Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 3, pp. 603–615
doi: https://doi.org/10.15407/mag19.03.603

On Multiply Warped Product Gradient Ricci Soliton

Tamalika Dutta, Sampa Pahan, X. Chen, and Arindam Bhattacharyya

The object of the present paper is to study the gradient Ricci soliton multiply warped product. We prove that when the manifold is complete, then the potential function depends only on the base and the fiber must be an Einstein manifold. We also present the necessary and sufficient conditions for constructing a gradient Ricci soliton multiply warped product.

Key words: gradient Ricci soliton, warped product, multiply warped product

Mathematical Subject Classification 2020: 53C24, 53C25, 53C21

1. Introduction

Warped product spaces play an important role in general theory of relativity. The (singly) warped product $B \times_b F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \to (0, \infty)$ is a product manifold of form $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_b F$ is the manifold $B \times F$ with the Riemannian structure such that $\|X\|^2 = \|\pi_*(X)\|^2 + (b \circ \pi)^2 \|\sigma_*(X)\|^2$ for any vector field X on M. The metric tensor $g = g_B \oplus b^2 g_F$. The pair (B, g_B) is called the base manifold, (F, g_F) is the fiber manifold and b is the warping function. The concept of warped product was introduced by Bishop and O'Neill [2] to construct examples of complete Riemannian manifolds with negative sectional curvature. Currently, the warped products have been very useful for studying Einstein type manifolds, see, e.g., [3, 8, 9, 15, 16].

Multiply warped products are generalizations of singly warped products. A multiply warped product (M, g) is the product manifold $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \ldots \times_{f_m} F_m$ with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \ldots \oplus f_m^2 g_{F_m}$, where for each $i \in \{1, 2, \ldots, m\}$, $f_i : B \to (0, \infty)$ are smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. For instance, if $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold, then M is known as the multiply generalized Robertson–Walker space-time.

In 1982, R.S. Hamilton [10] introduced a concept of the Ricci flow and proved its existence. This concept was developed to answer Thurston's Geometrization

Conjecture which says that each closed 3-manifold admits a geometric decomposition. A special solution to the Ricci flow is called a Ricci soliton [10] if it is represented only by a one-parameter group of diffeomorphisms and scalings. Ricci solitons are characterized by the equation

$$\frac{1}{2}\mathcal{L}_X g + \mathrm{Ric} = \rho g,$$

where $\mathcal{L}_X$ is the Lie derivative, Ric is the Ricci tensor of the Riemannian metric g, X is a vector field and ρ is a scalar. The Ricci soliton is said to be shrinking, steady or expanding if ρ is positive, zero or negative, respectively. Also, we know that a Ricci soliton is a natural generalization of an Einstein metric.

When the vector field X is the gradient of a smooth function h on M, we call (M, g) as a gradient Ricci soliton. For a gradient Ricci soliton, the equation will be

$$\mathrm{Ric} + \nabla^2 h = \rho g. \tag{1.1}$$

In this case, h is called the potential function of the Ricci soliton, ∇ is the Levi-Civita connection of g, and $\rho \in \mathbb{R}$. If ∇h is a Killing vector field, i.e., $\nabla^2 h = 0$, then (M, g) becomes an Einstein manifold.

Simple examples of gradient Ricci solitons can be obtained by considering $\mathbb{R}^n$ with the canonical metric g_0. It is well known that the pair $(\mathbb{R}^n, g_0)$ is a gradient shrinker with a potential function given by $h(x) = \frac{|x|^2}{4}$, $\mathrm{Ric} + \nabla^2 h = \frac{1}{2} g_0$. This is called Gaussian shrinking soliton.

In [9], the authors presented the necessary and sufficient conditions for constructing a gradient Ricci soliton on a warped product assuming that the potential function was lifted from the base. The last assumption was later removed in [3] under the requirement that the soliton was complete. In our case, we consider the gradient Ricci soliton on multiply warped product and we follow the same ideas as in [9]. There is also a well-known fact that there are examples of Ricci solitons on multiply warped product manifolds in [12]. We have first observed that when the manifold is complete, then the potential function for a gradient Ricci soliton multiply warped product depends only on the base and the fibers are Einstein manifolds. We have also shown the existence criteria for the gradient Ricci soliton multiply warped product.

