

Controllability Problems for the Heat Equation with Variable Coefficients on a Half-Axis Controlled by the Neumann Boundary Condition

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In the paper, the problems of controllability and approximate controllability are studied for the control system $w_t = \frac{1}{\rho}(kw_x)_x + \gamma w$, $\left(\sqrt{\frac{k}{\rho}}w_x\right)\Big|_{x=0} = u$, $x > 0$, $t \in (0, T)$, where u is a control, $u \in L^\infty(0, T)$. It is proved that any initial state of the control system is not controllable to the origin except the zero initial state in a given time $T > 0$. However, each initial state of the control system is approximately controllable to any target state in a given time $T > 0$. Due to transformation operator generated by the equation data ρ , k , γ , the main results are obtained from their analogues obtained earlier in the case of constant coefficients ($\rho = k = 1$, $\gamma = 0$). Applying this operator is a focal point of the paper. The results are illustrated by examples.

Key words: heat equation, controllability, approximate controllability

Mathematical Subject Classification 2020: 93B05, 35K05, 35B30

1. Introduction

Controllability problems for the heat equation were studied in a number of papers (see, e.g. [1–5, 10, 11, 14, 15, 17, 19, 20, 24–33]). However, the number of papers dealing with *the heat equation in bounded domains* is much more than the number of papers dealing with *the heat equation in unbounded domains* (see, e.g. [2, 3, 10, 11, 17, 25–27, 30, 31]). In particular, it seems these problems were investigated for *the heat equation with variable coefficients in the principal part of the differential operator on unbounded domains* only in [12].

The paper deals with the controllability problems for the heat equation with variable coefficients on a half-axis controlled by the Neumann boundary condition. Consider the following control system:

$$w_t = \frac{1}{\rho}(kw_x)_x + \gamma w, \quad x \in (0, +\infty), \quad t \in (0, T), \quad (1.1)$$

$$\left(\sqrt{\frac{k}{\rho}}w_x\right)\Big|_{x=0} = u, \quad t \in (0, T), \tag{1.2}$$

$$w(\cdot, 0) = w^0, \quad x \in (0, +\infty). \tag{1.3}$$

Here $T > 0$ is a constant; $\rho, k, \gamma,$ and w^0 are given functions; $u \in L^\infty(0, T)$ is a control. We assume $\rho, k \in C^1[0, +\infty)$ are positive on $[0, +\infty), (\rho k) \in C^2[0, +\infty), (\rho k)'(0) = 0$. Consider the even extensions of ρ, k, γ . Since we reduce control system (1.1)–(1.3) considered in $[0, +\infty) \times [0, T]$ to a control system considered in $\mathbb{R} \times [0, T]$, we denote these extensions by the same symbols $\rho, k, \gamma,$ respectively, and always assume ρ, k, γ are defined in \mathbb{R} throughout the paper. Denote

$$\sigma(x) = \int_0^x \sqrt{\rho(\xi)/k(\xi)} d\xi, \quad x \in \mathbb{R}. \tag{1.4}$$

We assume

$$\sigma(x) \rightarrow +\infty \quad \text{as } x \rightarrow +\infty. \tag{1.5}$$

Put $Q_2(\rho, k) = \sqrt{k/\rho}(Q_1(\rho, k))' + (Q_1(\rho, k))^2, Q_1(\rho, k) = \sqrt{k/\rho}(k\rho)'/(4k\rho)$. We also assume

$$Q_2(\rho, k) - \gamma \in L^\infty(0, +\infty) \cap C^1[0, +\infty) \tag{1.6}$$

and

$$\sqrt{\frac{\rho}{k}}(Q_2(\rho, k) - \gamma) \sigma \in L^1(0, +\infty). \tag{1.7}$$

We consider control system (1.1)–(1.3) in modified Sobolev spaces (see Section 2).

We recall that (1.1)–(1.3) is null-controllable in a given time $T > 0$ if for any initial state w^0 , we can find a control $u \in L^\infty(0, T)$ such that the state of the solution to the control system at $t = T$ satisfies the condition $w(\cdot, T) = 0$. Also, system (1.1)–(1.3) is approximately controllable to any target state in a given time $T > 0$ if for each initial state w^0 and each neighbourhood of a target state w^T there exists a control $u \in L^\infty(0, T)$ such that the end state of the solution to the control system (at $t = T$) belongs to this neighbourhood of w^T .

In [12], controllability problems for the heat equation with variable coefficients on a half-axis controlled by the Dirichlet boundary condition are studied. The general methods applied in the present paper are similar to those from paper [12]. But for the case of the Neumann boundary condition, different spaces and operators are used that caused different technique of proofs of main results.

Theorems 2.6 and 2.7 (see Section 2 below) are the main result of the paper. It is proved that each initial state of the control system is approximately controllable to any target state in a given time $T > 0$, i.e. for each initial state and each neighbourhood of a target state, there exists a control $u \in L^\infty(0, T)$ for which the end state of the solution to the control system (at $t = T$) is in this neighbourhood (Theorem 2.7). In the case of constant coefficients ($\rho = k = 1, \gamma = 0$), the result of this theorem has been obtained earlier in [11]. In the case of variable coefficients, this result is similar to those of papers [6–9] for the wave equation

with variable coefficients on a half-axis controlled either by the Dirichlet or by the Neumann boundary condition. However, the methods for obtaining the results are essentially different because of entirely different nature of the heat and wave equations. They are compared below. In Theorem 2.6, it is proved that any initial state of the control system cannot be targeted to the origin with the help of a control $u \in L^\infty(0, T)$ except the case of the zero initial state; in other words, it is proved that any non-zero initial state is not controllable to the origin. In the case of constant coefficients ($\rho = k = 1, \gamma = 0$), the result of this theorem has been obtained earlier in [11]. The case of constant coefficients and the Dirichlet boundary control was investigated in [10]. In [25], the null-controllability was studied for the heat equation with constant coefficients on a half-axis controlled by the Dirichlet boundary condition with a control of the class L^2 . To this aid, the initial state of the problem was developed into the Fourier series with respect to an orthogonal basis, which can be reduced, in fact, to the orthogonal basis of Hermite functions in $L^2(0, +\infty)$. The null-controllability problem was reduced to a power-moment problem with coefficients determined by the Fourier coefficients of the initial state, and conditions for solvability to this problem were obtained in Theorem 3.1 of [25]. Thus the null-controllability or the lack of the null-controllability can be established by analysing the growth of the Fourier coefficients. In fact, the results on the lack of the null-controllability in [10, 11] were obtained by using Theorem 3.1 of [25]. Note that the result on the lack of the null-controllability seems to contradict to results of paper [20]. But the null-controllability problems are different in [10, 11] and [20]. The difference is in the spaces where the problems were studied. In [20], initial states of exponential growth were admitted, but only square-integrable functions are considered in [10, 11]. The difference causes an essential distinction in the behaviour of solutions to the heat equation in these two cases. Similar difference and its effect was also discussed in [25].

Note that the results for the heat equation on a half-axis significantly differ from those on a bounded interval because of essential difference between the behaviour of differential operators considered in bounded or unbounded domains. In particular, it is well-known that each initial state is controllable to the origin for any $T > 0$ for the heat equation controlled by the boundary condition on a bounded interval (see, e.g. [31]) while each non-zero initial condition is not controllable to the origin for this equation on a half-axis [10, 11, 25].

To study control system (1.1)–(1.3), we use a transformation operator and modified Sobolev spaces, which are associated with the equation data (ρ, k, γ) , introduced and studied in [6–9]. Indeed, these operator and spaces are main tools in the study of this system. Note that applying a similar transformation operator and similar modified Sobolev spaces, which are associated with the equation data (ρ, k, γ) , controllability problems for equation (1.1) controlled by the Dirichlet boundary condition were investigated in [12].

For $s = -1, 0, 1$, the space \mathbb{H}^s is the space of the Sobolev type constructed by analogy with the classical Sobolev space $H^s = H^s(\mathbb{R})$. Namely in the space \mathbb{H}^s , the basic space $L^2(\mathbb{R})$ and the derivative d/dx are replaced by the space $L_\rho^2(\mathbb{R})$ of

functions square-integrable with the weight ρ and the linearly deformed derivative $\mathbb{D}_{\rho k} = \alpha d/dx + \beta$, where α and β are determined by ρ and k , $s = -1, 0, 1$. The subspace of the even functions (distributions, see the definitions on pp. 620, 622) in \mathbb{H}^s (or H^s) is denoted by $\widehat{\mathbb{H}}^s$ (or \widehat{H}^s), $s = -1, 0, 1$. The transformation operator $\widehat{\mathbb{T}} : \widehat{H}^{-1} \rightarrow \widehat{\mathbb{H}}^{-1}$ together with the spaces $\widehat{\mathbb{H}}^s$, $s = -1, 0, 1$, introduced and studied in [6–9], allows to extend the controllability results obtained earlier in the case of constant coefficients ($\rho = k = 1$, $\gamma = 0$) to the case of variable coefficients ρ, k, γ . The definitions of $\widehat{\mathbb{T}}$, $\widehat{\mathbb{H}}^s$, and \widehat{H}^s are given below in Section 2.

The operator $\widehat{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces \widehat{H}^s and $\widehat{\mathbb{H}}^s$. Moreover, it is one-to-one mapping between the set of the solutions to (1.1)–(1.3) with constant coefficients ($\rho = k = 1$, $\gamma = 0$) where $u = u^{110} \in L^\infty(0, T)$ and the set of the solutions to this problem with variable coefficients ρ, k, γ where $u = u^{\rho k \gamma} \in L^\infty(0, T)$ (see below Theorems 3.3 and 3.6). Note that u^{110} and $u^{\rho k \gamma}$ are different generally speaking. The proofs of the main results of the paper are based on Theorems 3.3 and 3.6 proved in Section 3. The control system with variable coefficients ρ, k, γ replicates the controllability properties of the control system with constant coefficients ($\rho = k = 1$, $\gamma = 0$) and vice versa.

