcal{D}_m)\int_0^1 b_j(t)\,dt\right| < \varepsilon\omega(\mathcal{D}_m)$$

for each j. We conclude

$$\left|I - \omega(\mathcal{D}_m)\left(\sum_{j=0}^{N_m}\int_0^1 b_j(t)\,dt\right)\right| < (1 + \omega(M))\varepsilon. \qquad (2.12)$$

Define smooth Hamiltonian functions $K, K', K_0, K_0' : S^2 \times [0,1] \to \mathbb{R}$ by

$$K(x,t) = \sum_{j=0}^{N_m}(E - b_j(t))h_{m,j}(x),$$

$$K'(x,t) = \sum_{j=0}^{N_m}(-E - a_j(t))h_{m,j}(x)$$

and

$$K_0(x,t) = \sum_{j=0}^{N_m}(E - b_j(t))h_m(x) = \left((N_m+1)E - \sum_{j=0}^{N_m}b_j(t)\right)h_m(x),$$

$$K_0'(x,t) = \sum_{j=0}^{N_m}(-E - a_j(t))h_m(x) = \left(-(N_m+1)E - \sum_{j=0}^{N_m}a_j(t)\right)h_m(x).$$

Similarly as in the proof of Lemma 2.3, applying the version of Lemma 2.1 for the universal cover $\mathrm{Ham}(M,\omega)$ as described in Section 2.6.1, we conclude that

$$d_{\mathrm{H}}([(\phi_{K_0}^t)], [(\phi_K^t)])\,,\ d_{\mathrm{H}}([(\phi_{K_0'}^t)], [(\phi_{K'}^t)]) < 3\omega(\mathcal{D}_m) < \varepsilon.$$

It is easy to see that we have

$$\int_0^1 \int_M K(x,t)\omega \, dt = \int_0^1 \int_M K_0(x,t)\omega \, dt$$

and

$$\int_0^1 \int_M K'(x,t)\omega \, dt = \int_0^1 \int_M K'_0(x,t)\omega \, dt.$$

Hence by the Hofer–Lipschitz and Independence of Hamiltonian properties of c_m we get

$$|c_m(K) - c_m(K_0)|, \ |c_m(K') - c_m(K'_0)| < \varepsilon.$$

Also, by (2.10), and by the Normalization and Independence of Hamiltonian properties we have

$$c_m(\overline{K_0}) = c_m(-K_0) \leqslant \left((N_m + 1)E - 1 - \sum_{j=0}^{N_m} \int_0^1 b_j(t) \, dt \right) c_m(-h_m),$$

$$c_m(K'_0) \leqslant \left((N_m + 1)E - 1 + \sum_{j=0}^{N_m} \int_0^1 a_j(t) \, dt \right) c_m(-h_m).$$

Therefore we conclude

$$c_m(\overline{K}) < \left((N_m + 1)E - 1 - \sum_{j=0}^{N_m} \int_0^1 b_j(t) \, dt \right) c_m(-h_m) + \varepsilon =: A_m,$$

$$c_m(K') < \left((N_m + 1)E - 1 + \sum_{j=0}^{N_m} \int_0^1 a_j(t) \, dt \right) c_m(-h_m) + \varepsilon =: B_m.$$

Note that

$$A_m + B_m = \left(2(N_m + 1)E - 2 + \sum_{j=0}^{N_m} \int_0^1 (a_j(t) - b_j(t)) \, dt \right) c_m(-h_m) + 2\varepsilon$$

$$< (2N_m E + 2E - 2 - (N_m + 1)\varepsilon)c_m(-h_m) + 2\varepsilon.$$

Therefore, if m is large enough then in view of (2.11), we get $A_m + B_m < -2E + 4\varepsilon$. Define Hamiltonian functions $H_1, H_2 : S^2 \times [0,1] \to \mathbb{R}$ by $H_1 = K\sharp H$ and $H_2 = K'\sharp H$. Then, like in the proof of Lemma 2.3, it is easy to see that we have $\|H_1\|, \|H_2\| \leqslant E$. In particular we get $-E \leqslant c_m(H_1), c_m(H_2) \leqslant E$. Moreover, we have $c_m(H) \leqslant c_m(H_1) + c_m(\overline{K}) < c_m(H_1) + A_m$ and $c_m(H_2) \leqslant c_m(H) + c_m(K') < c_m(H) + B_m$.

Let us summarize what we have:

- $A_m + B_m < -2E + 4\varepsilon$,
- $-E \leqslant c_m(H_1), c_m(H_2) \leqslant E$,

- $c_m(H) < c_m(H_1) + A_m,$
- $c_m(H_2) < c_m(H) + B_m.$

This implies that $c_m(H)$ lies in the interval $(-E - B_m, E + A_m)$ whose length is $(E + A_m) - (-E - B_m) = 2E + A_m + B_m < 4\varepsilon$. In particular,

$$|c_m(H) - (E + A_m)| < 4\varepsilon.$$

We have

$$E + A_m = E + \left((N_m + 1)E - 1 - \sum_{j=0}^{N_m} \int_0^1 b_j(t)\,dt\right) c_m(-h_m) + \varepsilon$$

$$= (N_m c_m(-h_m) + 1)E + (E - 1)c_m(-h_m)$$

$$+ \left(I - \omega(\mathcal{D}_m) \sum_{j=0}^{N_m} \int_0^1 b_j(t)\,dt\right)(c_m(-h_m)/\omega(\mathcal{D}_m))$$

$$- ((\omega(M))^{-1} + c_m(-h_m)/\omega(\mathcal{D}_m))I + \varepsilon + (\omega(M))^{-1}I.$$

Hence by (2.11), (2.12) and the Locality property we have

$$|E + A_m - (\omega(M))^{-1}I| < (3 + (\omega(M))^{-1})\varepsilon$$

provided that m is large enough. Therefore we get

$$|c_m(H) - (\omega(M))^{-1}I| < (7 + (\omega(M))^{-1})\varepsilon$$

for large enough m. Since $\varepsilon > 0$ is arbitrary, this shows (1.1).

2.4. Proof of Theorem 1.8. Denote the natural projection homomorphism by $\pi : \mathrm{FHomeo}(S^2) \to \mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$. For proving the theorem it is enough to find a group homomorphism $\Phi : l^\infty \to \mathrm{FHomeo}(S^2)$ such that for each $s \in l^\infty$ we have

$$\|\pi \circ \Phi(s)\|_{\mathcal{H}} = \limsup_{k \to \infty} |s_k| \tag{2.13}$$

and such that $\Phi(c_0) \subset \mathrm{Hameo}(S^2)$. Indeed, then $\pi \circ \Phi$ naturally descends to an isometric embedding $l^\infty/c_0 \to \mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$. After constructing Φ, the proof of the equality (2.13) will be divided into proving the inequalities "$\geqslant$" and "$\leqslant$". For the first inequality we will use properties of the functionals $c_{k,B}^0$, in particular the Calabi property stated in Theorem 1.7. For the opposite one, we will apply a soft argument which uses Lemma 2.1.

Recall (see the beginning of Section 2, right after the formulation of Lemma 2.1) that for each integer $m \geqslant 3$ we have chosen rational numbers $\frac{1}{m+1} < B_m < B_m' < \frac{1}{m}$, and denoted $C_m = (1 - 2B_m)/(m - 1)$ and $C_m' = (1 - 2B_m')/(m - 1)$. Also, we denoted $\sigma_m = c_{m,B_m}^0$, $\sigma_m' = c_{m,B_m'}^0$, and $\tau_m = \sigma_m - \sigma_m'$. For each $m \geqslant 3$ choose some $A_m \in (B_m', \frac{1}{m})$ and consider the open disc
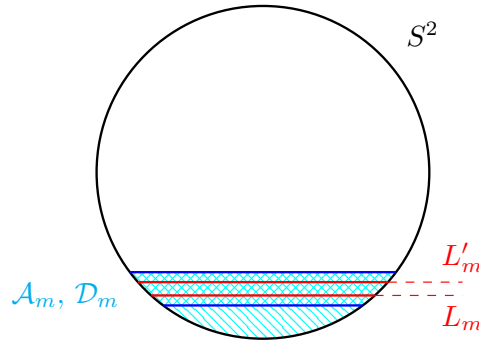
$$\mathcal{D}_m := (x_3)^{-1}([-1/2, -1/2 + A_m)).$$

Fig. 2.1
The annulus $\mathcal{A}_m$, the disc $\mathcal{D}_m$, and the circles L_m and L'_m.

Let

$$\mathcal{A}_m := (x_3)^{-1}((-1/2 + 1/(m+1), -1/2 + A_m)) \subset \mathcal{D}_m \subset S^2$$

denote the spherical annulus, and let $L_m := (x_3)^{-1}(-1/2 + B_m)$ and $L'_m := (x_3)^{-1}(-1/2 + B'_m)$ be circles inside $\mathcal{A}_m$.

