Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 4, pp. 719–765
doi: https://doi.org/10.15407/mag19.04.719

Qualitative Analysis of Nonregular Differential-Algebraic Equations and the Dynamics of Gas Networks

Maria Filipkovska

Conditions for the existence, uniqueness and boundedness of global solutions, as well as ultimate boundedness of solutions, and conditions for the blow-up of solutions of nonregular semilinear differential-algebraic equations have been obtained. An example demonstrating the application of the obtained results has been considered. Isothermal models of gas networks have been proposed as applications.

Key words: nonregular differential-algebraic equation, degenerate differential equation, singular pencil, gas network, global solvability, boundedness of solutions, blow-up, dissipativity

Mathematical Subject Classification 2020: 34A09, 34A12, 34C11, 34D23, 15A22

1. Introduction

This paper deals with systems of equations which can be represented as a differential-algebraic equation (DAE) of the form

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \tag{1.1}$$

where A, B are linear operators from $\mathbb{R}^n$ into $\mathbb{R}^m$ or $m \times n$-matrices. Various systems consisting of ordinary differential equations (ODEs) (or partial differential equations (PDEs), which after applying spatial discretization become ODEs) and of algebraic equations (not containing a derivative) can be written in this form. Note that this type of DAEs includes underdetermined and overdetermined systems of equations. DAEs of the form (1.1) are commonly referred to as nonregular (or singular) semilinear DAEs. In general, they belong to the class of ODEs unsolved for the higher derivative of the unknown function and are also called descriptor systems or degenerate differential equations.

In the present paper, conditions for the existence, uniqueness and boundedness of global solutions, as well as ultimate boundedness of solutions, and conditions for the blow-up of solutions of nonregular semilinear DAEs are obtained.

These conditions are presented both in the general form (Sections 3–7) and in certain particular cases (Section 8) which are convenient for practical application.

DAEs arise from the modelling of various systems and processes in control problems, gas industry, mechanics, radio engineering, chemical kinetics, economics and other fields (see, e.g., [2, 5, 17, 18, 25]). The use of DAEs in electrical circuit modelling is described in detail in [18] (see also [7, 8, 10, 11, 17, 22–25]). Besides electrical networks, DAEs are also used in modelling other objects whose structure is described by directed graphs, e.g., gas and neural networks. In [8,11] nonlinear electrical circuits described by singular (nonregular) semilinear DAEs have been considered. The present paper is focused on the DAEs describing the dynamics of gas networks in the isothermal case. The theorems obtained in this paper allow one to carry out the qualitative analysis of the dynamics of gas networks described by DAEs of the form (1.1). The description of gas network models, including the construction of models in the form of DAEs, is presented in [1, 2, 5, 14–16]. Generally, the dynamics of a gas flow in a pipeline (for a single pipe) is modelled by PDEs, namely, by the isothermal Euler equations in the case considered in Section 9.1, and by the equation of state for gases, which is an algebraic equation. We apply the spatial discretization (described, e.g., in [2, 15]) for the isothermal Euler equations, which leads to a semilinear DAE. A similar discretization is used to obtain a DAE which describes the dynamics of flows in gas networks (Section 9.2). This DAE arises from a system of differential and algebraic equations which has been presented in [16].

Most of the works on DAEs are related to the study of regular DAEs: their structure, index, local solvability, the Lyapunov stability of their equilibrium positions and the development of numerical methods for solving them. Much fewer works deal with nonregular DAEs in general and with the global solvability of DAEs in particular. Nonregular DAEs have been studied by using the concept of the "strangeness index" of a pair of matrices (or matrix functions) and a DAE in [17]. We use the (different) concept of an index only for a regular block of the characteristic pencil of the DAE (1.1) (see Section 2). To solve a singular linear time-invariant DAE, one usually uses the Weierstrass-Kronecker canonical form (see [13]) of a singular matrix pencil associated with the DAE. The solvability of nonregular time-varying linear DAEs with the use of a generalized canonical form and the application of the least squares method for their numerical solution have been studied in [3]. In [4], the conditions for the solvability of the Cauchy problem for a nonregular time-varying linear DAE with the use of a generalized Green operator have been found. The conditions for the Lagrange stability and instability of nonregular semilinear DAEs, which are a particular case of the conditions obtained in this paper, have been presented in [8]. The local solvability of nonregular semilinear DAEs in Banach spaces has been studied in [23]. Also, in [23] the decomposition of a singular pencil into regular and purely singular components, which was called the RS-splitting of the pencil, has been presented.

In this paper, we use the special block form of a singular operator pencil [9,11], which consists of the singular and regular blocks where zero and invertible blocks are separated out (see Section 2.2). The results from [13], related to singular matrix pencils, were used when constructing this block form. The presented block form is used to reduce the DAE with the singular characteristic pencil to a system of ODEs and algebraic equations (see Section 2.3). We also use differential inequalities for the Lyapunov type functions, the spectral projectors introduced in [22] and Yoshizawa's method [26]. The main differential inequalities used in the work are described in Section 2.1. An example demonstrating the application of the obtained results is given in Section 10.

The notations and definitions given below will be used in the present paper.

The following notations will be used: I_X is the identity operator in the space X; $\mathcal{A}^{(-1)}$ is the semi-inverse operator of an operator $\mathcal{A}$ (A^{-1} is the inverse operator of A); $\mathrm{Ker}(A)$ is the kernel of an operator A; $\mathcal{R}(A)$ is the range of an operator A; D^c is the complement of a set D; $\overline{D}$ is the closure of a set D; $\mathrm{L}(X,Y)$ is the space of continuous linear operators from X to Y; $\mathrm{L}(X,X) = \mathrm{L}(X)$, and similarly, $C((a,b),(a,b)) = C(a,b)$; $L_1 \dot{+} L_2$ is the direct sum of the linear spaces L_1 and L_2; δ_{ij} is the Kronecker delta; X' is the conjugate space of X (it is also called an adjoint or dual space); A^{T} is the transposed operator (i.e., the adjoint operator acting in real linear spaces to which the transposed matrix correspond) or the transposed matrix; $\|\cdot\|$ denotes some norm in a finite-dimensional space (it will be clear from the context in which one), unless it is explicitly stated which norm is considered; both $A \subset B$ and $A \subseteq$ mean that A is a subset of B, i.e., A can be a proper subset of B ($A \neq B$) or be equal to B; if A is a proper subset of B, we write $A \subsetneqq B$; $\partial_x := \partial/\partial x$ denotes the partial derivative with respect to x. Often, a function f is denoted by the same symbol $f(x)$ as its value at the point x in order to explicitly indicate its argument (or arguments), but it will be clear from the context what exactly is meant.

In what follows, a convex set containing a point $x_0 \in X$ that is contained in a ball $\{x \in X \mid \|x - x_0\| \leq \delta\}$ (where $\delta \geq 0$) or coincides with it will be called a *neighborhood* of the point x_0 and will be denoted by $N_\delta(x_0)$ (in particular, it is possible that $N_\delta(x_0) = \{x_0\}$ and in this case the neighborhood is degenerate). A neighborhood of some point that is an open (respectively, closed) set will be called an *open* (respectively, *closed*) neighborhood. By $U_\delta(x_0)$ and $\overline{N_\delta}(x_0)$ we denote the open neighborhood and closed neighborhood, respectively. Note that $\overline{U_\delta}(x_0)$ denotes the closure of the open neighborhood $U_\delta(x_0)$ (accordingly, $\delta > 0$). Sometimes we will denote a neighborhood (respectively, open neighborhood, closed neighborhood) of the point x_0 simply by $N(x_0)$ (respectively, $U(x_0)$, $\overline{N}(x_0)$), without indicating the radius of the ball which contains it.

In addition, if the variable t belongs to the interval $[a,b] \subset \mathbb{R}$, $a \neq b$, then by an open neighborhood $U_\delta(a)$ of the point a we mean a semi-open interval $[a, a + \delta)$, $0 < \delta < b - a$, and, similarly, by an open neighborhood $U_\delta(b)$ we mean a semi-open interval $(b - \delta, b]$, $0 < \delta < b - a$.

Let $f\colon J \to Y$ where J is an interval in $\mathbb{R}$ and Y is a normed linear space. If $J = [a,b)$, $b \leq +\infty$ ($J = (a,b]$, $a \geq -\infty$), then the derivative of the function f at the point a (at the point b) is understood as the derivative on the right (on the left) at this point (see, e.g., [20]). If the function $f\colon [a,b) \to Y$ is continuously differentiable on (a,b) and in addition the derivative of f on the right exists at a

and is continuous from the right at a, then f is said to belong $C^1([a,b),Y)$.

2. Problem statement, definitions and preliminary constructions

Consider an implicit differential equation

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \tag{2.1}$$

where A, $B \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^m)$, $t_+ \geq 0$. In the case when $m \neq n$ or $m = n$ and the operator A is noninvertible (degenerate), the equation (2.1) is called a *differential-algebraic equation* (*DAE*) or *degenerate differential equation*. In the DAE terminology, equations of the form (2.1) are called *semilinear*. For the considered equation, the initial condition (Cauchy condition) is given in the form

$$x(t_0) = x_0 \qquad (t_0 \geq t_+). \tag{2.2}$$

A DAE that contains a linear part $\dfrac{d}{dt}[Ax] + Bx$ such that the pencil $\lambda A + B$ is singular (see Definition 2.1) is called *singular* or *nonregular* (or *irregular* [8]). The pencil $\lambda A + B$ corresponding to this linear part is called *characteristic*.

If $\operatorname{rank}(\lambda A + B) = m < n$, then the DAE (2.1) corresponds to an underdetermined system of equations (that is, the number of equations is less than the number of unknowns).

If $\operatorname{rank}(\lambda A + B) = n < m$, then the DAE (2.1) corresponds to an overdetermined system of equations (that is, the number of equations is greater than the number of unknowns).

The function $x(t)$ is called a *solution of the equation* (2.1) *on* $[t_0, t_1)$, $t_1 \leq \infty$, if $x \in C([t_0,t_1), \mathbb{R}^n)$, $(Ax) \in C^1([t_0,t_1), \mathbb{R}^m)$ and $x(t)$ satisfies (2.1) on $[t_0,t_1)$. If the function $x(t)$ additionally satisfies the initial condition (2.2), then it is called a *solution of the initial value problem* (*IVP*) or *the Cauchy problem* (2.1), (2.2).

A solution $x(t)$ (of an equation or inequality) is called *global* if it exists on the whole interval $[t_0,\infty)$ (where t_0 is an initial value).

A solution $x(t)$ is called *Lagrange stable* if it is global and bounded, i.e., $x(t)$ exists on $[t_0,\infty)$ and $\sup_{t\in[t_0,\infty)} \|x(t)\| < \infty$.

A solution $x(t)$ *has a finite escape time* (or *is blow-up in finite time*) and is called *Lagrange unstable* if it exists on some finite interval $[t_0,\tau)$ and is unbounded, i.e., there exists $\tau > t_0$ ($\tau < \infty$) such that $\lim_{t\to\tau-0} \|x(t)\| = \infty$.

The equation (2.1) is called *Lagrange stable* (respectively, *unstable*) *for the initial point* (t_0,x_0) if the solution of the IVP (2.1), (2.2) is Lagrange stable (respectively, unstable) for this initial point.

The equation (2.1) is called *Lagrange stable* (respectively, *unstable*) if each solution of the IVP (2.1), (2.2) is Lagrange stable (respectively, unstable) (i.e., the equation is Lagrange stable (unstable) for each consistent initial point).

Solutions of an equation are called *ultimately bounded* if there exists a constant $C > 0$ (not depending on the choice of initial values) and for each solution

$x(t)$ with initial values t_0, x_0 there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < C$ for all $t \in [t_0 + \tau, \infty)$. If at the same time the number τ does not depend on the choice of t_0 (i.e., $\tau = \tau(x_0)$), then the solutions are called *uniformly ultimately bounded*.

The equation (2.1) is called *ultimately bounded* or *dissipative* (respectively, *uniformly ultimately bounded* or *uniformly dissipative*) if for any consistent initial point (t_0, x_0) there exists a global solution of the IVP (2.1), (2.2) and all solutions are ultimately bounded (respectively, uniformly ultimately bounded).

2.1. Remarks on differential inequalities. Here we give brief information about the existence of positive solutions (of different types) for differential inequalities which will be used below. Consider two differential inequalities:

$$\frac{dv}{dt} \leq \chi(t, v), \tag{2.3}$$
$$\frac{dv}{dt} \geq \chi(t, v), \tag{2.4}$$

where $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$. A scalar function $v \in C^1([t_0, \infty), \mathbb{R})$ which is positive and satisfies the differential inequality (2.3) (or (2.4)) on $[t_0, \infty)$ $(t_0 \geq t_+)$ is called a *positive solution* of this inequality on $[t_0, \infty)$. Let

$$\chi(t, v) = k(t)\, U(v), \tag{2.5}$$

where $k \in C([t_+, \infty), \mathbb{R})$ and $U \in C(0, \infty)$ (that is, $U \in C((0, \infty), \mathbb{R})$ is a positive function), then the inequalities (2.3) and (2.4) take the form

$$\frac{dv}{dt} \leq k(t)\, U(v), \tag{2.6}$$
$$\frac{dv}{dt} \geq k(t)\, U(v), \tag{2.7}$$

respectively, and the following statements hold (see, e.g., [19]):

- if $\int_c^\infty U^{-1}(v)\, dv = \infty$ ($c > 0$ is some constant), then the inequality (2.6) does not have positive solutions with finite escape time;
- if $\int_c^\infty U^{-1}(v)\, dv = \infty$ and $\int_{t_0}^\infty k(t) dt < \infty$ ($t_0 \geq t_+$ is some number), then the inequality (2.6) does not have unbounded positive solutions for $t \in [t_+, \infty)$;
- if $\int_c^\infty U^{-1}(v)\, dv < \infty$ and $\int_{t_0}^\infty k(t) dt = \infty$, then the inequality (2.7) does not have global (i.e., defined on $[t_+, \infty)$) positive solutions.

2.2. Block form of a singular pencil, the corresponding direct decompositions of spaces and projectors. The results from [11], [9] which will be used hereinafter are given below. The detailed description of these results can be found in [9] (where results from [11] have been generalized).

Let A, B be linear operators mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ or $\mathbb{C}^n$ into $\mathbb{C}^m$; by A, B we also denote $m \times n$-matrices corresponding to the operators A, B (with respect to some bases in $\mathbb{R}^n$, $\mathbb{R}^m$ or $\mathbb{C}^n$, $\mathbb{C}^m$ respectively). Consider the operator pencil

$\lambda A+B$, where λ is a complex parameter. The *rank of an operator pencil* $\lambda A+B$ is the dimension of its range. The *rank of a matrix pencil* $\lambda A+B$ is the largest among the orders of the pencil minors that do not vanish identically [13]. It equals the maximum number of columns (or rows) of the pencil that are linearly independent set of vectors for some $\lambda=\lambda_0$. Clearly, the ranks of the operator pencil and the corresponding matrix pencil coincide.