The investigations of multiply warped product gradient Ricci soliton help us to characterize the relation between the potential function of the gradient Ricci soliton and the multiply warped product.

2. Preliminaries

Throughout this paper, we will consider M to be connected, Hausdorff, paracompact and smooth. For an arbitrary n-dimensional pseudo-Riemannian manifold (M, g) and a smooth function $f : M \to \mathbb{R}$, we have that H^f and Δf denote the Hessian $(0, 2)$ tensor and the Laplace-Beltrami operator of f. We follow both the notation and the terminology of [14]. Moreover, we assume that all warping functions are non-constant.

Following the ideas of Bishop and O'Neill, the authors of [5,7,17,18] state the covariant derivative formulas for multiply warped products in the following way.

Lemma 2.1. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a pseudo-Riemannian multiply warped product with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$. Also, let $X, Y \in \chi(B)$ and $V \in \chi(F_i), W \in \chi(F_j)$, where $\chi(B)$, $\chi(F_i)$, $\chi(F_j)$ are the set of all vector fields on B, F_i, F_j, respectively. Then*

1. $\nabla_X Y$ *is the lift of* $\nabla_X^B Y$;
2. $\nabla_X V = \nabla_V X = \dfrac{X(f_i)}{f_i} V$;
3. $\nabla_V W = \begin{cases} 0 & \text{if } i \neq j \\ \nabla_V^{F_i} W - \dfrac{g(V,W)}{f_i} \nabla_B f_i & \text{if } i = j. \end{cases}$

Lemma 2.2. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a pseudo-Riemannian multiply warped product with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$. Let $\psi : B \to \mathbb{R}$ be a smooth function. Then*

1. $\nabla(\psi \circ \pi) = \nabla_B \psi$,
2. $\Delta(\psi \circ \pi) = \Delta_B \psi + \displaystyle\sum_{k=1}^{m} \frac{s_k}{f_k} g_B(\nabla_B \psi, \nabla_B f_k)$,

where ∇ and Δ denote the gradient and the Laplace–Beltrami operator on M.

Lemma 2.3. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a pseudo-Riemannian multiply warped product with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$. Also, let $X, Y, Z \in \chi(B)$ and $V \in \chi(F_i), W \in \chi(F_j)$. Then*

1. $\mathrm{Ric}(X,Y) = \mathrm{Ric}_B(X,Y) - \displaystyle\sum_{i=1}^{m} \frac{s_i}{f_i} H_B^{f_i}(X,Y)$,
2. $\mathrm{Ric}(X,V) = 0$,
3. $\mathrm{Ric}(V,W) = \begin{cases} 0 & \text{if } i \neq j, \\ \mathrm{Ric}_{F_i}(V,W) - \Bigg(\dfrac{\Delta_B f_i}{f_i} + (s_i - 1) \dfrac{\|\nabla_B f_i\|_B^2}{f_i^2} \\ \qquad + \displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \Bigg) g(V,W) & \text{if } i = j, \end{cases}$

where $H_B^{f_i} = \nabla_B^2 f_i$ denotes the Hessian operator on B and s_i stands for the dimension of F_i.

3. Gradient Ricci soliton in multiply warped product spaces

In this section we generalize the result of Corollary 2.2 of [3].

Proposition 3.1. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a complete Riemannian multiply warped product with the metric $\tilde{g} = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus$*

$\cdots \oplus f_m^2 g_{F_m}$, where $f_i : B \to (0,\infty)$ are non-constant warping functions for $i \in \{1, 2, \dots, m\}$. If $(M, \tilde{g})$ is a multiply warped product gradient Ricci soliton, then the potential function depends only on the base.