The last result also holds true for the wave equation on a half-axis [6–9]. But the proofs are essentially different for the cases of the wave and heat equations. Applying the operator $\widehat{\mathbb{T}}^{-1}$ to a solution to the equation with variable coefficients ρ, k, γ and a control $u = u^{\rho k \gamma} \in L^\infty(0, T)$, we obtain a solution to the equation with the constant coefficients $\rho = k = 1$, $\gamma = 0$ and a control $u = u^{110} \in L^\infty(0, T)$ different from the control $u^{\rho k \gamma}$. To find and to estimate the control u^{110} , we have to solve an integral equation of the form

$$u^{110}(t) = f(t) + \int_0^t P(t - \xi)u^{110}(\xi) d\xi, \quad t \in [0, T]. \tag{1.8}$$

In the case of the wave equation, it has been proved that f and P are bounded on $[0, T]$ [6–9]. Therefore, the integral operator in the right-hand side of (1.8) is of the Hilbert–Schmidt type. Hence, the Fredholm alternative together with the generalized Gronwall theorem can be applied to solve (1.8) in $L^2(0, T)$ and estimate the solution u^{110} in $L^\infty(0, T)$ when we deal with the wave equation [6–9]. In the case of the heat equation, it has been proved that f and $\sqrt{(\cdot)}P$ are bounded on $[0, T]$ (hence, $P(\xi) = O(1/\sqrt{\xi})$ as $\xi \rightarrow 0^+$) [12]. That is why the integral operator in the right-hand side of (1.8) is not of the Hilbert–Schmidt type, and the Fredholm alternative is not applicable in the general case. The Banach fixed-point theorem is also not applicable in general case. That is why the method of successive approximations has been used to construct a solution to (1.8) on $[0, T]$. Then the Banach fixed-point theorem has been applied in L^2 -space on small intervals to prove the uniqueness of the solution [12]. This result is recalled in Lemma 3.5 below.

Since the control system with variable coefficients ρ, k, γ replicates the controllability properties of the control system with constant coefficients ($\rho = k = 1$, $\gamma = 0$), we obtain the controllability properties of the first control system from the controllability properties of the second one by applying the operator $\widehat{\mathbb{T}}$, i.e.

we obtain Theorems 2.6 and 2.7 by applying Theorems 3.3 and 3.6 in Section 2.

The obtained results are illustrated by examples in Section 4.

2. Spaces, operators, and main results

Let us give definitions of the spaces used in the paper.

Let $\Omega = (0, +\infty)$ or $\Omega = \mathbb{R}$. Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions whose support is compact and is contained in Ω . For $\varphi \in L^2_{\text{loc}}(\Omega)$ we consider $\varphi' \in \mathcal{D}'(\Omega)$.

By H^p , $p = 0, 1$, denote the Sobolev spaces:

$$H^0 = L^2(\mathbb{R}), \quad H^1 = \{\varphi \in L^2(\mathbb{R}) \mid \varphi' \in L^2(\mathbb{R})\}$$

with the norm

$$\|\varphi\|^p = \left(\sum_{m=0}^p \binom{p}{m} \left(\|\varphi^{(m)}\|_{L^2(\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in H^p,$$

and $H^{-p} = (H^p)^*$ with the norm associated with the strong topology of the adjoint space. We have $H^0 = L^2(\mathbb{R}) = (H^0)^*$. By $\langle f, \varphi \rangle$, denote the value of a distribution $f \in H^{-p}$ on a test function $\varphi \in H^p$, $p = 0, 1$.

A distribution $f \in H^{-p}$ is said to be even if $\langle f, \varphi(\cdot) \rangle = \langle f, \varphi(-(\cdot)) \rangle$, $\varphi \in H^p$, $p = 0, 1$.

By \widehat{H}^l , denote the subspace of all even distributions in H^l , $l = -1, 0, 1$. It is easy to see that \widehat{H}^l is a closed subspace of H^l , $l = -1, 0, 1$.

Let $\varphi \in L^2_{\text{loc}}(\Omega)$. We define the derivative $\mathbb{D}_{\rho k}$ by the rule

$$\mathbb{D}_{\rho k} \varphi = \sqrt{\frac{k}{\rho}} \varphi' + Q_1(\rho, k) \varphi.$$

If, in addition, $\mathbb{D}_{\rho k} \varphi \in L^2_{\text{loc}}(\Omega)$ and $(\mathbb{D}_{\rho k} \varphi)' \in L^2_{\text{loc}}(\Omega)$ (the derivative $(\cdot)'$ is considered in $\mathcal{D}'(\Omega)$), we can consider $\mathbb{D}_{\rho k}^2 \varphi$. Then $\varphi'' \in \mathcal{D}'(\Omega)$ and

$$\mathbb{D}_{\rho k}^2 \varphi = \frac{1}{\rho} (k\varphi')' + Q_2(\rho, k) \varphi.$$

Obviously, $\mathbb{D}_{\rho k}^m \varphi = \varphi^{(m)}$ if $\rho = k = 1$, $m = 0, 1$.

Denote

$$L^2_{\rho}(\Omega) = \{f \in L^2_{\text{loc}}(\Omega) \mid \sqrt{\rho} f \in L^2(\Omega)\}$$

with the norm

$$\|f\|_{L^2_{\rho}(\Omega)} = \|\sqrt{\rho} f\|_{L^2(\Omega)} = \left(\int_{\Omega} |f(x)|^2 \rho(x) dx \right)^{1/2}, \quad f \in L^2_{\rho}(\Omega).$$

For $p = 0, 1$, consider also the spaces \mathring{H}^p :

$$\mathring{H}^0 = L^2_{\rho}(0, +\infty), \quad \mathring{H}^1 = \{\varphi \in L^2_{\rho}(0, +\infty) \mid \mathbb{D}_{\rho k} \varphi \in L^2_{\rho}(0, +\infty) \text{ and } \varphi(0^+) = 0\}$$

with the norm

$$\|\varphi\|^{p_0} = \left(\sum_{m=0}^p \binom{p}{m} \left(\|\mathbb{D}_{\rho k}^m \varphi\|_{L^2_\rho(0,+\infty)} \right)^2 \right)^{1/2}, \quad \varphi \in \mathring{\mathbb{H}}^p,$$

and the dual space $\mathring{\mathbb{H}}^{-p} = \left(\mathring{\mathbb{H}}^p\right)^*$ with the norm associated with the strong topology of the adjoint space. Evidently, $\mathring{\mathbb{H}}^0 = L^2_\rho(0,+\infty) = \left(\mathring{\mathbb{H}}^0\right)^*$. By $\langle\langle g, \varphi \rangle\rangle^\circ$, denote the value of a distribution $g \in \mathring{\mathbb{H}}^{-p}$ on a test function $\varphi \in \mathring{\mathbb{H}}^p$, $p = 0, 1$. In particular, we have

$$\langle\langle g, \varphi \rangle\rangle^\circ = \langle g, \varphi \rangle_{L^2_\rho(0,+\infty)} = \int_0^\infty g(x)\varphi(x)\rho(x) dx, \quad g \in \mathring{\mathbb{H}}^0, \varphi \in \mathring{\mathbb{H}}^0.$$

Put

$$\langle\langle \mathbb{D}_{\rho k} f, \varphi \rangle\rangle^\circ = - \langle\langle f, \mathbb{D}_{\rho k} \varphi \rangle\rangle^\circ, \quad f \in \mathring{\mathbb{H}}^0, \varphi \in \mathring{\mathbb{H}}^1.$$

For $p = 0, 1$, consider also the modified Sobolev spaces \mathbb{H}^p introduced and studied in [7–9]:

$$\mathbb{H}^0 = L^2_\rho(\mathbb{R}), \quad \mathbb{H}^1 = \{ \varphi \in L^2_\rho(\mathbb{R}) \mid \mathbb{D}_{\rho k} \varphi \in L^2_\rho(\mathbb{R}) \}$$

with the norm

$$\|\varphi\|^p = \left(\sum_{m=0}^p \binom{p}{m} \left(\|\mathbb{D}_{\rho k}^m \varphi\|_{L^2_\rho(\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in \mathbb{H}^p, \quad p = 0, 1,$$

and the dual space $\mathbb{H}^{-p} = \left(\mathbb{H}^p\right)^*$ with the norm associated with the strong topology of the adjoint space. By $\langle\langle f, \varphi \rangle\rangle$, denote the value of a distribution $f \in \mathbb{H}^{-p}$ on a test function $\varphi \in \mathbb{H}^p$, $p = 0, 1$. Evidently, $\mathbb{H}^0 = L^2_\rho(\mathbb{R}) = \left(\mathbb{H}^0\right)^*$ and

$$\langle\langle f, \varphi \rangle\rangle = \langle f, \varphi \rangle_{L^2_\rho(\mathbb{R})} = \int_{-\infty}^\infty f(x)\varphi(x)\rho(x) dx, \quad f \in \mathbb{H}^0, \varphi \in \mathbb{H}^0.$$

Put

$$\langle\langle \mathbb{D}_{\rho k} f, \varphi \rangle\rangle = - \langle\langle f, \mathbb{D}_{\rho k} \varphi \rangle\rangle, \quad f \in \mathbb{H}^0, \varphi \in \mathbb{H}^1.$$

For $\rho = k = 1$, we have $\mathbb{H}^m = H^m$, $m = -1, 0, 1$. In [7], it has been proved that $\mathbb{H}^m \subset \mathbb{H}^n$ is dense continuous embedding, $-1 \leq n \leq m \leq 1$, and $\mathcal{D} \subset \mathbb{H}^p \subset \mathbb{H}^{-p} \subset \mathcal{D}'$ are dense continuous embeddings, $p = 0, 1$, where $\mathcal{D} = \mathcal{D}(\mathbb{R})$. However, the relation between the Schwartz space \mathcal{S} and \mathbb{H}^p essentially depends on ρ and k . For example, if $\rho = k$ then

$$\varphi \in \mathbb{H}^p \Leftrightarrow \sqrt{\rho} \varphi \in H^p, \quad p = -1, 0, 1.$$

If $\rho(x) = k(x) = \cosh x$, $x \in \mathbb{R}$, then

$$\mathcal{S} \not\subset \mathbb{H}^p \quad \text{and} \quad \mathbb{H}^{-p} \not\subset \mathcal{S}', \quad p = 0, 1.$$

If $\rho(x) = k(x) = \operatorname{sech} x$, $x \in \mathbb{R}$, then

$$\mathcal{S} \subset \mathbb{H}^p \quad \text{and} \quad \mathbb{H}^{-p} \subset \mathcal{S}', \quad p = 0, 1.$$

A distribution $f \in \mathbb{H}^{-p}$ is said to be even if $\langle\langle f, \varphi(\cdot) \rangle\rangle = \langle\langle f, \varphi(-\cdot) \rangle\rangle$, $\varphi \in \mathbb{H}^p$, $p = 0, 1$.

