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Fractal Transformation of Krein–Feller Operators

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We consider a fractal transformed doubly reflected Brownian motion with state space being a Cantor-like set. By applying the theory of fractal transformations as developped by Barnsley, et al., together with an application of a generalised Taylor expression we show that its infinitesimal generator is given in terms of a second order measure geometric derivative $\frac{d}{d\mu}\frac{d}{d\mu}$ as introduced by Freiberg and Zähle. Furthermore we investigate its connection to the well known classical Krein–Feller operator $\frac{d}{d\mu}\frac{d}{dx}$ which is the generator of a so called "gap-diffusion".

Key words: measure geometric Krein–Feller-operator, Cantor-like sets, infinitesimal generator, gap-diffusion, fractal transformation

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1. Introduction

In [10] Freiberg defined the second order differential operator $\frac{d}{d\mu}\frac{d}{d\nu}$ with respect to finite atomless Borel measures μ and ν with compact supports and $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu) \subseteq \mathbb{R}$ as a generalisation of the well-known Krein–Feller operator of the form $\frac{d}{d\mu}\frac{d}{dx}$ which was previously studied in [9] and [17].

Thus, when choosing $\nu = \lambda$, where λ denotes the one-dimensional Lebesgue measure, the operator allows an interpretation as the infinitesimal generator of a so called quasi- (or gap-) diffusion (cf. [3, 7, 16]). Applying the more general framework of Dirichlet forms, it is shown in [12] that also $\frac{d}{d\mu}\frac{d}{d\nu}$ is an infinitesimal generator of a strong Markovian stochastic process with almost surely continuous paths on $\operatorname{supp}(\mu)$. In the case that μ equals a Cantor type measure the spectral asymptotics of $\frac{d}{d\mu}\frac{d}{dx}$ was obtained in [14] — and generalized later in [11] — where the square root of the eigenvalues of the operator imposed with Dirichlet boundary conditions can be regarded (up to a multiplicative constant) as the eigenfrequencies of a vibrating string with (singular) mass distribution according to μ (cf. [1]).

Instead, choosing $\nu = \mu$ the operator can be regarded as a Laplacian on certain compact (possibly fractal) subsets of the real line. Correspondingly, a harmonic calculus and spectral asymptotics of $\frac{d}{d\mu}\frac{d}{d\mu}$ were developed in [13]. Moreover, eigenvalues and eigenfunctions of Dirichlet respectively. Neumann boundary problems involving this operator were explicitly calculated in [19] and determined to be a composition of the appropriated classical trigonometric functions composed with a phase space transformation induced by the distribution function of μ. In the following elaboration we are concerned with a strong Markovian stochastic process possessing the operator $\frac{d}{d\mu}\frac{d}{d\mu}$ as its infinitesimal generator.

In Section 2 we briefly define $\frac{d}{d\mu}\frac{d}{d\nu}$ as a second order derivative with respect to the measures μ and ν and deduce a generalised Taylor expression. In Section 3 we illustrate how fractal transformations act on the class of functions defined on the attractors of two iterated function systems (IFS) with the same number of similitudes whereas in Section 4 we elaborate how these fractal transformations act on the class of derivatives with respect to the invariant measures with respect to the underlying IFSs. We then consider in Section 5 the connections to stochastic processes. In Subsection 5.1 we firstly recall the construction of the doubly reflected Brownian motion with state space being the unit interval [0,1]. In Theorem 5.9 its infinitesimal generator is given in terms of the second order differential operator $\frac{d}{dx}\frac{d}{dx}$ with Neumann boundary conditions. In Subsection 5.2 we then apply suitable fractal transformations on the doubly reflected Brownian motion such that the resulting process has state space being a Cantor-like set. We summarise its properties and define a semigroup of operators related to this process. The main result in Theorem 5.15 then claims that the infinitesimal generator of the associated semigroup is given in terms of $\frac{d}{d\mu}\frac{d}{d\mu}$ with generalised Neumann boundary conditions where μ is the invariant measure with respect to the IFS having the Cantor-like state space as its attractor. In order to prove the assertion we apply the generalised Taylor expression derived in Section 2. We finally conclude in Section 6 by sketching the construction of a stochastic process having infinitesimal generator of the form $\frac{d}{d\mu}\frac{d}{d\nu}$ and discuss how our approach is connected to already established results involving space and time change of a Brownian motion.

2. Measure geometric Krein–Feller operators

In the following section we define a derivative of a function with respect to a measure.

We follow the ideas of Freiberg [10], Arzt [1], Minorics [22] and Ehnes [6].

Definition 2.1. Let ν and μ be two atomless Borel probability measures on $[0,1]$ with $\mathrm{supp}(\mu) \subseteq \mathrm{supp}(\nu)$ and $0,1 \in \mathrm{supp}(\mu)$. Let $L^2(\nu) := L^2(\mathrm{supp}(\nu),\nu)$ and $L^2(\mu) := L^2(\mathrm{supp}(\mu),\mu)$. Define the space

$$\mathcal{D}_1^\nu := \left\{ f : [0,1] \to \mathbb{R} \,\middle|\, \exists f^\nu \in L^2(\nu) \, : \, f(x) = f(0) + \int_0^x f^\nu(y) d\nu(y) \, , \, x \in [0,1] \right\}.$$

Then the operator $\frac{d}{d\nu} : \mathcal{D}_1^\nu \to L^2(\nu), f \mapsto f^\nu$ will be referred to as the *ν-derivative*. We will write $\frac{d}{d\nu} f = \frac{df}{d\nu} := f^\nu$ for the ν-derivative of f. Furthermore we define the space

$$\mathcal{D}_2^{\mu,\nu} := \left\{ f \in \mathcal{D}_1^\nu \,\middle|\, \exists f^\mu \in L^2(\mu) : \right.$$

$$\left.\frac{d}{d\nu}f(x) = \frac{d}{d\nu}f(0) + \int_0^x f^\mu(y)\,d\mu(y),\ x \in [0,1]\right\}.$$

The operator $\frac{d}{d\mu}\frac{d}{d\nu} \colon \mathcal{D}_2^{\mu,\nu} \to L^2(\mu), f \mapsto f^\mu$ will then be called *μ-ν-derivative* (or *generalised measure geometric Krein-Feller operator*). We will write $\frac{d}{d\mu}\frac{d}{d\nu}f = \frac{d}{d\mu}\left(\frac{d}{d\nu}f\right) =: f^\mu$ for the μ-ν-derivative of f.

Remark 2.2.

(i) For any $f \in \mathcal{D}_2^{\mu,\nu}$ we obtain by Fubini's theorem the following representation (cf. [10, Remark 2.5])

$$\begin{aligned} f(x) = f(0) &+ \frac{d}{d\nu}f(0)F_\nu(x) \\ &+ \int_0^x (F_\mu(y) - F_\mu(x))\frac{d}{d\mu}\frac{d}{d\nu}f(y)d\mu(y) \quad (x \in [0,1]), \end{aligned} \tag{2.1}$$

where F_ν and F_μ denote the cumulative distribution functions of ν and μ.

(ii) In the case $\nu = \mu$ in definition 2.1 we write $\mathcal{D}_2^{\mu,\mu} =: \mathcal{D}_2^\mu$ and $\frac{d}{d\mu}\frac{d}{d\mu} =: \frac{d^2}{d\mu^2}$.

A detailed survey of analytical properties of derivatives with respect to a measure can be found in [10] and [22].

In the rest of this chapter we assume $\mu = \nu$.

Analogously to the classical case we derived the following mid-value theorem as an auxiliary result.

Lemma 2.3. *Let μ be an atomless Borel probability measure on $[0,1]$. Let $f, g \colon [0,1] \to \mathbb{R}$ be continuous and $[c,d] \subseteq [0,1]$. Then there exists $\tau \in [c,d]$ such that*

$$\int_c^d f(x)g(x)d\mu(x) = f(\tau)\int_c^d g(x)d\mu(x).$$

By an application of Cauchy–Schwarz inequality (cf. [1, Proposition 2.1.6]) we know that $\mathcal{D}_1^\mu \subseteq C([0,1])$ and one can verify easily by definition that $f \in \mathcal{D}_1^\mu$ is constant on $[0,1] \setminus \operatorname{supp}(\mu)$ and so f is defined uniquely by its values on $\operatorname{supp}(\mu)$.