Proof. Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a complete multiply warped product gradient Ricci soliton with the potential function given by $\psi : M \to \mathbb{R}$. Then we proceed as follows. Suppose by contradiction that there is $i \in \{1, 2, \dots, m\}$ such that ψ is not constant on the fiber F_i. Write $M = \tilde{B} \times F_i$, with $\tilde{B} = B \times F_1 \times \cdots \times F_{i-1} \times F_{i+1} \times \cdots \times F_m$, and consider the metric $\tilde{g} = g_{\tilde{B}} \times {f_i}^2 g_i$, where

$$g_{\tilde{B}} = g_B \oplus {f_1}^2 g_1 \oplus \cdots \oplus {f_{i-1}}^2 g_{i-1} \oplus {f_{i+1}}^2 g_{i+1} \oplus \cdots {f_m}^2 g_m.$$

Thus, $M = \tilde{B} \times_{f_i} F_i$ is a warped product Ricci soliton with M complete. Using Corollary 2.2 of [3], we conclude that ψ is constant on F_i, which is a contradiction. The conclusion is that ψ depends only on the base. $\square$

Proposition 3.2. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a complete Riemannian multiply warped product with the metric $\tilde{g} = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$, where $f_i : B \to (0,\infty)$ are non-constant warping functions for $i \in \{1, 2, \dots, m\}$. If $(M, \tilde{g})$ is a multiply warped product gradient Ricci soliton, then the fibers are Einstein manifolds.*

Proof. Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a complete multiply warped product with the potential function $\psi : B \to \mathbb{R}$ which depends only on the base. If $V \in \chi(F_i), W \in \chi(F_j)$ for $i = j$, then from Lemma 2.3 we have

$$\begin{aligned} \mathrm{Ric}(V, W) = \mathrm{Ric}_{F_i}(V, W) - \Bigg(&\frac{\Delta_B f_i}{f_i} + (s_i - 1)\frac{\|\nabla_B f_i\|_B^2}{f_i^2} \\ &+ \sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \Bigg) f_i^2 g_{F_i}(V, W). \end{aligned} \tag{3.1}$$

Now, for $V \in \chi(F_i)$ and $W \in \chi(F_j)$, we have from (1.1)

$$\mathrm{Ric}_{\tilde{g}}(V, W) + {H_{\tilde{g}}}^{\psi}(V, W) = \rho \tilde{g}(V, W). \tag{3.2}$$

Substituting (3.2) in (3.1), we have

$$\begin{aligned} \mathrm{Ric}_{F_i}(V, W) = \Bigg(\rho f_i^2 + f_i^2 \Bigg(&\frac{\Delta_B f_i}{f_i} + (s_i - 1)\frac{\|\nabla_B f_i\|_B^2}{f_i^2} \\ &+ \sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \Bigg) \Bigg) g_{F_i}(V, W) - {H_{\tilde{g}}}^{\psi}(V, W). \end{aligned} \tag{3.3}$$

Again, it is known by Proposition 3.1 that ψ depends only on the base. Also, we have

$$\nabla_V (\nabla_{\tilde{g}} \psi) = \frac{(\nabla_{g_B} \psi)(f_i)}{f_i} V.$$

Therefore,

$$H_{\tilde{g}}^{\psi}(V,W) = \frac{(\nabla_{g_B}\psi)(f_i)}{f_i}\tilde{g}(V,W) = f_i^2 g_{F_i}(V,W)\frac{(\nabla_{g_B}\psi)(f_i)}{f_i}.$$

Thus,

$$H_{\tilde{g}}^{\psi}(V,W) = f_i(\nabla_{g_B}\psi)(f_i)g_{F_i}(V,W). \tag{3.4}$$

Then, putting the value of (3.4) in equation (3.3), we derive

$$\begin{aligned}\mathrm{Ric}_{F_i}(V,W) = \Bigg(\rho f_i^2 + f_i^2\Bigg(\frac{\Delta_B f_i}{f_i} + (s_i-1)\frac{\|\nabla_B f_i\|_B^2}{f_i^2} \\ + \sum_{\substack{k=1\\k\neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k}\Bigg) - f_i(\nabla_{g_B}\psi)(f_i)\Bigg)g_{F_i}(V,W).\end{aligned}$$

Hence the proof follows. □

4. Existence of multiply warped product gradient Ricci soliton

In this section, we are proving some interesting result to establish the existence of multiply warped product gradient Ricci soliton.

Let M be a gradient Ricci soliton multiply warped product with a potential function ψ as the lift of a smooth function on B. Let $\tilde{\phi} = \phi \circ \pi$ be the lift of a smooth function ϕ on B. By [9], we get $\psi = \tilde{\phi}$. Now, we have the following proposition.