For each $m \geqslant 3$, pack M by open topological discs $\mathcal{D}_{m,j}$ $(0 \leqslant j \leqslant m-1)$ of equal areas that have smooth boundaries and mutually disjoint closures such that $\mathcal{D}_{m,0} = \mathcal{D}_m$ and such that the diameters of $\mathcal{D}_{m,0}, \ldots, \mathcal{D}_{m,m-1}$ are bounded from above by $o(1)$ when $m \to \infty$. Choose area-preserving diffeomorphisms $\phi_{m,j} : \mathcal{D}_{m,0} \to \mathcal{D}_{m,j}$, for $0 \leqslant j \leqslant m-1$ (where $\phi_{m,0}$ is taken to be the identity map).

In order to prepare for the construction, we inductively choose a sequence of indices $m_1 < m_2 < \ldots$ and also choose a sequence of smooth functions $h_k : S^2 \to \mathbb{R}$ such that each h_k depends only on the x_3-coordinate and is supported in $\mathcal{A}_{m_k}$. Moreover, some auxiliary Hamiltonian functions $F_{k,s} \in C^\infty(S^2 \times [0,1])$ (for $k \in \mathbb{N}$ and $s \in [-1,1]^k$) will be introduced, and later they will be useful in estimating the Hofer norm. The following properties of the construction will hold:

(1)   For every $k > 1$ and $1 \leqslant i < k$ we have $|m_i \tau_{m_k}(h_i)| \leqslant 2^{-k}$.
(2)   For each $k \geqslant 1$ and $s \in [-1,1]^k$, the Hamiltonian function $F_{k,s}$ is normalized, i.e. for every $t \in [0,1]$ we have $\int_{S^2} F_{k,s}(x,t)\omega = 0$.
(3)   For every $k \geqslant 1$ and $s \in [-1,1]^k$ we have

$$\|F_{k,s}\| \leqslant \max_{1 \leqslant i \leqslant k}(1 + 2^{-i} - 2^{-k-1})|s_i|.$$

(4)   For each $k \geqslant 1$ there exists a constant $C_k > 0$ such that for each $s \in [-1,1]^k$ and $t \in [0,1]$, the function $F_{k,s}(\cdot, t) : S^2 \to \mathbb{R}$ is $(C_k \max_{1 \leqslant i \leqslant k} |s_i|)$-Lipschitz.

Now we pass to the construction itself. At the first step we choose $m_1 \geqslant 3$ arbitrarily and then choose smooth functions $h'_1, h''_1 : S^2 \to [0,1]$ that depend only on the coordinate x_3 such that their disjoint supports are contained in the annulus $\mathcal{A}_{m_1}$ and for which $h'_1|_{L_{m_1}} \equiv 1$, $h'_1|_{L'_{m_1}} \equiv 0$, $h''_1|_{L_{m_1}} \equiv 0$, $h''_1|_{L'_{m_1}} \equiv 1$, and $\int_{S^2} h'_1 \omega = \int_{S^2} h''_1 \omega$. Then define $h_1 : S^2 \to \mathbb{R}$ by $h_1 := h'_1 - h''_1$, and for each

$s \in [-1, 1]$ define $F_{1,s} \in C^\infty(S^2 \times [0,1])$ as

$$F_{1,s}(x,t) = s_1 \sum_{j=0}^{m_1-1} (\phi_{m_1,j})_* h_1(x).$$

Functions h_1 and $F_{1,s}$ clearly satisfy properties (1)–(4). This finishes the first step.

Now we describe a step k when $k > 1$. In the previous step we have constructed the family $(F_{k-1,s}(\cdot, \cdot))_{s \in [-1,1]^{k-1}}$ of functions in $C^\infty(S^2 \times [0,1])$. By the previous step of the construction and property (4), there exists a constant C_{k-1} such that for every $t \in [0,1]$ and $s \in [-1,1]^{k-1}$, the function $F_{k-1,s}(\cdot, t) : S^2 \to \mathbb{R}$ is $(C_{k-1} \max_{1 \leqslant i \leqslant k-1} |s_i|)$-Lipschitz. Also, by property (2) from the previous step, for each $s \in [-1,1]^{k-1}$ the Hamiltonian $F_{k-1,s}$ in normalized, that is

$$\int_{S^2} F_{k-1,s}(x,t)\omega = 0 \tag{2.14}$$

for every $t \in [0,1]$. Moreover, by property (3) from the previous step, we have

$$\|F_{k-1,s}\| \leqslant \max_{1 \leqslant i \leqslant k-1} (1 + 2^{-i} - 2^{-k})|s_i|. \tag{2.15}$$

Hence we can choose $m_k > 2m_{k-1}$ sufficiently large such that for every $s \in [-1,1]^{k-1}$, $t \in [0,1]$ and j we have

$$\operatorname*{osc}_{\mathcal{D}_{m_k,j}} F_{k-1,s}(\cdot, t) = \sup_{\mathcal{D}_{m_k,j}} F_{k-1,s}(\cdot, t) - \inf_{\mathcal{D}_{m_k,j}} F_{k-1,s}(\cdot, t) \leqslant \frac{\|s\|_\infty}{2^{k+6}}, \tag{2.16}$$

and also denoting by Z_k the complement of the union $\cup_{j=0}^{m_k-1} \mathcal{D}_{m_k,j}$, we have

$$\int_{Z_k} |F_{k-1,s}(x,t)|\omega \leqslant \frac{\|s\|_\infty}{2^{k+5}} \tag{2.17}$$

for every $t \in [0,1]$. Moreover, we may assume that m_k is sufficiently large so that the property (1) holds, i.e. we have $|m_i \tau_{m_k}(h_i)| \leqslant 2^{-k}$ for every $1 \leqslant i \leqslant k-1$ (since each constructed h_i depends only on the x_3-coordinate, for verifying that condition it is enough to refer only to the Lagrangian control property).

For each $s \in [-1,1]^{k-1}$ and j, choose a smooth function $a_{k,j}(\cdot; s) : [0,1] \to \mathbb{R}$ such that

$$\inf_{\mathcal{D}_{m_k,j}} F_{k-1,s}(\cdot, t) - \frac{\|s\|_\infty}{2^{k+6}} \leqslant a_{k,j}(t; s) \leqslant \sup_{\mathcal{D}_{m_k,j}} F_{k-1,s}(\cdot, t) + \frac{\|s\|_\infty}{2^{k+6}}. \tag{2.18}$$

The dependence of $a_{k,j}(t; s)$ on $s \in [-1,1]^{k-1}$ is not required to be smooth or even continuous. An important property of $a_{k,j}$ is the boundedness which is uniform in s, and it follows from (2.18). By (2.14) and (2.17) we have

$$\left| \sum_{j=0}^{m_k-1} \int_{\mathcal{D}_{m_k,j}} F_{k-1,s}(x,t)\omega \right| \leqslant \frac{\|s\|_\infty}{2^{k+5}},$$

hence by (2.16) and (2.18), we get

$$\omega(\mathcal{D}_{m_k}) \left| \sum_{j=0}^{m_k-1} a_{k,j}(t;s) \right| \leqslant \frac{\|s\|_\infty}{2^{k+4}}$$

for every $t \in [0,1]$. Therefore the average

$$A_k(t;s) := \frac{1}{m_k} \sum_{j=0}^{m_k-1} a_{k,j}(t;s)$$

verifies

$$|A_k(t;s)| \leqslant \frac{\|s\|_\infty}{2^{k+4} m_k \omega(\mathcal{D}_{m_k})} \leqslant \frac{\|s\|_\infty}{2^{k+3}}. \tag{2.19}$$

Denoting $\tilde{a}_{k,j}(t;s) := a_{k,j}(t;s) - A_k(t;s)$, we have

$$\sum_{j=0}^{m_k-1} \tilde{a}_{k,j}(t;s) = 0 \tag{2.20}$$

for every t and s. Moreover, the functions $\tilde{a}_{k,j}(\cdot;s)$ are bounded, uniformly in $0 \leqslant j < m_k$ and $s \in [-1,1]^k$.

Choose smooth functions $h'_k, h''_k : S^2 \to [0,1]$ that depend only on the coordinate x_3 such that their disjoint supports are contained in the annulus $\mathcal{A}_{m_k}$ and for which $h'_k|_{L_{m_k}} \equiv 1$, $h'_k|_{L'_{m_k}} \equiv 0$, $h''_k|_{L_{m_k}} \equiv 0$, $h''_k|_{L'_{m_k}} \equiv 1$, and $\int_{S^2} h'_k \omega = \int_{S^2} h''_k \omega$.