We define for $k \in \{1,2\}$ the space C_μ^k to consist of all functions $f \in \mathcal{D}_k^\mu$ such that $\frac{d^m}{d\mu^m}f \in L^2(\mu)$ $(1 \le m \le k)$ is represented by a continuous function that is linear on $[0,1] \setminus \operatorname{supp}(\mu)$.

From lemma 2.3 we derive the next auxiliary result.

Lemma 2.4. *Assume that $f \in C_\mu^2$ and $[c,x] \subseteq [0,1]$. Then there exists $\xi \in [c,x]$ such that*

$$\int_c^x (F_\mu(x) - F_\mu(y))\frac{d^2}{d\mu^2}f(y)d\mu(y) = \frac{d^2}{d\mu^2}f(\xi)\frac{(F_\mu(x) - F_\mu(c))^2}{2}.$$

Together with equation (2.1) the previous lemma immediately gives us a generalised second-order Taylor formula.

Corollary 2.5. *Assume that $f \in C^2_\mu$ and $[c, x] \subseteq [0, 1]$. Then there exists $\xi \in [c, x]$ such that*

$$f(x) = f(c) + \frac{d}{d\mu} f(c)(F_\mu(x) - F_\mu(c)) + \frac{d^2}{d\mu^2} f(\xi) \frac{(F_\mu(x) - F_\mu(c))^2}{2}.$$

3. Fractal transformations

In this section we are going to present the notion of fractal transformations as in [2]. Further we give assumptions on the IFSs being used in all the following sections.

In the following we are interested in iterated function systems (IFS) of type

$$S := \{[0, 1] \mid s_1, \ldots, s_N\},$$

where $N \in \mathbb{N}$, $N \geq 2$, and $s_i \colon [0, 1] \to [0, 1]$ $(i = 1, \ldots, N)$ are contractions, i.e. $|s_i(x) - s_i(y)| \leq \lambda|x - y|$ for all $x, y \in [0, 1]$ and for some $\lambda \in [0, 1)$. Further we impose the following assumptions

(A.1) the s_i are increasing functions;

(A.2) the contractions satisfy an ascending order, i.e.

$$0 = s_1(0) \leq s_1(1) \leq s_2(0) \leq s_2(1) \leq \cdots \leq s_N(0) \leq s_N(1) = 1.$$

From [15] we know that for any such an IFS there exists a unique non-empty compact set A_S satisfying $A_S = \bigcup_{i=1}^N s_i(A_S)$. The set A_S will be called *attractor* of the IFS S. If the ascending order in (A.2) is strictly less then the emerging attractor will be a *Cantor-like* set.

Now let $F := \{[0, 1] \mid f_1, \ldots, f_N\}$ and $G := \{[0, 1] \mid g_1, \ldots, g_N\}$ be two IFSs with the same number of contractions satisfying the above assumptions (A.1) and (A.2). Let A_F and A_G be their attractors. We are now going to introduce the notion of fractal transformations as in [2].

Definition 3.1. Let $\{1, \ldots, N\}^{\mathbb{N}}$ denote the *code-space*.

We define the *coding maps* $\pi_F \colon \{1, \ldots, N\}^{\mathbb{N}} \to A_F$ and $\pi_G \colon \{1, \ldots, N\}^{\mathbb{N}} \to A_G$ respectively as

$$\pi_F(\sigma) := \lim_{k\to\infty} f_{\sigma_1} \circ \cdots \circ f_{\sigma_k}(x) \qquad (\sigma \in \{1, \ldots, N\}^{\mathbb{N}},\ x \in [0, 1]).$$
$$\pi_G(\rho) := \lim_{k\to\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_k}(y) \qquad (\rho \in \{1, \ldots, N\}^{\mathbb{N}},\ y \in [0, 1]).$$

Further we define the *section* of π_F to be the map $\tau_F \colon A_F \to \{1, \ldots, N\}^{\mathbb{N}}$ that satisfies $\pi_F \circ \tau_F = \mathrm{id}_{A_F}$. Analogously we define the *section* of π_G to be the map $\tau_G \colon A_G \to \{1, \ldots, N\}^{\mathbb{N}}$ that satisfies $\pi_G \circ \tau_G = \mathrm{id}_{A_G}$. We then define the fractal transformations

$$T_{FG} \colon A_F \to A_G, \qquad T_{FG}(x) := \pi_G \circ \tau_F(x) \quad (x \in A_F),$$
$$T_{GF} \colon A_G \to A_F, \qquad T_{GF}(y) := \pi_F \circ \tau_G(y) \quad (y \in A_G).$$

Remark 3.2.

(i) The section in above definition is not necessarily defined uniquely. Therefore we will always use $\tau_F(x) := \min \pi_F^{-1}(x)$ $(x \in A_F)$ and $\tau_G(y) := \min \pi_G^{-1}(y)$ $(y \in A_G)$ (where the minimum is with respect to the lexicographic order, i.e. we have $\rho > \sigma$ if $\rho \neq \sigma$ and $\rho_k > \sigma_k$ where k is the least integer satisfying $\rho_k \neq \sigma_k$).

(ii) If T_{FG} is a homeomorphism, then we will call it a *fractal homeomorphism* and in particular it then holds $(T_{FG})^{-1} = T_{GF}$.

For a given IFS S with contractions $s_1, \dots, s_N$ and a given probability vector $p = (p_1, p_2, \dots, p_N)$ there exists a unique Borel probability measure μ_S supported on the attractor A_S that is *invariant* under the IFS S in the sense that

$$\mu_S(B) = \sum_{i=1}^{N} p_i \mu_S(s_i^{-1}(B)) \quad (B \in \mathcal{B}([0,1])),$$

where $\mathcal{B}([0,1])$ denotes the Borel measurable subsets of $[0,1]$.

If the IFS S consists of similitudes and satisfies the *open set condition* and if we choose $p_i = c_i^D$ $(i = 1, \dots, N)$ where c_i denotes the scaling ratio of the i-th similitude s_i and where D denotes the Hausdorff dimension of the invariant set, then the unique invariant Borel probability measure is given by the normalised D-dimensional Hausdorff measure supported on A_S. For the theory of invariant measures we refer to [15].

Example 3.3. Consider the IFSs

$$F := \left\{ [0,1] \,\middle|\, f_1(x) = \frac{1}{2}x, f_2(x) = \frac{1}{2}x + \frac{1}{2} \right\} \text{ and}$$
$$G := \left\{ [0,1] \,\middle|\, g_1(x) = \frac{1}{3}x, g_2(x) = \frac{1}{3}x + \frac{2}{3} \right\}.$$

The contraction maps of these IFSs are increasing and satisfy the ascending order and so the assumptions (A.1) and (A.2) are fulfilled.

For the IFS F the attractor A_F is given by the unit interval $[0,1]$. For the IFS G the unique non-empty compact set $\mathcal{C}$ satisfying $\mathcal{C} = g_1(\mathcal{C}) \cup g_2(\mathcal{C})$ is called *Cantor set*.

For the unit interval the Hausdorff dimension equals 1 and for the Cantor set the Hausdorff dimension equals $\frac{\ln(2)}{\ln(3)}$. The corresponding invariant measures with respect to the same probability vector $p = (1/2, 1/2)$ are the one-dimensional Lebesgue measure $\lambda^1|_{[0,1]}$ (denoted by λ for short) supported on $[0,1]$ and the invariant measure supported on the Cantor set will be called *Cantor measure* and denoted by μ.

The corresponding fractal transformation $T_{FG} \colon [0,1] \to \mathcal{C}$ is a fractal homeomorphism.

We again consider IFSs F and G with the properties (A.1) and (A.2) stated at the beginning of the section. Then the corresponding attractors A_F and A_G

are non-overlapping with respect to its IFSs. (For the notion of *non-overlapping* sets see [2, Definition 2.5]).

Therefor we have the following transformation of invariant measures under fractal transformations (cf. [2, Theorem 2.4]).