Definition 2.1 ([8,11])**.** A pencil of operators (or matrices) $\lambda A+B$ is called *regular* if $n=m=\operatorname{rank}(\lambda A+B)$; otherwise, i.e., if $n\neq m$ or $n=m$ and $\operatorname{rank}(\lambda A+B)<n$, the pencil is called *singular* or *nonregular* (*irregular*).

For $m\times n$ matrices A, B, this definition is equivalent to that given in [13], namely, the pencil $\lambda A+B$ is called *regular* if $n=m$ and $\det(\lambda A+B)\not\equiv 0$, and *singular* otherwise ($n\neq m$ or $n=m$ and $\det(\lambda A+B)\equiv 0$). Definition 2.1 is also equivalent to the following (cf. [9]). An operator pencil $\lambda A+B\colon \mathbb{C}^n\to\mathbb{C}^m$ is called *regular* if the set of its regular points $\rho(A,B)=\{\lambda\in\mathbb{C}\mid(\lambda A+B)^{-1}\in \mathrm{L}(\mathbb{C}^m,\mathbb{C}^n)\}$ is not empty, and *singular* if $\rho(A,B)=\varnothing$. A pencil $\lambda A+B$ of the real operators $A,B\colon\mathbb{R}^n\to\mathbb{R}^m$ is called *regular* if the set of regular points $\rho(\hat{A},\hat{B})$ of its complex extension $\lambda\hat{A}+\hat{B}\in\mathrm{L}(\mathbb{C}^n,\mathbb{C}^m)$ ($\hat{A}$, $\hat{B}\colon\mathbb{C}^n\to\mathbb{C}^m$ are the complex extensions of A, B respectively) is not empty, and *singular* if $\rho(\hat{A},\hat{B})=\varnothing$. Then the regular points λ of the complex extension $\lambda\hat{A}+\hat{B}$ are called *regular points* of the pencil $\lambda A+B$ (since for these points the resolvent $(\lambda A+B)^{-1}$ exists). Recall that the ranks of the pencil $\lambda A+B$ and its complex extension $\lambda\hat{A}+\hat{B}$ coincide.

In what follows, we will consider linear operators A, $B\colon\mathbb{R}^n\to\mathbb{R}^m$. Instead of the real operators we can consider the complex operators A, $B\colon\mathbb{C}^n\to\mathbb{C}^m$, for which Proposition 2.2 (see below) remains true, but when constructing direct decompositions of the form (2.8) for the complex spaces $\mathbb{C}^n$, $\mathbb{C}^m$ and the corresponding projectors, it is necessary to replace transposition by Hermitian conjugation everywhere.

Let $A\colon X\to Y$ be a linear operator and X_0, Y_0 be some subspaces in X, Y respectively. The pair of subspaces $\{X_0,Y_0\}$ is said to be *invariant* under the operator A if $A\colon X_0\to Y_0$, i.e., $AX_0\subseteq Y_0$ (cf. [23]; in the case when $X=Y$ and $X_0=Y_0$, this is the classical definition of invariance [13]).

Recall the following definition: A linear space L is decomposed into the *direct sum* $L=L_1\dot{+}L_2$ of the subspaces $L_1\subseteq L$ and $L_2\subseteq L$ if $L_1\cap L_2=\{0\}$ and $L_1+L_2=\{x_1+x_2\mid x_1\in L_1,x_2\in L_2\}=L$, or, equivalently, if every $x\in L$ can be uniquely represented in the form $x=x_1+x_2$ where $x_i\in L_i$, $i=1,2$ (see, e.g., [6, p. 309]). The representation $L=L_1\dot{+}L_2$ is also called a *direct decomposition* of the space L.

Since the direct (Cartesian) product $L_1\times L_2$ is the direct sum of the spaces $L_1\times\{0\}$ and $\{0\}\times L_2$, where 0 from L_2 and L_1 respectively, then it can be identified with the direct sum $L_1\dot{+}L_2$ by identifying $L_1\times\{0\}$ with L_1 and $\{0\}\times L_2$ with L_2. *Thus, below, when indicating the block structures of operators, we identify direct sums and the corresponding direct products of subspaces for convenience of notation.*

Proposition 2.2 (see [9,11]). *For operators A, $B\colon \mathbb{R}^n \to \mathbb{R}^m$, which form a singular pencil $\lambda A + B$, there exist the decompositions of the spaces $\mathbb{R}^n$, $\mathbb{R}^m$ into the direct sums of subspaces (which can always be constructed)*

$$\mathbb{R}^n = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_r, \quad \mathbb{R}^m = Y_s \dot{+} Y_r = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_r, \tag{2.8}$$

with respect to which A, B have the block structures

$$A = \begin{pmatrix} A_s & 0 \\ 0 & A_r \end{pmatrix},\ B = \begin{pmatrix} B_s & 0 \\ 0 & B_r \end{pmatrix}\colon X_s \dot{+} X_r \to Y_s \dot{+} Y_r \ \ (X_s \times X_r \to Y_s \times Y_r), \tag{2.9}$$

where $A_s = A\big|_{X_s}$, $B_s = B\big|_{X_s}\colon X_s \to Y_s$ *and* $A_r = A\big|_{X_r}$, $B_r = B\big|_{X_r}\colon X_r \to Y_r$, *i.e., the pair of "singular" subspaces $\{X_s, Y_s\}$ and the pair of "regular" subspaces $\{X_r, Y_r\}$ are invariant under the operators A, B (i.e., $A, B\colon X_s \to Y_s$, $A, B\colon X_r \to Y_r$), and the blocks A_s, B_s, which are called singular, have the block structure*

$$A_s = \begin{pmatrix} A_{\rm gen} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_s = \begin{pmatrix} B_{\rm gen} & B_{\rm und} \\ B_{\rm ov} & 0 \end{pmatrix}\colon X_{s_1} \dot{+} X_{s_2} \to Y_{s_1} \dot{+} Y_{s_2}$$
$$(X_{s_1} \times X_{s_2} \to Y_{s_1} \times Y_{s_2}), \tag{2.10}$$

where the operator $A_{\rm gen}\colon X_{s_1} \to Y_{s_1}$ has the inverse $A_{\rm gen}^{-1} \in \mathrm{L}(Y_{s_1}, X_{s_1})$ (if $X_{s_1} \neq \{0\}$), $B_{\rm gen}\colon X_{s_1} \to Y_{s_1}$, $B_{\rm und}\colon X_{s_2} \to Y_{s_1}$, and $B_{\rm ov}\colon X_{s_1} \to Y_{s_2}$. If $\operatorname{rank}(\lambda A + B) = m < n$, *then the structure of the singular blocks takes the form*

$$A_s = \begin{pmatrix} A_{\rm gen} & 0 \end{pmatrix},\ B_s = \begin{pmatrix} B_{\rm gen} & B_{\rm und} \end{pmatrix}\colon X_{s_1} \dot{+} X_{s_2} \to Y_s \ \ (X_{s_1} \times X_{s_2} \to Y_s) \tag{2.11}$$

and $Y_{s_1} = Y_s$, $Y_{s_2} = \{0\}$ in (2.8), and if $\operatorname{rank}(\lambda A + B) = n < m$, *then the structure of the singular blocks takes the form*

$$A_s = \begin{pmatrix} A_{\rm gen} \\ 0 \end{pmatrix},\ B_s = \begin{pmatrix} B_{\rm gen} \\ B_{\rm ov} \end{pmatrix}\colon X_s \to Y_{s_1} \dot{+} Y_{s_2} \ \ (X_s \to Y_{s_1} \times Y_{s_2}) \tag{2.12}$$

and $X_{s_1} = X_s$, $X_{s_2} = \{0\}$ in (2.8). The direct decompositions of spaces (2.8) generate the pair S, P, the pair F, Q, the pair S_1, S_2 and the pair F_1, F_2 of the mutually complementary projectors (i.e., $S + P = I_{\mathbb{R}^n}$, $S^2 = S$, $P^2 = P$, $SP = PS = 0$; $F + Q = I_{\mathbb{R}^m}$, $F^2 = F$, $Q^2 = Q$, $FQ = QF = 0$; $S_1 + S_2 = S$, $S_i S_j = \delta_{ij} S_i$; $F_1 + F_2 = F$, $F_i F_j = \delta_{ij} F_i$)

$$S\colon \mathbb{R}^n \to X_s, \qquad P\colon \mathbb{R}^n \to X_r, \qquad F\colon \mathbb{R}^m \to Y_s, \qquad Q\colon \mathbb{R}^m \to Y_r, \tag{2.13}$$
$$S_i\colon \mathbb{R}^n \to X_{s_i}, \qquad F_i\colon \mathbb{R}^m \to Y_{s_i}, \qquad i = 1, 2, \tag{2.14}$$

where $F_1 = F$, $F_2 = 0$ if $\operatorname{rank}(\lambda A + B) = m < n$, *and $S_1 = S$, $S_2 = 0$ if* $\operatorname{rank}(\lambda A + B) = n < m$, *which have the properties*

$$FA = AS, \qquad FB = BS, \qquad QA = AP, \qquad QB = BP, \tag{2.15}$$
$$AS_2 = 0, \qquad F_2 A = 0, \qquad F_2 B S_2 = 0. \tag{2.16}$$

The converse assertion that there exist the pairs of mutually complementary projectors (2.13), (2.14) satisfying (2.15), (2.16) which generate the direct decompositions of spaces (2.8) is also true.

The method for constructing the subspaces from the decompositions (2.8) *and the corresponding projectors* (2.13), (2.14) *is described in* [8, Section 3] *and in detail in* [9, Section 3]. First we construct the singular subspaces X_s, Y_s, X_{s_i}, Y_{s_i}, $i = 1, 2$, and the corresponding projectors, then we construct the regular subspaces X_r, Y_r and the corresponding projectors. For the construction of the singular spaces certain collections of linearly independent solutions of the equations $(\lambda A + B)x = 0$ and $(\lambda A^T + B^T)y = 0$ are used. Further, if the regular block $\lambda A_r + B_r$ from (2.17) is a regular pencil of index not higher than 1 (see the definition below), we construct the regular subspaces X_i, Y_i, $i = 1, 2$, from the decompositions (2.28) and the projectors (2.30), which are described below.

With respect to the decompositions $\mathbb{R}^n = X_s \dot{+} X_r$, $\mathbb{R}^m = Y_s \dot{+} Y_r$ (see (2.8)) the singular pencil $\lambda A + B$ of the operators $A, B\colon \mathbb{R}^n \to \mathbb{R}^m$ takes the block form

$$\lambda A + B = \begin{pmatrix} \lambda A_s + B_s & 0 \\ 0 & \lambda A_r + B_r \end{pmatrix}, \ A_s, B_s\colon X_s \to Y_s,\ A_r, B_r\colon X_r \to Y_r, \quad (2.17)$$

where the regular block $\lambda A_r + B_r$ is a regular pencil and the singular block $\lambda A_s + B_s$ is a purely singular pencil, i.e., it is impossible to separate out a regular block in this pencil. If $X_r = \{0\}$, $Y_r = \{0\}$, then the regular block $\lambda A_r + B_r$ is absent and $\lambda A + B = \lambda A_s + B_s$ is a purely singular pencil.

In [9], extensions of the operators from the block representations (2.9), (2.10), (2.11), (2.12) to $\mathbb{R}^n$ and the corresponding semi-inverse operators have been introduced. These operators are described below and used in subsequent sections.

Extensions of the operators A_s, A_r, B_s, B_r from (2.9) to $\mathbb{R}^n$ are introduced as follows:

$$\mathcal{A}_s = FA, \quad \mathcal{A}_r = QA, \quad \mathcal{B}_s = FB, \quad \mathcal{B}_r = QB. \quad (2.18)$$

Then the operators $\mathcal{A}_s, \mathcal{B}_s, \mathcal{A}_r, \mathcal{B}_r \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ act so that $\mathcal{A}_s, \mathcal{B}_s\colon \mathbb{R}^n \to Y_s$, $\mathcal{A}_r, \mathcal{B}_r\colon \mathbb{R}^n \to Y_r$ $(\mathcal{A}_s, \mathcal{B}_s\colon X_s \to Y_s,\ \mathcal{A}_r, \mathcal{B}_r\colon X_r \to Y_r)$ and $X_r \subset \mathrm{Ker}(\mathcal{A}_s)$, $X_r \subset \mathrm{Ker}(\mathcal{B}_s)$, $X_s \subset \mathrm{Ker}(\mathcal{A}_r)$, $X_s \subset \mathrm{Ker}(\mathcal{B}_r)$ and

$$\mathcal{A}_s\big|_{X_s} = A_s, \quad \mathcal{A}_r\big|_{X_r} = A_r, \quad \mathcal{B}_s\big|_{X_s} = B_s, \quad \mathcal{B}_r\big|_{X_r} = B_r. \quad (2.19)$$

In the general case, when $\mathrm{rank}(\lambda A + B) < n$ and $\mathrm{rank}(\lambda A + B) < m$, the spaces $\mathbb{R}^n$, $\mathbb{R}^m$ have the decompositions (2.8) and, accordingly, the singular subspaces are decomposed into the direct sums $X_s = X_{s_1} \dot{+} X_{s_2}$, $Y_s = Y_{s_1} \dot{+} Y_{s_2}$ with respect to which the operators (singular blocks) A_s, B_s have the structure (2.10), and extensions of the operators (blocks) from (2.10) to $\mathbb{R}^n$ are introduced as follows:

$$\mathcal{A}_{\mathrm{gen}} = F_1 A, \quad \mathcal{B}_{\mathrm{gen}} = F_1 B S_1, \quad \mathcal{B}_{\mathrm{und}} = F_1 B S_2, \quad \mathcal{B}_{\mathrm{ov}} = F_2 B S_1 \quad (2.20)$$

(notice that $F_1 A = A S_1 = FA$). Then $\mathcal{A}_{\mathrm{gen}}, \mathcal{B}_{\mathrm{gen}}, \mathcal{B}_{\mathrm{und}}, \mathcal{B}_{\mathrm{ov}} \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ act so that $\mathcal{A}_{\mathrm{gen}} \mathbb{R}^n = \mathcal{A}_{\mathrm{gen}} X_{s_1} = Y_{s_1}$ $(X_{s_2} \dot{+} X_r = \mathrm{Ker}(\mathcal{A}_{\mathrm{gen}}))$, $\mathcal{B}_{\mathrm{gen}}\colon \mathbb{R}^n \to Y_{s_1}$, $X_{s_2} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{\mathrm{gen}})$, $\mathcal{B}_{\mathrm{und}}\colon \mathbb{R}^n \to Y_{s_1}$, $X_{s_1} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{\mathrm{und}})$, and $\mathcal{B}_{\mathrm{ov}}\colon \mathbb{R}^n \to Y_{s_2}$, $X_{s_2} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{\mathrm{ov}})$, and

$$\mathcal{A}_{\mathrm{gen}}\big|_{X_{s_1}} = A_{\mathrm{gen}}, \quad \mathcal{B}_{\mathrm{gen}}\big|_{X_{s_1}} = B_{\mathrm{gen}}, \quad \mathcal{B}_{\mathrm{und}}\big|_{X_{s_2}} = B_{\mathrm{und}}, \quad \mathcal{B}_{\mathrm{ov}}\big|_{X_{s_1}} = B_{\mathrm{ov}}. \quad (2.21)$$