By $\widehat{\mathbb{H}}^s$, denote the subspace of all even distributions (or functions) in \mathbb{H}^s , $s = -1, 0, 1$. The even extension of a function from $\mathring{\mathbb{H}}^s$ belongs to $\widehat{\mathbb{H}}^s$, $s = 0, 1$ (see [9]). The restriction of a function from $\widehat{\mathbb{H}}^0$ to $[0, +\infty)$ belongs to $\mathring{\mathbb{H}}^0$. However, there exist functions from $\widehat{\mathbb{H}}^1$ whose restrictions do not belong to $\mathring{\mathbb{H}}^1$. Therefore, there exist distributions from $\mathring{\mathbb{H}}^{-1}$ which cannot be extended to the space $\widehat{\mathbb{H}}^1$. But due to the following important theorem proved in [8, Theorem 3.12], the distribution generated by the derivative $\mathbb{D}_{\rho k}^2 f_+ \in \mathring{\mathbb{H}}^{-1}$ of a function $f_+ \in \mathring{\mathbb{H}}^1$ can be extended to the space $\widehat{\mathbb{H}}^1$.

Theorem 2.1. *Let $f_+ \in \mathring{\mathbb{H}}^1$, $\varphi \in \widehat{\mathbb{H}}^1$ and f be the even extension of f_+ . If $(\mathbb{D}_{\rho k} f_+)(0^+) \in \mathbb{R}$, then the distribution $\mathbb{D}_{\rho k}^2 f_+ \in \mathring{\mathbb{H}}^{-1}$ can be extended to the even distribution $F \in \widehat{\mathbb{H}}^{-1}$ such that*

$$\langle\langle F, \varphi \rangle\rangle = \langle\langle \mathbb{D}_{\rho k}^2 f, \varphi \rangle\rangle + 2\sqrt{(\rho k)(0)} (\mathbb{D}_{\rho k} f_+)(0^+) \varphi(0).$$

Put

$$q = Q_2(\rho, k) - \gamma. \quad (2.1)$$

Due to (1.6), $q \in L^\infty(0, +\infty) \cap C^1[0, +\infty)$. Note that q is defined in \mathbb{R} and $q \in C^1(-\infty, 0] \cup C^1[0, +\infty)$, but q' may have a jump at $x = 0$.

We will use the transformation operator $\widehat{\mathbb{T}} = \mathbf{S}\widehat{\mathbb{T}}_r : \widehat{H}^{-1} \rightarrow \widehat{\mathbb{H}}^{-1}$ to investigate controllability problems for system (1.1)–(1.3). The operators \mathbf{S} and $\widehat{\mathbb{T}}_r$ have been introduced and studied in [8, 9]. A description and some properties of the operators \mathbf{S} and $\widehat{\mathbb{T}}_r$ are given in Section 3.

Theorem 2.2 ([8, 9]). *The following assertions hold.*

- (i) *The operator $\widehat{\mathbb{T}}$ is an isomorphism of \widehat{H}^m and $\widehat{\mathbb{H}}^m$, $m = -1, 0, 1$.*
- (ii) *$\widehat{\mathbb{T}}\delta = \sqrt[4]{(\rho k)(0)}\delta$.*
- (iii) *If $g \in \widehat{H}^1$ and $g'(0^+) \in \mathbb{R}$, then $(\mathbb{D}_{\rho k} \widehat{\mathbb{T}}g)(0^+) \in \mathbb{R}$ and*

$$(\mathbb{D}_{\rho k}^2 - q) \widehat{\mathbb{T}}g - 2\sqrt{(\rho k)(0)} (\mathbb{D}_{\rho k} \widehat{\mathbb{T}}g)(0^+) \delta = \widehat{\mathbb{T}} \left(\frac{d^2}{d\xi^2} g - 2g'(0^+) \delta \right).$$

- (iv) *If $f \in \widehat{\mathbb{H}}^1$ and $(\mathbb{D}_{\rho k} f)(0^+) \in \mathbb{R}$, then $(\widehat{\mathbb{T}}^{-1} f)'(0^+) \in \mathbb{R}$ and*

$$\begin{aligned} \frac{d^2}{d\xi^2} \widehat{\mathbb{T}}^{-1} f - 2(\widehat{\mathbb{T}}^{-1} f)'(0^+) \delta \\ = \widehat{\mathbb{T}}^{-1} \left((\mathbb{D}_{\rho k}^2 - q) f - 2\sqrt{(\rho k)(0)} (\mathbb{D}_{\rho k} f)(0^+) \delta \right). \end{aligned}$$

Here δ is the Dirac distribution at the point $x_0 = 0$.

2.1. Main results. Consider control system (1.1)–(1.3). We suppose that $(\frac{d}{dt})^p w : [0, T] \rightarrow \mathring{\mathbb{H}}^{1-2p}$, $p = 0, 1$; $w^0 \in \mathring{\mathbb{H}}^1$. One can easily see that equation (1.1) can be rewritten in the form

$$w_t = \mathbb{D}_{\rho k}^2 w - qw, \quad t \in (0, T), \tag{2.2}$$

and condition (1.2) is equivalent to the condition

$$(\mathbb{D}_{\rho k} w)(0, \cdot) = u, \quad t \in (0, T). \tag{2.3}$$

Let $w^T \in \mathring{\mathbb{H}}^1$. Consider the steering condition for system (1.1)–(1.3)

$$w(\cdot, T) = w^T, \quad x \in (0, +\infty). \tag{2.4}$$

Let $w(\cdot, t), w^0 \in \mathring{\mathbb{H}}^1$ and let $W(\cdot, t), W^0$ be their even extensions with respect to x , respectively, $t \in [0, T]$. Let q be defined by (2.1). If w is a solution to control system (1.1)–(1.3), then using Theorem 2.1 and taking into account (2.2) and (2.3), we conclude that W is a solution to the system

$$W_t = \mathbb{D}_{\rho k}^2 W - qW - 2\sqrt{(\rho k)(0)}u\delta, \quad \text{in } \mathbb{R} \times (0, T), \tag{2.5}$$

$$W(\cdot, 0) = W^0, \quad \text{in } \mathbb{R}, \tag{2.6}$$

where $(\frac{d}{dt})^l W : [0, T] \rightarrow \widehat{\mathbb{H}}^{1-2l}$, $l = 0, 1$, $W^0 \in \widehat{\mathbb{H}}^1$, δ is the Dirac distribution with respect to x . Let $W(\cdot, t), W^0 \in \widehat{\mathbb{H}}^1$ and let $w(\cdot, t), w^0$ be their restrictions to $(0, +\infty)$ with respect to x , respectively, $t \in [0, T]$. If W is a solution to control system (2.5), (2.6), then due to Corollary 3.4 (see Section 3 below),

$$(\mathbb{D}_{\rho k} w)(0, \cdot) = (\mathbb{D}_{\rho k} W)(0^+, \cdot) = u \quad \text{a.e. on } (0, T). \tag{2.7}$$

Hence, w is a solution to control system (1.1)–(1.3).

Let $w^T \in \mathring{\mathbb{H}}^1$ and let $W^T \in \widehat{\mathbb{H}}^1$ be its even extension with respect to x . It is easy to see that $w(\cdot, T) = w^T$ iff $W(\cdot, T) = W^T$.

Thus, control systems (1.1)–(1.3) and (2.5), (2.6) are equivalent. Taking into account this equivalence, we will further consider system (2.5), (2.6).

Let $T > 0$, $W^0 \in \widehat{\mathbb{H}}^1$. By $\mathcal{R}_T^{\rho k \gamma}(W^0)$, denote the set of all states $W^T \in \widehat{\mathbb{H}}^1$ for which there exists a control $u^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution W to (2.5), (2.6) with $u = u^{\rho k \gamma}$ and $W(\cdot, T) = W^T$.

Definition 2.3. A state $W^0 \in \widehat{\mathbb{H}}^1$ is said to be controllable to a state $W^T \in \widehat{\mathbb{H}}^1$ with respect to system (2.5), (2.6) in a given time $T > 0$ if $W^T \in \mathcal{R}_T^{\rho k \gamma}(W^0)$.

In other words, a state $W^0 \in \widehat{\mathbb{H}}^1$ is controllable to a state $W^T \in \widehat{\mathbb{H}}^1$ with respect to system (2.5), (2.6) in a given time $T > 0$ if there exists a control $u^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution W to (2.5), (2.6) with $u = u^{\rho k \gamma}$ and $W(\cdot, T) = W^T$.

Definition 2.4. A state $W^0 \in \widehat{\mathbb{H}}^1$ is said to be approximately controllable to a state $W^T \in \widehat{\mathbb{H}}^1$ with respect to system (2.5), (2.6) in a given time $T > 0$ if $W^T \in \overline{\mathcal{R}_T^{\rho k \gamma}(W^0)}$, where the closure is considered in the space $\widehat{\mathbb{H}}^1$.

In other words, a state $W^0 \in \widehat{\mathbb{H}}^1$ is approximately controllable to a state $W^T \in \widehat{\mathbb{H}}^1$ with respect to system (2.5), (2.6) in a given time $T > 0$ if for each $\varepsilon > 0$, there exists a control $u_\varepsilon^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution W_ε to (2.5), (2.6) with $u = u_\varepsilon^{\rho k \gamma}$ and $\|W_\varepsilon(\cdot, T) - W^T\|^1 < \varepsilon$.

Thus, the main goal of the paper is to investigate whether the state W^0 is controllable (approximately controllable) to a target state W^T with respect to system (2.5), (2.6) in a given time T .

To this aid, consider the control system with the simplest heat operator (system (2.5), (2.6) with $\rho = k = 1, \gamma = 0$):

$$Z_t = Z_{\xi\xi} - 2u\delta, \quad \text{in } \mathbb{R} \times (0, T), \quad (2.8)$$

$$Z(\cdot, 0) = Z^0, \quad \text{in } \mathbb{R}, \quad (2.9)$$

where $u \in L^\infty(0, T)$ is a control, $u = u^{110}$, $(\frac{d}{dt})^l Z : [0, T] \rightarrow \widehat{H}^{1-2l}$, $l = 0, 1$, $Z^0 \in \widehat{H}^1$. Let $Z^T \in \widehat{H}^1$. Consider also the steering condition for this system:

$$Z(\cdot, T) = Z^T, \quad \text{in } \mathbb{R}.$$

Control system (2.8), (2.9) has been investigated in [11]. In particular, it has been proved therein that

$$Z_x(0^+, \cdot) = u, \quad \text{a.e. on } (0, T). \quad (2.10)$$

Using Theorems 3.3 and 3.6 (see Section 3 below), we obtain the following theorem.