Proposition 3.4. *Let F and G be two IFSs with the same number of similitudes. Suppose the attractors A_F and A_G to be non-overlapping with respect to the given IFSs, and let μ_F and μ_G be the invariant measures with respect to the same probability vector. Then we have with the fractal transformations T_{FG} and T_{GF}*

$$\mu_F \circ T_{GF} = \mu_G \quad \textit{and} \quad \mu_G \circ T_{FG} = \mu_F.$$

We now want to transform functions defined on A_F to functions defined on A_G and vice versa.

Let $L^2(\mu_F) := L^2(A_F, \mu_F)$ and $L^2(\mu_G) := L^2(A_G, \mu_G)$ denote the space of equivalence classes of square-integrable functions on A_F and A_G with respect to the invariant measures μ_F and μ_G respectively. Define the scalar products

$$\langle \Psi_F, \Phi_F \rangle_F := \int_{A_F} \Psi_F(x)\Phi_F(x) d\mu_F(x),$$
$$\langle \Psi_G, \Phi_G \rangle_G := \int_{A_G} \Psi_G(y)\Phi_G(y) d\mu_G(y)$$

for $\Psi_F, \Phi_F \in L^2(\mu_F)$ and $\Psi_G, \Phi_G \in L^2(\mu_G)$.
Then $(L^2(\mu_F), \langle\cdot,\cdot\rangle_F)$ and $(L^2(\mu_G), \langle\cdot,\cdot\rangle_G)$ are Hilbert spaces.

Definition 3.5. Define the linear operators $U_{FG} : L^2(\mu_F) \to L^2(\mu_G)$ and $U_{GF} : L^2(\mu_G) \to L^2(\mu_F)$ to be

$$(U_{FG}\phi_F)(x) := \phi_F(T_{GF}(x)) \qquad (\phi_F \in L^2(\mu_F),\, x \in A_G),$$
$$(U_{GF}\phi_G)(y) := \phi_G(T_{FG}(y)) \qquad (\phi_G \in L^2(\mu_G),\, y \in A_F).$$

With notations and conditions as in previous definition it is known the following (cf. [2, Theorem 4.1]).

Proposition 3.6. (i) *$U_{FG} : L^2(\mu_F) \to L^2(\mu_G)$ and $U_{GF} : L^2(\mu_G) \to L^2(\mu_F)$ are isometries;*

(ii) *$U_{FG} \circ U_{GF} = id_{L^2(\mu_F)}$ and $U_{GF} \circ U_{FG} = id_{L^2(\mu_G)}$;*

(iii) *$\langle \psi_G, U_{FG}\phi_F \rangle_G = \langle U_{GF}\psi_G, \phi_F \rangle_F \quad (\psi_G \in L^2(\mu_G), \phi_F \in L^2(\mu_F))$.*

4. Fractal transformation of derivatives

We now can formulate how the derivative with respect to an invariant measure transforms under fractal transformations.

Observe that $f \in U_{FG}(\mathcal{D}_1^{\mu_F})$ is only defined on A_G. Therefore let $\overline{U_{FG}(\mathcal{D}_1^{\mu_F})}^{\text{lin}}$ denote the set of all functions from $U_{FG}(\mathcal{D}_1^{\mu_F})$ that are extended linearly on $[0,1] \setminus A_G$.

Theorem 4.1. *Let F and G be two IFSs with non-overlapping attractors $A_F, A_G \subseteq [0,1]$ with $0,1 \in A_F \cap A_G$ and invariant measures μ_F and μ_G with respect to the same probability vector. Let the fractal transformation $T_{FG}\colon A_F \to A_G$ be a bijection. Then we have $\overline{U_{FG}(\mathcal{D}_1^{\mu_F})}^{lin} = \mathcal{D}_1^{\mu_G}$ with*

$$\frac{d}{d\mu_G}(U_{FG}f) = \left(U_{FG} \circ \frac{d}{d\mu_F} \circ U_{GF}\right)(U_{FG}f) \quad (f \in \mathcal{D}_1^{\mu_F})$$

in the weak sense of the definition of a derivative with respect to a measure (definition 2.1).

Proof. We know that $\mathcal{D}_1^{\mu_F} \subset C([0,1],\mathbb{R}) \subset L^2(\mu_F)$, therefore we can apply the operator U_{FG}. Let $f \in \mathcal{D}_1^{\mu_F}$ and $x \in [0,1]$. As $U_{FG}f$ is determined by its values on A_G it is enough to consider $x \in A_G$. By virtue of Proposition 3.4 and the statement (ii) of Proposition 3.6 we deduce

$$\begin{aligned} U_{FG}f(x) = f(T_{GF}x) &= f(0) + \int_0^{T_{GF}x} \frac{d}{d\mu_F} f(y)\, d\mu_F(y) \\ &= f(0) + \int_0^{T_{GF}x} \frac{d}{d\mu_F} f(y)\, d\mu_F(y) \\ &\qquad - f(0) + f(T_{GF}0) - \int_0^{T_{GF}0} \frac{d}{d\mu_F} f(y)\, d\mu_F(y). \end{aligned}$$

Since

$$f(0) + f(T_{GF}0) - \int_0^{T_{GF}0} \frac{d}{d\mu_F} f(y) d\mu_F(y) = 0,$$

we get

$$\begin{aligned} U_{FG}f(x) &= f(T_{GF}0) + \int_{T_{GF}0}^{T_{GF}x} \frac{d}{d\mu_F} f(y) d\mu_F(y) \\ &= (U_{FG}f)(0) + \int_0^x \frac{d}{d\mu_F} f(T_{GF}y) d\mu_F \circ (T_{GF})^{-1}(y) \\ &= (U_{FG}f)(0) + \int_0^x (U_{FG} \circ \frac{d}{d\mu_F} f)(y) d\mu_F \circ (T_{FG})(y) \\ &= (U_{FG}f)(0) + \int_0^x (U_{FG} \circ \frac{d}{d\mu_F} \circ U_{GF})(U_{FG}f)(y) d\mu_G(y) \end{aligned}$$

so the linear extension of $U_{FG}f$ is in $\mathcal{D}_1^{\mu_G}$ and $\frac{d}{d\mu_G}(U_{FG}f) = (U_{FG} \circ \frac{d}{d\mu_F} \circ U_{GF})(U_{FG}f)$. If T_{FG} is bijective it remains to show that for any $g \in \mathcal{D}_1^{\mu_G}$ there exists $f \in \mathcal{D}_1^{\mu_F}$ such that $U_{FG}f = g$. As T_{FG} is bijective we have $T_{GF}^{-1} = T_{FG}$. Setting $f := U_{GF}g \in \mathcal{D}_1^{\mu_F}$ we obtain $U_{FG}f = f \circ T_{GF} = (g \circ T_{FG}) \circ T_{GF} = g$ as desired. The statement about the derivative with respect to μ_F follows similarly as in the calculation before. □

Remark 4.2. The same way one can show that $\overline{U_{FG}(\mathcal{D}_2^{\mu_F})}^{\text{lin}} = \mathcal{D}_2^{\mu_G}$, $\overline{U_{FG}(C_{\mu_F}^k)}^{\text{lin}} = C_{\mu_G}^k$ $(k \in \{1,2\})$ and

$$\frac{d^2}{d\mu_G^2}(U_{FG}f) = \left(U_{FG} \circ \frac{d^2}{d\mu_F^2} \circ U_{GF}\right)(U_{FG}f) \quad (f \in \mathcal{D}_2^{\mu_F})$$

in the weak sense of Definition 2.1.

We now consider the special case that the attractor of the IFS F is given by the unit interval $[0,1]$ and that the invariant measure is given by $\mu_F = \lambda^1|_{[0,1]} = \lambda$. We again assume the contractions of the IFS to be increasing (A.1) and satisfying an ascending ordering (A.2). Further we assume that the fractal transformation T_{FG} is a fractal homeomorphism. Then we have the following local representation of the μ_G-derivative (cf. [2, Theorem 5.1])

Proposition 4.3. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuously differentiable function and define $g := U_{FG} f$. Then*

$$(U_{FG} \circ \frac{d}{dx} \circ U_{GF}) g(y_0) \frac{d}{d\mu_G} g(y_0) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{F_{\mu_G}(y) - F_{\mu_G}(y_0)} \quad (y_0 \in A_G).$$

The above Proposition 4.3 states that under the given assumptions the μ_G-derivative is given as conjugation of the classical derivative $\frac{d}{dx}$ via the fractal transformations.