In the case when $\operatorname{rank}(\lambda A+B)=m<n$, the singular subspace X_s is decomposed into the direct sum $X_s = X_{s_1}\dot{+}X_{s_2}$ with respect to which the operators (singular blocks) A_s, B_s have the structure (2.11), and extensions of the operators (blocks) from (2.11) to $\mathbb{R}^n$ are introduced as follows:

$$\mathcal{A}_{\text{gen}} = AS_1, \quad \mathcal{B}_{\text{gen}} = BS_1, \quad \mathcal{B}_{\text{und}} = BS_2. \tag{2.22}$$

Then $\mathcal{A}_{\text{gen}}, \mathcal{B}_{\text{gen}}, \mathcal{B}_{\text{und}} \in \mathrm{L}(\mathbb{R}^n,\mathbb{R}^m)$ act so that $\mathcal{A}_{\text{gen}}\mathbb{R}^n = \mathcal{A}_{\text{gen}}X_{s_1} = Y_s$ ($X_{s_2}\dot{+}X_r = \operatorname{Ker}(\mathcal{A}_{\text{gen}})$), $\mathcal{B}_{\text{gen}}\colon \mathbb{R}^n \to Y_s$, $X_{s_2}\dot{+}X_r \subset \operatorname{Ker}(\mathcal{B}_{\text{gen}})$, $\mathcal{B}_{\text{und}}\colon \mathbb{R}^n \to Y_s$, $X_{s_1}\dot{+}X_r \subset \operatorname{Ker}(\mathcal{B}_{\text{und}})$, and

$$\mathcal{A}_{\text{gen}}\big|_{X_{s_1}} = A_{\text{gen}}, \quad \mathcal{B}_{\text{gen}}\big|_{X_{s_1}} = B_{\text{gen}}, \quad \mathcal{B}_{\text{und}}\big|_{X_{s_2}} = B_{\text{und}}. \tag{2.23}$$

In the case when $\operatorname{rank}(\lambda A+B)=n<m$, the singular subspace Y_s is decomposed into the direct sum $Y_s = Y_{s_1}\dot{+}Y_{s_2}$ with respect to which the operators (singular blocks) A_s, B_s have the structure (2.12), and extensions of the operators (blocks) from (2.12) to $\mathbb{R}^n$ are introduced as follows:

$$\mathcal{A}_{\text{gen}} = F_1A, \quad \mathcal{B}_{\text{gen}} = F_1B, \quad \mathcal{B}_{\text{ov}} = F_2B. \tag{2.24}$$

Then $\mathcal{A}_{\text{gen}}, \mathcal{B}_{\text{gen}}, \mathcal{B}_{\text{ov}} \in \mathrm{L}(\mathbb{R}^n,\mathbb{R}^m)$ act so that $\mathcal{A}_{\text{gen}}\mathbb{R}^n = \mathcal{A}_{\text{gen}}X_s = Y_{s_1}$ ($X_r = \operatorname{Ker}(\mathcal{A}_{\text{gen}})$), $\mathcal{B}_{\text{gen}}\colon \mathbb{R}^n \to Y_{s_1}$, $X_r \subset \operatorname{Ker}(\mathcal{B}_{\text{gen}})$, $\mathcal{B}_{\text{ov}}\colon \mathbb{R}^n \to Y_{s_2}$, $X_r \subset \operatorname{Ker}(\mathcal{B}_{\text{ov}})$,

$$\mathcal{A}_{\text{gen}}\big|_{X_s} = A_{\text{gen}}, \quad \mathcal{B}_{\text{gen}}\big|_{X_s} = B_{\text{gen}}, \quad \mathcal{B}_{\text{ov}}\big|_{X_s} = B_{\text{ov}}. \tag{2.25}$$

Remark 2.3 ([9]). The extension $\mathcal{A}_{\text{gen}}^{(-1)} \in \mathrm{L}(\mathbb{R}^m,\mathbb{R}^n)$ of the operator A_{gen}^{-1} to $\mathbb{R}^m$ that satisfies the properties

$$\mathcal{A}_{\text{gen}}^{(-1)}\mathcal{A}_{\text{gen}} = S_1, \quad \mathcal{A}_{\text{gen}}\,\mathcal{A}_{\text{gen}}^{(-1)} = F_1, \quad \mathcal{A}_{\text{gen}}^{(-1)} = S_1\mathcal{A}_{\text{gen}}^{(-1)}, \tag{2.26}$$

where $F_1 = F$ if $\operatorname{rank}(\lambda A+B)=m<n$ and $S_1 = S$ if $\operatorname{rank}(\lambda A+B)=n<m$, is the *semi-inverse* operator of $\mathcal{A}_{\text{gen}}$, i.e., $\mathcal{A}_{\text{gen}}^{(-1)}\mathbb{R}^m = \mathcal{A}_{\text{gen}}^{(-1)}Y_{s_1} = X_{s_1}$ ($Y_{s_2}\dot{+}Y_r = \operatorname{Ker}(\mathcal{A}_{\text{gen}}^{(-1)})$) and $A_{\text{gen}}^{-1} = \mathcal{A}_{\text{gen}}^{(-1)}\big|_{Y_{s_1}}$ (the definition of a semi-inverse operator can be found in [6]). Thus, the semi-inverse operator $\mathcal{A}_{\text{gen}}^{(-1)}$ of $\mathcal{A}_{\text{gen}}$ is defined by the relations (2.26). Also, they enable one to find the form of $\mathcal{A}_{\text{gen}}^{(-1)}$ (or A_{gen}^{-1}), using the form of the projectors. Note that $\mathcal{A}_{\text{gen}}^{(-1)}F_1 = S_1\mathcal{A}_{\text{gen}}^{(-1)} = \mathcal{A}_{\text{gen}}^{(-1)}$ (where $F_1 = F$ if $\operatorname{rank}(\lambda A+B)=m<n$ and $S_1 = S$ if $\operatorname{rank}(\lambda A+B)=n<m$).

Consider a regular pencil $\lambda A_r + B_r$ of operators A_r, $B_r\colon X_r \to Y_r$ acting in finite-dimensional spaces ($\dim X_r = \dim Y_r$). We assume that either $\lambda=\infty$ is a removable singular point of the resolvent $(\lambda A_r + B_r)^{-1}$, or A_r is invertible. Thus, we assume that there exist constants $C_1, C_2 > 0$ such that

$$\big\|(\lambda A_r + B_r)^{-1}\big\| \le C_1, \quad |\lambda| \ge C_2. \tag{2.27}$$

If A_r is noninvertible and (2.27) holds (hence, $\mu=0$ is a simple pole of the resolvent $(A_r+\mu B_r)^{-1}$), then $\lambda A_r + B_r$ is a regular pencil of *index 1*. Note that if $A_r = 0$ and there exists B_r^{-1}, then $\lambda A_r + B_r \equiv B_r$ can be considered as a regular pencil of index 1. If A_r is invertible (hence, $\mu = 0$ is a regular point of $A_r + \mu B_r$), then $\lambda A_r + B_r$ is a regular pencil of *index 0*. Thus, if $\lambda A_r + B_r$ is a regular pencil and (2.27) holds, then $\lambda A_r + B_r$ is a regular pencil of *index not higher than 1* (cf. [8, 9]).

Remark 2.4. If the regular block $\lambda A_r + B_r$ from (2.17) is a regular pencil of index not higher than 1 (i.e., satisfies (2.27)), then there exists the pair $\tilde{P}_j : X_r \to X_j$, $j = 1, 2$, and the pair $\tilde{Q}_j : Y_r \to Y_j$, $j = 1, 2$, of mutually complementary projectors which generate the direct decompositions

$$X_r = X_1 \dot{+} X_2, \quad Y_r = Y_1 \dot{+} Y_2 \tag{2.28}$$

such that the pairs of subspaces X_1, Y_1 and X_2, Y_2 are invariant under A_r, B_r ($A_r, B_r : X_j \to Y_j$, $j = 1, 2$), i.e., $\tilde{Q}_j A_r = A_r \tilde{P}_j$, $\tilde{Q}_j B_r = B_r \tilde{P}_j$, and the restricted operators $A_j = A_r|_{X_j} : X_j \to Y_j$, $B_j = B_r|_{X_j} : X_j \to Y_j$, $j = 1, 2$, are such that $A_2 = 0$ ($\tilde{Q}_2 A_r = 0$) and there exist $A_1^{-1} \in \mathrm{L}(Y_1, X_1)$ (if $X_1 \neq \{0\}$) and $B_2^{-1} \in \mathrm{L}(Y_2, X_2)$ (if $X_2 \neq \{0\}$). For a regular pencil of operators, the pairs of projectors with the specified properties were introduced in [22]. With respect to the direct decompositions (2.28) the operators A_r, B_r have the block structure

$$A_r = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \; B_r = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} : X_1 \dot{+} X_2 \to Y_1 \dot{+} Y_2 \; (X_1 \times X_2 \to Y_1 \times Y_2), \tag{2.29}$$

where A_1 and B_2 are invertible (if $X_1 \neq \{0\}$ and $X_2 \neq \{0\}$ respectively).

Thus, if $\lambda A_r + B_r$ is a regular pencil of index not higher than 1, then there exist the direct decompositions of the regular spaces (2.28) with respect to which A_r and B_r have the block structure (2.29).

The projectors $\tilde{P}_j$ and $\tilde{Q}_j$ can be calculated by using contour integration [22, p. 2005] or defined by the formulas (28) from [9]. In addition, for a regular pencil of index 1 one can obtain projectors onto the subspaces from the decompositions (2.28) without using the formulas from [22] or the formulas [9, (28)] as described in [9, Remark 3, p. 44–45].

Introduce the extensions P_j, Q_j of the projectors $\tilde{P}_j$, $\tilde{Q}_j$ to $\mathbb{R}^n$, $\mathbb{R}^m$, respectively, so that $X_j = P_j \mathbb{R}^n$, $Y_j = Q_j \mathbb{R}^m$, $j = 1, 2$ (where X_j, Y_j from (2.28)) [9]. Then the extended projectors

$$P_j : \mathbb{R}^n \to X_j, \quad Q_j : \mathbb{R}^m \to Y_j, \quad j = 1, 2, \tag{2.30}$$

have the properties of the original ones, i.e., P_1, P_2 and Q_1, Q_2 are two pairs of mutually complementary projectors ($P_i P_j = \delta_{ij} P_i$, $P_1 + P_2 = P$, $Q_i Q_j = \delta_{ij} Q_i$, $Q_1 + Q_2 = Q$) and $Q_j A = A P_j$, $Q_j B = B P_j$, $Q_2 A = 0$. The properties of the operators $A_j = A|_{X_j} : X_j \to Y_j$ and $B_j = B|_{X_j} : X_j \to Y_j$, $j = 1, 2$, are also retained, and extensions of the operators A_j, B_j to $\mathbb{R}^n$ are introduced as follows:

$$\mathcal{A}_j = Q_j A, \quad \mathcal{B}_j = Q_j B, \quad j = 1, 2. \tag{2.31}$$

Then the operators $\mathcal{A}_j, \mathcal{B}_j \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ act so that $\mathcal{A}_1\mathbb{R}^n = \mathcal{A}_1 X_1 = Y_1$ $(X_2\dot{+}X_s = \mathrm{Ker}(\mathcal{A}_1))$, $\mathcal{A}_2 = 0$, $\mathcal{B}_1 \colon \mathbb{R}^n \to Y_1$, $X_2\dot{+}X_s \subset \mathrm{Ker}(\mathcal{B}_1)$, and $\mathcal{B}_2\mathbb{R}^n = \mathcal{B}_2 X_2 = Y_2$ $(X_1\dot{+}X_s = \mathrm{Ker}(\mathcal{B}_2))$, and

$$\mathcal{A}_j\big|_{X_j} = A_j, \quad \mathcal{B}_j\big|_{X_j} = B_j, \quad j = 1, 2. \tag{2.32}$$

Remark 2.5 ([9]). The extension $\mathcal{A}_1^{(-1)} \in \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ of the operator A_1^{-1} to $\mathbb{R}^m$ that satisfies the properties

$$\mathcal{A}_1^{(-1)}\mathcal{A}_1 = P_1, \quad \mathcal{A}_1\mathcal{A}_1^{(-1)} = Q_1, \quad \mathcal{A}_1^{(-1)} = P_1\mathcal{A}_1^{(-1)}, \tag{2.33}$$

is the semi-inverse operator of $\mathcal{A}_1$, i.e., $\mathcal{A}_1^{(-1)}\mathbb{R}^m = \mathcal{A}_1^{(-1)}Y_1 = X_1$ $(Y_2\dot{+}Y_s = \mathrm{Ker}(\mathcal{A}_1^{(-1)}))$ and $A_1^{-1} = \mathcal{A}_1^{(-1)}\big|_{Y_1}$. The semi-inverse operator $\mathcal{B}_2^{(-1)} \in \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ of $\mathcal{B}_2$, i.e., $\mathcal{B}_2^{(-1)}\mathbb{R}^m = \mathcal{B}_2^{(-1)}Y_2 = X_2$ $(Y_1\dot{+}Y_s = \mathrm{Ker}(\mathcal{B}_2^{(-1)}))$ and $B_2^{-1} = \mathcal{B}_2^{(-1)}\big|_{Y_2}$, is defined in a similar way as

$$\mathcal{B}_2^{(-1)}\mathcal{B}_2 = P_2, \quad \mathcal{B}_2\mathcal{B}_2^{(-1)} = Q_2, \quad \mathcal{B}_2^{(-1)} = P_2\mathcal{B}_2^{(-1)}. \tag{2.34}$$

Note that $\mathcal{A}_1^{(-1)}Q_1 = P_1\mathcal{A}_1^{(-1)} = \mathcal{A}_1^{(-1)}$ and $\mathcal{B}_2^{(-1)}Q_2 = P_2\mathcal{B}_2^{(-1)} = \mathcal{B}_2^{(-1)}$. The relations (2.33) and (2.34) enable one to find the form of $\mathcal{A}_1^{(-1)}$ and $\mathcal{B}_2^{(-1)}$ (or A_1^{-1}, B_2^{-1}), using the form of the projectors.