Now let $s = (s_1, \dots, s_k) \in [-1,1]^k$, and denote $s' = (s_1, \dots, s_{k-1}) \in [-1,1]^{k-1}$. If $s_k = 0$ then we set $F_{k,s} := F_{k-1,s'}$. Otherwise, define the Hamiltonian function

$$
\begin{aligned}
&G_{k,s}(x,t) \\
&= \sum_{j=0}^{m_k-1} \left((s_k - \tilde{a}_{k,j}(t;s'))(\phi_{m_k,j})_* h'_k(x) + (-s_k - \tilde{a}_{k,j}(t;s'))(\phi_{m_k,j})_* h''_k(x) \right),
\end{aligned}
$$

and then set

$$F_{k,s} := G_{k,s} \sharp F_{k-1,s'}.$$

By (2.20) and by the choice of h'_k and h''_k we get that $G_{k,s}$ is normalized, and hence $F_{k,s}$ is normalized as well. This shows the property (2). Moreover, since for every $(x,t) \in S^2 \times [0,1]$ we have

$$F_{k,s}(x,t) = G_{k,s}(x,t) + F_{k-1,s'}((\phi_{G_{k,s}}^t)^{-1}(x),t), \tag{2.21}$$

by the definition of $G_{k,s}$ and by the boundedness of the functions $\tilde{a}_{k,j}(\cdot,s)$ which is uniform in $0 \leqslant j < m_k$ and $s \in [-1,1]^k$, it follows that the property (4) holds: there exists a constant $C_k > 0$ such that for each $s \in [-1,1]^k$ and $t \in [0,1]$, the function $F_{k,s}(\cdot,t) : S^2 \to \mathbb{R}$ is $(C_k \max_{1 \leqslant i \leqslant k} |s_i|)$-Lipschitz.

Consider the case when $s_k \neq 0$. By (2.21), for $x \notin \cup_j \mathcal{D}_{m_k,j}$ we have $F_{k,s}(x,t) = F_{k-1,s'}(x,t)$. Otherwise, if $x \in \mathcal{D}_{m_k,j}$ for some j, then in the case when x lies outside the supports of $(\phi_{m_k,j})_* h'_k$ and $(\phi_{m_k,j})_* h''_k$, we get the same conclusion $F_{k,s}(x,t) = F_{k-1,s'}(x,t)$. It remains to treat separately the cases of $x \in \mathrm{supp}((\phi_{m_k,j})_* h'_k)$ and $x \in \mathrm{supp}((\phi_{m_k,j})_* h''_k)$, where our aim is to estimate $F_{k,s}(x,t)$. Assuming $x \in \mathrm{supp}((\phi_{m_k,j})_* h'_k)$, we get $x \notin \mathrm{supp}((\phi_{m_k,j})_* h''_k)$, and hence for a given $t \in [0,1]$, denoting $\lambda = (\phi_{m_k,j})_* h'_k(x) \in [0,1]$ and $y = (\phi^t_{G_{k,s}})^{-1}(x) \in \mathcal{D}_{m_k,j}$, we have

$$
\begin{aligned}
F_{k,s}(x,t) &= \lambda(s_k - \tilde{a}_{k,j}(t;s')) + F_{k-1,s'}(y,t) \\
&= \lambda(s_k - a_{k,j}(t;s') + A_k(t;s')) + F_{k-1,s'}(y,t) \\
&= (1-\lambda)F_{k-1,s'}(y,t) + \lambda s_k + \lambda(F_{k-1,s'}(y,t) - a_{k,j}(t;s') + A_k(t;s')).
\end{aligned}
$$

But by (2.18) and (2.19) we have

$$
|F_{k-1,s'}(y,t) - a_{k,j}(t;s') + A_k(t;s')| \leqslant \frac{\|s\|_\infty}{2^{k+2}}.
$$

Hence by (2.15) we conclude

$$
\begin{aligned}
|F_{k,s}(x,t)| &\leqslant \max_{1\leqslant i\leqslant k}(1 + 2^{-i} - 2^{-k})|s_i| + 2^{-k-2}\|s\|_\infty \\
&\leqslant \max_{1\leqslant i\leqslant k}(1 + 2^{-k-2})(1 + 2^{-i} - 2^{-k})|s_i| \leqslant \max_{1\leqslant i\leqslant k}(1 + 2^{-i} - 2^{-k-1})|s_i|.
\end{aligned}
$$

We obtained this estimate under the assumption of $x \in \mathrm{supp}((\phi_{m_k,j})_* h'_k)$. The case of $x \in \mathrm{supp}((\phi_{m_k,j})_* h''_k)$ is completely analogous, and the same conclusion follows. Thus we get

$$
\|F_{k,s}\| \leqslant \max_{1\leqslant i\leqslant k}(1 + 2^{-i} - 2^{-k-1})|s_i|. \tag{2.22}
$$

Now if $s_k = 0$, then (2.22) holds as well, by the inductive assumption. This shows the property (3). Finally, define the function $h_k \in C^\infty(S^2)$ by $h_k := h'_k - h''_k$. This finishes the step k.

Note that if we have $s \in [-1,1]^k$ and $s_k = 0$, then denoting $s' := (s_1, \ldots, s_{k-1}) \in [-1,1]^{k-1}$, by the construction we have $F_{k,s} \equiv F_{k-1,s'}$. Hence for each $s \in l^\infty$ which has only a finite number of non-zero coordinates, we can define $F_s(x,t) := F_{\ell,\sigma}(x,t)$ where $\sigma = (s_1, \ldots, s_\ell) \in [-1,1]^\ell$, and ℓ is such that $s_k = 0$ for $k > \ell$.

For every $s = (s_1, s_2, \ldots) \in l^\infty$ denote $^k s := (s_1, s_2, \ldots, s_k, 0, 0, \ldots) \in l^\infty$ to be the element of l^∞ whose first k coordinates coincide with those of s and whose remaining coordinates are 0. Moreover, we define $s^k := s - {}^{(k-1)}s = (0, \ldots, 0, s_k, s_{k+1}, \ldots) \in l^\infty$.

For each $s \in l^\infty$ which has only a finite number of non-zero coordinates, define $\Phi(s) \in \mathrm{Ham}(S^2)$ to be the time-1 map of the autonomous Hamiltonian $\sum m_k s_k h_k$. Then, for every $s \in l^\infty$ define

$$
\Phi(s) := (C^0)\lim_{k\to\infty}\Phi(^k s) \in \overline{\mathrm{Ham}}(S^2).
$$

We claim that for each $s \in l^\infty$ with $\|s\|_\infty \leqslant 1$ we in fact have $\Phi(s) \in \mathrm{FHomeo}(S^2)$ and moreover

$$\|\Phi(s)\|_{\mathcal{H}} \leqslant \sup_k (1 + 2^{-k})|s_k| + 3 \sum_{s_k \neq 0} \frac{1}{m_k}. \tag{2.23}$$

For proving this, it is enough to show that for every $s \in l^\infty$, $\|s\|_\infty \leqslant 1$ such that only a finite number of its coordinates are non-zero, we have

$$\|\Phi(s)\|_{\mathrm{H}} \leqslant \max_k (1 + 2^{-k})|s_k| + 3 \sum_{s_k \neq 0} \frac{1}{m_k}. \tag{2.24}$$

Indeed, having (2.24), for any $s \in l^\infty$, $\|s\|_\infty \leqslant 1$, we get

$$\Phi(s) = (C^0) \lim_{\ell \to \infty} \Phi(^\ell s),$$

and

$$\|\Phi(^\ell s)\|_{\mathrm{H}} \leqslant \sup_k (1 + 2^{-k})|s_k| + 3 \sum_{s_k \neq 0} \frac{1}{m_k}$$

for each $\ell \geqslant 1$, hence $\Phi(s) \in \mathrm{FHomeo}(S^2)$, and (2.23) follows just by the definition of the norm $\| \cdot \|_{\mathcal{H}}$ on $\mathrm{FHomeo}(S^2)$.

To show (2.24), it is enough to prove that

$$d_{\mathrm{H}}(\Phi(s), \phi^1_{F_s}) \leqslant 3 \sum_{s_k \neq 0} \frac{1}{m_k}. \tag{2.25}$$

Indeed, then (2.24) will readily follow from (2.25) and (2.22). Now, let us prove (2.25) by induction in $\ell := \max(\{0\} \cup \{k \,|\, s_k \neq 0\})$. The case of $\ell = 0$ is clear since then both sides of (2.25) vanish. In the case of $\ell = 1$ we have $\Phi(s) = \phi^1_{m_1 s_1 h_1}$, whereas

$$F_{1,s}(x,t) = s_1 \sum_{j=0}^{m_1 - 1} (\phi_{m_1, j})_* h_1(x).$$

Hence a direct application of Lemma 2.1 yields

$$d_{\mathrm{H}}(\Phi(s), \phi^1_{F_s}) = d_{\mathrm{H}}(\phi^1_{m_1 s_1 h_1}, \phi^1_{F_{1,s}}) < \frac{3}{m_1}$$

showing (2.25). If $\ell > 1$ then denoting $\sigma := (s_1, \ldots, s_\ell) \in [-1,1]^\ell$, $\sigma' := (s_1, \ldots, s_{\ell-1}) \in [-1,1]^{\ell-1}$ and $s' := {}^{(\ell-1)}s = (s_1, \ldots, s_{\ell-1}, 0, 0, \ldots) \in l^\infty$, from the induction hypothesis it follows that

$$d_{\mathrm{H}}(\Phi(s'), \phi^1_{F_{s'}}) \leqslant 3 \sum_{s_k \neq 0,\, k < \ell} \frac{1}{m_k}.$$

In addition we have

$$\Phi(s) = \phi^1_{m_\ell s_\ell h_\ell} \Phi(s')$$

and

$$\phi^1_{F_s} = \phi^1_{F_{\ell,\sigma}} = \phi^1_{G_{\ell,\sigma}} \phi^1_{F_{\ell-1,\sigma'}} = \phi^1_{G_{\ell,\sigma}} \phi^1_{F_{s'}}$$

where

$$G_{\ell,\sigma}(x,t)$$
$$= \sum_{j=0}^{m_\ell-1} \left((s_\ell - \tilde{a}_{\ell,j}(t;\sigma'))(\phi_{m_\ell,j})_* h'_\ell(x) + (-s_\ell - \tilde{a}_{\ell,j}(t;\sigma'))(\phi_{m_\ell,j})_* h''_\ell(x) \right)$$

as before. By the construction we always have

$$\sum_{j=0}^{m_\ell-1} \tilde{a}_{\ell,j}(t;\sigma') = 0,$$

hence a direct application of Lemma 2.1 yields

$$d_{\mathrm{H}}(\phi^1_{G_{\ell,\sigma}}, \phi^1_{m_\ell s_\ell h_\ell}) < \frac{3}{m_\ell}.$$

Hence we conclude (2.25) by the triangle inequality.