5. Fractal transformed doubly reflected Brownian motion

The construction of a fractal transformed doubly reflected Brownian motion is due to Ehnes in [5].

5.1. Doubly reflected Brownian motion.

Let us recall the definition of a Brownian motion.

Definition 5.1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A stochastic process $B = (B_t)_{t \geq 0}$ with $B_t\colon (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ $(t \geq 0)$ is a *Brownian motion* if

(i) $\mathbb{P}(B_0 = 0) = 1$;

(ii) for $0 \leq s_0 < \cdots < s_n$ $(n \in \mathbb{N})$ the increments $B_{s_1} - B_{s_0}, \ldots, B_{s_n} - B_{s_{n-1}}$ are stochastically independent;

(iii) for $0 \leq s < t$ we have $B_t - B_s \sim \mathcal{N}(0, t-s)$;

(iv) the trajectories of B are continuous $\mathbb{P}$-almost surely.

Remark 5.2. With $\mathbb{P}^x$ $(x \in [0,1])$ we denote the probability measure such that $(B_t - x)_{t \geq 0}$ is a Brownian motion. Moreover we equip the probability space with the natural filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ induced by B, i.e. $\mathcal{F}_t = \sigma(B_s \mid s \leq t)$.

Definition 5.3. Let $h : \mathbb{R} \to [0,1]$ be defined by $h(x) := \mathbf{1}_{(-1,1)}(x)(1 - |x|)$. Then we define the *reflection map* $\phi : \mathbb{R} \to [0,1]$ by

$$\phi(x) := \sum_{n \in \mathbb{Z}} h(x + 2n - 1).$$

Definition 5.4. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let ϕ be the previous reflection map. Then the process $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ defined by

$$\tilde{B}_t := \phi(B_t) \quad (t \geq 0)$$

is called *doubly reflected Brownian motion.*

In [5, Proposition 4.7 and Proposition 4.33] is shown the following

Theorem 5.5. *The doubly reflected Brownian motion $\tilde{B}$ is a $[0,1]$-valued, $\mathcal{F}$-adapted, strong Markovian stochastic process.*

Definition 5.6. Define for $u \in C([0,1])$, $x \in [0,1]$ and $t \in [0,\infty)$

$$P_t u(x) := \mathbb{E}^x \left[u\left(\tilde{B}_t\right)\right].$$

With the notation from the previous definition we have the following.

Theorem 5.7.

(i) $(P_t)_{t\geq 0}$ *defines a semigroup of operators on* $C([0,1])$*;*

(ii) P_t *is strongly continuous on* $C([0,1])$*, i.e.* $\lim_{t\to 0}\|P_t u - u\|_\infty = 0$ *for* $u \in C([0,1])$*.*

Proof. The proof follows as in [24, Lemma 7.1. and Proposition 7.3. (f)].

(i) By the linearity of the expectation it follows that P_t is a linear operator. Now let $u \in C([0,1])$. Then $\tilde{u} := u \circ \phi \in C([0,1])$ and $\tilde{u}$ is bounded. We then infer by Lebesgue's dominated convergence theorem that for $y \in [0,1]$

$$\begin{aligned}\lim_{x\to y} P_t u(x) &= \lim_{x\to y} \mathbb{E}^x[u(\tilde{B}_t)] = \lim_{x\to y} \mathbb{E}^x[\tilde{u}(B_t)] \\ &= \lim_{x\to y} \mathbb{E}[\tilde{u}(B_t + x)] = \mathbb{E}[\tilde{u}(B_t + y)] = P_t u(y),\end{aligned}$$

and this shows that $P_t u(\cdot) \in C([0,1])$ for any $t \geq 0$. By the Markov-property we infer that for $s,t \geq 0$ and $u \in C([0,1])$

$$\begin{aligned}P_{t+s}u(x) &= \mathbb{E}^x\left[u(\tilde{B}_{t+s})\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[u(\tilde{B}_{t+s})|\mathcal{F}_s\right]\right] \\ &= \mathbb{E}^x\left[\mathbb{E}^{\tilde{B}_s}\left[u(\tilde{B}_t)\right]\right] = \mathbb{E}^x\left[P_t u(\tilde{B}_s)\right] = P_s P_t u(x)\end{aligned}$$

and so $(P_t)_{t\geq 0}$ has the semigroup property.

(ii) As the reflection map ϕ is uniformly continuous we have for any $u \in C([0,1])$ that $\tilde{u} := u \circ \phi$ is uniformly continuous on $[0,1]$. Then for given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have

$$|\tilde{u}(x) - \tilde{u}(y)| \leq \varepsilon.$$

Thus we have with $\tilde{u}|_{[0,1]} = u$ that

$$\begin{aligned}\|P_t u - u\|_\infty &= \sup_{x\in[0,1]} |\mathbb{E}^x[u(\tilde{B}_t)] - u(x)| = \sup_{x\in[0,1]} |\mathbb{E}^x[\tilde{u}(B_t)] - \tilde{u}(x)| \\ &\leq \sup_{x\in[0,1]} \mathbb{E}^x\left[|\tilde{u}(B_t) - \tilde{u}(x)|\right] \\ &= \sup_{x\in[0,1]} \left(\int_{\{|B_t - x|<\delta\}} |\tilde{u}(B_t) - \tilde{u}(x)| d\mathbb{P}^x \right.\end{aligned}$$

$$+\int_{\{|B_t-x|\geq\delta\}}|\tilde{u}(B_t)-\tilde{u}(x)|d\mathbb{P}^x\Bigg)$$
$$\leq\varepsilon\sup_{x\in[0,1]}\mathbb{P}^x\left(|B_t-x|<\delta\right)+2\|\tilde{u}\|_\infty\sup_{x\in[0,1]}\mathbb{P}^x\left(|B_t-x|\geq\delta\right)$$
$$\leq\varepsilon+2\|\tilde{u}\|_\infty\mathbb{P}\left(|B_t|\geq\delta\right).$$

Since $(B_t)_{t\geq 0}$ is uniformly stochastically continuous, i.e.

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}\mathbb{P}^x\left(|B_t-x|>\delta\right)=0,\quad \delta>0,$$

(cf. [24, Lemma 7.2]), we get $\limsup_{t\to 0}\|P_tu-u\|_\infty\leq\varepsilon$. Letting ε tend to zero then gives the assertion. □

Definition 5.8. Let $C(E)$ denote the continuous functions on a compact set E. Let $(P_t)_{t\geq 0}$ be a strongly continuous semigroup on $C(E)$. Define

$$Au:=\lim_{t\to 0}\frac{P_tu-u}{t}\quad\text{where the limit is taken with respect to }\|\cdot\|_\infty$$

and

$$D_A:=\left\{u\in C(E)\,\middle|\,\exists g\in C(E):\lim_{t\to 0}\left\|\frac{P_tu-u}{t}-g\right\|_\infty=0\right\}.$$

Then A is called the *infinitesimal generator* with domain D_A of the semigroup $(P_t)_{t\geq 0}$.

It is readily known (cf. [25, p.65]) that the infinitesimal generator of the semigroup $(P_t)_{t\geq 0}$ is given by the *Neumann–Laplacian* as explained in the following theorem.

Theorem 5.9. *Let $\tilde{B}=(\tilde{B}_t)_{t\geq 0}$ denote the doubly reflected Brownian motion and $(P_t)_{t\geq 0}$ (as in Definition 5.6) its associated semigroup. Further denote by A its infinitesimal generator. Then for $f\in D_A$*

$$Af=\frac{1}{2}\frac{d^2}{dx^2}u$$

with domain $D_A=C^{2,N}([0,1]):=\{u\in C^2([0,1])\,|\,\frac{d}{dx}u(0)=\frac{d}{dx}u(1)=0\}$.

5.2. Fractal transformed doubly reflected Brownian motion. Let F and G be two IFSs with the same number of increasing contractions satisfying the assumptions (A.1) and (A.2) from Section 3 on the ascending ordering. Further assume that the fractal transformation T_{FG} is bijective. We assume that the attractor of the IFS F is given by the unit interval $[0,1]$ and that the invariant measure μ_F is given by $\mu_F=\lambda^1|_{[0,1]}=\lambda$. The invariant measure supported on A_G will be denoted by μ for short.