The decompositions (2.8) and (2.28) together give the decomposition of $\mathbb{R}^n$ into the direct sum of subspaces

$$\mathbb{R}^n = X_s\dot{+}X_r = X_{s_1}\dot{+}X_{s_2}\dot{+}X_1\dot{+}X_2 \tag{2.35}$$

with respect to which any element $x \in \mathbb{R}^n$ can be *uniquely represented* (the uniqueness of the representation follows from the definition of a direct sum of subspaces) in the form

$$x = x_s + x_r = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2} \quad (x_s = x_{s_1} + x_{s_2}, \;\; x_r = x_{p_1} + x_{p_2}), \tag{2.36}$$

where $x_s = Sx \in X_s$, $x_r = Px \in X_r$, $x_{s_i} = S_i x \in X_{s_i}$, $x_{p_i} = P_i x \in X_i$, $i = 1, 2$.

In what follows, it is assumed that the specified correspondence between the subscript of an element from the subspace present in the decomposition (2.35) (or a component from the representation (2.36)) and the subspace to which this element belongs is always fulfilled, i.e., the element x_{s_i} $(i = 1, 2)$ belongs to X_{s_i} because it has the subscript s_i $(i = 1, 2)$, the element x_{p_j} belongs to X_j $(j = 1, 2)$, and so on. Thus, we will not always explicitly indicate belonging to one of the subspaces introduced in (2.35), when the element has one of the subscripts given in (2.36), can be any element from the corresponding subspace, and it is clear from the context what exactly is meant.

Similarly, the decompositions (2.8) and (2.28) together also give the decomposition of $\mathbb{R}^m$ into the direct sum of subspaces

$$\mathbb{R}^m = Y_s\dot{+}Y_r = Y_{s_1}\dot{+}Y_{s_2}\dot{+}Y_1\dot{+}Y_2, \tag{2.37}$$

with respect to which any element $y \in \mathbb{R}^m$ can be uniquely represented as

$$y = y_s + y_r = y_{s_1} + y_{s_2} + y_{p_1} + y_{p_2} \quad (y_s = y_{s_1} + y_{s_2}, \;\; y_r = y_{p_1} + y_{p_2}), \tag{2.38}$$

where $y_s = Fy \in Y_s$, $y_r = Qy \in Y_r$, $y_{s_i} = F_i y \in Y_{s_i}$ and $y_{p_i} = Q_i y \in Y_i$, $i = 1, 2$.

2.3. Reduction of a DAE with the singular characteristic pencil to a system of ordinary differential and algebraic equations. The information given in Section 2.2 is used below.

Consider the DAE (2.1) with the singular characteristic pencil $\lambda A + B$ that has the regular block $\lambda A_r + B_r$ (see (2.17)) of index not higher than 1.

Applying the projectors F_1, Q_1, Q_2, F_2 from (2.14), (2.30) to the equation (2.1) and using their properties, we obtain the equivalent system

$$\frac{d}{dt}(F_1 A S_1 x) + F_1 B S x = F_1 f(t,x), \tag{2.39}$$

$$\frac{d}{dt}(Q_1 A P_1 x) + Q_1 B P_1 x = Q_1 f(t,x), \tag{2.40}$$

$$Q_2 B P_2 x = Q_2 f(t,x), \tag{2.41}$$

$$F_2 B S_1 x = F_2 f(t,x). \tag{2.42}$$

Using the representation (2.36), the operators (2.20), (2.31) and the semi-inverse operators $\mathcal{A}_{\rm gen}^{(-1)}$, $\mathcal{A}_1^{(-1)}$ and $\mathcal{B}_2^{(-1)}$ (the method of their calculation is indicated in Section 2.2), we obtain the following system equivalent to (2.39)–(2.42):

$$\frac{d}{dt}x_{s_1} = \mathcal{A}_{\rm gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{\rm gen}x_{s_1} - \mathcal{B}_{\rm und}x_{s_2}\big), \tag{2.43}$$

$$\frac{d}{dt}x_{p_1} = \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big), \tag{2.44}$$

$$0 = \mathcal{B}_2^{(-1)} Q_2 f(t,x) - x_{p_2}, \tag{2.45}$$

$$0 = F_2 f(t,x) - \mathcal{B}_{\rm ov}x_{s_1}, \tag{2.46}$$

where $x_{s_i} = S_i x \in X_{s_i}$, $x_{p_i} = P_i x \in X_i$, $i = 1, 2$, and the representation of x in the form $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$ (see (2.36)) is unique.

Thus, the singular semilinear DAE (2.1) has been reduced to the equivalent system (2.43)–(2.46) of ordinary differential equations (ODEs) and algebraic equations (AEs). Instead of the system (2.43)–(2.46) one can also obtain the equivalent system with the restricted operators.

By $V'_{(2.43),(2.44)}$ we denote the derivative of a function $V \in C^1([t_+,\infty)\times D_{s_1}\times D_{p_1},\mathbb{R})$, where $D_{s_1}\times D_{p_1}\subset X_{s_1}\times X_1$ is some open set, along the trajectories of the system (2.43), (2.44), which has the form

$$\begin{aligned} V'_{(2.43),(2.44)}(t,x_{s_1},x_{p_1}) &= \frac{\partial V}{\partial t}(t,x_{s_1},x_{p_1}) + \frac{\partial V}{\partial(x_{s_1},x_{p_1})}(t,x_{s_1},x_{p_1})\Upsilon(t,x) \\ &= \frac{\partial V}{\partial t}(t,x_{s_1},x_{p_1}) + \frac{\partial V}{\partial x_{s_1}}(t,x_{s_1},x_{p_1})\left[\mathcal{A}_{\rm gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{\rm gen}x_{s_1} - \mathcal{B}_{\rm und}x_{s_2}\big)\right] \\ &+ \frac{\partial V}{\partial x_{p_1}}(t,x_{s_1},x_{p_1})\left[\mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big)\right], \end{aligned} \tag{2.47}$$

$$\Upsilon(t,x) = \begin{pmatrix} \mathcal{A}_{\rm gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{\rm gen}x_{s_1} - \mathcal{B}_{\rm und}x_{s_2}\big) \\ \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big) \end{pmatrix} \tag{2.48}$$

($\Upsilon(t,x)$ consists of the right-hand sides of the equations (2.43), (2.44)). As usual, $\dfrac{\partial}{\partial(x_{s_1},x_{p_1})} = \left(\dfrac{\partial}{\partial x_{s_1}}, \dfrac{\partial}{\partial x_{p_1}}\right)$.

When proving theorems, we will use the representation of an element $x \in \mathbb{R}^n$ in the form (2.36) (with respect to the direct sum of subspaces (2.35)) and its corresponding representation in the form $x = (x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$ (with respect to the corresponding direct product of subspaces). The correspondence between these representations is established below and, in general, is obvious.

Taking into account that the sum of subspaces in (2.35) is direct and, accordingly, any element $x \in \mathbb{R}^n$ can be uniquely represented as (2.36), one can identify an ordered collection $(x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) \in X_{s_1} \times X_{s_2} \times X_1 \times X_2$ (which is assumed to be a column vector) with the corresponding element $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2} \in \mathbb{R}^n = X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$. A norm in the space $X_{s_1} \times X_{s_2} \times X_1 \times X_2$ is defined so that the norms of any element of the form $x = x_{s_1}$ and the corresponding element (ordered collection) $x = (x_{s_1}, 0, 0, 0)$ from $X_{s_1} \times X_{s_2} \times X_1 \times X_2$ coincide and, similarly, the norms of the elements $x = x_{s_2}$, $x = x_{p_1}$, $x = x_{p_2}$ and the corresponding ordered collections $x = (0, x_{s_2}, 0, 0)$, $x = (0, 0, x_{p_1}, 0)$, and $x = (0, 0, 0, x_{p_2})$ coincide. In addition, norms in $X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$ and $X_{s_1} \times X_{s_2} \times X_1 \times X_2$ are defined so that they coincide for any element x. Obviously, $\dim(X_{s_1} \times X_{s_2} \times X_1 \times X_2) = n$ and the space $\mathbb{R}^n = X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$ is isomorphic to the space $X_{s_1} \times X_{s_2} \times X_1 \times X_2$. Thus, the representations $x = (x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$ and $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$, where $x_{s_i} \in X_{s_i}$, $x_{p_i} \in X_i$, $i = 1, 2$, define the same element x which we will write in the form of the ordered collection (column vector) or sum of the components x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}.

In a similar way, an ordered collection (column vector) $y = (y_{s_1}, y_{s_2}, y_{p_1}, y_{p_2}) \in Y_{s_1} \times Y_{s_2} \times Y_1 \times Y_2$ can be identified with the corresponding element $y = y_{s_1} + y_{s_2} + y_{p_1} + y_{p_2} \in \mathbb{R}^m = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_1 \dot{+} Y_2$.

Consider one more representation of a vector $x \in \mathbb{R}^n$, which allows one to reduce the DAE (2.1) to an equivalent system of ODEs and AEs with the operators restricted to the subspaces from (2.35) (cf. [8]). Denote the dimensions of the subspaces from the decomposition (2.35) as $\dim X_{s_1} = b$, $\dim X_{s_2} = l$, $\dim X_1 = a$ and $\dim X_2 = d$ ($b + l + a + d = n$, $\dim X_s = b + l$, $\dim X_r = a + d$). Further, we choose some bases $\{s_j\}_{j=1}^{b}$, $\{s_{b+j}\}_{j=1}^{l}$, $\{p_j\}_{j=1}^{a}$ and $\{p_{a+j}\}_{j=1}^{d}$ of the subspaces X_{s_1}, X_{s_2}, X_1 and X_2, respectively. The union of these bases is a basis of the space $\mathbb{R}^n = \mathbb{R}^b \times \mathbb{R}^l \times \mathbb{R}^a \times \mathbb{R}^d$, and with respect to this basis each vector $x \in \mathbb{R}^n$ ($x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$) can be written in the form of the column vector $x = (w^T, \xi^T, z^T, u^T)^T$, where $w \in \mathbb{R}^b$, $\xi \in \mathbb{R}^l$, $z \in \mathbb{R}^a$ and $u \in \mathbb{R}^d$ are column vectors consisting of the coordinates of the vector x with respect to the chosen bases in the subspaces X_{s_1}, X_{s_2}, X_1 and X_2 respectively. The specified one-to-one correspondence between X_{s_1}, X_{s_2}, X_1, X_2 and $\mathbb{R}^b$, $\mathbb{R}^l$, $\mathbb{R}^a$, $\mathbb{R}^d$ (between each x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2} and each w, ξ, z, u), respectively, defines the linear operators $S_b \colon \mathbb{R}^b \to X_{s_1}$, $S_l \colon \mathbb{R}^l \to X_{s_2}$, $P_a \colon \mathbb{R}^a \to X_1$, $P_d \colon \mathbb{R}^d \to X_2$ establishing an isomorphism between the spaces, which have the inverse $S_b^{-1} \colon X_{s_1} \to \mathbb{R}^b$, $S_l^{-1} \colon X_{s_2} \to \mathbb{R}^l$, $P_a^{-1} \colon X_1 \to \mathbb{R}^a$ and $P_d^{-1} \colon X_2 \to \mathbb{R}^d$. Then we restrict the operators in the equations (2.39)–(2.42) to the subspaces X_{s_1}, X_{s_2},

X_1, X_2, make the change of variables

$$x_{s_1} = S_b\, w, \quad x_{s_2} = S_l\, \xi, \quad x_{p_1} = P_a\, z, \quad x_{p_2} = P_d\, u,$$

and transform the system (2.39)–(2.42) into the following system (equivalent to the DAE (2.1) and similar to the one in [8]):

$$\frac{d}{dt}w = S_b^{-1} A_{\rm gen}^{-1} \left(F_1 \tilde{f}(t,w,\xi,z,u) - B_{\rm gen} S_b\, w - B_{\rm und} S_l\, \xi \right), \tag{2.49}$$

$$\frac{d}{dt}z = P_a^{-1} A_1^{-1} \left(Q_1 \tilde{f}(t,w,\xi,z,u) - B_1 P_a\, z \right), \tag{2.50}$$

$$0 = P_d^{-1} B_2^{-1} Q_2 \tilde{f}(t,w,\xi,z,u) - u, \tag{2.51}$$

$$0 = F_2 \tilde{f}(t,w,\xi,z,u) - B_{\rm ov} S_b\, w, \tag{2.52}$$

where $A_{\rm gen}$, A_1, B_2 are defined in Section 2.2, $\tilde{f}(t,w,\xi,z,u) = f(t, S_b\, w + S_l\, \xi + P_a\, z + P_d\, u)$ and the projectors F_i, Q_i $(i = 1,2)$ on the subspaces Y_{s_i}, Y_i are considered as the operators from $\mathbb{R}^m$ into Y_{s_i}, Y_i, respectively (i.e., $F_i \in {\rm L}(\mathbb{R}^m, Y_{s_i})$, $Q_i \in {\rm L}(\mathbb{R}^m, Y_i)$), that have the same projection properties as the projectors $F_i \in {\rm L}(\mathbb{R}^m)$, $Q_i \in {\rm L}(\mathbb{R}^m)$ defined in Section 2.2, i.e., $F_i y = F_i y_{s_i} = y_{s_i} \in Y_{s_i}$ and $Q_i y = Q_i y_{p_i} = y_{p_i} \in Y_i$ $(i = 1,2)$ for any $y \in \mathbb{R}^m$ (see the representation (2.38)). For convenience, we keep the previous notation for these operators.

In what follows, when considering an equation with the restricted (induced) operators, where the restricted operators are understood as $A_{\rm gen}$, $B_{\rm gen}$, $B_{\rm und}$, $B_{\rm ov}$, A_i, B_i, $i = 1,2$, $A_{\rm gen}^{-1}$, A_1^{-1} and B_2^{-1}, the projectors F_i, Q_i $(i = 1,2)$ are considered as the operators $F_i \in {\rm L}(\mathbb{R}^m, Y_{s_i})$, $Q_i \in {\rm L}(\mathbb{R}^m, Y_i)$ having the same projection properties as the projectors F_i, Q_i defined in Section 2.2 (see, e.g., the comments to the system (2.49)–(2.52) for details). In general, the projectors F_i, Q_i by definition belong to ${\rm L}(\mathbb{R}^m)$ (see Section 2.2), and Y_{s_i}, Y_i are their ranges, respectively $(\,{\rm Ker}\, F_i = (\mathbb{R}^m \setminus Y_{s_i}) \cup \{0\},\; {\rm Ker}\, Q_i = (\mathbb{R}^m \setminus Y_i) \cup \{0\}\,)$. Since, in fact, the described differences are formal and become significant only in the transition from the operators to the corresponding matrices, then we keep the same notations for F_i, Q_i $(i = 1,2)$ in all cases.

For clarity, note that if we choose some basis $\{e_j\}_{j=1}^{m-d}$ of $Y_s \dot{+} Y_1$ and some basis $\{q_j\}_{j=1}^{d}$ of Y_2 (notice that $\dim Y_2 = \dim X_2 = d$), and we take the basis of $\mathbb{R}^m$ as the union of these bases, i.e., in the form $\{e_1, \dots, e_{m-d}, q_1, \dots, q_d\}$, then the matrix corresponding to the mentioned operator $Q_2 \in {\rm L}(\mathbb{R}^m, Y_2)$ with respect to the chosen bases in $\mathbb{R}^m$ and Y_2 will have the form $Q_2 = \begin{pmatrix} 0 & I_{Y_2} \end{pmatrix}$, where 0 is the null $d \times m - d$ matrix and I_{Y_2} is the identity $d \times d$ matrix corresponding to the identity operator I_{Y_2} with respect to the chosen basis of Y_2.