We have proved (2.23). Now as a corollary we get that for every $s \in l^\infty$ we have $\Phi(s) \in \mathrm{FHomeo}(S^2)$ and

$$\|\Phi(s)\|_{\mathcal{H}} \leqslant \sup_k (1 + 2^{-k})|s_k| + 3(\|s\|_\infty + 1) \sum_{s_k \neq 0} \frac{1}{m_k}. \tag{2.26}$$

Indeed, if $s \in l^\infty$ then let N to be equal to the integer part of $\|s\|_\infty$, and denote $\tilde{s} := s/(N+1)$. Then clearly $\|\tilde{s}\|_\infty \leqslant 1$, hence $\Phi(\tilde{s}) \in \mathrm{FHomeo}(S^2)$ and so $\Phi(s) = (\Phi(\tilde{s}))^{N+1} \in \mathrm{FHomeo}(S^2)$, and moreover by (2.23) we conclude

$$\|\Phi(s)\|_{\mathcal{H}} \leqslant (N+1)\|\Phi(\tilde{s})\|_{\mathcal{H}} \leqslant (N+1)\sup_k (1+2^{-k})|\tilde{s}_k| + 3(N+1) \sum_{\tilde{s}_k \neq 0} \frac{1}{m_k}$$
$$\leqslant \sup_k (1+2^{-k})|s_k| + 3(\|s\|_\infty + 1) \sum_{s_k \neq 0} \frac{1}{m_k}.$$

Recall that we denoted by $\pi : \mathrm{FHomeo}(S^2) \to \mathcal{G} = \mathrm{FHomeo}(S^2)/\mathrm{Hameo}(S^2)$ the natural projection homomorphism. For every $s \in l^\infty$ and every $\ell \geqslant 1$ we have

$$\|\pi \circ \Phi(s)\|_{\mathcal{H}} = \|\pi \circ \Phi(s^\ell)\|_{\mathcal{H}},$$

and by (2.26) we have

$$\limsup_{\ell \to \infty} \|\pi \circ \Phi(s^\ell)\|_{\mathcal{H}} \leqslant \limsup_{\ell \to \infty} \|\Phi(s^\ell)\|_{\mathcal{H}} \leqslant \limsup_{\ell \to \infty} |s_\ell|.$$

Thus we get the estimate

$$\|\pi \circ \Phi(s)\|_{\mathcal{H}} \leqslant \limsup_{\ell \to \infty} |s_\ell| \tag{2.27}$$

for every $s \in l^\infty$. In order to conclude (2.13) it remains to show the opposite inequality, and for this we will apply properties of the functionals $c^0_{k,B}$.

One important property of the functionals $\tau_m = \sigma_m - \sigma'_m = c^0_{m,B_m} - c^0_{m,B'_m}$ is that they are C^0-continuous on $\mathrm{Ham}(S^2)$ and that they extend by continuity to $\overline{\mathrm{Ham}}(S^2)$. Consider any $s \in l^\infty$ having a finite number of non-zero coordinates. Then by the Lagrangian control property for c^0_{m,B_m} and c^0_{m,B'_m} we get

$$\tau_{m_k}(\Phi(s)) = \tau_{m_k}\left(\sum_i m_i s_i h_i\right) = \sum_{i=1}^\infty m_i s_i \tau_{m_k}(h_i)$$

$$= m_k s_k \tau_{m_k}(h_k) + \sum_{i=1}^{k-1} m_i s_i \tau_{m_k}(h_i).$$

Now, we have $m_k \tau_{m_k}(h_k) = 2$ and $|m_i \tau_{m_k}(h_i)| \leqslant 2^{-k}$ for $1 \leqslant i \leqslant k-1$, hence we get

$$\tau_{m_k}(\Phi(s)) \geqslant 2s_k - 2^{-k} k \|s\|_\infty.$$

Now, if $\psi \in \mathrm{Ham}(M, \omega)$ is arbitrary, then

$$\tau_{m_k}(\Phi(s)\psi^{-1}) = \sigma_{m_k}(\Phi(s)\psi^{-1}) - \sigma'_{m_k}(\Phi(s)\psi^{-1})$$

$$\geqslant (\sigma_{m_k}(\Phi(s)) - \sigma_{m_k}(\psi)) - (\sigma'_{m_k}(\Phi(s)) + \sigma'_{m_k}(\psi^{-1}))$$

$$= \tau_{m_k}(\Phi(s)) - (\sigma_{m_k}(\psi) + \sigma'_{m_k}(\psi^{-1}))$$

$$\geqslant 2s_k - 2^{-k} k \|s\|_\infty - (\sigma_{m_k}(\psi) + \sigma'_{m_k}(\psi^{-1})).$$

As a corollary, for any $s \in l^\infty$, any $\ell \geqslant k \geqslant 1$, and every $\psi \in \mathrm{Ham}(S^2)$ we get

$$\tau_{m_k}(\Phi(^\ell s)\psi^{-1}) \geqslant 2s_k - 2^{-k} k \|s\|_\infty - (\sigma_{m_k}(\psi) + \sigma'_{m_k}(\psi^{-1})).$$

But then the C^0 continuity of τ_{m_k} implies

$$\tau_{m_k}(\Phi(s)\psi^{-1}) \geqslant 2s_k - 2^{-k} k \|s\|_\infty - (\sigma_{m_k}(\psi) + \sigma'_{m_k}(\psi^{-1})). \tag{2.28}$$

Now let $\varphi \in \mathrm{Hameo}(S^2)$, and choose a sequence of $\psi_j \in \mathrm{Ham}(S^2)$ such that $\varphi = (C^0)\lim_{j\to\infty} \psi_j$ and $d_{\mathrm{H}}(\psi_i, \psi_j) \leqslant 1/i$ for $j \geqslant i \geqslant 1$. Then by (2.28) and by the Hofer Lipschitz and Independence of Hamiltonian properties of σ_m, σ'_m, we get

$$\tau_{m_k}(\Phi(s)\psi_j^{-1}) \geqslant 2s_k - 2^{-k} k \|s\|_\infty - (\sigma_{m_k}(\psi_i) + \sigma'_{m_k}(\psi_i^{-1})) - 2/i.$$

for $j \geqslant i \geqslant 1$. Taking $j \to \infty$, by the C^0 continuity of τ_{m_k} we get

$$\tau_{m_k}(\Phi(s)\varphi^{-1}) \geqslant 2s_k - 2^{-k} k \|s\|_\infty - (\sigma_{m_k}(\psi_i) + \sigma'_{m_k}(\psi_i^{-1})) - 2/i. \tag{2.29}$$

But by the Calabi property stated in Theorem 1.7, for each $i \geqslant 1$ we have

$$\lim_{m\to\infty} \sigma_m(\psi_i) + \sigma'_m(\psi_i^{-1}) = \lim_{m\to\infty} c^0_{m,B_m}(\psi_i) + c^0_{m,B'_m}(\psi_i^{-1}) = 0.$$

Hence from (2.29), by taking $k \to \infty$, we get

$$\limsup_{k\to\infty} \tau_{m_k}(\Phi(s)\varphi^{-1}) \geqslant 2\limsup_{k\to\infty} s_k - 2/i$$

for every $i \geqslant 1$, and therefore we finally conclude that the inequality

$$\limsup_{k \to \infty} \tau_{m_k}(\Phi(s)\varphi^{-1}) \geqslant 2 \limsup_{k \to \infty} s_k \tag{2.30}$$

holds for every $s \in l^\infty$ and $\varphi \in \mathrm{Hameo}(S^2)$. Now, we claim that given such s and φ, we have

$$\|\Phi(s)\varphi^{-1}\|_{\mathcal{H}} \geqslant \limsup_{k \to \infty} s_k. \tag{2.31}$$

Indeed, assume that we have a sequence $\phi_i \in \mathrm{Ham}(S^2)$ that C^0 converges to $\Phi(s)\varphi^{-1}$. Let $\delta > 0$ be arbitrary, and then by (2.30) we can find some ℓ such that

$$\tau_{m_\ell}(\Phi(s)\varphi^{-1}) \geqslant 2 \limsup_{k \to \infty} s_k - \delta.$$