Definition 5.10. Let F and G be the two previous IFSs. Let $T_{FG} : [0,1] \to A_G$ be the fractal transformation and $\tilde{B}$ the doubly reflected Brownian motion. Then the process $T\tilde{B} = (T\tilde{B}_t)_{t\geq 0}$ defined by

$$T\tilde{B}_t := T_{FG}(\tilde{B}_t) \quad (t \geq 0)$$

is called *fractal transformed doubly reflected Brownian motion*.

It is known from [5, Proposition 5.2 and Proposition 5.20] that

Theorem 5.11. *$T\tilde{B}$ is an A_G-valued, $(\mathcal{F}_t)_{t\geq 0}$-adapted, strong Markovian stochastic process.*

We now define a semigroup of operators related to the fractal transformed doubly reflected Brownian motion by conjugation of the semigroup of the doubly reflected Brownian motion.

Definition 5.12. Let $(P_t)_{t\geq 0}$ be the semigroup associated to the doubly reflected Brownian motion. We define for $u \in U_{FG}(C([0,1]))$, $x \in A_G$ and $t \geq 0$

$$Q_t u(x) := U_{FG} \circ P_t \circ U_{GF} u(x).$$

Remark 5.13. Since for any $u \in U_{FG}\,(C([0,1]))$ there exists $f \in C([0,1])$ with $u = U_{FG} f$ we have for $x \in A_G$ and $t \geq 0$ the following representation

$$\left(Q_t u\right)(x) = \mathbb{E}^{T_{GF}x}\left[u\left(T\tilde{B}_t\right)\right],$$

i.e. the expectation of $T_{FG}\tilde{B}_t$ under the condition that $B_0 = T_{GF}x$.

Immediately from the definition and the corresponding properties of the semigroup of the doubly reflected Brownian motion we have the following:

Theorem 5.14.

(i) *$(Q_t)_{t\geq 0}$ defines a semigroup of operators on $U_{FG}(C([0,1])))$;*
(ii) *Q_t is strongly continuous on $U_{FG}(C([0,1]))$.*

We now state the main result of this section in which we want to present an application of Corollary 2.5 resembling the method in [24, Example 7.9].

Theorem 5.15. *Denote*

$$C^{2,N}([0,1]) = \{u \in C^2([0,1]) \mid \frac{d}{dx}u(0) = \frac{d}{dx}u(1) = 0\}.$$

Then we have for all $u \in C^{2,N}_\mu := \overline{U_{FG}(C^{2,N}([0,1]))}^{lin}$

$$\lim_{t\to 0} \frac{Q_t u - u}{t} = \frac{1}{2}\frac{d^2}{d\mu^2}u \quad \textit{where the limit is with respect to } \|\cdot\|_\infty.$$

Proof. Let $u \in C_\mu^{2,N}$. Then we have $u \in C_\mu^2$ and u satisfies the μ-Neumann boundary conditions since $u = U_{FG}f$ for some $f \in C^{2,N}([0,1])$ and for $x \in \{0,1\}$ we have $T_{GF}(x) = x$, so we get

$$\frac{d}{d\mu}u(x) = U_{FG} \circ \frac{d}{dx}f(x) = \frac{d}{dx}f(T_{GF}x) = \frac{d}{dx}f(x) = 0 \quad (x \in \{0,1\}).$$

Let $t > 0$. Write as abbreviation $y := y(t,\omega) := T_{FG}\tilde{B}_t \in A_G$. Let $x \in A_G$. We apply the generalised Taylor formula (Corollary 2.5) on u around x:

$$u(y) = u(x) + \frac{d}{d\mu}u(x)(F_\mu(y) - F_\mu(x)) + \frac{1}{2}\frac{d^2}{d\mu^2}u(\xi)(F_\mu(y) - F_\mu(x))^2$$

for some $\xi = \xi(t,\omega) \in ([x,y] \cup [y,x])$. Inserting this into the operator Q_t we have an expansion of the semigroup as follows

$$\begin{aligned} Q_t u(x) &= \mathbb{E}^{T_{GF}x}[u(y)] \\ &= \mathbb{E}^{T_{GF}x}\left[u(x) + \frac{d}{d\mu}u(x)(F_\mu(y) - F_\mu(x)) + \frac{1}{2}\frac{d^2}{d\mu^2}u(\xi)(F_\mu(y) - F_\mu(x))^2\right] \\ &= u(x) + \frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}(F_\mu(y) - F_\mu(x)) \\ &\qquad + \mathbb{E}^{T_{GF}x}\left[\frac{1}{2}\frac{d^2}{d\mu^2}u(\xi)(F_\mu(y) - F_\mu(x))^2\right]. \end{aligned}$$

As $\mu = \lambda \circ T_{FG}^{-1} = \lambda \circ T_{GF}$ by Proposition 3.6 we have

$$\begin{aligned} F_\mu(y) - F_\mu(x) &= F_\mu(T_{FG}\tilde{B}_t) - F_\mu(x) = \mu([x, T_{FG}\tilde{B}_t]) \\ &= \lambda([T_{GF}x, \tilde{B}_t]) = \tilde{B}_t - T_{GF}x \end{aligned}$$

(together with the convention $\lambda([T_{GF}x, \tilde{B}_t]) = -\lambda([\tilde{B}_t, T_{GF}x])$ if $T_{GF}x > \tilde{B}_t$) and we can write

$$Q_t u(x) = u(x) + \frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right] + \mathbb{E}^{T_{GF}x}\left[\frac{1}{2}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\right].$$

So we have the following

$$\begin{aligned} \left|\frac{1}{t}(Q_t u(x) - u(x)) - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right| &= \left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right. \\ &\qquad \left. + \frac{1}{2t}\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right| \\ &\leq \left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| \\ &\qquad + \left|\frac{1}{2t}\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right|. \end{aligned} \tag{5.1}$$

First we consider the case that $x \notin \{0,1\}$. Then we can choose $\delta := \min\{T_{GF}x, 1 - T_{GF}x\} > 0$ and denote by $\mathrm{B}_\delta(T_{GF}x) := \{y \in \mathbb{R} \mid y - T_{GF}x| < \delta\}$.

For the above summands in (5.1) we can estimate as follows; for the first summand in (5.1) we have

$$\begin{aligned}\left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| &\leq \left\|\frac{d}{d\mu}u\right\|_\infty \frac{1}{t}\left|\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| \\ &\leq \left\|\frac{d}{d\mu}u\right\|_\infty \frac{1}{t}\left|\int_{\{B_t\in \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right. \\ &\qquad \left. + \int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right|.\end{aligned}$$

Since $\tilde{B}_t = B_t$, we get

$$\begin{aligned}&\left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| \\ &\qquad \leq \left\|\frac{d}{d\mu}u\right\|_\infty \frac{1}{t}\left|\int_{\{B_t\in \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right. \\ &\qquad\qquad + \int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x} \\ &\qquad\qquad - \left(\int_{\{B_t\in \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right. \\ &\qquad\qquad \left.\left. + \int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right)\right|.\end{aligned}$$

Taking into account

$$\begin{aligned}\int_{\{B_t\in \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x} + \int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x} \\ = \mathbb{E}^{T_{GF}x}\left[B_t - T_{GF}x\right] = \mathbb{E}\left[B_t\right] = 0\end{aligned}$$

we obtain

$$\begin{aligned}&\left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| \\ &\qquad = \left\|\frac{d}{d\mu}u\right\|_\infty \frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right. \\ &\qquad\qquad \left. - \int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right| \\ &\qquad \leq \left\|\frac{d}{d\mu}u\right\|_\infty \frac{1}{t}\left(\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right|\right. \\ &\qquad\qquad \left. + \left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right|\right). \qquad (5.2)\end{aligned}$$

For the first summand in (5.2) we estimate

$$\frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right| \le \frac{1}{t}\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}|\tilde{B}_t - T_{GF}x|\,d\mathbb{P}^{T_{GF}x}.$$