Theorem 2.5. *Let $T > 0$, $W^0 \in \widehat{\mathbb{H}}^1$, $W^T \in \widehat{\mathbb{H}}^1$, $Z^0 = \widehat{\mathbb{T}}^{-1}W^0$, $Z^T = \widehat{\mathbb{T}}^{-1}W^T$. Then*

- (i) $\mathcal{R}_T^{\rho k \gamma}(W^0) = \widehat{\mathbb{T}}(\mathcal{R}_T^{110}(Z^0))$.
- (ii) *A state Z^0 is controllable to a state Z^T with respect to system (2.8), (2.9) in a time T iff a state W^0 is controllable to a state W^T with respect to system (2.5), (2.6) in this time T .*
- (iii) *A state Z^0 is approximately controllable to a state Z^T with respect to system (2.8), (2.9) in a time T iff a state W^0 is approximately controllable to a state W^T with respect to system (2.5), (2.6) in this time T .*

Thus, control system (2.5), (2.6) with a general heat operator replicates the controllability properties of control system (2.8), (2.9) with the simplest heat operator and vice versa.

The main results of the paper are the following two theorems.

Theorem 2.6. *If a state $W^0 \in \widehat{\mathbb{H}}^1$ is controllable to the origin with respect to system (2.5), (2.6) in a time $T > 0$, then $W^0 = 0$.*

The lack of null-controllability means the lack of controls of the class $L^\infty(0, T)$ for which the solution W to system (2.5), (2.6) hits the origin at $t = T$ (i.e. satisfies the condition $W(\cdot, T) = 0$) except the case of the zero initial condition ($W^0 = 0$).

Theorem 2.7. *Each state $W^0 \in \widehat{\mathbb{H}}^1$ is approximately controllable to any target state $W^T \in \widehat{\mathbb{H}}^1$ with respect to system (2.5), (2.6) in a given time $T > 0$.*

In other words, Theorem 2.7 asserts that for each initial state $W^0 \in \widehat{\mathbb{H}}^1$ and each neighbourhood of a target state $W^T \in \widehat{\mathbb{H}}^1$, there exists a control $u \in L^\infty(0, T)$ such that the end state $W(\cdot, T)$ of the solution W to control system (2.5), (2.6) belongs to the neighbourhood of W^T .

In the case $\rho = k = 1, \gamma = 0$ these theorems have been proved in [11]. By using Theorem 2.5, we obtain Theorems 2.6 and 2.7.

Taking into account the algorithm given in [11, Section 7], one can construct piecewise constant controls solving the approximate controllability problem for system (2.8), (2.9). Hence, using Theorem 3.3, one can obtain controls solving the approximate controllability problem for system (2.5), (2.6) (see Section 3 below).

3. The transformation operator $\widehat{\mathbb{T}}$ and its application to a control system

In this section, we recall some properties of the operator $\widehat{\mathbb{T}}$ and apply it to control system (2.5), (2.6). We have $\widehat{\mathbb{T}} = \mathbf{S}\widehat{\mathbb{T}}_r : \widehat{H}^{-1} \rightarrow \widehat{\mathbb{H}}^{-1}$.

The operator $\mathbf{S} : H^{-1} \rightarrow \mathbb{H}^{-1}$ has been introduced and studied in [8, 9].

Theorem 3.1 ([8, 9]). *The following assertions hold.*

- (i) *The operator \mathbf{S} is an isometric isomorphism of H^m and \mathbb{H}^m , $m = -1, 0, 1$;*
- (ii) $\mathbb{D}_{\rho k} \mathbf{S}\psi = \mathbf{S} \frac{d}{d\lambda} \psi, \psi \in H^m, m = 0, 1$;
- (iii) $\langle\langle f, \varphi \rangle\rangle = \langle \mathbf{S}^{-1} f, \mathbf{S}^{-1} \varphi \rangle, f \in \mathbb{H}^{-m}, \varphi \in \mathbb{H}^m, m = 0, 1$;
- (iv) $\mathbf{S}\delta = \sqrt[4]{(\rho k)(0)}\delta$.

In particular, we have

$$\mathbf{S}\psi = \frac{\psi \circ \sigma}{\sqrt[4]{\rho k}}, \quad \psi \in H^0, \quad \text{and} \quad \mathbf{S}^{-1}\varphi = \left(\sqrt[4]{\rho k} \varphi \right) \circ \sigma^{-1}, \quad \varphi \in \mathbb{H}^0,$$

where $\psi \circ \sigma = \psi(\sigma(x))$, σ is defined by (1.4). It follows from (1.4), (1.5) that σ is an odd increasing invertible function and $\sigma(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$.

Put

$$r(\lambda) = (q \circ \sigma^{-1})(\lambda) = ((Q_2(\rho, k) - \gamma) \circ \sigma^{-1})(\lambda), \quad \lambda \in [0, +\infty). \tag{3.1}$$

Due to (1.6) and (1.7), we have

$$r \in L^\infty(0, +\infty) \cap C^1[0, +\infty) \quad \text{and} \quad \lambda r \in L^1(0, +\infty). \tag{3.2}$$

Consider the operator $\widehat{\mathbf{T}}_r : \widehat{H}^{-1} \rightarrow \widehat{H}^{-1}$. This operator is the extension to \widehat{H}^{-1} of the well-known transformation operator of the Sturm–Liouville problem (see, e.g. [23, Chap. 3]). The complete description of the extension and its application to the wave equation with variable coefficients have been given in [8, 9, 21].

Theorem 3.2 ([8, 9]). *The following assertions hold.*

- (i) *The operator $\widehat{\mathbf{T}}_r$ is an automorphism of \widehat{H}^m , $m = -1, 0, 1$.*
(ii) *If $g \in \widehat{H}^1$ and $g'(0^+) \in \mathbb{R}$, then $(\widehat{\mathbf{T}}_r g)'(0^+) \in \mathbb{R}$ and*

$$\left(\frac{d^2}{d\lambda^2} - r \right) \widehat{\mathbf{T}}_r g - 2 (\widehat{\mathbf{T}}_r g)'(0^+) \delta = \widehat{\mathbf{T}}_r \left(\frac{d^2}{d\xi^2} g - 2g'(0^+) \delta \right).$$

- (iii) *If $f \in \widehat{H}^1$ and $f'(0^+) \in \mathbb{R}$, then $(\widehat{\mathbf{T}}_r^{-1} f)'(0^+) \in \mathbb{R}$ and*

$$\frac{d^2}{d\xi^2} \widehat{\mathbf{T}}_r^{-1} f - 2 (\widehat{\mathbf{T}}_r^{-1} f)'(0^+) \delta = \widehat{\mathbf{T}}_r^{-1} \left(\left(\frac{d^2}{d\lambda^2} - r \right) f - 2f'(0^+) \delta \right).$$

- (iv) $\widehat{\mathbf{T}}_r \delta = \delta$.

In particular, we have

$$\begin{aligned} (\widehat{\mathbf{T}}_r g)(\lambda) &= g(\lambda) + \int_{|\lambda|}^{\infty} K(|\lambda|, \xi) g(\xi) d\xi, & \lambda \in \mathbb{R}, g \in \widehat{H}^0, \\ (\widehat{\mathbf{T}}_r^{-1} f)(\xi) &= f(\xi) + \int_{|\xi|}^{\infty} L(|\xi|, \lambda) f(\lambda) d\lambda, & \xi \in \mathbb{R}, f \in \widehat{H}^0, \end{aligned}$$

where, according to [23, Chap. 3], the kernel $K \in C^2(\Omega)$ is a unique solution to the system

$$\begin{cases} K_{y_1 y_1} - K_{y_2 y_2} = r(y_1) K, & \text{in } \Omega, \\ K(y_1, y_1) = \frac{1}{2} \int_{y_1}^{\infty} r(\xi) d\xi, & y_1 > 0, \\ \lim_{y_1 + y_2 \rightarrow \infty} K_{y_1}(y_1, y_2) = \lim_{y_1 + y_2 \rightarrow \infty} K_{y_2}(y_1, y_2) = 0, & \text{in } \Omega, \end{cases} \quad (3.3)$$

$\Omega = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > y_1 > 0\}$, and the kernel $L \in C^2(\Omega)$ is determined by the following equation

$$L(y_1, y_2) + K(y_1, y_2) + \int_{y_1}^{y_2} L(y_1, \xi) K(\xi, y_2) d\xi = 0, \quad \text{in } \Omega. \quad (3.4)$$

We also need the following estimates proved in [23, Chap. 3]:

$$|K(y_1, y_2)| \leq M_0 \sigma_0 \left(\frac{y_1 + y_2}{2} \right), \quad \text{in } \Omega, \quad (3.5)$$

$$|K_{y_1}(y_1, y_2)| \leq \frac{1}{4} \left| r \left(\frac{y_1 + y_2}{2} \right) \right| + M_1 \sigma_0 \left(\frac{y_1 + y_2}{2} \right), \quad \text{in } \Omega, \quad (3.6)$$

where $M_0 > 0$, $M_1 > 0$ are constants, and

$$\sigma_0(x) = \int_x^\infty |r(\xi)|d\xi, \quad x > 0. \tag{3.7}$$

In the following theorems, the application of the transformation operator $\widehat{\mathbb{T}}$ to a control system is considered.