But then by the C^0 continuity and by the Hofer Lipschitz property of τ_{m_ℓ} we get

$$2 \liminf_{i \to \infty} \|\phi_i\|_{\mathrm{H}} \geqslant \lim_{i \to \infty} \tau_{m_\ell}(\phi_i) = \tau_{m_\ell}(\Phi(s)\varphi^{-1}) \geqslant 2 \limsup_{k \to \infty} s_k - \delta,$$

and since $\delta > 0$ is arbitrary, we conclude (2.31). It remains to notice that for a given $s \in l^\infty$ and $\varphi \in \mathrm{Hameo}(S^2)$, substituting $-s$ and φ^{-1} into (2.31), by $(\Phi(-s)\varphi)^{-1} = \varphi^{-1}\Phi(s) = \varphi^{-1}(\Phi(s)\varphi^{-1})\varphi$ we get

$$\|\Phi(s)\varphi^{-1}\|_{\mathcal{H}} = \|\Phi(-s)\varphi\|_{\mathcal{H}} \geqslant \limsup_{k \to \infty}(-s_k),$$

and together with (2.31) this implies

$$\|\Phi(s)\varphi^{-1}\|_{\mathcal{H}} \geqslant \limsup_{k \to \infty} |s_k|$$

for every $s \in l^\infty$ and $\varphi \in \mathrm{Hameo}(S^2)$, which means that we have

$$\|\pi \circ \Phi(s)\|_{\mathcal{H}} \geqslant \limsup_{k \to \infty} |s_k|$$

for every $s \in l^\infty$, and this finishes the proof of (2.13).

It remains to show that the image $\Phi(c_0)$ is contained in $\mathrm{Hameo}(S^2)$. This will be done for a special choice of the sequence m_k and of functions h'_k and h''_k. On step k of the construction, we now additionaly choose a sufficiently thin neigbourhood W_k of $L_{m_k} \cup L'_{m_k}$ in $\mathcal{A}_{m_k}$. Then, for each k and each $\eta \in (0,1]$ denote by $\mathcal{J}(k,\eta)$ the collection consisting of all the indices j such that the disc $\mathcal{D}_{m_k,j}$ lies inside $S^2 \cap \{x_3 < -1/2 + \eta\}$ but does not intersect the union $\cup_{\ell=1}^{k-1} \cup_{j=0}^{m_\ell - 1} \phi_{m_\ell,j}(W_\ell)$. We require that our consequent choices of m_k and W_k are such that for every $\eta \in (0,1]$ we have

$$|\mathcal{J}(k,\eta)| > m_k \eta / 2 \tag{2.32}$$

when k is large enough. Recall that the area of the spherical cap $S^2 \cap \{x_3 < -1/2 + \eta\}$ equals η, and that the area of a disc $\mathcal{D}_m$ behaves asymptotically like $1/m$. Hence it is enough on each step k to choose the neighborhood W_k to be so thin that the area of the intersection of the union

$\cup_{\ell=1}^{k} \cup_{j=0}^{m_\ell - 1} \phi_{m_\ell, j}(W_\ell)$ with the spherical cap $S^2 \cap \{x_3 < -1/2 + \eta\}$ remains to be less than $\frac{1}{2}\omega(S^2 \cap \{x_3 < -1/2 + \eta\}) = \eta/2$ for each $\eta \in (0, 1]$. Indeed, this allows on further steps k' to choose $m_{k'}$ to be large enough so that the requirement (2.32) is eventually met. Once m_k and W_k are chosen on a step k, we choose the functions h'_k and h''_k as before but impose an additional requirement that their supports lie in W_k. As before, we put $h_k = h'_k - h''_k$.

Now assume that we have some $s \in c_0$, and let us show that $\Phi(s) \in \mathrm{Hameo}(S^2)$. We can find a sequence $m'_k \leqslant m_k$ of natural numbers such that $\lim_{k\to\infty} m'_k/m_k = 0$ and at the same time the sequence $s'_k := m_k s_k/m'_k$ converges to 0 when $k \to \infty$. By (2.32), it is possible to choose a sequence $\eta_k \in (0, 1]$ and find an index k_0 such that $\lim_{k\to\infty} \eta_k = 0$ and such that $m'_k \leqslant |\mathcal{J}(k, \eta_k)|$ for $k \geqslant k_0$. For every $k \geqslant k_0$ choose a subset $0 \in J_k \subset \mathcal{J}(k, \eta_k)$ that has exactly m'_k elements.

For each $k \geqslant k_0$, consider the autonomous Hamiltonian function

$$G_k := s'_k \sum_{j \in J_k} (\phi_{m_k, j})_* h_k.$$

By Lemma 2.1, and Remark 2.2 applied to the surface $S^2 \cap \{x_3 < -1/2 + \eta_k\}$, one can find a normalized Hamiltonian function $H_k \in C^\infty(S^2 \times [0, 1])$ compactly supported in $S^2 \cap \{x_3 < -1/2 + \eta_k\}$ such that $\phi^1_{m_k s_k h_k} = \phi^1_{m'_k s'_k h_k} = \phi^1_{G_k}(\phi^1_{H_k})^{-1}$ and such that

$$\|H_k\| < \frac{3}{m_k}. \tag{2.33}$$

Since the supports of the G_k's are mutually disjoint and are "converging" to the south pole of S^2, and since $\lim_{k\to\infty} \|G_k\|_\infty = 0$, it follows that

$$\Psi := (C^0) \lim_{k\to\infty} \phi^1_{G_k} \circ \phi^1_{G_{k-1}} \circ \cdots \circ \phi^1_{G_{k_0}}$$

is an element of $\mathrm{Hameo}(S^2)$ being the time-1 map of a continuous flow of homeomorphisms generated by the continuous (autonomous) Hamiltonian $\sum_{k=k_0}^{\infty} G_k$. Therefore it is enough to prove that $\Psi^{-1} \circ \Phi(s) \in \mathrm{Hameo}(S^2)$, which is equivalent to $(\Phi(s^{k_0}))^{-1} \circ \Psi \in \mathrm{Hameo}(S^2)$. We have

$$(\Phi(s^{k_0}))^{-1} \circ \Psi = (C^0) \lim_{k\to\infty} \Phi_k^{-1} \Psi_k,$$

where

$$\Psi_k = \phi^1_{G_k} \circ \phi^1_{G_{k-1}} \circ \cdots \circ \phi^1_{G_{k_0}}$$

and

$$\Phi_k = \phi^1_{m_k s_k h_k} \circ \cdots \circ \phi^1_{m_{k_0} s_{k_0} h_{k_0}}$$

for $k \geqslant k_0$. Denoting $\Phi_{k_0-1} := \mathbb{1}$, for each $k \geqslant k_0$ we have

$$\Phi_k^{-1} \Psi_k = (\Phi_{k-1}^{-1}(\phi^1_{m_k s_k h_k})^{-1}\phi^1_{G_k}\Phi_{k-1}) \circ \cdots \circ (\Phi_{k_0-1}^{-1}(\phi^1_{m_{k_0} s_{k_0} h_{k_0}})^{-1}\phi^1_{G_{k_0}}\Phi_{k_0-1})$$

$$= (\Phi_{k-1}^{-1}\phi^1_{H_k}\Phi_{k-1}) \circ \cdots \circ (\Phi_{k_0-1}^{-1}\phi^1_{H_{k_0}}\Phi_{k_0-1}). \tag{2.34}$$

Choose a smooth non-decreasing function $c : [0,1] \to [0,1]$ such that $c(t) = 0$ for t close to 0 and $c(t) = 1$ for t close to 1, and for each j define $c_j : [\frac{j-1}{j}, \frac{j}{j+1}] \to \mathbb{R}$ by $c_j(t) = c((j+1)(jt - j + 1))$. Now, for each $k \geqslant k_0$, define the Hamiltonian function $\widetilde{H}_k \in C^\infty(S^2 \times [0,1])$ by $\widetilde{H}_k(x,t) = 0$ when $t \in [0, \frac{k_0-1}{k_0}] \cup [\frac{k}{k+1}, 1]$, and

$$
\begin{aligned}
\widetilde{H}_k(x,t) &= c_j'(t) H_j(\Phi_{j-1}(x), c_j(t)) \\
&= j(j+1)c'((j+1)(jt - j + 1))H_j(\Phi_{j-1}(x), c((j+1)(jt - j + 1)))
\end{aligned}
$$

for $t \in [\frac{j-1}{j}, \frac{j}{j+1}]$ and $k_0 \leqslant j \leqslant k$. By (2.33) and by the rapid growth of the sequence (m_k) (we have chosen the sequence so that in particular $m_k > 2m_{k-1}$ for each $k > 1$), the sequence $(\widetilde{H}_k)$ converges in $\|\cdot\|$ to a continuous function $\widetilde{H} : S^2 \times [0,1] \to \mathbb{R}$. Since the sequence $(\widetilde{H}_k)$ of Hamiltonian functions stabilizes on $S^2 \times [0,t]$, for every $t \in [0,1)$, the function $\widetilde{H}$ is smooth on $S^2 \times [0,1)$, hence the time-t map $\phi_{\widetilde{H}}^t$ is well defined for $t \in [0,1)$. We claim the convergence

$$(C^0) \lim_{t \to 1} \phi_{\widetilde{H}}^t = (\Phi(s^{k_0}))^{-1} \circ \Psi. \tag{2.35}$$

To see this, first note that we have that convergence along the sequence $t_k = \frac{k}{k+1}$, since by (2.34) and by the definition of $\widetilde{H}$ we have $\phi_{\widetilde{H}}^{t_k} = \Phi_k^{-1} \Psi_k$ for $k \geqslant k_0$. Therefore (2.35) follows from the fact that for each $k \geqslant k_0$, the flow

$$\phi_{\widetilde{H}}^{t_{k-1}+\tau} \circ (\phi_{\widetilde{H}}^{t_{k-1}})^{-1} = \Phi_{k-1}^{-1} \phi_{H_k}^{c_k(t_{k-1}+\tau)} \Phi_{k-1},$$

($\tau \in [0, \frac{1}{k(k+1)}]$) converges to the identity when $k \to \infty$, uniformly in τ. The latter holds since the Hamiltonian H_k is supported in the spherical cap $S^2 \cap \{x_3 < -1/2 + \eta_k\}$ which is invariant under Φ_{k-1}, and we have $\lim_{k \to \infty} \eta_k = 0$ (that is, that spherical cap "shrinks" into the south pole of S^2 when k converges to infinity).