Since $|\tilde{B}_t - T_{GF}x \le 1$, we obtain

$$\begin{aligned}\frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right| &\le \frac{1}{t}\int_{\{B_t\notin \mathrm{B}_\delta(0)\}} 1\,d\mathbb{P}\\ &= \frac{1}{t}\mathbb{P}(\{B_t\notin \mathrm{B}_\delta(0)\}) = \frac{1}{t}\mathbb{P}(\{|B_t|\ge\delta\})\\ &= \frac{2}{t}\,\mathbb{P}\left(\{B_t\ge\delta\}\right) = \frac{2}{t}\,\mathbb{P}\left(\left\{\frac{1}{\sqrt{t}}B_t \ge \frac{\delta}{\sqrt{t}}\right\}\right) = \frac{2}{t}\,\frac{1}{\sqrt{2\pi}}\int_{\frac{\delta}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}\,dx.\end{aligned}$$

Taking into account

$$\int_{\frac{\delta}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}\,dx \le \frac{\sqrt{t}}{\delta}e^{-\frac{\delta^2}{2t}}$$

(cf. [24] Lemma 10.5.), we deduce

$$\frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(\tilde{B}_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right| \le \frac{2}{\delta\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{\delta^2}{2t}} \xrightarrow{t\to 0} 0$$

and for the second term in (5.2) we calculate

$$\begin{aligned}\frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(T_{GF}x)\}}(B_t - T_{GF}x)\,d\mathbb{P}^{T_{GF}x}\right| &= \frac{1}{t}\left|\int_{\{B_t\notin \mathrm{B}_\delta(0)\}} B_t\,d\mathbb{P}\right|\\ &= \frac{1}{t}\frac{1}{\sqrt{2\pi t}}\left|\int_{-\infty}^{-\delta} xe^{-\frac{x^2}{2t}}\,dx + \int_{\delta}^{\infty} xe^{-\frac{x^2}{2t}}\,dx\right| = 0\end{aligned}$$

because of the symmetry of the integrands. In the case $x\in\{0,1\}$ the first term in (5.1) involving a first order μ-derivative vanishes by the Neumann boundary conditions. Therefore we have shown

$$\lim_{t\to 0}\sup_{x\in A_G}\left|\frac{1}{t}\frac{d}{d\mu}u(x)\mathbb{E}^{T_{GF}x}\left[\tilde{B}_t - T_{GF}x\right]\right| = 0.$$

Now we are going to estimate the second term in (5.1). In this case we choose

$$\delta' := \begin{cases}\min\{T_{GF}x, 1-T_{GF}x\} & , x\in A_G\setminus\{0,1\}\\ 1 & , x\in\{0,1\}\end{cases}.$$

We then calculate in a similar manner setting $t = \mathbb{E}^{T_{GF}x}\left[(B_t - T_{GF}x)^2\right]$

$$\left|\frac{1}{2t}\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right|$$

$$
\begin{aligned}
&= \frac{1}{2t}\left|\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - t\frac{d^2}{d\mu^2}u(x)\right| \\
&= \frac{1}{2t}\left|\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - \mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(x)\left(B_t - T_{GF}x\right)^2\right]\right| \\
&= \frac{1}{2t}\left|\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right.
\end{aligned}
$$

Since $(\tilde{B}_t - T_{GF}x)^2 = (\tilde{B}_t - T_{GF}x)^2$, we get

$$
\begin{aligned}
&\left|\frac{1}{2t}\mathbb{E}^{T_{GF}x}\left[\frac{d^2}{d\mu^2}u(\xi)\left(\tilde{B}_t - T_{GF}x\right)^2\right] - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right| \\
&\qquad + \int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x} \\
&\qquad - \int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\left(B_t - T_{GF}x\right)^2\, d\mathbb{P}^{T_{GF}x} \\
&\qquad - \int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\left(B_t - T_{GF}x\right)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&= \frac{1}{2t}\left|\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)(B_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&\qquad + \frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right. \\
&\qquad - \left.\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\left(B_t - T_{GF}x\right)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&\le \frac{1}{2t}\left|\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)(B_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&\qquad + \frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&\qquad + \frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\left(B_t - T_{GF}x\right)^2\, d\mathbb{P}^{T_{GF}x}\right|. \qquad (5.3)
\end{aligned}
$$

We are going to show that the last two integrals in (5.3) vanish for $t \to 0$ uniformly in $x \in A_G$. Taking into account $(\tilde{B}_t - T_{GF}x)^2 \le 1$ for the second summand in (5.3), we observe

$$
\begin{aligned}
&\frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(\xi)(\tilde{B}_t - T_{GF}x)^2\, d\mathbb{P}^{T_{GF}x}\right| \\
&\qquad \le \left\|\frac{d^2}{d\mu^2}u\right\|_\infty \frac{1}{2t}\mathbb{P}^{T_{GF}x}(\{B_t \notin \mathrm{B}_{\delta'}(T_{GF}x)\}) \\
&\qquad \le \left\|\frac{d^2}{d\mu^2}u\right\|_\infty \frac{1}{2t}\mathbb{P}(\{|B_t| \ge \delta\}) \le \left\|\frac{d^2}{d\mu^2}u\right\|_\infty \frac{1}{2t}2\mathbb{P}(\{B_t \ge \delta'\}) \\
&\qquad \le \left\|\frac{d^2}{d\mu^2}u\right\|_\infty \frac{1}{\delta'}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{\delta'^2}{2t}} \quad \xrightarrow{t\to 0} 0
\end{aligned}
$$

and for the third summand in (5.3) we estimate

$$
\begin{aligned}
&\frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\,(B_t-T_{GF}x)^2\;d\mathbb{P}^{T_{GF}x}\right|\\
&\qquad\le\left\|\frac{d^2}{d\mu^2}u\right\|_\infty\frac{1}{2t}\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}(B_t-T_{GF}x)^2\;d\mathbb{P}^{T_{GF}x}\\
&\qquad\le\left\|\frac{d^2}{d\mu^2}u\right\|_\infty\frac{1}{2t}\int_{\{B_t\notin \mathrm{B}_{\delta'}(0)\}}{B_t}^2\,d\mathbb{P}=\left\|\frac{d^2}{d\mu^2}u\right\|_\infty\frac{1}{2t}\frac{2}{\sqrt{2\pi t}}\int_{\delta'}^{\infty}e^{-\frac{x^2}{2t}}\,dx\\
&\qquad=\frac{2}{\sqrt{2\pi t}}\frac{1}{t}\sqrt{t}\int_{\frac{\delta'}{\sqrt{t}}}^{\infty}ty^2e^{-\frac{y^2}{2}}\,dy=\frac{2}{\sqrt{2\pi}}\int_{\frac{\delta'}{\sqrt{t}}}^{\infty}y^2e^{-\frac{y^2}{2}}\,dy\\
&\qquad=\frac{2}{\sqrt{2\pi}}\left(\frac{\delta'}{\sqrt{t}}e^{-\frac{\delta'^2}{2t}}+\int_{\frac{\delta'}{\sqrt{t}}}^{\infty}e^{-\frac{y^2}{2}}\,dy\right).
\end{aligned}
$$

Taking into account

$$\int_{\frac{\delta'}{\sqrt{t}}}^{\infty}e^{-\frac{y^2}{2}}\,dy\le\frac{\sqrt{t}}{\delta'}e^{-\frac{\delta'^2}{2t}}$$

we conclude

$$
\begin{aligned}
&\frac{1}{2t}\left|\int_{\{B_t\notin \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{d^2}{d\mu^2}u(x)\,(B_t-T_{GF}x)^2\;d\mathbb{P}^{T_{GF}x}\right|\\
&\qquad\le\frac{2}{\sqrt{2\pi}}\left(\frac{\delta'}{\sqrt{t}}e^{-\frac{\delta'^2}{2t}}+\frac{\sqrt{t}}{\delta'}e^{-\frac{\delta'^2}{2t}}\right)\xrightarrow{t\to 0}0
\end{aligned}
$$

because

$$\frac{\delta'}{\sqrt{t}}e^{-\frac{\delta'^2}{2t}}\xrightarrow{t\to 0}0\quad\text{and}\quad\frac{\sqrt{t}}{\delta'}e^{-\frac{\delta'^2}{2t}}\xrightarrow{t\to 0}0.$$