Proposition 4.1. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a Riemannian multiply warped product with $dim(B) = n$, $dim(F_i) = s_i$, $i = 1, 2, \ldots, m$, and let ϕ be a smooth function on B such that $(M, \tilde{g})$ is a gradient Ricci soliton with a potential function $\psi = \tilde{\phi}$. Then we have on B*

$$2\rho\phi - |\nabla\phi|^2 + \Delta\phi + \sum_{i=1}^{m}\frac{s_i}{f_i}\nabla\phi(f_i) = c$$

for some constants ρ and c.

Proof. We have the gradient Ricci soliton equation as

$$\mathrm{Ric} + H^{\psi} = \rho g.$$

Taking the trace of the above equation, we get

$$r + \Delta\psi = \rho k,$$

where

$$k = n + \sum_{i=1}^{m} s_i.$$

For the Riemannian case, Hamilton [11] proved that

$$2\rho\psi - |\nabla\psi|^2 + \Delta\psi = c$$

for some constant c. So, the above equation can be written as

$$2\rho\tilde{\phi} - |\nabla\tilde{\phi}|^2 + \Delta\tilde{\phi} = c. \tag{4.1}$$

Also,

$$\widetilde{\nabla\phi} = \nabla\tilde{\phi},$$

and from Lemma 2.2 we have

$$\Delta\tilde{\phi} = \Delta\phi + \sum_{i=1}^{m} \frac{s_i}{f_i} \nabla\phi(f_i). \tag{4.2}$$

Substituting (4.2) in (4.1), we obtain

$$2\rho\phi - |\nabla\phi|^2 + \Delta\phi + \sum_{i=1}^{m} \frac{s_i}{f_i} \nabla\phi(f_i) = c$$

for some constants ρ and c. □

Proposition 4.2. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a Riemannian multiply warped product, and let ϕ be a smooth function on B such that $(M, \tilde{g})$ is a gradient Ricci soliton with the potential function $\psi = \tilde{\phi}$. Then we have*

$$\mathrm{Ric}_B = \rho g_B - {H_B}^{\phi} + \sum_{i=1}^{m} \frac{s_i}{f_i} H_B^{f_i}$$

and

$$\mathrm{Ric}_{F_i} = \xi_i g_{F_i},$$

where

$$\begin{aligned} \xi_i &= \rho f_i^2 - f_i \nabla\phi(f_i) \\ &\quad + f_i^2 \left(\frac{\Delta_B f_i}{f_i} + (s_i - 1) \frac{\|\nabla_B f_i\|_B^2}{f_i^2} + \sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \right). \end{aligned} \tag{4.3}$$

Proof. For $X, Y \in \chi(B)$, from Lemma 2.3, we have

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_B(X, Y) - \sum_{i=1}^{m} \frac{s_i}{f_i} H_B^{f_i}(X, Y). \tag{4.4}$$

Using (1.1) and the fact ${H_M}^{\psi}(X, Y) = {H_B}^{\phi}(X, Y)$ in (4.4), we obtain

$$\mathrm{Ric}_B(X, Y) = \rho g_B(X, Y) - {H_B}^{\phi}(X, Y) + \sum_{i=1}^{m} \frac{s_i}{f_i} H_B^{f_i}(X, Y).$$

This proves the first part of the proposition.

Since we are assuming $\psi = \tilde{\phi}$, the proof of the second part is the same as in Proposition 3.2. □

Now we can rewrite any $(0,2)$ tensor T on M as a $(1,1)$ tensor by

$$g(T(Z),Y)=T(Z,Y),\quad Y,Z\in\chi(M).$$

Then we have

$$\operatorname{div}(\phi T)=\phi\operatorname{div}T+T(\nabla\phi,.)$$

and

$$\nabla(\phi T)=\phi(\nabla T)+d\phi\otimes T,\quad \phi\in C^{\infty}(M).$$

Hence,

$$\operatorname{div}(H_B^{\phi})=\operatorname{Ric}(\nabla\phi,\cdot)+d(\Delta\phi)$$

and

$$\frac{1}{2}d|\nabla\phi|^2=H_B^{\phi}(\nabla\phi,\cdot).$$

These identities will be used in our next proposition.