Theorem 3.3. *Let Z be a solution to (2.8), (2.9) with $u = u^{110}$, where $u^{110} \in L^\infty(0, T)$, $Z^0 \in \widehat{H}^1$. Let $W(\cdot, t) = (\widehat{\mathbb{T}}Z)(\cdot, t)$, $t \in [0, T]$, $W^0 = \widehat{\mathbb{T}}Z^0$. Then W is a solution to system (2.5), (2.6) with the control $u = u^{\rho k \gamma}$,*

$$u^{\rho k \gamma}(t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(u^{110}(t) + \int_0^\infty K_{y_1}(0, x)Z(x, t)dx - \frac{1}{2}Z(0^+, t) \int_0^\infty r(\xi)d\xi \right), \quad t \in [0, T], \tag{3.8}$$

where K is a solution to (3.3), r is defined by (3.1). Besides, (2.7) holds and

$$\|W(\cdot, t)\|^1 \leq E_0 \|Z(\cdot, t)\|^1, \quad t \in [0, T], \tag{3.9}$$

$$\|u^{\rho k \gamma}\|_{L^\infty(0, T)} \leq G_0(T) \|u^{110}\|_{L^\infty(0, T)} + E_1 \|Z^0\|^1, \tag{3.10}$$

where $E_0 > 0$ and $E_1 > 0$ are constants independent of T ,

$$G_0(T) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(1 + (T + 3) \left(\frac{2\sqrt{\sigma_0(0)}}{\sqrt{\pi}} \sqrt{R_0 + M_1^2 R} + \frac{\sigma_0(0)}{\sqrt{2\pi}} \right) \right),$$

M_1 is the constant from (3.6), σ_0 is defined by (3.7), and

$$R = \int_0^\infty \xi |r(\xi)|d\xi, \quad R_0 = \frac{1}{16} \|r\|_{L^\infty(0, +\infty)}. \tag{3.11}$$

Proof. The first part of this theorem is proved similarly to the first part of the corresponding theorem in [8, 9] ([8, Theorem 6.12], [9, Theorem 4.2]). Applying Theorem 2.2 (iii), we obtain the first assertion of the theorem.

Taking into account Theorem 3.1 (ii), (2.10), (3.3), and (3.8) we obtain

$$\begin{aligned} (\mathbb{D}_{\rho k}W)(0^+, t) &= (\mathbb{D}_{\rho k}\widehat{\mathbb{T}}Z)(0^+, t) = \mathbf{S} \left(\widehat{\mathbb{T}}_r Z \right)'(0^+, t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(Z_x(0^+, t) \right. \\ &\quad \left. + \int_0^\infty K_{y_1}(0, x)Z(x, t)dx - K(0, 0)Z(0^+, t) \right) = u^{\rho k \gamma}(t), \quad t \in [0, T]. \end{aligned}$$

Thus, (2.7) is valid.

It follows from Theorem 2.2 (i) that there exists a constant $E_0 > 0$ such that (3.9) holds.

To complete the proof, it remains to prove (3.10). Due to (3.6), we obtain from (3.8)

$$\|u^{\rho k \gamma}\|_{L^\infty(0,T)} \leq \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(\|u^{110}\|_{L^\infty(0,T)} + \frac{\|Z(\cdot, t)\|^0}{\sqrt{2}} \sqrt{\int_0^\infty \left| \frac{1}{4} r \left(\frac{x}{2} \right) + M_1 \sigma_0 \left(\frac{x}{2} \right) \right|^2 dx + \frac{1}{2} \sigma_0(0) |Z(0^+, t)|} \right), \quad t \in [0, T].$$

Since $\|Z(\cdot, t)\|^0 \leq \|Z(\cdot, t)\|^1$ ([16, Chap. 1]), $t \in [0, T]$, we get from here that

$$\|u^{\rho k \gamma}\|_{L^\infty(0,T)} \leq \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(\|u^{110}\|_{L^\infty(0,T)} + \|Z(\cdot, t)\|^1 \sqrt{R_0 \sigma_0(0) + M_1^2 \sigma_0(0) R} + \frac{1}{2} \sigma_0(0) |Z(0^+, t)| \right). \quad (3.12)$$

For $Z \in \widehat{H}^1$ we have $Z(0^+, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty (\mathcal{F}Z)(\sigma, t) d\sigma$, $t \in [0, T]$, where $\mathcal{F} : H^0 \rightarrow H^0$ is the Fourier transform operator, and $\mathcal{F}H^1 = H_1$, $H_1 = \{f \in H^0 \mid (1 + |\sigma|^2)^{1/2} f \in H^0\}$, $\|f\|_1 = \|(1 + |\sigma|^2)^{1/2} f\|_0$, $f \in H_1$ (see, e.g. [16, Chap. 1]). Hence,

$$\begin{aligned} |Z(0^+, t)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^\infty \sqrt{1 + \sigma^2} (\mathcal{F}Z)(\sigma, t) \frac{d\sigma}{\sqrt{1 + \sigma^2}} \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \|(\mathcal{F}Z)(\cdot, t)\|_1 \sqrt{\int_{-\infty}^\infty \frac{d\sigma}{1 + \sigma^2}} = \frac{1}{\sqrt{2}} \|Z(\cdot, t)\|^1, \quad t \in [0, T]. \end{aligned} \quad (3.13)$$

Substituting (3.13) into (3.12), we get

$$\begin{aligned} \|u^{\rho k \gamma}\|_{L^\infty(0,T)} &\leq \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(\|u^{110}\|_{L^\infty(0,T)} + \|Z(\cdot, t)\|^1 \left(\sqrt{\sigma_0(0)} \sqrt{R_0 + M_1^2 R} + \frac{\sigma_0(0)}{2\sqrt{2}} \right) \right), \quad t \in [0, T]. \end{aligned} \quad (3.14)$$

Using formula (13) from [11], we have

$$(\mathcal{F}Z)(\sigma, t) = e^{-\sigma^2 t} (\mathcal{F}Z^0)(\sigma) - \sqrt{\frac{2}{\pi}} \int_0^t e^{-(t-\xi)\sigma^2} u^{110}(\xi) d\xi, \quad \sigma \in \mathbb{R}, \quad t \in [0, T].$$

It is easy to obtain from here

$$\begin{aligned} \|Z(\cdot, t)\|^1 &= \|(\mathcal{F}Z)(\cdot, t)\|_1 \leq \|\mathcal{F}Z^0\|_1 + \frac{2(T+3)}{\sqrt{\pi}} \|u^{110}\|_{L^\infty(0,T)} \\ &= \|Z^0\|^1 + \frac{2(T+3)}{\sqrt{\pi}} \|u^{110}\|_{L^\infty(0,T)}, \quad t \in [0, T]. \end{aligned} \quad (3.15)$$

Substituting (3.15) into (3.14), we get

$$\|u^{\rho k \gamma}\|_{L^\infty(0,T)} \leq \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(\|u^{110}\|_{L^\infty(0,T)} + \left(\|Z^0\|^1 + \frac{2(T+3)}{\sqrt{\pi}} \|u^{110}\|_{L^\infty(0,T)} \right) \times \left(\sqrt{\sigma_0(0)(R_0 + M_1^2 R)} + \frac{\sigma_0(0)}{2\sqrt{2}} \right) \right).$$

The theorem is proved. □

Corollary 3.4. *Let W be a solution to (2.5), (2.6) with $u = u^{\rho k \gamma}$, where $W^0 \in \widehat{\mathbb{H}}^1$ and $u \in L^\infty(0, T)$. Then $(\mathbb{D}_{\rho k} W)(0^+, \cdot) = u$ a.e. on $[0, T]$, i.e. (2.7) holds.*

Proof. Put $Z(\cdot, t) = (\widehat{\mathbb{T}}^{-1} W)(\cdot, t)$, $t \in [0, T]$ and apply the operator $\widehat{\mathbb{T}}^{-1}$ to (2.5). Due to Theorem 2.2 (iv), we obtain

$$Z_t(\cdot, t) = Z_{\xi\xi}(\cdot, t) - 2Z_x(0^+, t)\delta + 2\sqrt{(\rho k)(0)} \left((\mathbb{D}_{\rho k} W)(0^+, t) - u^{\rho k \gamma}(t) \right) \widehat{\mathbb{T}}^{-1}\delta, \quad t \in [0, T].$$

Using Theorem 2.2 (ii), we get

$$Z_t(\cdot, t) = Z_{\xi\xi}(\cdot, t) - 2 \left(Z_x(0^+, t) - \sqrt[4]{(\rho k)(0)} (\mathbb{D}_{\rho k} W)(0^+, t) + \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) \right) \delta, \quad t \in [0, T].$$

Thus, Z is a solution to system (2.8), (2.9) with $Z^0 = \widehat{\mathbb{T}}^{-1} W^0$ and with the control $u = u^{110}$,

$$u^{110}(t) = Z_x(0^+, t) - \sqrt[4]{(\rho k)(0)} (\mathbb{D}_{\rho k} W)(0^+, t) + \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t), \quad t \in [0, T].$$

Due to (2.10), we get $(\mathbb{D}_{\rho k} W)(0^+, t) = u^{\rho k \gamma}(t)$, $t \in [0, T]$. □

To prove the next theorem, we need the following lemma proved in [12].

Lemma 3.5. *Let*

$$|f(t)| \leq N_0 \quad \text{and} \quad |P(t)| \leq \frac{N_1}{\sqrt{\pi t}}, \quad t \in (0, T],$$

where $N_0 > 0$ and $N_1 > 0$ are constants. Then there exists a unique solution $v \in L^\infty(0, T)$ to equation

$$v(t) = f(t) + \int_0^t v(\xi) P(t - \xi) d\xi, \quad t \in [0, T], \tag{3.16}$$

and

$$\|v\|_{L^\infty(0,T)} \leq N_0 \left(1 + 2N_1 \sqrt{\frac{T}{\pi}} e^{N_1^2 T} \right). \tag{3.17}$$

Theorem 3.6. Let W be a solution to (2.5), (2.6) with $u = u^{\rho k \gamma}$, where $u^{\rho k \gamma} \in L^\infty(0, T)$, $W^0 \in \widehat{\mathbb{H}}^1$. Let $Z(\cdot, t) = (\widehat{\mathbb{T}}^{-1}W)(\cdot, t)$, $t \in [0, T]$, $Z^0 = \widehat{\mathbb{T}}^{-1}W^0$. Then Z is a solution to system (2.8), (2.9) with the control $u = u^{110}$,

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)}u^{\rho k \gamma}(t) + \frac{1}{2}\sqrt[4]{(\rho k)(0)}W(0^+, t) \int_0^\infty r(\mu)d\mu \\ + \int_0^\infty L_{y_1}(0, x) (\mathbf{S}^{-1}W)(x, t)dx, \quad t \in [0, T], \quad (3.18)$$

where L is determined by (3.4), r is defined by (3.1). In addition,

$$\|Z(\cdot, t)\|^1 \leq E_2 \|W(\cdot, t)\|^1, \quad t \in [0, T], \quad (3.19)$$

$$\|u^{110}\|_{L^\infty(0, T)} \leq G_1(T) \left(\|u^{\rho k \gamma}\|_{L^\infty(0, T)} + E_3 \|W^0\|^1 \right), \quad (3.20)$$

where $E_2 > 0$ and $E_3 > 0$ are constants independent of T ,

$$G_1(T) = \sqrt[4]{(\rho k)(0)}e^{(\sigma_0(0)+2M_1R)^2T} \left(1 + 2\sqrt{\frac{T}{\pi}}(\sigma_0(0) + 2M_1R) \right),$$

M_1 is the constant from (3.6), σ_0 is defined by (3.7), R is defined by (3.11).