3. Global solvability of singular (nonregular) semilinear DAEs

Remark 3.1. We introduce the manifold

$$L_{t_*} = \{(t,x) \in [t_*, \infty) \times \mathbb{R}^n \mid (F_2 + Q_2)[Bx - f(t,x)] = 0\}, \tag{3.1}$$

where $t_* \geq t_+$. The manifold (3.1) is defined by the equations (2.41) (or $Q_2[Bx - f(t,x)] = 0$) and (2.42) (or $F_2[Bx - f(t,x)] = 0$) and can be represented as

$$L_{t_*} = \{(t,x) \in [t_*, \infty) \times \mathbb{R}^n \mid (t,x) \text{ satisfies the equations (2.41), (2.42) } \}.$$

The initial values t_0, x_0 satisfying the *consistency condition* $(t_0, x_0) \in L_{t_+}$ (L_{t_+} has the form (3.1) where $t_* = t_+$) are called *consistent initial values*, and, accordingly, the initial point $(t_0, x_0) \in L_{t_+}$ is called a *consistent initial point*.

It is clear that the graph of a solution of the IVP (2.1), (2.2) as well as the initial point (t_0, x_0) must lie in the manifold L_{t_0}.

Theorem 3.2. *Let $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^m)$ and $\lambda A + B$ be a singular pencil of operators such that its regular block $\lambda A_r + B_r$ from (2.17) has the index not higher than 1. Let the following conditions be fulfilled:*

1. *For any fixed $t \in [t_+, \infty)$, $x_{s_1} \in X_{s_1}$, $x_{s_2} \in D_{s_2}$, where $D_{s_2} \subset X_{s_2}$ is some set, and $x_{p_1} \in X_1$, there exists a unique $x_{p_2} \in X_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$.*
2. *There exists the partial derivative $\frac{\partial}{\partial x} f \in C([t_+, \infty) \times \mathbb{R}^n, \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m))$. For any fixed t_*, $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$ such that $(t_*, x_*) \in L_{t_+}$ and $x^*_{s_2} \in D_{s_2}$, the operator Φ_{t_*, x_*} defined by*
$$\Phi_{t_*,x_*} := \left[\frac{\partial Q_2 f}{\partial x}(t_*, x_*) - B\right] P_2 \colon X_2 \to Y_2 \tag{3.2}$$
is invertible.
3. *There exists a number $R > 0$, a function $V \in C^1([t_+, \infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ positive on $[t_+, \infty) \times D_{s_1} \times D_{p_1}$, where $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid \|(x_{s_1}, x_{p_1})\| > R\}$, and a function $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$ such that:*
   (a) *$V(t, x_{s_1}, x_{p_1}) \to \infty$ as $\|(x_{s_1}, x_{p_1})\| \to \infty$ uniformly in t on each finite interval $[a, b) \subset [t_+, \infty)$;*
   (b) *for each $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1}, x_{p_1})\| > R$, the inequality*
   $$V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \leq \chi\big(t, V(t, x_{s_1}, x_{p_1})\big), \tag{3.3}$$
   where $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1})$ has the form (2.47), is satisfied;
   (c) *the differential inequality (2.3), i.e., $dv/dt \leq \chi(t, v)$ $(t \in [t_+, \infty))$, does not have positive solutions with finite escape time.*

Then for each initial point $(t_0, x_0) \in L_{t_+}$, where $S_2 x_0 \in D_{s_2}$, the initial value problem (2.1), (2.2) has a unique global (i.e., on $[t_0, \infty)$) solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $\operatorname{rank}(\lambda A + B) < n$; *when* $\operatorname{rank}(\lambda A + B) = n$, *the component $S_2 x$ is absent.*

Remark 3.3. The operator defined (for fixed t_*, x_*) by the formula from (3.2), i.e.,

$$\widehat{\Phi}_{t_*,x_*} := \left[\frac{\partial Q_2 f}{\partial x}(t_*, x_*) - B\right] P_2, \tag{3.4}$$

belongs to $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ and acts so that $\widehat{\Phi}_{t_*,x_*}\colon \mathbb{R}^n \to Y_2$, $X_1 \dot{+} X_s \subset \mathrm{Ker}(\widehat{\Phi}_{t_*,x_*})$. Its restriction to X_2 is the operator $\Phi_{t_*,x_*} = \widehat{\Phi}_{t_*,x_*}\big|_{X_2}$ defined by (3.2). Since the operator (3.2) is invertible, then $\widehat{\Phi}_{t_*,x_*}\mathbb{R}^n = \widehat{\Phi}_{t_*,x_*}X_2 = Y_2$ ($X_s \dot{+} X_1 = \mathrm{Ker}(\widehat{\Phi}_{t_*,x_*})$), and the extension $\widehat{\Phi}^{(-1)}_{t_*,x_*} \in \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ of $\Phi^{-1}_{t_*,x_*}$ to $\mathbb{R}^m$ that satisfies the equalities $\widehat{\Phi}^{(-1)}_{t_*,x_*}\widehat{\Phi}_{t_*,x_*} = P_2$, $\widehat{\Phi}_{t_*,x_*}\widehat{\Phi}^{(-1)}_{t_*,x_*} = Q_2$, $\widehat{\Phi}^{(-1)}_{t_*,x_*} = P_2\widehat{\Phi}^{(-1)}_{t_*,x_*}$ is the semi-inverse operator of $\widehat{\Phi}_{t_*,x_*}$ (i.e., $\widehat{\Phi}^{(-1)}_{t_*,x_*}\mathbb{R}^m = \widehat{\Phi}^{(-1)}_{t_*,x_*}Y_2 = X_2$, $\Phi^{-1}_{t_*,x_*} = \widehat{\Phi}^{(-1)}_{t_*,x_*}\big|_{Y_2}$).

The proof of Theorem 3.2. As shown above, the DAE (2.1) is equivalent to the system (2.43)–(2.46), where the representation $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$ (see (2.36)), $x_{s_i} = S_i x \in X_{s_i}$, $x_{p_i} = P_i x \in X_i$, $i = 1, 2$, is uniquely determined for each $x \in \mathbb{R}^n$. Notice that the correspondence between $X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$ and $X_{s_1} \times X_{s_2} \times X_1 \times X_2$ (i.e., between the representations $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$ and $x = (x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$ where $x_{s_i} \in X_{s_i}$, $x_{p_i} \in X_i$, $i = 1, 2$) is established in Section 2.3.

Since $\mathcal{B}_2^{(-1)}Q_2\mathbb{R}^m = \mathcal{B}_2^{(-1)}Y_2 = X_2 = B_2^{-1}Y_2 = B_2^{-1}(Q_2\mathbb{R}^m)$ (recall that $B_2^{-1} = \mathcal{B}_2^{(-1)}\big|_{Y_2}$) and $Q_2 f(t, x) \in Y_2$ for any (t, x), then the equation (2.45) is equivalent to the equation

$$B_2^{-1}Q_2 f(t, x) - x_{p_2} = 0, \tag{3.5}$$

where the projector Q_2 on Y_2 is considered as the operator belonging to $\mathrm{L}(\mathbb{R}^m, Y_2)$, while its projection properties are retained, i.e., $Q_2 y = Q_2 y_{p_2} = y_{p_2} \in Y_2$ for any $y \in \mathbb{R}^m$, and, for convenience, its previous notation Q_2 does not change. Denote

$$\tilde{f}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) = f(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) = f(t, x)$$

and consider the mapping

$$\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) := B_2^{-1}Q_2\tilde{f}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) - x_{p_2}, \tag{3.6}$$

where $\Psi\colon [t_+, \infty) \times X_{s_1} \times X_{s_2} \times X_1 \times X_2 \to X_2$. Then (3.5) can be written as

$$\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) = 0, \tag{3.7}$$

and this equation is equivalent to the equation (2.45), as shown above. Obviously, $\Psi \in C([t_+, \infty) \times X_{s_1} \times X_{s_2} \times X_1 \times X_2,\ X_2)$ has continuous partial derivatives with respect to x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}, and its partial derivative with respect to x_{p_2} at the point $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2})$ has the form

$$\begin{aligned} W_{t_*,x_*} &:= \frac{\partial \Psi}{\partial x_{p_2}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2}) \\ &= B_2^{-1}Q_2\left[\frac{\partial Q_2 f}{\partial x}(t_*, x_*) - B\right]P_2\big|_{X_2} = B_2^{-1}\Phi_{t_*,x_*} \in \mathrm{L}(X_2), \end{aligned} \tag{3.8}$$

where $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$ and $\Phi_{t_*,x_*} \in \mathrm{L}(X_2, Y_2)$ is the operator defined by (3.2). Since for any fixed element $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ such that $x_{s_2} \in D_{s_2}$ the operator $\Phi_{t,x}$ (where $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$) has the inverse

$\Phi_{t,x}^{-1} \in \mathrm{L}(Y_2, X_2)$, then the operator $W_{t,x}$ also has the inverse $W_{t,x}^{-1} = \Phi_{t,x}^{-1} B_2 \in \mathrm{L}(X_2)$ for the indicated (t, x).

Note that a point $(t, x) \in [t_+, \infty) \times \mathbb{R}^n$ belongs to the manifold L_{t_+} if and only if it satisfies the equations (2.41), (2.42) or the equivalent equations, e.g., (2.45), (2.46) or (2.45), (3.7), where $x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2} = x$.

Take any fixed $t_* \in [t_+, \infty)$, $x_{s_1}^* \in X_{s_1}$, $x_{s_2}^* \in D_{s_2}$, $x_{p_1}^* \in X_1$. Then, by virtue of condition 1, there exists a unique $x_{p_2}^* \in X_2$ such that $(t_*, x_*) \in L_{t_+}$, where $x_* = x_{s_1}^* + x_{s_2}^* + x_{p_1}^* + x_{p_2}^*$. Fix this point (t_*, x_*) and note that the operator (3.8) has the inverse $W_{t_*,x_*}^{-1} \in \mathrm{L}(X_2)$ for it, as shown above. In addition, the function $\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$ has the continuous partial derivative with respect to $(x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$ at every point from $[t_+, \infty) \times X_{s_1} \times X_{s_2} \times X_1 \times X_2$. Using the implicit function theorems and fixed point theorems [20], we obtain that there exist open neighborhoods $U_{\delta_1}(t_*)$ (if $t_* = t_+$, then $U_{\delta_1}(t_+) := [t_+, t_+ + \delta_1))$, $U_{\delta_2}(x_{s_1}^*)$, $U_{\delta_4}(x_{p_1}^*)$, $U_\varepsilon(x_{p_2}^*)$, a neighborhood $N_{\delta_3}(x_{s_2}^*)$ (see the definitions in Section 1) and a unique function $x_{p_2} = \mu(t, x_{s_1}, x_{s_2}, x_{p_1}) \in C(N_\delta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*), U_\varepsilon(x_{p_2}^*))$, where $N_\delta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = U_{\delta_1}(t_*) \times U_{\delta_2}(x_{s_1}^*) \times N_{\delta_3}(x_{s_2}^*) \times U_{\delta_4}(x_{p_1}^*)$, which is continuously differentiable in x_{s_1}, x_{p_1} and such that $\mu(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = x_{p_2}^*$ and $\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, \mu(t, x_{s_1}, x_{s_2}, x_{p_1})) = 0$ for all $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in N_\delta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$, i.e., the function $\mu(t, x_{s_1}, x_{s_2}, x_{p_1})$ is a solution of the equation (3.7) with respect to x_{p_2}. Moreover, if the neighborhood $N_{\delta_3}(x_{s_2}^*)$ is open, then the function μ is continuously differentiable in x_{s_2} on $N_\delta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$ as well. Since the implicit function theorems [20] assume that the set of variables is open, then to prove the existence of an implicitly defined function with the above properties when $t_* = t_+$ (i.e., $U_{\delta_1}(t_+) = [t_+, t_+ + \delta_1))$ and when the set D_{s_2} is not open (accordingly, $N_{\delta_3}(x_{s_2}^*)$ can be not open), the fixed point theorems [20, Theorems 46, 46_2] as well as the proofs of the implicit function theorems [20, Theorems 25, 28] are used.

Thus, it is proved that in some neighborhood $N_r(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = U_{r_1}(t_*) \times U_{r_2}(x_{s_1}^*) \times N_{r_3}(x_{s_2}^*) \times U_{r_4}(x_{p_1}^*)$ of each (fixed) point $(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$ there exists a unique solution $x_{p_2} = \mu_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(t, x_{s_1}, x_{s_2}, x_{p_1})$ of the equation (3.7) and, hence, the equivalent equation (2.45), and this solution is continuous in $(t, x_{s_1}, x_{s_2}, x_{p_1})$, continuously differentiable in (x_{s_1}, x_{p_1}) and satisfies the equality $\mu_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = x_{p_2}^* \in D_{p_2}$, where the set $D_{p_2} \subset X_2$ is such that for each $x_{p_2} \in D_{p_2}$ there exists $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$. Recall that $x_{p_2}^*$ is uniquely determined for each such $(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$ by virtue of condition 1. Introduce a function

$$\eta \colon [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1 \to D_{p_2}$$

and define it by $\eta(t, x_{s_1}, x_{s_2}, x_{p_1}) = \mu_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(t, x_{s_1}, x_{s_2}, x_{p_1})$ at the point $(t, x_{s_1}, x_{s_2}, x_{p_1}) = (t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$ for each $(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$. Then the function $x_{p_2} = \eta(t, x_{s_1}, x_{s_2}, x_{p_1})$ is continuous in $(t, x_{s_1}, x_{s_2}, x_{p_1})$, continuously differentiable in (x_{s_1}, x_{p_1}) and satisfies the equation (2.45) as well as the equation (3.7), i.e., $\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, \eta(t, x_{s_1}, x_{s_2}, x_{p_1})) = 0$, for $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$. Let us prove the uniqueness of the function η. Assume that there exists another function $x_{p_2} = \zeta(t, x_{s_1}, x_{s_2}, x_{p_1})$ defined in the same way as the function η and, accordingly, having the same properties, but differing from η at some point $(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$. Then, due to condition 1, $\eta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = \zeta(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) = x_{p_2}^*$ (since $(t_*, x_{s_1}^* + x_{s_2}^* + x_{p_1}^* + x_{p_2}^*) \in L_{t_+}$), which contradicts the assumption. This holds for each point $(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*) \in [t_+, \infty) \times X_{s_1} \times D_{s_2} \times X_1$, and hence $\eta(t, x_{s_1}, x_{s_2}, x_{p_1}) \equiv \zeta(t, x_{s_1}, x_{s_2}, x_{p_1})$.