Proposition 4.3. *Let (B^n,g) be a pseudo-Riemannian manifold with smooth functions $f_i>0$ and ϕ satisfying*

$$\operatorname{Ric}+H^{\phi}=\rho g_B+\sum_{i=1}^{m}\frac{s_i}{f_i}H_i^f \tag{4.5}$$

and

$$2\rho\phi-|\nabla\phi|^2+\Delta\phi+\frac{s_i}{f_i}\nabla\phi(f_i)=c \tag{4.6}$$

for some constants $c,\rho\in\mathbb{R}$ and for each $i=1,\dots,m$. Then f_i and ϕ satisfy

$$\sum_{i=1}^{m}d(f_i(\Delta f_i)+\rho f_i^2+(s_i-1)|\nabla f_i|^2-f_i\nabla\phi(f_i))=0.$$

Proof. Taking the trace of equation (4.5), we get

$$r=n\rho+\sum_{i=1}^{m}\frac{s_i}{f_i}\Delta f_i-\Delta\phi, \tag{4.7}$$

where r is the scalar curvature of (B,g_B). Thus, from (4.7), we have

$$dr=-\sum_{i=1}^{m}\frac{s_i}{f_i^2}\Delta f_i df_i+\sum_{i=1}^{m}\frac{s_i}{f_i}d(\Delta f_i)-d(\Delta\phi). \tag{4.8}$$

In what follows, we will use the second contracted Bianchi identity, namely:

$$-\frac{1}{2}dr+\operatorname{div}(\operatorname{Ric})=0. \tag{4.9}$$

So, we compute the divergence on both sides of (4.5) to obtain

$$\operatorname{div}(\operatorname{Ric})=\sum_{i=1}^{m}s_i\left(\frac{1}{f_i}\operatorname{div}(H_i^f)-\frac{1}{f_i^2}(H_i^f)(\nabla f_i,\cdot)\right)-\operatorname{div}(H^{\phi}).$$

Therefore,

$$\begin{aligned}\operatorname{div}(\mathrm{Ric}) = \sum_{i=1}^{m} \frac{s_i}{f_i} \mathrm{Ric}(\nabla f_i, \cdot) + \sum_{i=1}^{m} \frac{s_i}{f_i} d(\Delta f_i) \\ - \mathrm{Ric}(\nabla \phi, \cdot) - d(\Delta \phi) - \sum_{i=1}^{m} \frac{s_i}{2f_i^2} d(|\nabla f_i|^2).\end{aligned} \tag{4.10}$$

From (4.5), we have

$$\mathrm{Ric}(\nabla f_i, \cdot) = \rho d f_i + \sum_{i=1}^{m} \frac{s_i}{2f_i} d(|\nabla f_i|^2) - (H^{\phi})(\nabla f_i, \cdot) \tag{4.11}$$

and

$$\mathrm{Ric}(\nabla \phi, \cdot) = \rho d\phi + \sum_{i=1}^{m} \frac{s_i}{f_i} (H_i^f)(\nabla \phi, \cdot) - \frac{1}{2} d(|\nabla \phi|^2). \tag{4.12}$$

Putting (4.11) and (4.12) into (4.10), after a brief simplification we get

$$\begin{aligned}\operatorname{div}(\mathrm{Ric}) = \sum_{i=1}^{m} \frac{s_i}{f_i} \rho d f_i + \sum_{i=1}^{m} \frac{s_i(s_i-1)}{2f_i^2} d(|\nabla f_i|^2) + \sum_{i=1}^{m} \frac{s_i}{f_i} d(\Delta f_i) - \rho d\phi \\ + \frac{1}{2} d(|\nabla \phi|^2) - d(\Delta \phi) - \sum_{i=1}^{m} \frac{s_i}{f_i} [(H^{\phi})(\nabla f_i, \cdot) + (H_i^f)(\nabla \phi, \cdot)].\end{aligned}$$

We know

$$d(\nabla \phi(f)) = (H^{\phi})(\nabla f, .) + (H^f)(\nabla \phi, \cdot).$$

Then

$$\begin{aligned}\operatorname{div}(\mathrm{Ric}) = \sum_{i=1}^{m} \frac{s_i}{f_i} \rho d f_i + \sum_{i=1}^{m} \frac{s_i(s_i-1)}{2f_i^2} d(|\nabla f_i|^2) + \sum_{i=1}^{m} \frac{s_i}{f_i} d(\Delta f_i) - \rho d\phi \\ + \frac{1}{2} d(|\nabla \phi|^2) - d(\Delta \phi) - \sum_{i=1}^{m} \frac{s_i}{f_i} d(\nabla \phi(f)).\end{aligned} \tag{4.13}$$