Because of (2.35), if we define $\phi_{\widetilde{H}}^1 := (\Phi(s^{k_0}))^{-1} \circ \Psi$, then $(\phi_{\widetilde{H}}^t)_{t \in [0,1]}$ is a continuous path of homeomorphisms. But then (2.35) also implies that $(\phi_{\widetilde{H}}^t)_{t \in [0,1]}$ is the C^0 limit of the sequence $(\phi_{\widetilde{H}_k}^t)_{t \in [0,1]}$ of smooth Hamiltonian flows (this can be readily seen from the definition of $\widetilde{H}_k$). Recall that we have also verified that the sequence $(\widetilde{H}_k)$ of Hamiltonians converges in the L^∞ norm (in particular in the $L^{(1,\infty)}$ norm) to $\widetilde{H}$. This shows that $(\Phi(s^{k_0}))^{-1} \circ \Psi = \phi_{\widetilde{H}}^1 \in \mathrm{Hameo}(S^2)$, and hence finishes the proof.

2.5. Proof of Lemma 2.1. Our proof of the lemma is based on an idea of Sikorav [10, Section 8.4]. Before passing to the proof, let us show the following auxiliary statement:

Claim 1. *Let D be a closed symplectic disc of area less than ε, let W be an open symplectic 2-disc, a let $\phi, \psi : D \to W$ be a pair of symplectic embeddings whose images do not intersect. Then there exists a Hamiltonian diffeomorphism $\Psi \in \mathrm{Ham}(W)$ satisfying $\Psi \circ \phi = \psi$, whose generating Hamiltonian is normalized, compactly supported in W, and has $L^{(1,\infty)}$ norm less than $\varepsilon/2$.*

Proof. First, one can easily find an example of such discs (of the same areas respectively) and symplectic embeddings as in the claim, for which statement of the claim holds. Indeed, one can take W' to be an open topological disc of a rectangular shape on the standard symplectic 2-plane, sharing the same area with W, and take ϕ', ψ' to be symplectic embeddings of D into W' whose disjoint images have nearly rectangular shape such that ψ' is a composition of ϕ' with a horizontal translation. Then for a suitable such choice of W', ϕ' and ψ', a cutoff of an appropriate linear planar Hamiltonian function will have $L^{(1,\infty)}$ norm less than $\varepsilon/2$ and will generate a Hamiltonian diffeomorphism Ψ' compactly supported in W' such that $\psi' = \Psi' \circ \phi'$.

Now look at the symplectic embeddings $\phi' \circ \phi^{-1} : \phi(D) \to W$ and $\psi' \circ \psi^{-1} : \psi(D) \to W$, and find a symplectomorphism $\Theta : W \to W'$ which extends them, meaning that $\Theta = \phi' \circ \phi^{-1}$ on $\phi(D)$ and $\Theta = \psi' \circ \psi^{-1}$ on $\psi(D)$. It remains to define $\Psi \in \mathrm{Ham}(W)$ by $\Psi = \Theta^{-1} \circ \Psi' \circ \Theta$. $\square$
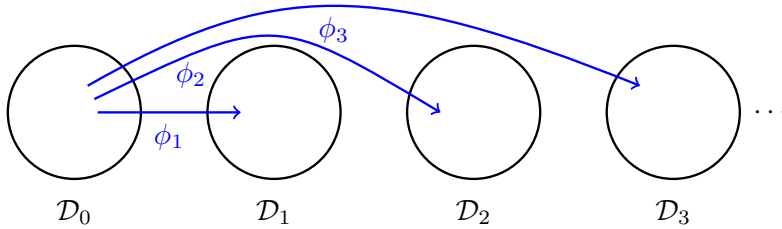

Fig. 2.2
The discs $\mathcal{D}_j$ and maps ϕ_j.

Let us now pass to the proof of the lemma. Denote $\ell = \lfloor \frac{m}{2} \rfloor$ and $\ell' = \lfloor \frac{m-1}{2} \rfloor$. By slightly decreasing the disc $\mathcal{D}_0$ and re-defining $\mathcal{D}_j := \phi_j(\mathcal{D}_0)$ for all j, we may assume that they all are topological discs with smooth boundaries and with mutually disjoint closures. We claim that there exists a Hamiltonian diffeomorphism $\Psi \in \mathrm{Ham}(M, \omega)$ such that:

- $\Psi \circ \phi_{2i} = \phi_{2i+1}$ for all $0 \leqslant i \leqslant \ell'$ (here $\phi_0 : \mathcal{D}_0 \to \mathcal{D}_0$ stands for the identity map).
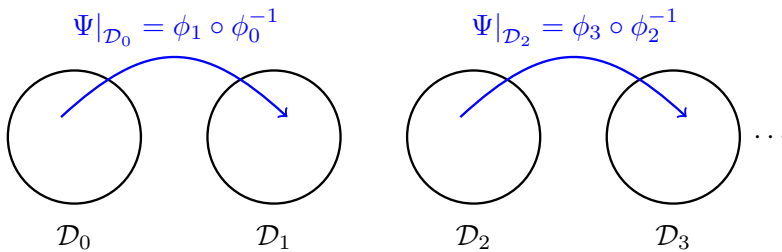
- $\|\Psi\|_{\mathrm{H}} < \varepsilon/2$.

Fig. 2.3
The map Ψ and the discs $\mathcal{D}_j$.

To see this, for each $0 \leqslant i \leqslant \ell'$, find a topological open disc $\widehat{\mathcal{D}}_i$ that compactly

contains both $\mathcal{D}_{2i}$ and $\mathcal{D}_{2i+1}$ and such that all the discs $\widehat{\mathcal{D}}_i$ are pairwise disjoint. By Claim 1 (stated at the beginning of the proof of the lemma), for each $\widehat{\mathcal{D}}_i$ we can find a Hamiltonian diffeomorphism $\Psi_i \in \mathrm{Ham}(M, \omega)$ satisfying $\Psi_i \circ \phi_{2i} = \phi_{2i+1}$, whose generating Hamiltonian is normalized, compactly supported in $\widehat{\mathcal{D}}_i$, and has $L^{(1,\infty)}$ norm less than $\varepsilon/2$. Now denote $\Psi = \Psi_0 \circ \cdots \circ \Psi_\ell$.
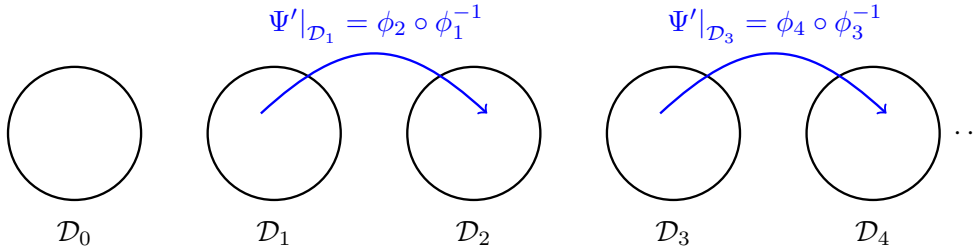
Fig. 2.4
The map Ψ' and the discs $\mathcal{D}_j$.