It remains to show that the first term in (5.3) including second order μ-derivatives vanishes uniformly in x as $t\to 0$. This can be achieved as follows

$$
\begin{aligned}
&\frac{1}{2}\left|\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\left(\frac{d^2}{d\mu^2}u(\xi)-\frac{d^2}{d\mu^2}u(x)\right)\frac{1}{t}(B_t-T_{GF}x)^2\,d\mathbb{P}^{T_{GF}x}\right|\\
&\qquad\le\frac{1}{2}\sqrt{\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\left(\frac{d^2}{d\mu^2}u(\xi)-\frac{d^2}{d\mu^2}u(x)\right)^2\,d\mathbb{P}^{T_{GF}x}}\\
&\qquad\quad\times\sqrt{\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{1}{t^2}(B_t-T_{GF}x)^4\,d\mathbb{P}^{T_{GF}x}}.
\end{aligned}\tag{5.4}
$$

Again we estimate separately. For the last term in (5.4)

$$
\begin{aligned}
&\sqrt{\int_{\{B_t\in \mathrm{B}_{\delta'}(T_{GF}x)\}}\frac{1}{t^2}(B_t-T_{GF}x)^4\,d\mathbb{P}^{T_{GF}x}}\le\sqrt{\mathbb{E}^{T_{GF}x}\left[\frac{1}{t^2}(B_t-T_{GF}x)^4\right]}\\
&\qquad=\sqrt{\mathbb{E}^{T_{GF}x}\left[\left(\frac{1}{t}\left(B_t-T_{GF}x\right)^2\right)^2\right]}=\sqrt{\mathbb{E}\left[\left(\frac{1}{t}\left(B_t\right)^2\right)^2\right]}
\end{aligned}
$$

$$= \sqrt{\mathbb{E}\left[\left(\frac{1}{t}\left(\sqrt{t}B_1\right)^2\right)^2\right]} = \sqrt{\mathbb{E}\left[\frac{t^2}{t^2}\right]} = 1.$$

Here $B_t \sim \sqrt{t}B_1$ has been used. For the first term in (5.4) we observe

$$\begin{aligned}
&\sqrt{\int_{\{B_t \in \mathrm{B}_{\delta'}(T_{GF}x)\}} \left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)^2 \, d\mathbb{P}^{T_{GF}x}} \\
&\qquad \le \sqrt{\int_\Omega \left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)^2 \, d\mathbb{P}^{T_{GF}x}} \\
&\qquad = \sqrt{\mathbb{E}^{T_{GF}x}\left[\left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)^2\right]}. \qquad (5.5)
\end{aligned}$$

Since

$$\left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)^2 \le 4\left\|\frac{d^2}{d\mu^2}u\right\|_\infty^2$$

and

$$\frac{d^2}{d\mu^2}u(\xi) \xrightarrow{t\to 0} \frac{d^2}{d\mu^2}u(x)$$

uniformly in x by the continuity of $\frac{d^2}{d\mu^2}u$ we now can apply Lebesgue's dominated convergence theorem to show that for (5.5) it holds that

$$\sqrt{\mathbb{E}^{T_{GF}x}\left[\left(\frac{d^2}{d\mu^2}u(\xi) - \frac{d^2}{d\mu^2}u(x)\right)^2\right]} \xrightarrow{t\to 0} 0.$$

Summarising all auxiliary estimates we have eventually shown that

$$\lim_{t\to 0} \sup_{x\in A_G} \left|\frac{1}{t}\left(Q_t u(x) - u(x)\right) - \frac{1}{2}\frac{d^2}{d\mu^2}u(x)\right| = 0. \qquad \square$$

Remark 5.16. From [8, section 2] it is readily known that the generator of a strongly continuous semigroup which is conjugated by a bijection is given by the corresponding conjugated infinitesimal generator defined on the transformed domain, i.e. if $\frac{1}{2}\frac{d^2}{dx^2}$ denotes the generator of the semigroup $(P_t)_{t\ge 0}$ with domain $C^{2,N}([0,1])$ (see theorem 5.9), then the generator of $(U_{FG} \circ P_t \circ U_{GF})_{t\ge 0}$ with domain $U_{FG}(C^{2,N}([0,1]))$ is given by $\frac{1}{2}U_{FG} \circ \frac{d^2}{dx^2} \circ U_{GF}$ as one can easily verify. Namely for $f \in C^{2,N}([0,1])$ and $u := U_{FG}f$ we have uniformly in $x \in A_G$ that

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}\left(Q_t u(x) - u(x)\right) &= \lim_{t\to 0} \frac{1}{t}\left(\left(U_{FG} \circ P_t \circ U_{GF}\right) U_{FG} f(x) - U_{FG} f(x)\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(P_t f\left(T_{GF}x\right) - f\left(T_{GF}x\right)\right)
\end{aligned}$$

$$= \frac{1}{2}\frac{d^2}{dx^2} f(T_{GF}x) = \frac{1}{2} U_{FG} \circ \frac{d^2}{dx^2} f(x)$$
$$= \frac{1}{2}\left(U_{FG} \circ \frac{d^2}{dx^2} \circ U_{GF} \right) u(x) = \frac{1}{2}\frac{d^2}{d\mu^2} u(x).$$

This observation coincides with the result from Theorem 5.15 that we derived by application of a generalised second order Taylor-formula.

6. Space and time change of a Brownian motion

In this section we want to sketch the construction of a stochastic process such that its associated semigroup has generator $\frac{d}{d\mu}\frac{d}{d\nu}$. The ideas can be found in [3] and [18]. Moreover we want to discuss its connections to the fractal transformed doubly reflected Brownian motion from Section 5.2.

Again we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion defined on the probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of the Brownian motion. For the subsequent construction we need the notion of the *local time* of a Brownian motion which is given by

$$\begin{aligned} l(t,x) = l(t,x,\omega) :=& \mathbb{P} - \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(B_s - x) ds \\ =& \mathbb{P} - \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \lambda(\{s \in [0,t] \,|\, B_s \in (x-\varepsilon, x+\varepsilon)\}) \end{aligned}$$

for $t \geq 0$ and $x \in \mathbb{R}$.

As in the Section 2 let ν and μ be two atomless Borel probability measures on $[0,1]$ with $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu)$ and $0,1 \in \operatorname{supp}(\mu)$.

Definition 6.1. Let $l(t,x) = l(t,x,\omega)$ $(t \geq 0, x \in \mathbb{R}, \omega \in \Omega)$ denote the local time of a standard Brownian motion. We define for $t \geq 0$

$$S_t := \int_{F_\nu(\operatorname{supp}(\mu))} l(t,x) d\mu \circ F_\nu^{-1}(x) \quad \text{and} \quad T_t := \inf\{u \geq 0 \,|\, S_t > t\}.$$

Then we set

$$X := \left((X_t)_{t\geq 0} := (B_{T_t})_{t\geq 0}, (\mathcal{F}_{T_t})_{t\geq 0}, \mathbb{P} \right)$$

and call X a *gap diffusion* with *speed measure* $\mu \circ F_\nu^{-1}$. Furthermore we define

$$Y := (Y_t)_{t\geq 0} := \left(\check{F}_\nu^{-1}(X_t) \right)_{t\geq 0},$$

where $\check{F}_\nu^{-1}(x) := \inf\{y \in [0,1] | F_\nu(y) \geq x\}$ denotes the generalised inverse of F_ν. We will call Y a gap diffusion with speed measure $\mu \circ F_\nu^{-1}$ and *scale measure* ν.

With the notations as in previous definition we have the following

Proposition 6.2.

(i) *for all* $t \geq 0$ *we have* $Y_t \in supp(\mu)$ $\mathbb{P}$- *almost surely*;

(ii) *X is a strong Markovian stochastic process;*
(iii) *for all $f \in C\left(supp(\mu)\right)$ the map $x \mapsto \mathbb{E}^x\left[f(X_t)\right]$ belongs to $C\left(supp(\mu)\right)$;*
(iv) *for $f \in C\left(supp(\mu)\right)$ and $x \in supp(\mu)$ we have $\lim_{t\to 0}\mathbb{E}^x\left[f(X_t)\right] = f(x)$.*

Proof. (i) From [7, Lemma 3.1] we know that $X_t \in \operatorname{supp}(\mu\circ F_\nu^{-1}) = F_\nu(\operatorname{supp}(\mu))$ $\mathbb{P}$- almost surely, thus $Y_t = \check{F}_\nu^{-1}(X_t) \in \operatorname{supp}(\mu)$ $\mathbb{P}$-almost surely for any $t \geq 0$.