Substituting (4.8) and (4.13) in (4.9), after a lengthy calculation we have

$$\begin{aligned} d\left(\sum_{i=1}^{m} f_i(\Delta f_i) + \sum_{i=1}^{m} \rho f_i^2 + \sum_{i=1}^{m} (s_i - 1)|\nabla f_i|^2 \right) \\ - 2\sum_{i=1}^{m} f_i d(\nabla \phi(f_i)) - \sum_{i=1}^{m} \frac{f_i^2}{s_i} d(\Delta \phi + 2\rho\phi - |\nabla \phi|^2) = 0.\end{aligned} \tag{4.14}$$

But by our hypothesis, we have

$$2\rho\phi - |\nabla \phi|^2 + \Delta \phi + \frac{s_i}{f_i} \nabla \phi(f_i) = c,$$

which gives

$$-\sum_{i=1}^{m} \frac{f_i^2}{s_i} d(\Delta\phi + 2\rho\phi - |\nabla\phi|^2) - \sum_{i=1}^{m} f_i d(\nabla\phi(f_i)) = -\sum_{i=1}^{m} \nabla\phi(f_i) df_i.$$

So, from (4.14), we get

$$\sum_{i=1}^{m} d(f_i(\Delta f_i) + \rho f_i^2 + (s_i - 1)|\nabla f_i|^2 - f_i \nabla\phi(f_i)) = 0,$$

which completes the proof. □

Now we will derive two main results using these propositions.

Theorem 4.4. *Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a gradient Ricci soliton multiply warped product with $\nabla_B f_i \perp \nabla_B f_k$ for all $i \neq k$. Assume the potential function satisfies $\psi = \tilde{\phi}$. If the soliton is steady or expanding and the warping functions reach both maximum and minimum on the base B, then M is a simply Riemannian product.*

Proof. Let $M = B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m$ be a gradient Ricci soliton with $\mathrm{Ric} + H^{\tilde{\phi}} = \rho g$, where ρ is constant.

Now, from Proposition 4.2, we have $\mathrm{Ric}_{F_i} = \xi_i g_{F_i}$, where

$$\xi_i = f_i^2 \left(\frac{\Delta_B f_i}{f_i} + (s_i - 1) \frac{\|\nabla_B f_i\|_B^2}{f_i^2} + \sum_{k=1, k\neq i}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \right) - f_i \nabla\phi(f_i) + \rho f_i^2.$$

Now we will assume that the functions f_i reach both maximum and minimum on the base B. At the maximum p_i and at the minimum q_i of f_i, by using equation (4.6), we get the following:

$$\xi_i = \rho {f_i}^2(p_i) + f_i(\Delta f_i)(p_i) \leq \rho {f_i}^2(p_i),$$
$$\xi_i = \rho {f_i}^2(q_i) + f_i(\Delta f_i)(q_i) \geq \rho {f_i}^2(q_i),$$

which implies, when $\rho < 0$, that ${f_i}^2(p_i) \leq \rho^{-1}\xi_i \leq {f_i}^2(q_i) \leq \underline{{f_i}^2(p_i)}$ for any $p \in B$. As a consequence, $\xi_i < 0$ and f_i is constant that equals $\sqrt{\rho^{-1}\xi_i}$. Now, if $\rho = 0$, then it follows that $\xi_i = 0$. Putting this into (4.3) gives

$$\tilde{L} f_i = \Delta f_i - \nabla u(f_i) = \frac{1}{f_i}(1 - s_i)|\nabla f_i|^2 \leq 0,$$

where

$$u = \phi - \sum_{\substack{k=1 \\ k\neq i}}^{m} s_k \ln(f_k).$$

Once one is assuming that the minimum is attained, the strong Maximum Principle implies that f_i is constant. Hence M is a simply Riemannian product. □

Remark 4.5. The strong Maximum Principle should be applied to $\tilde{L}$, not to the operator L used in [9], and $\tilde{L}$ is still an elliptic linear differential operator.