Choose any initial point $(t_0, x_0) \in L_{t_+}$, where $S_2 x_0 \in D_{s_2}$, and any function $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ satisfying the condition $\phi_{s_2}(t_0) = S_2 x_0$. Substitute the chosen function into η and denote $q(t, x_{s_1}, x_{p_1}) = \eta(t, x_{s_1}, \phi_{s_2}(t), x_{p_1})$. Further, we substitute the functions $x_{p_2} = q(t, x_{s_1}, x_{p_1})$ and $x_{s_2} = \phi_{s_2}(t)$ in (2.43), (2.44) and write the obtained system in the form

$$\frac{d}{dt}\omega = \widetilde{\Upsilon}(t, \omega), \tag{3.9}$$

where $\omega = \begin{pmatrix} x_{s_1} \\ x_{p_1} \end{pmatrix}$,

$$\widetilde{\Upsilon}(t, \omega) = \begin{pmatrix} \mathcal{A}_{\text{gen}}^{(-1)} \big[F_1 \tilde{f}(t, x_{s_1}, \phi_{s_2}(t), x_{p_1}, q(t, x_{s_1}, x_{p_1})) - \mathcal{B}_{\text{gen}} x_{s_1} - \mathcal{B}_{\text{und}} \phi_{s_2}(t)\big] \\ \mathcal{A}_1^{(-1)} \big[Q_1 \tilde{f}(t, x_{s_1}, \phi_{s_2}(t), x_{p_1}, q(t, x_{s_1}, x_{p_1})) - \mathcal{B}_1 x_{p_1}\big] \end{pmatrix}$$
$$= \Upsilon(t, x_{s_1} + \phi_{s_2}(t) + x_{p_1} + q(t, x_{s_1}, x_{p_1}))$$

($\Upsilon(t, x)$ is defined in (2.48)).

Due to the properties of $\tilde{f}$, q and ϕ_{s_2}, the function $\widetilde{\Upsilon}(t, \omega)$ is continuous in (t, ω) and continuously differentiable in ω on $[t_0, \infty) \times X_{s_1} \times X_1$. Consequently, there exists a unique solution $\omega = \omega(t)$ of (3.9) on some interval $[t_0, \beta)$ which satisfies the initial condition

$$\omega(t_0) = \omega_0, \quad \omega_0 = (x_{s_1,0}^{\mathrm{T}}, x_{p_1,0}^{\mathrm{T}})^{\mathrm{T}}, \quad x_{s_1,0} = S_1 x_0, \quad x_{p_1,0} = P_1 x_0. \tag{3.10}$$

This solution can be extended over a maximal interval of existence $[t_0, \beta) \subseteq [t_0, \infty)$ (i.e., the solution exists on $[t_0, \beta)$ and does not exist on a larger interval), and the extended solution $\omega(t)$ is a unique solution of the IVP (3.9), (3.10) on the whole interval $[t_0, \beta)$ (see, e.g., [21]).

Let us introduce the function $V(t, \omega) := V(t, x_{s_1}, x_{p_1})$, where $V(t, x_{s_1}, x_{p_1})$ follows the theorem condition 3. It follows from condition 3 that the derivative of V along the trajectories of the equation (3.9) satisfies the inequality

$$V'_{(3.9)}(t, \omega) = \frac{\partial V}{\partial t}(t, \omega) + \frac{\partial V}{\partial \omega}(t, \omega) \widetilde{\Upsilon}(t, \omega) \le \chi\big(t, V(t, \omega)\big) \tag{3.11}$$

for all $t \ge t_0$, $\|\omega\| > R$. Due to condition (c), the differential inequality (2.3), $t \ge t_0$, does not have positive solutions with finite escape time. Hence, by [19, Chapter IV, Theorem XIII] every solution of (3.9) exists on $[t_0, \infty)$ (or is defined in the future [19]), and, consequently, the solution $\omega(t) = (\omega_{s_1}(t)^{\mathrm{T}}, \omega_{p_1}(t)^{\mathrm{T}})^{\mathrm{T}}$ is global (i.e., the maximal interval of existence is $[t_0, \infty)$).

Thus, the functions $\omega_{s_1} \in C^1([t_0,\infty), X_{s_1})$, $\omega_{p_1} \in C^1([t_0,\infty), X_1)$ (the components of the solution $\omega(t)$) and $q(t,\omega_{s_1}(t),\omega_{p_1}(t)) = \eta(t,\omega_{s_1}(t),\phi_{s_2}(t),\omega_{p_1}(t))$ are a unique solution of the system (2.43), (2.44) and (2.45) on $[t_0,\infty)$, and the equation (2.46) is an identity since $\big(t,\omega_{s_1}(t)+\phi_{s_2}(t)+\omega_{p_1}(t)+q(t,\omega_{s_1}(t),\omega_{p_1}(t))\big) \in L_{t_0}$ for all $t \in [t_0,\infty)$. Therefore, the function

$$x(t) = \omega_{s_1}(t)+\phi_{s_2}(t)+\omega_{p_1}(t)+q(t,\omega_{s_1}(t),\omega_{p_1}(t))$$

is a unique solution of the IVP (2.1), (2.2) on $[t_0,\infty)$. The chosen function $\phi_{s_2} \in C([t_0,\infty), D_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2x_0$, which can be regarded as a functional parameter, uniquely defines the component $S_2x(t) = \phi_{s_2}(t)$ of the solution $x(t)$. If $\operatorname{rank}(\lambda A+B) = n \le m$, then $X_{s_2} = \{0\}$, $S_2 = 0$ and the component S_2x is absent. Since the initial point (t_0,x_0) was chosen arbitrarily, then it is proved that the IVP (2.1), (2.2) has a unique global solution $x(t)$ with the fixed component $S_2x(t) = \phi_{s_2}(t)$ (where ϕ_{s_2} is an arbitrary function belonging to $C([t_0,\infty), D_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2x_0$) for each initial point $(t_0,x_0) \in L_{t_+}$ where $S_2x_0 \in D_{s_2}$. □

A mapping $f(t,x)$ of a set $J\times D$, where J is an interval in $\mathbb{R}$, $D \subset X$ and X is a linear space, into a linear space Y is said to *satisfy locally a Lipschitz condition* (or to be *locally Lipschitz continuous*) *with respect to* x *on* $J\times D$ if for each (fixed) $(t_*,x_*) \in J\times D$ there exist open neighborhoods $U(t_*)$, $\widetilde{U}(x_*)$ of the points t_*, x_* and a constant $L \ge 0$ such that $\|f(t,x_1)-f(t,x_2)\| \le L\|x_1-x_2\|$ for any $t \in U(t_*)$, $x_1,x_2 \in \widetilde{U}(x_*)$.

Theorem 3.4. *Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^m)$ and $\lambda A+B$ be a singular pencil of operators such that its regular block λA_r+B_r from (2.17) has the index not higher than 1. Assume that conditions 1 and 3 of Theorem 3.2 hold and that condition 2 of Theorem 3.2 is replaced by the following:*

2. *A function $f(t,x)$ satisfies locally a Lipschitz condition with respect to x on $[t_+,\infty)\times\mathbb{R}^n$. For any fixed t_*, $x_* = x^*_{s_1}+x^*_{s_2}+x^*_{p_1}+x^*_{p_2}$ such that $(t_*,x_*) \in L_{t_+}$ and $x^*_{s_2} \in D_{s_2}$, there exists a neighborhood $N_\delta(t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}) = U_{\delta_1}(t_*)\times U_{\delta_2}(x^*_{s_1})\times N_{\delta_3}(x^*_{s_2})\times U_{\delta_4}(x^*_{p_1})$, an open neighborhood $U_\varepsilon(x^*_{p_2})$ (the numbers $\delta,\varepsilon>0$ depend on the choice of t_*, x_*) and an invertible operator $\Phi_{t_*,x_*} \in \mathrm{L}(X_2,Y_2)$ such that for each $(t,x_{s_1},x_{s_2},x_{p_1}) \in N_\delta(t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1})$ and each $x^i_{p_2} \in U_\varepsilon(x^*_{p_2})$, $i=1,2$, the mapping*

$$\widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}) := Q_2f(t,x_{s_1}+x_{s_2}+x_{p_1}+x_{p_2}) - B\big|_{X_2}x_{p_2}\colon [t_+,\infty)\times X_{s_1}\times X_{s_2}\times X_1\times X_2 \to Y_2 \quad (3.12)$$

satisfies the inequality

$$\|\widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x^1_{p_2}) - \widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x^2_{p_2}) - \Phi_{t_*,x_*}[x^1_{p_2}-x^2_{p_2}]\| \le c(\delta,\varepsilon)\|x^1_{p_2}-x^2_{p_2}\|, \quad (3.13)$$

where $c(\delta,\varepsilon)$ is such that $\lim_{\delta,\,\varepsilon\to 0} c(\delta,\varepsilon) < \|\Phi^{-1}_{t_*,x_*}\|^{-1}$.

Then for each initial point $(t_0,x_0)\in L_{t_+}$*, where* $S_2x_0\in D_{s_2}$*, the initial value problem* (2.1)*,* (2.2) *has a unique global solution* $x(t)$ *for which the choice of the function* $\phi_{s_2}\in C([t_0,\infty),D_{s_2})$ *with the initial value* $\phi_{s_2}(t_0)=S_2x_0$ *uniquely defines the component* $S_2x(t)=\phi_{s_2}(t)$ *when* $\operatorname{rank}(\lambda A+B)<n$*; when* $\operatorname{rank}(\lambda A+B)=n$*, the component* S_2x *is absent.*

Remark 3.5. If a function $f(t,x)$ has the partial derivative $\frac{\partial}{\partial x}f\in C([t_+,\infty)\times\mathbb{R}^n,\mathrm{L}(\mathbb{R}^n,\mathbb{R}^m))$, then the function (3.12) has the continuous partial derivatives with respect to x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2} on $[t_+,\infty)\times X_{s_1}\times X_{s_2}\times X_1\times X_2$ and $\dfrac{\partial\widetilde{\Psi}}{\partial x_{p_2}}(t_*,x_{s_1}^*,x_{s_2}^*,x_{p_1}^*,x_{p_2}^*)=\Phi_{t_*,x_*}$, where the operator Φ_{t_*,x_*} is defined by (3.2), $x_*=x_{s_1}^*+x_{s_2}^*+x_{p_1}^*+x_{p_2}^*$.

Corollary 3.6. *If the conditions of Theorem* 3.2 *are fulfilled, then the conditions of Theorem* 3.4 *are also fulfilled.*

Proof. Obviously, it follows from the existence of $\dfrac{\partial}{\partial x}f\in C([t_+,\infty)\times\mathbb{R}^n,\mathrm{L}(\mathbb{R}^n,\mathbb{R}^m))$ that $f(t,x)$ satisfies locally a Lipschitz condition with respect to x on $[t_+,\infty)\times\mathbb{R}^n$. Take Φ_{t_*,x_*} defined by (3.2) as the operator Φ_{t_*,x_*} appearing in condition 2 of Theorem 3.4. Then $\Phi_{t_*,x_*}=\dfrac{\partial\widetilde{\Psi}}{\partial x_{p_2}}(t_*,x_{s_1}^*,x_{s_2}^*,x_{p_1}^*,x_{p_2}^*)$, where $x_*=x_{s_1}^*+x_{s_2}^*+x_{p_1}^*+x_{p_2}^*$, and there exists $\Phi_{t_*,x_*}^{-1}\in\mathrm{L}(Y_2,X_2)$ by virtue of condition 2 of Theorem 3.2. It is readily verified that condition 2 of Theorem 3.4, where Φ_{t_*,x_*} is the operator (3.2), is satisfied. The rest of the conditions of Theorems 3.2 and Theorem 3.4 coincide. □

The proof of Theorem 3.4. We define the norm $\|\cdot\|$ in $X_{s_1}\dot{+}X_{s_2}\dot{+}X_1\dot{+}X_2$ as $\|x\|=\|x_{s_1}\|+\|x_{s_2}\|+\|x_{p_1}\|+\|x_{p_2}\|$, where we denote by $\|x_{s_1}\|=\|x_{s_1}\|_{X_{s_1}}$, $\|x_{s_2}\|=\|x_{s_2}\|_{X_{s_2}}$, $\|x_{p_1}\|=\|x_{p_1}\|_{X_1}$ and $\|x_{p_2}\|=\|x_{p_2}\|_{X_2}$ the norms of the components x_{s_1}, x_{s_2}, x_{p_1} and x_{p_2} in the subspaces X_{s_1}, X_{s_2}, X_1 and X_2, respectively. Taking into account the correspondence between $X_{s_1}\dot{+}X_{s_2}\dot{+}X_1\dot{+}X_2$ and $X_{s_1}\times X_{s_2}\times X_1\times X_2$ which is established in Section 2.3, the norm $\|x\|$ of $x\in X_{s_1}\times X_{s_2}\times X_1\times X_2$ is defined in the same way and coincides with the above-defined norm of the corresponding element $x\in X_{s_1}\dot{+}X_{s_2}\dot{+}X_1\dot{+}X_2$. Since for any norm $\|\cdot\|_{\mathbb{R}^n}$ in $\mathbb{R}^n$ the inequality $\|x\|_{\mathbb{R}^n}\le\|x_{s_1}\|_{\mathbb{R}^n}+\|x_{s_2}\|_{\mathbb{R}^n}+\|x_{p_1}\|_{\mathbb{R}^n}+\|x_{p_2}\|_{\mathbb{R}^n}$ holds (due to (2.36)), then the chosen norm is "maximal". Similarly, in $\mathbb{R}\times\mathbb{R}^n$ we use the norm $\|(t,x)\|=\|t\|+\|x_{s_1}\|+\|x_{s_2}\|+\|x_{p_1}\|+\|x_{p_2}\|$.

Consider the equation (3.7), that is, $\Psi(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2})=0$ where Ψ is defined by (3.6). Recall that this equation is equivalent to the equation (2.45) and that $B\big|_{X_2}=\mathcal{B}_2\big|_{X_2}=B_2$. The mapping (3.12) can be represented as

$$\begin{aligned}\widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2})&=Q_2\tilde{f}(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2})-B_2x_{p_2}\\&=B_2\Psi(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}),\end{aligned}$$

and we can rewrite the equation (3.7) in the form

$$x_{p_2}=\mathcal{N}(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}),\tag{3.14}$$

where $\mathcal{N}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) := x_{p_2} - \Phi^{-1}_{t_*,x_*} \widetilde{\Psi}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$. Recall that if $(t_*, x_*) \in L_{t_+}$, then (t_*, x_*) satisfies (3.7), i.e., $\Psi(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2}) = 0$ where $x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2} = x_*$.