Proof. Applying Theorem 2.2 (see 2.2 (i), 2.2 (iv)), and Corollary 3.4 we obtain (3.18) and (3.19). Let us prove (3.20). From (3.18), it follows that

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)}u^{\rho k \gamma}(t) + \frac{1}{2}(\widehat{\mathbf{T}}_r Z)(0^+, t) \int_0^\infty r(\mu)d\mu \\ + \int_0^\infty L_{y_1}(0, x) (\widehat{\mathbf{T}}_r Z)(x, t)dx = \sqrt[4]{(\rho k)(0)}u^{\rho k \gamma}(t) + \frac{1}{2}Z(0^+, t) \int_0^\infty r(\mu)d\mu \\ + \frac{1}{2} \int_0^\infty r(\mu)d\mu \int_0^\infty K(0, x)Z(x, t)dx + \int_0^\infty L_{y_1}(0, x)Z(x, t)dx \\ + \int_0^\infty Z(x, t) \int_0^x L_{y_1}(0, \xi)K(\xi, x)d\xi dx, \quad t \in [0, T].$$

By differentiating (3.4) with respect to y_1 , we get

$$-K_{y_1}(0, x) = L_{y_1}(0, x) + \frac{1}{2}K(0, x) \int_0^\infty r(\mu)d\mu + \int_0^x L_{y_1}(0, \xi)K(\xi, x)d\xi, \quad x > 0.$$

Therefore,

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)}u^{\rho k \gamma}(t) + \frac{1}{2}Z(0^+, t) \int_0^\infty r(\mu)d\mu \\ - \int_0^\infty K_{y_1}(0, x)Z(x, t)dx, \quad t \in [0, T].$$

(In fact, it is relation (3.8) from Theorem 3.3.)

Using formula (15) (for solution to (2.8), (2.9)) from [11], we have

$$Z(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) - \sqrt{\frac{2}{\pi}} \int_0^t u^{110}(\xi) \frac{e^{-\frac{x^2}{4(t-\xi)}}}{\sqrt{2(t-\xi)}} d\xi, \quad x \in \mathbb{R}, t \in [0, T].$$

Thus, we obtain

$$\begin{aligned} u^{110}(t) &= \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) + \frac{1}{2} \int_0^\infty r(\mu) d\mu \int_{-\infty}^\infty \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} Z^0(x) dx \\ &\quad - \frac{1}{2\sqrt{\pi}} \int_0^\infty r(\mu) d\mu \int_0^t \frac{u^{110}(\xi)}{\sqrt{t-\xi}} d\xi - \int_0^\infty K_{y_1}(0, x) \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) \right) dx \\ &\quad + \sqrt{\frac{2}{\pi}} \int_0^\infty K_{y_1}(0, x) \int_0^t u^{110}(\xi) \frac{e^{-\frac{x^2}{4(t-\xi)}}}{\sqrt{2(t-\xi)}} d\xi dx, \quad t \in [0, T]. \end{aligned} \tag{3.21}$$

Denote

$$\begin{aligned} f(t) &= \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) + \frac{1}{2} \int_0^\infty r(\mu) d\mu \int_{-\infty}^\infty \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} Z^0(x) dx \\ &\quad - \int_0^\infty K_{y_1}(0, x) \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) \right) dx, \quad t \in [0, T], \end{aligned} \tag{3.22}$$

$$P(t) = \frac{1}{\sqrt{\pi t}} \left(\int_0^\infty K_{y_1}(0, x) e^{-\frac{x^2}{4t}} dx - \frac{1}{2} \int_0^\infty r(\mu) d\mu \right), \quad t \in [0, T]. \tag{3.23}$$

Then (3.21) takes the form (3.16). Let us estimate f and P . We have

$$\begin{aligned} \left| \int_{-\infty}^\infty \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} Z^0(x) dx \right| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^\infty e^{-t\sigma^2} (\mathcal{F}Z^0)(\sigma) d\sigma \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \sqrt{1+\sigma^2} |\mathcal{F}Z^0|(\sigma) \frac{d\sigma}{\sqrt{1+\sigma^2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \|\mathcal{F}Z^0\|_1 \sqrt{\int_{-\infty}^\infty \frac{d\sigma}{1+\sigma^2}} = \frac{1}{\sqrt{2}} \|Z^0\|^1, \quad t \in [0, T]. \end{aligned} \tag{3.24}$$

According to (3.6), we get

$$\begin{aligned} \left(\|K_{y_1}(0, \cdot)\|_{L^2(0,+\infty)} \right)^2 &\leq \frac{1}{16} \int_0^\infty r^2\left(\frac{x}{2}\right) dx + M_1^2 \int_0^\infty \sigma_0^2\left(\frac{x}{2}\right) dx \\ &\leq 2\sigma_0(0) (R_0 + M_1^2 R), \end{aligned} \tag{3.25}$$

where R_0 is defined by (3.11). We also have

$$\left\| \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}} * Z^0 \right\|^0 = \left\| e^{-t(\cdot)^2} \mathcal{F}Z^0 \right\|^0 \leq \|\mathcal{F}Z^0\|^0 = \|Z^0\|^0 \leq \|Z^0\|^1, \quad t \in [0, T]. \tag{3.26}$$

Due to (3.25) and (3.26), we obtain

$$\begin{aligned} \left| \int_0^\infty K_{y_1}(0, x) \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) \right) dx \right| &\leq \frac{1}{\sqrt{2}} \|K_{y_1}(0, \cdot)\|_{L^2(0, +\infty)} \left\| \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}} * Z^0 \right\|^0 \\ &\leq \frac{1}{\sqrt{2}} \sqrt{2\sigma_0(0) (R_0 + M_1^2 R)} \|Z^0\|^1 \\ &= \sqrt{\sigma_0(0) (R_0 + M_1^2 R)} \|Z^0\|^1, \quad t \in [0, T]. \end{aligned} \quad (3.27)$$

With regard to (3.24), (3.27), and (3.19), we get

$$\begin{aligned} |f(t)| &\leq \sqrt[4]{(\rho k)(0)} \|u^{\rho k \gamma}\|_{L^\infty(0, T)} + \frac{1}{2\sqrt{2}} \sigma_0(0) \|Z^0\|^1 \\ &\quad + \sqrt{\sigma_0(0) (R_0 + M_1^2 R)} \|Z^0\|^1 \\ &\leq \sqrt[4]{(\rho k)(0)} \|u^{\rho k \gamma}\|_{L^\infty(0, T)} \\ &\quad + E_2 \left(\frac{\sigma_0(0)}{2\sqrt{2}} + \sqrt{\sigma_0(0) (R_0 + M_1^2 R)} \right) \|W^0\|^1, \quad t \in [0, T]. \end{aligned} \quad (3.28)$$

Taking into account (3.6), we obtain

$$\begin{aligned} |P(t)| &\leq \frac{1}{\sqrt{\pi t}} \left(\frac{1}{4} \int_0^\infty \left| r\left(\frac{x}{2}\right) \right| dx + M_1 \int_0^\infty \left| \sigma_0\left(\frac{x}{2}\right) \right| dx + \frac{1}{2} \sigma_0(0) \right) \\ &= \frac{\sigma_0(0) + 2M_1 R}{\sqrt{\pi t}}, \quad t \in [0, T]. \end{aligned} \quad (3.29)$$

Using (3.28) and (3.29) and applying Lemma 3.5, we conclude that there exists a unique solution to equation (3.16) (and, consequently, (3.21)). Moreover, using (3.17), we have

$$\begin{aligned} \|u^{110}\|_{L^\infty(0, T)} &\leq \left(1 + 2(\sigma_0(0) + 2M_1 R) \sqrt{\frac{T}{\pi}} \right) e^{(\sigma_0(0) + 2M_1 R)^2 T} \\ &\quad \times \left(\sqrt[4]{(\rho k)(0)} \|u^{\rho k \gamma}\|_{L^\infty(0, T)} + E_2 \left(\frac{\sigma_0(0)}{2\sqrt{2}} + \sqrt{\sigma_0(0) (R_0 + M_1^2 R)} \right) \|W^0\|^1 \right). \end{aligned}$$

The theorem is proved. \square

Due to Theorems 3.3 and 3.6, the operator $\widehat{\mathbb{T}}$ is not only a continuous one-to-one mapping between the spaces \widehat{H}^s and $\widehat{\mathbb{H}}^s$ (see Theorem 2.2) but also is one-to-one mapping between the set of the solutions to (1.1)–(1.3) with constant coefficients ($\rho = k = 1$, $\gamma = 0$) where $u = u^{110} \in L^\infty(0, T)$ and the set of the solutions to this problem with variable coefficients ρ, k, γ where $u = u^{\rho k \gamma} \in L^\infty(0, T)$. Note that u^{110} and $u^{\rho k \gamma}$ are different generally speaking.