Similarly, we can find a Hamiltonian diffeomorphism $\Psi' \in \mathrm{Ham}(M, \omega)$ such that:

- $\Psi' \circ \phi_{2i-1} = \phi_{2i}$ for all $1 \leqslant i \leqslant \ell$.
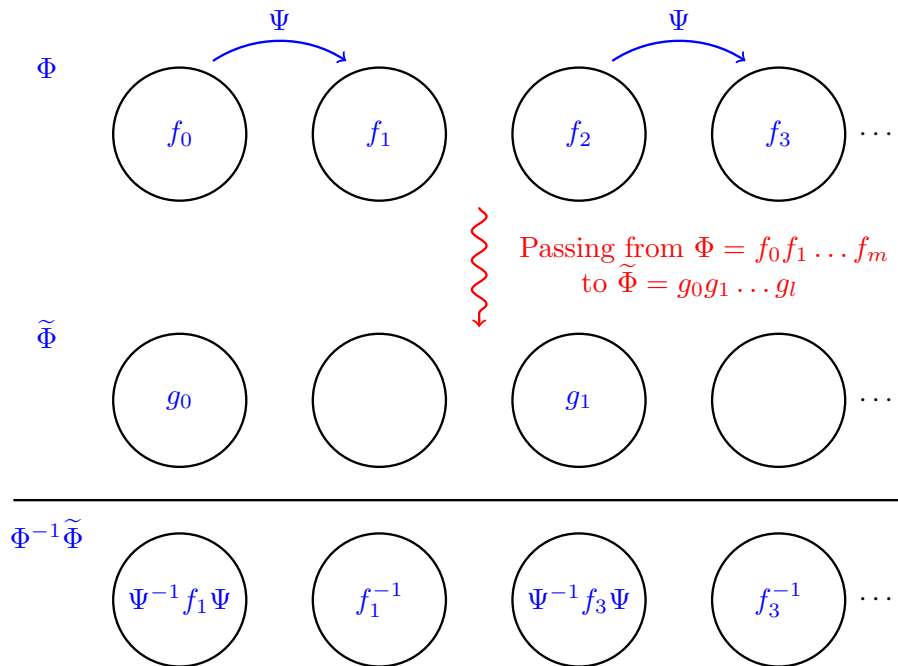- $\|\Psi'\|_{\mathrm{H}} < \varepsilon/2$.

Fig. 2.5
Passage from the map Φ to $\widetilde{\Phi}$.

For convenience we now introduce $g_0, g_1, \ldots, g_\ell \in \mathrm{Ham}(M, \omega)$ as follows. If $m = 2\ell + 1$ is odd, then define $g_i = f_{2i}\Psi^{-1}f_{2i+1}\Psi = f_{2i}(\phi_{2i+1}\phi_{2i}^{-1})^* f_{2i+1}$ for $0 \leqslant i \leqslant \ell$. If $m = 2\ell$ is even, then we put $g_i = f_{2i}\Psi^{-1}f_{2i+1}\Psi = f_{2i}(\phi_{2i+1}\phi_{2i}^{-1})^* f_{2i+1}$ for $0 \leqslant i \leqslant \ell - 1$, and moreover $g_\ell = f_{2\ell} = f_m$. Then

$\mathrm{supp}(g_i) \subset \mathcal{D}_{2i}$ for $0 \leqslant i \leqslant \ell$. Denote $\widetilde{\Phi} := g_0 g_1 \ldots g_\ell$. Then we have (here and below, the symbol $\prod$ stands for a composition of maps)

$$\Phi^{-1}\widetilde{\Phi} = \prod_{i=0}^{\ell'}(\Psi^{-1} f_{2i+1} \Psi f_{2i+1}^{-1}) = \left(\prod_{i=0}^{\ell'} f_{2i+1}\right)^{-1} \Psi^{-1} \left(\prod_{i=0}^{\ell'} f_{2i+1}\right) \Psi,$$

and hence

$$d_{\mathrm{H}}(\Phi, \widetilde{\Phi}) = \|\Phi^{-1}\widetilde{\Phi}\|_{\mathrm{H}} \leqslant 2\|\Psi\|_{\mathrm{H}} < \varepsilon. \tag{2.36}$$
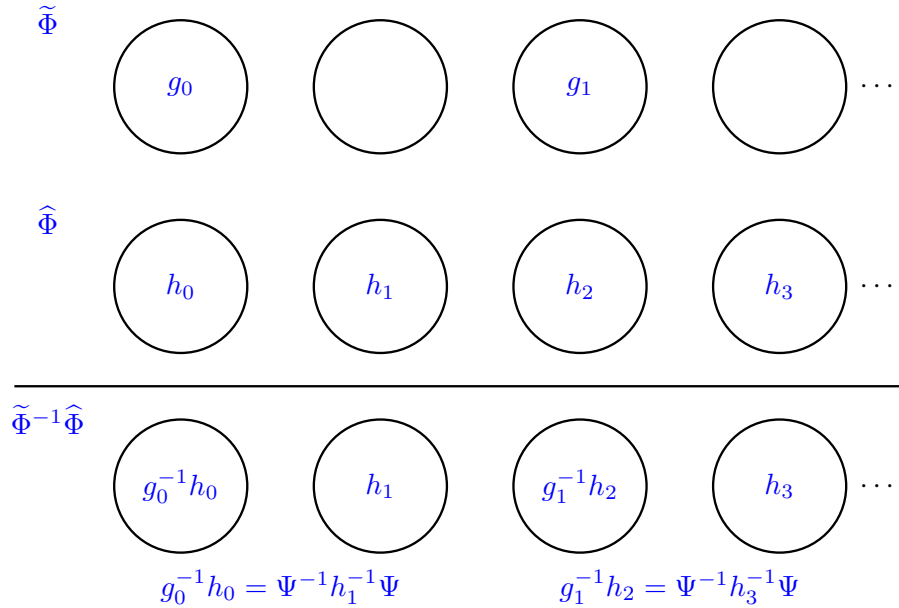

Fig. 2.6
The maps $\widetilde{\Phi}$, $\widehat{\Phi}$ and $\widetilde{\Phi}^{-1}\widehat{\Phi}$.

Recall that for each $0 \leqslant i \leqslant \ell$, the map g_i is supported in $\mathcal{D}_{2i}$, and is given by $g_i = f_{2i}(\phi_{2i+1}\phi_{2i}^{-1})^* f_{2i+1}$, unless $m = 2\ell$ is even and $i = \ell$ in which case we have $g_\ell = f_{2\ell} = f_m$. Define $\hat{g}_i := \phi_{2i}^* g_i$ for $0 \leqslant i \leqslant \ell$. Then put $\hat{h}_i := \prod_{j=i}^{\ell} \hat{g}_j$ for $0 \leqslant i \leqslant \ell$, and then define (see Section 1.3 for the definition of the pushforward operation that we use here) $h_{2i} = (\phi_{2i})_* \hat{h}_i$ for $0 \leqslant i \leqslant \ell$ and $h_{2i-1} = (\phi_{2i-1})_* \hat{h}_i^{-1}$ for $1 \leqslant i \leqslant \ell$. Put $\widehat{\Phi} := h_0 h_1 \cdots h_{2\ell}$. Then

$$\widetilde{\Phi}^{-1}\widehat{\Phi} = \left(\prod_{i=1}^{\ell} h_{2i-1}\right) \Psi^{-1} \left(\prod_{i=1}^{\ell} h_{2i-1}\right)^{-1} \Psi$$

and hence

$$d_{\mathrm{H}}(\widetilde{\Phi}, \widehat{\Phi}) = \|\widetilde{\Phi}^{-1}\widehat{\Phi}\|_{\mathrm{H}} \leqslant 2\|\Psi\|_{\mathrm{H}} < \varepsilon. \tag{2.37}$$

Finally, note that

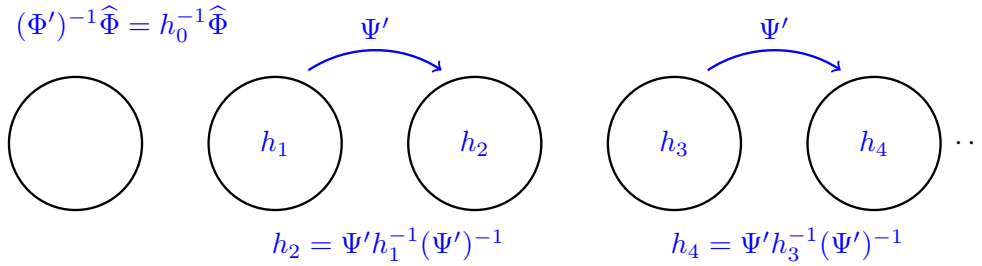$$h_0 = \hat{h}_0 = \prod_{j=0}^{\ell} \hat{g}_j = \prod_{i=0}^{m} \phi_i^* f_i = \Phi',$$

Fig. 2.7
The map $(\Phi')^{-1}\widehat{\Phi}$.

therefore (recall that Ψ' was chosen at the beginning of the proof)

$$(\Phi')^{-1}\widehat{\Phi} = h_0^{-1}\widehat{\Phi} = \prod_{i=1}^{2\ell} h_i = \left(\prod_{i=1}^{\ell} h_{2i-1}\right) \Psi' \left(\prod_{i=1}^{\ell} h_{2i-1}\right)^{-1} (\Psi')^{-1}$$

and so we get

$$d_{\mathrm{H}}(\widehat{\Phi}, \Phi') = \|(\Phi')^{-1}\widehat{\Phi}\|_{\mathrm{H}} \leqslant 2\|\Psi'\|_{\mathrm{H}} < \varepsilon. \tag{2.38}$$

Now, the inequalities (2.36), (2.37) and (2.38) imply (2.1), and this concludes the proof.