Lemma 3.7. *For any fixed elements $t_* \in [t_+, \infty)$, $x^*_{s_1} \in X_{s_1}$, $x^*_{s_2} \in D_{s_2}$, $x^*_{p_1} \in X_1$, $x^*_{p_2} \in X_2$ for which $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}) \in L_{t_+}$, there exists a neighborhood $N_r(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = U_{r_1}(t_*) \times U_{r_2}(x^*_{s_1}) \times N_{r_3}(x^*_{s_2}) \times U_{r_4}(x^*_{p_1})$ (where $N_{r_3}(x^*_{s_2}) \neq \{x^*_{s_2}\}$ if the neighborhood $N_{\delta_3}(x^*_{s_2})$ defined in condition 2 does not degenerate into the point $x^*_{s_2}$, i.e., $N_{\delta_3}(x^*_{s_2}) \neq \{x^*_{s_2}\}$), an open neighborhood $U_\rho(x^*_{p_2})$ and a unique function $x_{p_2} = \mu(t, x_{s_1}, x_{s_2}, x_{p_1}) \in C(N_r(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}), U_\rho(x^*_{p_2}))$ which satisfies the equality $\mu(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = x^*_{p_2}$ and a Lipschitz condition with respect to (x_{s_1}, x_{p_1}) (with respect to $(x_{s_1}, x_{s_2}, x_{p_1})$ if $N_{\delta_3}(x^*_{s_2}) \neq \{x^*_{s_2}\}$) on $N_r(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1})$ and is a solution of the equation (3.7) with respect to x_{p_2}, i.e., $\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, \mu(t, x_{s_1}, x_{s_2}, x_{p_1})) = 0$, for all $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in N_r(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1})$ (the numbers $r, \rho > 0$ depend on the choice of t_*, $x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2}$).*

Proof. It follows from condition 2 that for any fixed point $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}) \in L_{t_+}$ for which $x^*_{s_2} \in D_{s_2}$, there exists a closed neighborhood $\overline{N_{\tilde{\delta}}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = \overline{U_{\tilde{\delta}_1}}(t_*) \times \overline{U_{\tilde{\delta}_2}}(x^*_{s_1}) \times \overline{N_{\tilde{\delta}_3}}(x^*_{s_2}) \times \overline{U_{\tilde{\delta}_4}}(x^*_{p_1}) \subset N_\delta(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = U_{\delta_1}(t_*) \times U_{\delta_2}(x^*_{s_1}) \times N_{\delta_3}(x^*_{s_2}) \times U_{\delta_4}(x^*_{p_1})$, where $0 < \tilde{\delta}_i < \delta_i$, $i = 1, 2, 4$, $\tilde{\delta}_3 = \delta_3 = 0$ if $N_{\delta_3}(x^*_{s_2}) = \{x^*_{s_2}\}$ and $0 < \tilde{\delta}_3 < \delta_3$ (i.e., $\{x^*_{s_2}\} \neq \overline{N_{\tilde{\delta}_3}}(x^*_{s_2}) \subsetneqq N_{\delta_3}(x^*_{s_2})$) otherwise, and a closed neighborhood $\overline{U_{\tilde{\varepsilon}}}(x^*_{p_2}) \subset U_\varepsilon(x^*_{p_2})$ such that $\mathcal{N}$ is a contractive mapping with respect to x_{p_2} (uniformly in $(t, x_{s_1}, x_{s_2}, x_{p_1})$) on $\overline{N_{\tilde{\delta}}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) \times \overline{U_{\tilde{\varepsilon}}}(x^*_{p_2})$, i.e.,

$$\|\mathcal{N}(t, x_{s_1}, x_{s_2}, x_{p_1}, x^1_{p_2}) - \mathcal{N}(t, x_{s_1}, x_{s_2}, x_{p_1}, x^2_{p_2})\| \leq l \|x^1_{p_2} - x^2_{p_2}\|, \quad l < 1 \quad (3.15)$$

(l is a constant), for every $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in \overline{N_{\tilde{\delta}}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1})$, $x^i_{p_2} \in \overline{U_{\tilde{\varepsilon}}}(x^*_{p_2})$, $i = 1, 2$. Indeed, due to (3.13), there exist numbers $\tilde{\delta} \in (0, \delta)$ (accordingly, numbers $\tilde{\delta}_i \in (0, \delta_i)$, $i = 1, 2, 4$, and $\tilde{\delta}_3 \in (0, \delta_3]$ such that $\tilde{\delta}_3 = \delta_3 = 0$ if $N_{\delta_3}(x^*_{s_2}) = \{x^*_{s_2}\}$ and $0 < \tilde{\delta}_3 < \delta_3$ otherwise) and $\tilde{\varepsilon} \in (0, \varepsilon)$ such that for every $\alpha = (t, x_{s_1}, x_{s_2}, x_{p_1}) \in \overline{N_{\tilde{\delta}}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = \overline{U_{\tilde{\delta}_1}}(t_*) \times \overline{U_{\tilde{\delta}_2}}(x^*_{s_1}) \times \overline{N_{\tilde{\delta}_3}}(x^*_{s_2}) \times \overline{U_{\tilde{\delta}_4}}(x^*_{p_1})$ and every $x^1_{p_2}, x^2_{p_2} \in \overline{U_{\tilde{\varepsilon}}}(x^*_{p_2})$ the following holds:

$$\begin{aligned}\|\mathcal{N}(\alpha, x^1_{p_2}) - \mathcal{N}(\alpha, x^2_{p_2})\| &= \|x^1_{p_2} - x^2_{p_2} - \Phi^{-1}_{t_*,x_*}[\widetilde{\Psi}(\alpha, x^1_{p_2}) - \widetilde{\Psi}(\alpha, x^2_{p_2})\| \\ &\leq \|\Phi^{-1}_{t_*,x_*}\| c(\delta_0, \varepsilon_0) \|x^1_{p_2} - x^2_{p_2}\|,\end{aligned}$$

where $\|\Phi^{-1}_{t_*,x_*}\| c(\delta_0, \varepsilon_0) \leq l < 1$ for every $\delta_0 \in (0, \tilde{\delta}]$, $\varepsilon_0 \in (0, \tilde{\varepsilon}]$. This implies (3.15).

Choose a point $(t_*, x_*) = (t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2})$ such that $x^*_{s_2} \in D_{s_2}$ and $(t_*, x_*) \in L_{t_+}$, and fix it. As above, denote $\alpha = (t, x_{s_1}, x_{s_2}, x_{p_1})$, and denote $\alpha_* = (t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1})$, then $\mathcal{N}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) = \mathcal{N}(\alpha, x_{p_2})$.

Since $\widetilde{\Psi}$ is continuous on $[t_+, \infty) \times X_{s_1} \times X_{s_2} \times X_1 \times X_2$, then $\widetilde{\Psi}(\alpha, x^*_{p_2}) \to \widetilde{\Psi}(\alpha_*, x^*_{p_2}) = 0$ as $\alpha \to \alpha_*$, and therefore there exists a closed neighborhood

$\overline{N_{\delta_*}}(\alpha_*) = \overline{U_{\delta_1^*}}(t_*) \times \overline{U_{\delta_2^*}}(x_{s_1}^*) \times \overline{N_{\delta_3^*}}(x_{s_2}^*) \times \overline{U_{\delta_4^*}}(x_{p_1}^*) \subseteq \overline{N_{\tilde{\delta}}}(\alpha_*)$, where $\delta_* \in (0,\tilde{\delta}]$, $\delta_i^* \in (0,\tilde{\delta}_i]$, $i=1,2,4$, $\delta_3^* \in [0,\tilde{\delta}_3]$ and $\delta_3^* \neq 0$ if $N_{\delta_3}(x_{s_2}^*) \neq \{x_{s_2}^*\}$, such that $\|\Phi_{t_*,x_*}^{-1}\| \|\widetilde{\Psi}(\alpha, x_{p_2}^*)\| \leq (1-l)\tilde{\varepsilon}$ (where l is the constant from (3.15)) for every $\alpha \in \overline{N_{\delta_*}}(\alpha_*)$. Hence, for each (fixed) $\alpha \in \overline{N_{\delta_*}}(\alpha_*)$ and every $x_{p_2} \in \overline{U_{\tilde{\varepsilon}}}(x_{p_2}^*)$ we have $\|\mathcal{N}(\alpha, x_{p_2}) - x_{p_2}^*\| \leq \|\mathcal{N}(\alpha, x_{p_2}) - \mathcal{N}(\alpha, x_{p_2}^*)\| + \|\Phi_{t_*,x_*}^{-1}\| \|\widetilde{\Psi}(\alpha, x_{p_2}^*)\| \leq l\tilde{\varepsilon} + (1-l)\tilde{\varepsilon} = \tilde{\varepsilon}$. Thus, $\mathcal{N}(\alpha, x_{p_2})$ maps $\overline{U_{\tilde{\varepsilon}}}(x_{p_2}^*)$ into itself for each $\alpha \in \overline{N_{\delta_*}}(\alpha_*)$.

From the foregoing it follows that, by the fixed point theorems (see, e.g., [20, Theorems 46, 46_2]), the mapping $\mathcal{N}(\alpha, x_{p_2})$ as a function of x_{p_2}, depending on the parameter $\alpha = (t, x_{s_1}, x_{s_2}, x_{p_1})$, has a unique fixed point $\mu_\alpha = \mu(\alpha)$ (i.e., $\mathcal{N}(\alpha, \mu(\alpha)) = \mu(\alpha)$) in $\overline{U_{\tilde{\varepsilon}}}(x_{p_2}^*)$ for each $\alpha \in \overline{N_{\delta_*}}(\alpha_*) = \overline{N_{\delta_*}}(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$, which satisfies the equality $\mu(\alpha_*) = x_{p_2}^*$, and $\mu(\alpha)$ depends continuously on α. The continuity of the function $\mu\colon \overline{N_{\delta_*}}(\alpha_*) \to \overline{U_{\tilde{\varepsilon}}}(x_{p_2}^*)$ is proved in the same way as in [20, Theorem 46_2].

Let us prove that $\mu(\alpha) = \mu(t, x_{s_1}, x_{s_2}, x_{p_1})$ satisfies a Lipschitz condition with respect to (x_{s_1}, x_{p_1}) (with respect to $(x_{s_1}, x_{s_2}, x_{p_1})$ if $N_{\delta_3}(x_{s_2}^*)$ does not degenerate into the point $x_{s_2}^*$) on $N_r(\alpha_*) = N_r(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$, where the neighborhood $N_r(\alpha_*)$ is specified below ($\alpha_* = (t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$). Recall that here we use the notation $\tilde{f}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2}) = f(t,x)$ introduced in the proof of Theorem 3.2. Since the function $f(t,x)$ satisfies locally a Lipschitz condition with respect to x on $[t_+,\infty) \times \mathbb{R}^n$, then there exists an open neighborhood $U(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*, x_{p_2}^*) = \widehat{U}(t_*) \times \widetilde{U}(x_{s_1}^*, x_{s_2}^*, x_{p_1}^*, x_{p_2}^*)$ and a constant $L \geq 0$ such that

$$\begin{aligned} &\|\tilde{f}(t, x_{s_1}^1, x_{s_2}^1, x_{p_1}^1, x_{p_2}^1) - \tilde{f}(t, x_{s_1}^2, x_{s_2}^2, x_{p_1}^2, x_{p_2}^2)\| \\ &\qquad \leq L\|(x_{s_1}^1, x_{s_2}^1, x_{p_1}^1, x_{p_2}^1) - (x_{s_1}^2, x_{s_2}^2, x_{p_1}^2, x_{p_2}^2)\| \\ &\qquad = L\big(\|x_{s_1}^1 - x_{s_1}^2\| + \|x_{s_2}^1 - x_{s_2}^2\| + \|x_{p_1}^1 - x_{p_1}^2\| + \|x_{p_2}^1 - x_{p_2}^2\|\big) \end{aligned} \tag{3.16}$$

for any $(t, x_{s_1}^i, x_{s_2}^i, x_{p_1}^i, x_{p_2}^i) \in U(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*, x_{p_2}^*)$, $i=1,2$. Choose numbers $r \in (0,\delta_*]$, $r_i \in (0,\delta_i^*]$, $i=1,2,4$, a number $r_3 \in [0,\delta_3^*]$ such that $r_3 \neq 0$ if $N_{\delta_3}(x_{s_2}^*) \neq \{x_{s_2}^*\}$ and a number $\rho \in (0,\tilde{\varepsilon}]$ so that $N_r(\alpha_*) = U_{r_1}(t_*) \times U_{r_2}(x_{s_1}^*) \times N_{r_3}(x_{s_2}^*) \times U_{r_4}(x_{p_1}^*) \subset \overline{N_{\delta_*}}(\alpha_*)$, $U_\rho(x_{p_2}^*) \subset \overline{U_{\tilde{\varepsilon}}}(x_{p_2}^*)$, $N_r(\alpha_*) \times U_\rho(x_{p_2}^*) \subseteq U(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*, x_{p_2}^*)$ and μ maps $N_r(\alpha_*)$ into $U_\rho(x_{p_2}^*)$. Then, carrying out certain transformations and using (3.15), (3.16), we obtain that

$$\begin{aligned} \|\mu(t, x_{s_1}^1, x_{s_2}^1, x_{p_1}^1) - \mu(t, x_{s_1}^2, x_{s_2}^2, x_{p_1}^2)\| &\leq \widehat{L}\,\|(x_{s_1}^1, x_{s_2}^1, x_{p_1}^1) - (x_{s_1}^2, x_{s_2}^2, x_{p_1}^2)\| \\ &= \widehat{L}\big(\|x_{s_1}^1 - x_{s_1}^2\| + \|x_{s_2}^1 - x_{s_2}^2\| + \|x_{p_1}^1 - x_{p_1}^2\|\big), \end{aligned}$$

where

$$\widehat{L} = L\|\Phi_{t_*,x_*}^{-1}\|\,\|Q_2\|/(1-l) \geq 0,$$

for any $(t, x_{s_1}^i, x_{s_2}^i, x_{p_1}^i) \in N_r(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$, $i=1,2$, where $x_{s_2}^1 = x_{s_2}^2 = x_{s_2}^*$ if $N_{\delta_3}(x_{s_2}^*) = \{x_{s_2}^*\}$. Hence, $\mu(t, x_{s_1}, x_{s_2}, x_{p_1})$ satisfies a Lipschitz condition with respect to (x_{s_1}, x_{p_1}) (with respect to $(x_{s_1}, x_{s_2}, x_{p_1})$ if $N_{\delta_3}(x_{s_2}^*) \neq \{x_{s_2}^*\}$) on $N_r(t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*)$.