In [11, Section 7], piecewise constant controls $u_{N,l}^{110}$, $N, l \in \mathbb{N}$, solving the approximate controllability problem for system (2.8), (2.9), have been constructed. Moreover, the solution to this system with the controls $u_{N,l}^{110}$ has been obtained:

$$Z_{N,l}(\xi, t) = \frac{e^{-\frac{\xi^2}{4t}}}{2\sqrt{\pi t}} * Z^0(\xi)$$

$$-\sqrt{\frac{2}{\pi}} \int_0^t e^{-\frac{\xi^2}{4\tau}} \frac{u_{N,l}^{110}(t-\tau)}{\sqrt{2\tau}} d\tau, \quad N, l \in \mathbb{N}, \xi \in \mathbb{R}, t \in [0, T].$$

In addition, it has been proved that

$$\|Z^T - Z_{N,l}(\cdot, T)\|^1 \rightarrow 0 \quad \text{as } N \rightarrow \infty \text{ and } l \rightarrow \infty.$$

Therefore, according to Theorem 3.3, the controls

$$u_{N,l}^{\rho k \gamma}(t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(u_{N,l}^{110}(t) + \int_0^\infty K_{y_1}(0, \xi) Z_{N,l}(\xi, t) d\xi - \frac{1}{2} Z_{N,l}(0^+, t) \int_0^\infty r(\xi) d\xi \right), \quad t \in [0, T], N \in \mathbb{N}, l \in \mathbb{N},$$

solve the approximate controllability problem for system (2.5), (2.6) with $u = u_{N,l}^{\rho k \gamma}$. In addition, $u_{N,l}^{\rho k \gamma} \in L^\infty(0, T)$ due to Theorem 3.3. Moreover, $W_{N,l}(\cdot, t) = \widehat{\mathbb{T}} Z_{N,l}(\cdot, t)$, $t \in [0, T]$, and

$$\|W^T - W_{N,l}(\cdot, T)\|^1 \rightarrow 0 \quad \text{as } N \rightarrow \infty \text{ and } l \rightarrow \infty.$$

4. Examples

Example 4.1. Consider system (1.1)–(1.3) with

$$k(x) = \frac{(1 + 2|x|) \cosh x}{3}, \quad \rho(x) = \frac{12 \cosh x}{1 + 2|x|},$$

$$\gamma(x) = \frac{(1 + 2|x|) \tanh |x|}{36} + \frac{(1 + 2|x|)^2}{144} \left(1 + \frac{1}{\cosh^2 x} \right) - \frac{1}{4(1 + 2|x|)^3}, \quad x \in \mathbb{R}.$$

We have

$$Q_2(\rho, k) = \frac{(1 + 2|x|) \tanh |x|}{36} + \frac{(1 + 2|x|)^2}{144} \left(1 + \frac{1}{\cosh^2 x} \right),$$

$$q(x) = Q_2(\rho, k) - \gamma(x) = \frac{1}{4(1 + 2|x|)^3}, \quad x \in \mathbb{R}.$$

Due to (1.4), we get

$$\sigma(x) = \operatorname{sgn} x \ln(1 + 2|x|)^3, \quad x \in \mathbb{R}, \quad \text{and} \quad \sigma^{-1}(\lambda) = \frac{1}{2} \operatorname{sgn} \lambda \left(e^{\frac{|\lambda|}{3}} - 1 \right), \quad \lambda \in \mathbb{R}.$$

Let us consider system (2.8), (2.9) with $Z^0(x) = e^{-\frac{|x|}{2}}$ and with the steering condition $Z^T(x) = e^{-\frac{2|x|-T}{4}}$, $x \in \mathbb{R}$. Evidently,

$$Z(x, t) = e^{-\frac{2|x|-t}{4}}, \quad x \in \mathbb{R}, t \in [0, T],$$

is the unique solution to this system and the state Z^0 is controllable to the state Z^T with respect to system (2.8), (2.9) in the time T with the control

$$u^{110}(t) = -\frac{1}{2}e^{\frac{t}{4}}, \quad t \in [0, T].$$

Now consider system (2.5), (2.6) with the given q . According to Theorem 2.5 (ii), the state $W^0 = \widehat{\mathbb{T}}Z^0$ is controllable to the state $W^T = \widehat{\mathbb{T}}Z^T$ with respect to system (2.5), (2.6) in the time T . Moreover, due to Theorem 3.3, a control $u^{\rho k \gamma}$ solving controllability problem for system (2.5), (2.6) is defined by (3.8).

Let us find W^0 , W^T and $u^{\rho k \gamma}$ explicitly. Due to (3.1),

$$r(\lambda) = q \circ \sigma^{-1} = \frac{1}{4}e^{-\lambda}, \quad \lambda > 0.$$

The kernel of the transformation operator $\widehat{\mathbb{T}}_r$ has been found in [6, 21] for this r . We have

$$K(y_1, y_2) = \frac{e^{-\frac{y_1+y_2}{2}} I_1 \left(\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)} \right)}{4 \sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)}}, \quad y_2 > y_1 > 0, \quad (4.1)$$

where I_n is the modified Bessel function, $n = \overline{0, \infty}$.

It is well-known (see, e.g. [18, 9.6.28]) that

$$(y^{-n} I_n(y))' = y^{-n} I_{n+1}(y) \text{ and } (y^n I_n(y))' = y^n I_{n-1}(y), \quad y > 0, \quad n \in \mathbb{N}. \quad (4.2)$$

Due to the first of these formulae with $n = 1$, we obtain

$$\begin{aligned} K_{y_1}(y_1, y_2) &= -\frac{1}{8} e^{-\frac{y_1+y_2}{2}} \frac{I_1 \left(\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)} \right)}{\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)}} \\ &- \frac{1}{16} e^{-\frac{y_1+y_2}{2}} \frac{I_2 \left(\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)} \right)}{\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)}} \frac{e^{-\frac{y_1}{2}} \left(2e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)}{\sqrt{e^{-\frac{y_1}{2}} \left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}} \right)}}, \quad y_2 > y_1 > 0. \end{aligned}$$

Hence,

$$\begin{aligned} K_{y_1}(0, x) &= -\frac{e^{-\frac{x}{2}} I_1 \left(\sqrt{1 - e^{-\frac{x}{2}}} \right)}{8 \sqrt{1 - e^{-\frac{x}{2}}}} \\ &- \frac{e^{-\frac{x}{2}} I_2 \left(\sqrt{1 - e^{-\frac{x}{2}}} \right)}{16} \left(1 + \frac{1}{1 - e^{-\frac{x}{2}}} \right), \quad x > 0. \end{aligned}$$

Taking into account (3.8), we get

$$u^{\rho k \gamma}(t) = -\frac{e^{\frac{t}{4}}}{8\sqrt{2}} \left(5 + \int_0^\infty e^{-x} \frac{I_1\left(\sqrt{1 - e^{-\frac{x}{2}}}\right)}{\sqrt{1 - e^{-\frac{x}{2}}}} dx + \frac{1}{2} \int_0^\infty e^{-x} I_2\left(\sqrt{1 - e^{-\frac{x}{2}}}\right) \left(1 + \frac{1}{1 - e^{-\frac{x}{2}}}\right) dx \right), \quad t \in [0, T].$$

Substituting y for $\sqrt{1 - e^{-\frac{x}{2}}}$ and then integrating the first integral by parts, we get

$$u^{\rho k \gamma}(t) = -\frac{e^{\frac{t}{4}}}{8\sqrt{2}} \left(5 + 4 \int_0^1 (1 - y^2) I_1(y) dy + 2 \int_0^1 \left(\frac{1}{y} - y^3\right) I_2(y) dy \right) = -\frac{e^{\frac{t}{4}}}{8\sqrt{2}} \left(1 + 8 \int_0^1 y I_0(y) dy + 2 \int_0^1 \frac{I_2(y)}{y} dy - 2 \int_0^1 y^3 I_2(y) dy \right).$$

With regard to (4.2), we obtain

$$u^{\rho k \gamma}(t) = -\frac{e^{\frac{t}{4}}}{8\sqrt{2}} (1 + 8I_1(1) + 2I_1(1) - 1 - 2I_3(1)) = \frac{e^{\frac{t}{4}}}{4\sqrt{2}} (I_3(1) - 5I_1(1)), \quad t \in [0, T]. \tag{4.3}$$

According to the definition of the operator $\widehat{\mathbf{T}}$, we have

$$W(x, t) = (\widehat{\mathbf{S}}\widehat{\mathbf{T}}_r Z)(x, t) = \frac{(\widehat{\mathbf{T}}_r Z(\cdot, t)) (\operatorname{sgn} x \ln(1 + 2|x|)^3)}{\sqrt{2} \cosh x}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Due to the definition of the operator $\widehat{\mathbf{T}}_r$, we obtain

$$(\widehat{\mathbf{T}}_r Z)(\lambda, t) = e^{-\frac{2|\lambda| - t}{4}} + \int_{|\lambda|}^\infty \frac{e^{-\frac{|\lambda| + x}{2}}}{4} \frac{I_1\left(\sqrt{e^{-\frac{|\lambda|}{2}} \left(e^{-\frac{|\lambda|}{2}} - e^{-\frac{x}{2}}\right)}\right)}{\sqrt{e^{-\frac{|\lambda|}{2}} \left(e^{-\frac{|\lambda|}{2}} - e^{-\frac{x}{2}}\right)}} e^{-\frac{2x - t}{4}} dx, \quad \lambda \in \mathbb{R}, \quad t \in [0, T].$$

Replacing $\sqrt{e^{-\frac{|\lambda|}{2}} \left(e^{-\frac{|\lambda|}{2}} - e^{-\frac{x}{2}}\right)}$ by y in the integral, we get

$$(\widehat{\mathbf{T}}_r Z)(\lambda, t) = e^{-\frac{2|\lambda| - t}{4}} + e^{\frac{2|\lambda| + t}{4}} \int_0^{e^{-\frac{|\lambda|}{2}}} (e^{-|\lambda|} - y^2) I_1(y) dy = e^{-\frac{2|\lambda| - t}{4}} + e^{\frac{t}{4}} \left(2I_1\left(e^{-\frac{|\lambda|}{2}}\right) - e^{-\frac{|\lambda|}{2}} \right) = 2e^{\frac{t}{4}} I_1\left(e^{-\frac{|\lambda|}{2}}\right), \quad \lambda \in \mathbb{R}, \quad t \in [0, T]. \tag{4.4}$$

Thus,

$$\begin{aligned} W(x, t) &= e^{\frac{t}{4}} \sqrt{\frac{2}{\cosh x}} I_1 \left(e^{-\frac{1}{2} \ln(1+2|x|)^3} \right) \\ &= e^{\frac{t}{4}} \sqrt{\frac{2}{\cosh x}} I_1 \left(\frac{1}{(1+2|x|)^{3/2}} \right), \quad x \in \mathbb{R}, \quad t \in [0, T]. \end{aligned}$$

Hence,

$$W^0(x) = \sqrt{\frac{2}{\cosh x}} I_1 \left(\frac{1}{(1+2|x|)^{3/2}} \right), \quad x \in \mathbb{R}, \quad (4.5)$$

$$W^T(x) = e^{\frac{T}{4}} \sqrt{\frac{2}{\cosh x}} I_1 \left(\frac{1}{(1+2|x|)^{3/2}} \right), \quad x \in \mathbb{R}. \quad (4.6)$$

Thus, the initial state W^0 defined by (4.5) is controllable to the steering state W^T defined by (4.6) with respect to system (2.5), (2.6) in the time T by the control (4.3).