Theorem 4.6. *Let (B, g_B) be a Riemannian manifold with smooth functions $f_i > 0$ satisfying $\nabla_B f_i \perp \nabla_B f_k$ for all $i \neq k$, $\{i = 1, 2, \dots, m, i \neq k\}$ and ϕ satisfying*

$$\operatorname{Ric} + H^{\phi} = \rho g + \sum_{i=1}^{m} \frac{s_i}{f_i} H_i^f$$

and

$$2\rho\phi - |\nabla\phi|^2 + \Delta\phi + \sum_{i=1}^{m} \frac{s_i}{f_i} \nabla\phi(f_i) = c$$

for some constants $m, c, \rho \in \mathbb{R}$ with $m \neq 0$.

Let $F_1, F_2, \dots, F_m$ be Riemannian manifolds with metrics g_{F_i} whose Ricci tensors satisfy $\operatorname{Ric}_{F_i} = \xi_i g_{F_i}$, where

$$\xi_i = f_i^2 \left(\frac{\Delta_B f_i}{f_i} + (s_i - 1) \frac{\|\nabla_B f_i\|_B^2}{f_i^2} - f_i \nabla\phi(f_i) + \rho f_i^2 + \sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\nabla_B f_i, \nabla_B f_k)}{f_i f_k} \right).$$

Then $(B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m, \tilde{g})$ is a gradient Ricci soliton multiply warped product.

Proof. From our assumption, we have

$$\operatorname{Ric} + H^{\phi} = \rho g + \sum_{i=1}^{m} \frac{s_i}{f_i} H_i^f$$

and

$$2\rho\phi - |\nabla\phi|^2 + \Delta\phi + \sum_{i=1}^{m} \frac{s_i}{f_i} \nabla\phi(f_i) = c$$

for some constant c.

We also assume that $\nabla_B f_i$ and $\nabla_B f_k$, $\{i = 1, 2, \dots, m, i \neq k\}$ are orthogonal to each other, and from Proposition 4.3 we get that any ξ_i given by (4.5) is constant. Now we consider the Einstein manifolds (F_i, g_{F_i}) with the Ricci tensor $\operatorname{Ric}_{F_i} = \xi_i g_{F_i}$ and the warped product $(B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_m} F_m, \tilde{g})$ with $\tilde{g} = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$.

Let us consider three cases.

Case 1: Let $Y, Z \in \chi(B)$. It follows from the fact

$$H^{\phi}(Y, Z) = H^{\tilde{\phi}}(Y, Z), \quad H^{f}(Y, Z) = H^{\tilde{f}}(Y, Z),$$

from Lemma 2.3, where $\tilde{\phi}, \tilde{f}$ are lifts of ϕ and f, and the hypotheses (4.5), (4.6) of Proposition 4.3 that the fundamental equation $\operatorname{Ric} + H^{\tilde{\phi}} = \rho g$ is satisfied for all $Y, Z \in \chi(B)$.

Case 2: Let $Y \in \chi(B)$ and $V \in \chi(F_i)$. Using $\nabla\tilde{\phi} \in \chi(B)$ and Lemma 2.1, we can easily verify that

$$H^{\tilde{\phi}}(Y, V) = g(D_Y \nabla\tilde{\phi}, V) = 0.$$

So, from part 2 of Lemma 2.3, we can assert that the fundamental equation $\mathrm{Ric} + H^{\tilde{\phi}} = \rho g$ is satisfied.

Case 3: Let $V, W \in \chi(F_i)$. Using Lemma 2.3 and the definition of ξ_i, we have

$$\mathrm{Ric}(V, W) = \xi_i g_{F_i}(V, W) - f_i^2 \left(\frac{\Delta_B f_i}{f_i} + (s_i - 1)\frac{\|\nabla_B f_i\|_B^2}{f_i^2} \right) g_{F_i}(V, W)$$

since $\nabla_B f_i$ is orthogonal to $\nabla_B f_k$ for $\{i = 1, 2, \dots, m, i \neq k\}$. Therefore,

$$\begin{aligned}\mathrm{Ric}(V, W) &= (\xi_i - f_i \Delta_B f_i - (s_i - 1)\|\nabla_B f_i\|_B^2) g_{F_i}(V, W) \\ &= (\rho f_i^2 - f_i \nabla\phi(f_i)) g_{F_i}(V, W) = (\rho - \frac{1}{f_i}\nabla\phi(f_i)) g(V, W).\end{aligned}$$