(ii)–(iv) For these assertions we refer to [18, Theorem 4.8]. □

Due to the Markov property of the process $(Y_t)_{t\geq 0}$ the expression $\left(\mathbb{E}^x\left[f(Y_t)\right]\right)_{t\geq 0}$ $(x \in \operatorname{supp}(\mu))$ again defines a semigroup of operators for which its infinitesimal generator is stated in the following theorem.

Theorem 6.3. *Let $(Y_t)_{t\geq 0} = \left(\check{F}_\nu^{-1}(X_t)\right)_{t\geq 0}$ be the gap diffusion described in Definition 6.1 with speed-measure $\mu\circ F_\nu^{-1}$ and scale measure ν. Let A be the infinitesimal generator of the semigroup $\left(\mathbb{E}^x\left[f(Y_t)\right]\right)_{t\geq 0}$ $(x \in supp(\mu))$. Then for f in the domain of A there exists a continuous continuation (again denoted by f) in $\mathcal{D}_2^{\mu,\nu}$ such that*

$$f(x) = f(0) + \int_0^x \left(F_\mu(x) - F_\mu(y)\right) 2Af(y)d\mu(y) \quad (x \in \mathbb{R}),$$

i.e. $A = \frac{1}{2}\frac{d}{d\mu}\frac{d}{d\nu}$ and the Neumann boundary conditions $\frac{d}{d\nu}f(0) = \frac{d}{d\nu}f(1) = 0$ are satisfied.

Proof. See ([18], Theorem 4.11). □

Remark 6.4. Setting $\mu = \nu$ in Definition 6.1 gives a process Y such that its state space and the infinitesimal generator of its associated semigroup coincides with that of a fractal transformed doubly reflected Brownian motion.

Therefore we now want to briefly sketch the connection of the fractal transformed doubly reflected Brownian motion $T\tilde{B}$ from Definition 5.4 and the process Y from Definition 6.1. Assume that $\mu = \nu$ in Definition 6.1 and assume that μ is the invariant measure supported on the attractor A_G.

Again assume that $A_F = [0,1]$ and $\mu_F = \lambda$. Under the given assumptions on the IFSs F and G, i.e. the increasing contraction maps (A.1), that are ordered ascendingly (A.2), gives that the fractal transformation $T_{GF}\colon A_G \to [0,1]$ is essentially the cumulative distribution function F_μ restricted on A_G and $\check{F}_\mu^{-1}$ coincides with $T_{FG}\colon [0,1] \to A_G$. Hence the fractal transformed doubly reflected Brownian motion just evolves by the transformation of a doubly reflected Brownian motion via the cumulative distribution function F_μ; compare to the definition of the process Y by transformation of the gap diffusion X with speed measure $\mu\circ F_\mu^{-1} = \lambda|_{[0,1]}$ by $\check{F}_\mu^{-1}$ as in Definition 6.1.

In our setting the transformation via fractal transformations is essentially a transformation via the distribution function of the measure μ supported on the attractor A_G and known results from classical analysis on $[0,1]$ can be transferred via a transformation with F_μ to results on a Cantor-like set A_G.

For more examples on this we refer to [2, 18, 20, 21].

References

[1] P. Arzt, *Eigenvalues of Measure Theoretic Laplacians on Cantor-like Sets*, PhD thesis, Universität Siegen, 2014.

[2] C. Bandt, M. Barnsley, M. Hegland, and A. Vince, *Conjugacies provided by fractal transformations I : Conjugate measures, Hilbert spaces, orthogonal expansions, and flows, on self-referential spaces*, preprint, https://arxiv.org/abs/1409.3309.

[3] G. Burkhardt, *Über Quasidiffusionen als Zeittransformationen des Wienerschen Prozesses*, PhD thesis, TU Dresden, 1983.

[4] E.B. Dynkin, *Markov Processes*, **I**, **II**, Springer-Verlag Berlin, Heidelberg, 1965.

[5] T. Ehnes, *Eigenschaften einer fraktaltransformierten doppelt-reflektierten Brownschen Bewegung*, Master's thesis, Universität Stuttgart, 2017.

[6] T. Ehnes, *Stochastic Partial Differential Equations on Cantor-like Sets*, PhD thesis, Universität Stuttgart, 2020.

[7] E. Ekström, D. Hobson, S. Janson, and J. Tysk, *Can time-homogeneous diffusions produce any distribution?*, Probab. Theory Related Fields **155** (2013), 493–520.

[8] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[9] W. Feller, *Generalized second order differential operators and their lateral conditions*, Illinois J. Math. **1** (1957), 459–504.

[10] U. Freiberg, *Analytical properties of measure geometric Krein–Feller-operators on the real line*, Math. Nachr. **260** (2003), 34–47.

[11] U. Freiberg, *Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets*, Forum Math. **17** (2005), 87–104.

[12] U. Freiberg, *Dirichlet forms on fractal subsets of the real line*, Real Anal. Exchange **30**, 2004/2005, 589–604.

[13] U. Freiberg and M. Zähle, *Harmonic calculus on fractals — A measure geometric approach I*, Potential Anal. **16** (2002), 265–277.

[14] T. Fujita, *A fractional dimension, self–similarity and a generalized diffusion operator*, Probabilistic Methods in Mathematical Physics, Proc. of Taniguchi International Symp., 1987, 83–90.

[15] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[16] K. Itô and H. P. McKean, *Diffusion processes and their sample paths*, 2^{nd} ed., Springer, Berlin, Heidelberg, New York, 1974.

[17] I.S. Kac and G. Kreĭn, *On the spectral functions of the string*, Amer. Math. Soc. Transl., **103**, 1974, 19–102.

[18] M. Kesseböhmer, A. Niemann, T. Samuel, and H. Weyer, *Generalised Krein–Feller operators and Liouville Brownian motion via transformation of measure spaces*, preprint, https://arxiv.org/abs/1909.08832v3.

[19] M. Kesseböhmer, T. Samuel, and H. Weyer, *A note on measure-geometric Laplacians*, preprint, https://arxiv.org/abs/1411.2491v1

[20] H. Kunze, D. La Torre, F. Mendivil, and E. R. Vrscay, *Differential Equations Using Generalized Derivatives on Fractals,* Recent Developments in Mathematical, Statistical and Computational Sciences (Eds. D.M. Kilgour, H. Kunze, R. Makarov, R. Melnik, X. Wang), Springer, 2019, 81–91.

[21] H. Kunze, D. La Torre, F. Mendivil, and E. R. Vrscay, *Self-similarity of solutions to integral and differential equations with respect to a fractal measure,* Fractals **27** (2019), 1950014.

[22] L. Minorics, *Eigenwertapproximation von Krein–Feller-Operatoren bezüglich singulärer invarianter Wahrscheinlichkeitsmaße*, Master's thesis, Universität Stuttgart, 2016.

[23] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 2005.

[24] R.L. Schilling and L. Partzsch, *Brownian Motion: An Introduction to Stochastic Processes*, DeGruyter, Berlin-Boston, 2012.

[25] A. Winter and J. Swart, *Markov Processes: Theory and Examples*, Lecture notes, 2013. Available from: https://www.uni-due.de/~hm0112/teaching/markovprocesses-19ss/sw20.pdf

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Фрактальне перетрорення операторів Крейна–Феллера

Max Menzel and Uta Freiberg

Ми розглядаємо фрактально перетворений броунівський рух з подвійним відбиттям з простором станів, що є множиною, подібною до канторової. Застосовуючи теорію фрактальних перетворень, розвинуту Барнслі та ін., а також узагальнений вираз Тейлора, ми доводимо, що його інфінітезимальний генератор задається в термінах геометричної похідної другого порядку за мірою $\frac{d}{d\mu}\frac{d}{d\mu}$, яку було розглянуто Фрайберґом і Целе. Крім того, ми досліджуємо його зв'язок з добре відомим класичним оператором Крейна–Феллера $\frac{d}{d\mu}\frac{d}{dx}$, який є генератором так званої "щілинної дифузії" ("gap-diffusion").

Ключові слова: геометричний оператор міри Крейна–Феллера, множини, подібні до канторової, інфінітезимальний генератор, фрактальне перетворення, щілинна дифузія (gap-diffusion)