Example 4.2. Let

$$k(x) = \frac{4+x^2}{3+|x|}, \quad \rho(x) = (4+x^2)(3+|x|), \quad \gamma(x) = \frac{12-|x|^3}{(3+|x|)^3(4+x^2)^2}, \quad x \in \mathbb{R}.$$

Consider approximate controllability problem for system (1.1)–(1.3), (2.4), where $T = 1/2$, $w^0 = 0$, $u = u^{\rho k \gamma}$, and

$$w^T(x) = \frac{1}{\sqrt{4+x^2}} \cosh \frac{x(|x|+6)}{2\sqrt{2T}} e^{-\frac{x^2(|x|+6)^2}{16T} - \frac{1}{4}}, \quad x \in \mathbb{R}.$$

It is easy to see that

$$Q_2(\rho, k) = \frac{12-|x|^3}{(3+|x|)^3(4+x^2)^2}, \quad x \in \mathbb{R}.$$

Therefore, $q(x) = Q_2(\rho, k) - \gamma(x) = 0$ in \mathbb{R} . We obtain

$$\sigma(x) = \frac{1}{2}x(|x|+6), \quad x \in \mathbb{R}, \quad \text{and} \quad \sigma^{-1}(\lambda) = \operatorname{sgn} \lambda \left(\sqrt{2|\lambda|+9} - 3 \right), \quad \lambda \in \mathbb{R}.$$

We have $W^0 = w^0$ and $W^T = w^T$ in \mathbb{R} . Consider control system (2.5), (2.6) with $q = 0$, $W^0 = 0$, and with the steering condition

$$W(x, T) = W^T(x) = \frac{1}{\sqrt{4+x^2}} \cosh \frac{x(|x|+6)}{2} e^{-\frac{x^2(|x|+6)^2}{8} - \frac{1}{4}}, \quad x \in \mathbb{R}.$$

Let us investigate whether the state W^0 is approximately controllable to a target state W^T with respect to system (2.5), (2.6) in the time $T = 1/2$.

According to (3.1), $r = 0$ in \mathbb{R} . Hence, $\widehat{\mathbf{T}}_r = \text{Id}$, and the transformation operator $\widehat{\mathbf{T}}$ takes the form $\widehat{\mathbf{T}} = \mathbf{S}$. Denote

$$\begin{aligned} Z(\cdot, t) &= \left(\widehat{\mathbf{T}}^{-1}W\right)(\cdot, t) = (\mathbf{S}^{-1}W)(\cdot, t), \quad t \in [0, 1/2], \\ Z^0 &= \widehat{\mathbf{T}}^{-1}W^0 = \mathbf{S}^{-1}W^0, \quad Z^T = \widehat{\mathbf{T}}^{-1}W^T = \mathbf{S}^{-1}W^T. \end{aligned}$$

Due to Theorem 3.6, Z is the solution to system (2.8), (2.9) with

$$u = u^{110} = \sqrt[4]{(\rho k)(0)}u^{\rho k \gamma} = 2u^{\rho k \gamma}, \quad Z^0 = 0,$$

and with the steering condition

$$Z(\xi, T) = Z^T(\xi) = \cosh \xi e^{-\frac{\xi^2}{2} - \frac{1}{4}}, \quad \xi \in \mathbb{R}.$$

Controllability problems for this system have been considered in Example 4 in [11]. Controls solving the approximate controllability problem for system (2.8), (2.9) have been found in the form

$$u_{N,l}^{110} = \sum_{p=0}^N U_{p,l}^N, \quad N \in \mathbb{N},$$

where $U_{p,l}^N \in \mathbb{R}$ is a constant, $p = \overline{0, N}$, l depends on N , $N \in \mathbb{N}$. The end states $Z_{N,l}^T$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \exists l \in \mathbb{N} \quad \|Z^T - Z_{N,l}^T\|^1 \leq \varepsilon$$

have been found in the form

$$Z_{N,l}^T(\xi) = -\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}} e^{-\frac{\xi^2}{4\tau}} \frac{u_{N,l}^{110} \left(\frac{1}{2} - \tau\right)}{\sqrt{2\tau}} d\tau, \quad \xi \in \mathbb{R}.$$

Applying Theorem 3.3, we conclude that controls $u_{N,l}^{\rho k \gamma} = \frac{1}{2}u_{N,l}^{110}$ solve the approximate controllability problem for given system (2.5), (2.6). Moreover,

$$W_{N,l}^T(x) = \frac{1}{\sqrt{4+x^2}} Z_{N,l}^T\left(\frac{1}{2}x(|x|+6)\right), \quad x \in \mathbb{R},$$

and for ε , N , and l mentioned above, we get

$$\|W^T - W_{N,l}^T\|^1 \leq E_0\varepsilon,$$

where E_0 is the constant from estimate (3.9). The graphs of $u_{N,l}^{\rho k \gamma}$ and $W_{N,l}^T$ see in Figs. 4.1, 4.2.

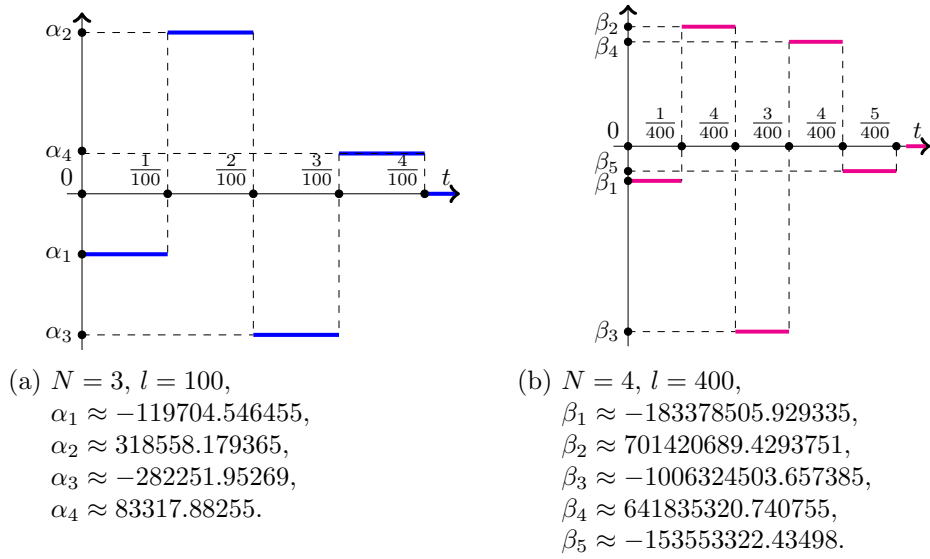


Fig. 4.1: The controls $u_{N,l}^{\rho k \gamma}$.

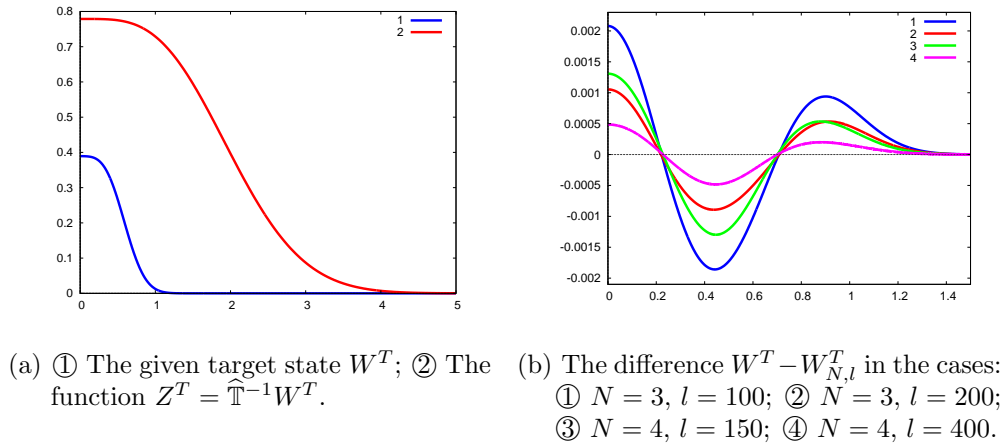


Fig. 4.2: The influence of the controls $u = u_{N,l}^{\rho k \gamma}$ on the end state $W_{N,l}^T$ of the solution to (2.5), (2.6).

Acknowledgments. The authors are partially supported by “Pauli Ukraine Project”, funded in the WPI Thematic Program “Quantum equations and experiments” (2021/2022). The second author is partially supported by the National Academy of Sciences of Ukraine (Grant No. 0122U111111).

The authors are very grateful to the anonymous referee for his/her valuable comments and remarks.

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**Проблеми керованості для рівняння
теплопровідності зі змінними коефіцієнтами на
півосі, керованого крайовою умовою Ноймана**

Larissa Fardigola and Kateryna Khalina

У роботі досліджено проблеми керованості та наближеної керованості для керованої системи $w_t = \frac{1}{\rho} (kw_x)_x + \gamma w$, $\left(\sqrt{\frac{k}{\rho}} w_x\right)\Big|_{x=0} = u$, $x > 0$, $t \in (0, T)$, де u є керуванням, $u \in L^\infty(0, T)$. Доведено, що жодний початковий стан (крім нульового) не може бути скерованим до нуля за заданий час $T > 0$. Проте, кожний початковий стан керованої системи є наближено керованим у будь-який цільовий стан за заданий час $T > 0$. Завдяки оператору перетворення, породженому параметрами рівняння ρ , k , γ основні результати роботи одержано з їх аналогів для випадку сталих коефіцієнтів ($\rho = k = 1$, $\gamma = 0$). Застосування цього оператора є ключовим моментом роботи. Результати проілюстровано прикладами.

Ключові слова: рівняння теплопровідності, керованість, наближена керованість