2.6. Additional remarks

2.6.1. An adaptation of Lemma 2.1 to $\widetilde{\mathrm{Ham}}(M, \omega)$. Lemma 2.1 holds also for the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$. More precisely, let (M, ω), $\varepsilon > 0$, discs $\mathcal{D}_0, \ldots, \mathcal{D}_m \subset M$ and symplectic diffeomorphisms $\phi_j : \mathcal{D}_0 \to \mathcal{D}_j$ be as in the statement of Lemma 2.1. Furthermore, let $(f_j^t)_{t \in [0,1]}$ be Hamiltonian flows on M, $(0 \leqslant j \leqslant m)$ such that the flow (f_j^t) is compactly supported in $\mathcal{D}_j$ for every j. Define the Hamiltonian flows (Φ_t), (Φ_t') by

$$\Phi_t = f_0^t f_1^t \cdots f_m^t, \quad \Phi_t' = f_0^t \Pi_{j=1}^m \phi_j^* f_j^t.$$

Then for the representatives $[(\Phi_t)], [(\Phi_t')] \in \widetilde{\mathrm{Ham}}(M, \omega)$ of the flows $(\Phi_t), (\Phi_t')$ we have

$$d_{\mathrm{H}}([(\Phi_t)], [(\Phi_t')]) < 3\varepsilon. \tag{2.39}$$

Our proof of Lemma 2.1 transfers almost verbatim to that case. As before, we pick a Hamiltonian diffeomorphisms Ψ and Ψ' with the same properties, and define flows (g_i^t) in an analogous way. Namely, if $m = 2\ell + 1$ is odd then $g_i^t = f_{2i}^t \Psi^{-1} f_{2i+1}^t \Psi$ for $0 \leqslant i \leqslant \ell$, and if $m = 2\ell$ is even then $g_i^t = f_{2i}^t \Psi^{-1} f_{2i+1}^t \Psi$ for $0 \leqslant i \leqslant \ell - 1$ and moreover $g_\ell^t = f_{2\ell}^t = f_m^t$. Then we define the flow $\widetilde{\Phi}_t := g_0^t g_1^t \cdots g_\ell^t$, and we get

$$(\Phi_t)^{-1}\widetilde{\Phi}_t = \left(\prod_{i=0}^{\ell'} f_{2i+1}^t\right)^{-1} \Psi^{-1} \left(\prod_{i=0}^{\ell'} f_{2i+1}^t\right) \Psi.$$

At this point we wish to conclude that we have

$$d_{\mathrm{H}}([(\Phi_t)], [(\widetilde{\Phi}_t)]) = \|[((\Phi_t)^{-1}\widetilde{\Phi}_t)]\|_{\mathrm{H}} < \varepsilon. \tag{2.40}$$

Let us explain why we are able to do that. Denote

$$\Theta_t = \prod_{i=0}^{\ell'} f_{2i+1}^t$$

and choose a Hamiltonian flow (Ψ_t) of Hofer length less than $\varepsilon/2$ such that $\Psi_1 = \Psi$. Then the Hofer length of the Hamiltonian flow $(\Theta_1^{-1}\Psi_t^{-1}\Theta_1\Psi_t)_{t\in[0,1]}$ is less than ε. Hence in order to show the inequality

$$d_{\mathrm{H}}([(\Phi_t)], [(\widetilde{\Phi}_t)]) = \|[((\Phi_t)^{-1}\widetilde{\Phi}_t)]\|_{\mathrm{H}} = \|[(\Theta_t^{-1}\Psi_1^{-1}\Theta_t\Psi_1)]\|_H < \varepsilon$$

it would be enough to verify that

$$[(\Theta_t^{-1}\Psi_1^{-1}\Theta_t\Psi_1)] = [(\Theta_1^{-1}\Psi_t^{-1}\Theta_1\Psi_1)] \in \widetilde{\mathrm{Ham}}(M,\omega).$$

But that is a general fact which holds for universal covers of Lie groups, and it follows from looking at the homotopy $F_{s,t} := \Theta_t^{-1}\Psi_s^{-1}\Theta_t\Psi_s$, and from observing that $F_{s,0} = F_{0,t} = \mathbb{1}$.

We have shown the inequality (2.40). Then, in a completely analogous way, we define the flows (h_i^t) and then define the flow $(\widehat{\Phi}_t)$, showing that $d_{\mathrm{H}}([(\widetilde{\Phi}_t)], ([(\widehat{\Phi}_t)]) < \varepsilon$ and $d_{\mathrm{H}}([(\widehat{\Phi}_t)], ([(\Phi'_t)]) < \varepsilon$, and this implies (2.39), finishing the proof.

2.6.2. Some properties of l^∞/c_0. Let us explain how one can isometrically embed the normed group $(l^\infty, \|\cdot\|_\infty)$ into the normed group $(l^\infty/c_0, \|\cdot\|_\infty)$, and moreover isometrically embed $(C(X), \|\cdot\|_\infty)$ into $(l^\infty, \|\cdot\|_\infty)$ (for a separable topological space X). Decompose the set of indices $k \in \mathbb{N}$ into a countable union of countable sets: $\mathbb{N} = I_1 \cup I_2 \cup \cdots$. Then define the map $\alpha : l^\infty \to l^\infty$ by $\alpha((s_k)) = (s'_i)$ where $s'_i = s_k$ when $i \in I_k$. The composition of α with the natural homomorphism $l^\infty \to l^\infty/c_0$ is an isometric embedding of l^∞ into l^∞/c_0. Now, if X is a separable topological space and $(x_i)_{i\in\mathbb{N}}$ is a dense sequence in X, then the map $C(X) \to l^\infty$ which sends $f \in C(X)$ to the sequence $(f(x_i))_{i\in\mathbb{N}}$, is an isometric embedding.

Now we explain why there is no isometric group embedding of $(l^\infty/c_0, \|\cdot\|_\infty)$ into $(l^\infty, \|\cdot\|_\infty)$. Assume on the contrary that such an embedding $\iota : l^\infty/c_0 \to l^\infty$ exists. Consider the set J consisting of all infinite sequences $\alpha = (\alpha_0, \alpha_1, \ldots)$ where $\alpha_i \in \{1, 2\}$ for each i. For every $\alpha \in J$ define a subset I_α consisting of all integers of the form $\sum_{i=0}^k \alpha_i 3^i$ for all $k \geqslant 0$, and then define $s^\alpha = [(s_j^\alpha)] \in l^\infty/c_0$ where $s_j^\alpha = 1$ for $j \in I_\alpha$ and $s_j^\alpha = 0$ otherwise. For every distinct $\alpha, \beta \in J$, the intersection $I_\alpha \cap I_\beta$ is finite. Therefore for every $\alpha_1, \ldots, \alpha_m \in J$ and $t_1, \ldots, t_m \in \mathbb{R}$ we have

$$\|t_1 s^{\alpha_1} + \ldots + t_m s^{\alpha_m}\|_\infty = \max_\ell |t_\ell|.$$

Now for each pair of integers $i, k \geqslant 1$ consider the subset $J_{i,k} \subset J$ that consists of all $\alpha \in J$ for which the absolute value of the i-th coordinate of $\iota(s^\alpha) \in l^\infty$ is greater than $1/k$. Then $J_{i,k}$ is a finite set of cardinality $|J_{i,k}| < k$. Indeed, otherwise taking some distinct $\alpha_1, \ldots, \alpha_k \in J_{i,k}$, for a suitable choice of $\epsilon_i \in \{-1, 1\}$, the i-th coordinate of $\sum_{j=1}^{k} \epsilon_j \iota(s^{\alpha_j})$ is greater than 1 and consequently

$$\|\iota(\sum_{j=1}^{k} \epsilon_j s^{\alpha_j})\|_\infty = \|\sum_{j=1}^{k} \epsilon_j \iota(s^{\alpha_j})\|_\infty > 1,$$

which contradicts $\|\sum_{j=1}^{k} \epsilon_j s^{\alpha_j}\|_\infty = 1$. This shows that $J_{i,k}$ is a finite set and hence the union $\cup_{i,k} J_{i,k}$ is countable. Since J is uncountable we therefore can find some $\alpha \in J$ that does not belong to any of $J_{i,k}$, which means that s^α belongs to the kernel of ι, and so the kernel is non-trivial as stated. A similar argument shows that every continuous group homomorphism between $(l^\infty/c_0, \|\cdot\|_\infty)$ and $(l^\infty, \|\cdot\|_\infty)$ has a non-trivial kernel (in fact, such a homomorphism is always linear over $\mathbb{R}$ and the kernel must be an infinite dimensional linear subspace of l^∞/c_0).

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Про дві визначні групи гомеоморфізмів, що зберігають площу

Lev Buhovsky

Ми доводимо, що на симплектичній сфері група гамільтонових гомеоморфізмів в сенсі Ога і Мюллера є власною нормальною підгрупою гамільтонових гомеоморфізмів зі скінченною енергією. Більш того, ми знаходимо скінченновимірні плоскі модулі, наділені природною псевдометрикою Гофера, у факторгрупі цих груп.

Ключові слова: гамільтонові гомеоморфізми, гіпотеза Фаті, метрика Гофера, спектральні інваріанти