Again, from (4.4), we get

$$H^{\tilde{\phi}}(V, W) = \frac{\nabla\phi(f_i)}{f_i} g_{F_i}(V, W). \tag{4.15}$$

Combining equations (4.5) and (4.6), we conclude that the fundamental equation is satisfied, which completes the proof. □

Acknowledgment. The authors wish to express their sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

References

[1] C. Wolf, *A mathematical model for the propagation of a hantavirus in structured populations*, Discrete Contin. Dyn. Syst. Ser. B **4** (2004), 1065–1089.

[2] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49.

[3] V. Borges and K. Tenenblat, *Ricci almost solitons on semi-Riemannian warped products*, Mathematische Nachrichten (2022), 295, 1–22.

[4] X. Cao, *Compact gradient shrinking Ricci solitons with positive curvature operator*, J. Geom. Anal. **17** (2007), 425–433.

[5] J. Choi, *Multiply warped products with nonsmooth metrics*, J. Math. Phys. **41**, (2000), 8163–8169.

[6] B. Chow, S.C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci Flow: Techniques and Applications Part I: Geometric Aspects. Mathematical surveys and Monographs*, Amer. Math. Soc., Providence, RI, 2007.

[7] F. Dobarro and B. Ünal, *Curvature of multiply warped products*, J. Geom. Phys. **55** (2005), 75–106.

[8] D. Dumitru, *Special multiply Einstein warped products with an affine connection*, Int. J. Geom. Meth. Modern Phys. **15** (2018), 1850107.

[9] F.E.S. Feitosa, A.A. Freitas Filho, and J.N.V. Gomes, *On the construction of gradient Ricci soliton warped product*, Nonlinear Anal. **161** (2017), 30–43.

[10] R.S. Hamilton, *Three manifold with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[11] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry(Cambridge, MA, 1993), International Press, Cambridge, MA, **2** (1995), 7–136.

[12] T. Ivey, *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. **122** (1994), 241–245.

[13] B.H. Kim, S.D. Lee, J.H. Choi, and Y.O. Lee, *On warped product spaces with a certain Ricci condition*, Bull. Korean Math. Soc. **50**, (2013), 1683–1691.

[14] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York (1983).

[15] S. Pahan, B. Pal, and A. Bhattacharyya, *On Einstein warped products with a quarter-symmetric connection*, Int. J. Geom. Meth. Modern Phys. **14**, (2017), 1750050.

[16] R.S. Pina and M.L. Sousa, *Gradient Ricci solitons with structure of warped product*, Results. Math. **71** (2017), 825–840.

[17] B. Ünal, *Doubly Warped Products*, Ph. D. Thesis, University of Missouri-Columbia, 2000.

[18] B. Ünal, *Multiply Warped Products*, J. Geom. Phys. **34** (2000), 287–301.

Received November 17, 2021, revised January 21, 2023.

Tamalika Dutta,
Department of Mathematics, Raja Rammohun Roy Mahavidyalaya Radhanagar-712406, India,
E-mail: tamalika.bagnan@gmail.com

Sampa Pahan,
Department of Mathematics, Mrinalini Datta Mahavidyapith, Kolkata-700051, India,
E-mail: sampapahan25@gmail.com

X. Chen,
College of Science , China University, Beijing-102249, China,
E-mail: xmchen@cup.edu.cn

Arindam Bhattacharyya,
Department of Mathematics, Jadavpur University, Kolkata-700032, India,
E-mail: bhattachar1968@yahoo.co.in

Про градієнтний солітон Річчі, що є множинно викривленим добутком

Tamalika Dutta, Sampa Pahan, X. Chen, and Arindam Bhattacharyya

Метою роботи є вивчення градієнтного солітону Річчі, що є множинно викривленим добутком. Ми доводимо, що коли многовид є повним, то тоді потенціальна функція залежить лише від бази, а шар повинен бути енштейновим многовидом. Також ми наводимо необхідні та достатні умови для побудови градієнтного солітону Річчі, що є множинно викривленим добутком.

Ключові слова: солітон Річчі, викривлений добуток, множинно викривлений добуток