Since the equations (3.7) and (3.14) are equivalent, the lemma is proved. □

Due to condition 1 of Theorem 3.2, for any fixed $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) \in [t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1$ there exists a unique $x^*_{p_2} \in X_2$ such that $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}) \in L_{t_+}$. Further, it follows from Lemma 3.7 that in some neighborhood $N_r(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = U_{r_1}(t_*) \times U_{r_2}(x^*_{s_1}) \times N_{r_3}(x^*_{s_2}) \times U_{r_4}(x^*_{p_1})$ of each (fixed) point $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) \in [t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1$ there exists a unique solution $x_{p_2} = \mu_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(t, x_{s_1}, x_{s_2}, x_{p_1})$ of the equation (3.7), and this solution is continuous in $(t, x_{s_1}, x_{s_2}, x_{p_1})$, satisfies a Lipschitz condition with respect to (x_{s_1}, x_{p_1}) and the equality $\mu_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = x^*_{p_2} \in D_{p_2}$, where the set $D_{p_2} \subset X_2$ is such that for each $x_{p_2} \in D_{p_2}$ there exists $(t, x_{s_1}, x_{s_2}, x_{p_1}) \in [t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$. As in the proof of Theorem 3.2, we introduce a function $\eta\colon [t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1 \to D_{p_2}$ and define it by $\eta(t, x_{s_1}, x_{s_2}, x_{p_1}) = \mu_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(t, x_{s_1}, x_{s_2}, x_{p_1})$ at the point $(t, x_{s_1}, x_{s_2}, x_{p_1}) = (t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1})$ for each $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) \in [t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1$. Then the function $x_{p_2} = \eta(t, x_{s_1}, x_{s_2}, x_{p_1})$ is continuous in $(t, x_{s_1}, x_{s_2}, x_{p_1})$, satisfies locally a Lipschitz condition with respect to (x_{s_1}, x_{p_1}) on $[t_+,\infty) \times X_{s_1} \times D_{s_2} \times X_1$ and is a unique solution of the equation (3.7) (i.e., $\Psi(t, x_{s_1}, x_{s_2}, x_{p_1}, \eta(t, x_{s_1}, x_{s_2}, x_{p_1})) = 0$) as well as the equation (2.45) with respect to x_{p_2}. The uniqueness of η is proved in the same way as the uniqueness of the function η in the proof of Theorem 3.2.

Choose any initial point $(t_0, x_0) \in L_{t_+}$, where $S_2x_0 \in D_{s_2}$, and any function $\phi_{s_2} \in C([t_0,\infty), D_{s_2})$ satisfying the condition $\phi_{s_2}(t_0) = S_2x_0$. We substitute the function $x_{s_2} = \phi_{s_2}(t)$ into η, denote $q(t, x_{s_1}, x_{p_1}) = \eta(t, x_{s_1}, \phi_{s_2}(t), x_{p_1})$, and then we substitute the functions $x_{s_2} = \phi_{s_2}(t)$ and $x_{p_2} = q(t, x_{s_1}, x_{p_1})$ in (2.43), (2.44). We write the obtained system in the form (3.9). Due to the properties of $\tilde{f}(t, x_{s_1}, x_{s_2}, x_{p_1}, x_{p_2})$, $\eta(t, x_{s_1}, x_{s_2}, x_{p_1})$ (and, accordingly, $q(t, x_{s_1}, x_{p_1})$) and $\phi_{s_2}(t)$, the function $\widetilde{\Upsilon}(t,\omega)$ is continuous in (t,ω) and satisfies locally a Lipschitz condition with respect to ω on $[t_0,\infty) \times X_{s_1} \times X_1$. Consequently, there exists a unique solution $\omega = \omega(t)$ of (3.9) on some interval $[t_0, \beta)$ which satisfies the initial condition (3.10) (this follows from, e.g., [21, Theorem 1]).

The subsequent proof coincides with the proof of Theorem 3.2 (see the part of the proof after (3.10)). □

4. Lagrange stability of singular semilinear DAEs

Theorem 4.1. *Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^m)$ and $\lambda A + B$ be a singular pencil of operators such that its regular block $\lambda A_r + B_r$ from (2.17) has the index not higher than 1. Assume that condition 1 of Theorem 3.2 holds and condition 2 of Theorem 3.2 or condition 2 of Theorem 3.4 holds. Let the following condition be satisfied:*

3. *There exists a number $R > 0$, a function $V \in C^1([t_+,\infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ positive on $[t_+,\infty) \times D_{s_1} \times D_{p_1}$, where $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid \|(x_{s_1}, x_{p_1})\| > R\}$, and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:*
   (a) *$V(t, x_{s_1}, x_{p_1}) \to \infty$ as $\|(x_{s_1}, x_{p_1})\| \to \infty$ uniformly in t on $[t_+,\infty)$;*
   (b) *for each $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1}, x_{p_1})\| > R$, the inequality (3.3) is satisfied;*

(c) *the differential inequality* (2.3), *i.e.*, $\dfrac{dv}{dt} \le \chi(t,v)$ $(t \in [t_+,\infty))$, *does not have unbounded positive solutions for* $t \in [t_+,\infty)$.

Then for each initial point $(t_0,x_0) \in L_{t_+}$, *where* $S_2x_0 \in D_{s_2}$, *the initial value problem* (2.1), (2.2) *has a unique global solution* $x(t)$ *for which the choice of the function* $\phi_{s_2} \in C([t_0,\infty),D_{s_2})$ *with the initial value* $\phi_{s_2}(t_0) = S_2x_0$ *uniquely defines the component* $S_2x(t) = \phi_{s_2}(t)$ *when* $\operatorname{rank}(\lambda A + B) < n$.

Let, in addition to the above conditions, the following conditions also hold:

4. *For all* $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, *for which* $x_{s_2} \in D_{s_2}$ *and* $\|x_{s_1} + x_{s_2} + x_{p_1}\| \le M < \infty$ (M *is an arbitrary constant*), *the inequality*
$$\|x_{p_2}\| \le K_M < \infty$$
or the inequality
$$\|Q_2 f(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2})\| \le K_M < \infty,$$
where $K_M = K(M)$ *is a constant, holds.*
5. $\|F_2 f(t,x)\| < \infty$ *for all* $(t,x) \in L_{t_+}$ *such that* $S_2x \in D_{s_2}$ *and* $\|x\| \le C < \infty$ (C *is an arbitrary constant*).

Then, for the initial points $(t_0,x_0) \in L_{t_+}$ *where* $S_2x_0 \in D_{s_2}$ *and any function* $\phi_{s_2} \in C([t_0,\infty),D_{s_2})$ *satisfying the relations* $\phi_{s_2}(t_0) = S_2x_0$ *and* $\sup_{t\in[t_0,\infty)} \|\phi_{s_2}(t)\| < \infty$, *the equation* (2.1), *where* $S_2x = \phi_{s_2}(t)$, *is Lagrange stable*; *when* $\operatorname{rank}(\lambda A + B) = n < m$, *the component* S_2x *is absent.*

Remark 4.2. If condition 3 of Theorem 4.1 holds, then condition 3 of Theorem 3.2 holds. This is easily verified since conditions (a) and (c) of Theorem 4.1 imply conditions (a) and (c) of Theorem 3.2, respectively, and conditions (b) of Theorem 4.1 and (b) of Theorem 3.2 coincide. Note that condition 3 of Theorem 3.2 must also hold for Theorem 3.4.

The proof of Theorem 4.1. We will carry out the proof, assuming that condition 2 of Theorem 3.2 holds. If we replace it by condition 2 of Theorem 3.4, then in the proof of the present theorem it will be necessary to replace "Theorem 3.2" by "Theorem 3.4".

Considering Remark 4.2, we conclude that all conditions of Theorem 3.2 hold. Consequently, for an arbitrary initial point $(t_0,x_0) \in L_{t_+}$, where $S_2x_0 \in D_{s_2}$, there exists a unique solution $x(t)$ of the IVP (2.1), (2.2) on $[t_0,\infty)$, such that $S_2x(t) = \phi_{s_2}(t)$ where $\phi_{s_2} \in C([t_0,\infty),D_{s_2})$ is some chosen function with the initial value $\phi_{s_2}(t_0) = S_2x_0$. Thus, the existence of a global solution of the IVP (2.1), (2.2) is proved.

Let us prove the Lagrange stability. As shown in the proof of Theorem 3.2, the solution of the IVP (2.1), (2.2) can be represented in the form $x(t) = \omega_{s_1}(t) + \phi_{s_2}(t) + \omega_{p_1}(t) + q(t,\omega_{s_1}(t),\omega_{p_1}(t))$. It is assumed that the function $\phi_{s_2}(t)$ defining the component $S_2x(t)$ of the solution $x(t)$ was chosen so that $\sup_{t\in[t_0,\infty)} \|\phi_{s_2}(t)\| < \infty$. This is fulfilled due to the requirements of the present theorem and, obviously, does not affect the proof of Theorem 3.2. It follows from condition 3 that the derivative of the function V along the trajectories of the equation (3.9) satisfies the inequality (3.11) for all $t \geq t_0$, $\|\omega\| > R$, and the differential inequality (2.3) does not have unbounded positive solutions for $t \in [t_+, \infty)$. Then by [19, Chapter IV, Theorem XV] the equation (3.9) is Lagrange stable. Consequently, $\sup_{t\in[t_0,\infty)} \|\omega(t)\| < \infty$. Hence there exists a constant $M > 0$ such that

$$\|\omega_{s_1}(t) + \phi_{s_2}(t) + \omega_{p_1}(t)\| \leq M, \qquad t \in [t_0, \infty). \tag{4.1}$$

Recall that the function $q(t, \omega_{s_1}(t), \omega_{p_1}(t)) = \eta(t, \omega_{s_1}(t), \phi_{s_2}(t), \omega_{p_1}(t))$ is a solution of the equation (2.45), as well as (3.7), with respect to the variable x_{p_2}. Denote $u(t) := q(t, \omega_{s_1}(t), \omega_{p_1}(t))$. Therefore,

$$u(t) = \mathcal{B}_2^{(-1)} Q_2 f(t, \omega_{s_1}(t) + \phi_{s_2}(t) + \omega_{p_1}(t) + u(t)). \tag{4.2}$$

Then from (4.1), condition 4 and the boundedness of the norm of the operator $\mathcal{B}_2^{(-1)} \in \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ it follows that there exists a constant $K_M = K(M)$ (depending on the constant M, in general) such that $\|u(t)\| \leq K_M$ for all $t \in [t_0, \infty)$.

It follows from the above that $\|x(t)\| \leq M + K_M < \infty$ for all $t \in [t_0, \infty)$, i.e., the solution $x(t)$ is bounded on $[t_0, \infty)$ and, therefore, is Lagrange stable. Condition 5 ensures the correctness of the equality (2.46), which is equivalent to the equality $F_2 B x(t) = F_2 f(t, x(t))$. Thus, the theorem is proved. □

5. Lagrange instability of singular semilinear DAEs (the blow-up of solutions in finite time)

Theorem 5.1. *Let $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^m)$ and $\lambda A + B$ be a singular pencil of operators such that its regular block $\lambda A_r + B_r$ from (2.17) has the index not higher than 1. Assume that condition 1 of Theorem 3.2 holds and condition 2 of Theorem 3.2 or condition 2 of Theorem 3.4 holds. Let the following conditions hold*:

3. *There exists a region $\Omega_1 \subset X_{s_1} \times X_1$ such that the vector $(S_1 x(t), P_1 x(t))$ consisting of the components $S_1 x(t)$, $P_1 x(t)$ of any solution $x(t)$ with the initial point $(t_0, x_0) \in L_{t_+}$, where $(S_1 x_0, P_1 x_0) \in \Omega_1$ and $S_2 x_0 \in D_{s_2}$, remains all the time in Ω_1 (i.e., remains in Ω_1 for all t from the maximal interval of existence of the solution).*
4. *There exists a function $V \in C^1([t_+, \infty) \times \Omega_1, \mathbb{R})$ positive on $[t_+, \infty) \times \Omega_1$ and a function $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$ such that:*
   (a) *for each $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $(x_{s_1}, x_{p_1}) \in \Omega_1$, the inequality*

   $$V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \geq \chi\big(t, V(t, x_{s_1}, x_{p_1})\big), \tag{5.1}$$

   where $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1})$ has the form (2.47), is satisfied;

   (b) *the differential inequality (2.4), i.e., $dv/dt \geq \chi(t, v)$ $(t \in [t_+, \infty))$, does not have global positive solutions.*

Then for each initial point $(t_0, x_0) \in L_{t_+}$*, where* $S_2x_0 \in D_{s_2}$ *and* $(S_1x_0, P_1x_0) \in \Omega_1$*, the initial value problem* (2.1), (2.2) *has a unique solution* $x(t)$ *for which the choice of the function* $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ *with the initial value* $\phi_{s_2}(t_0) = S_2x_0$ *uniquely defines the component* $S_2x(t) = \phi_{s_2}(t)$ *when* $\operatorname{rank}(\lambda A + B) < n$ (*when* $\operatorname{rank}(\lambda A + B) = n < m$*, the component* S_2x *is absent*)*, and this solution is Lagrange unstable* (*has a finite escape time*)*.*

Proof. It is proved in the same way as in Theorem 3.2 (or 3.4) that there exists the unique solution $\omega(t)$ of the IVP (3.10), (3.9) on the interval $[t_0, \beta)$. In addition, it follows from the proof of Theorem 3.2 (or 3.4) that there exists a unique solution $x(t) = \omega_{s_1}(t) + \phi_{s_2}(t) + \omega_{p_1}(t) + q(t, \omega_{s_1}(t), \omega_{p_1}(t))$ of the IVP (2.1), (2.2) on the maximal interval of existence $[t_0, \beta)$. Further, it is assumed that the initial point (t_0, x_0) for the solution mentioned above has been chosen so that condition 3 is satisfied. Then the initial value $\omega_0 = (x_{s_1,0}^{\mathrm{T}}, x_{p_1,0}^{\mathrm{T}})^{\mathrm{T}}$ from condition (3.10) belongs to the region Ω_1, which is defined in condition 3. Therefore, the solution $\omega(t)$ of (3.9) remains all the time in Ω_1. By virtue of condition 4, $V'_{(3.9)}(t, \omega) \geq \chi\big(t, V(t, \omega)\big)$ for all $t \geq t_0$, $\omega \in \Omega_1$, and the inequality (2.4), $t \geq t_0$, does not have global positive solutions. Hence, using the theorem [19, Chapter IV, Theorem XIV], we obtain that the solution $\omega(t)$ has a finite escape time, i.e., it is defined on some finite interval $[t_0, T)$ and $\lim_{t \to T-0} \|\omega(t)\| = +\infty$. Consequently, $[t_0, \beta) = [t_0, T)$ and the solution $x(t)$ has a finite escape time. Accordingly, it is Lagrange unstable. □

The statement of Theorem 5.1 means that (2.1) is Lagrange unstable for the initial points $(t_0, x_0) \in L_{t_+}$ for which $S_2x_0 \in D_{s_2}$ and $(S_1x_0, P_1x_0) \in \Omega_1$.

6. Dissipativity (ultimate boundedness) of singular semilinear DAEs

Below, we will use the notation $(z, w)_H := (H(t)z, w)$ for a scalar product with the weight $H(t)$.

An operator function $H : J \to \mathrm{L}(X)$, where X is a finite-dimensional linear or Hilbert space and $J \subseteq \mathbb{R}$ is an interval, is called *self-adjoint* if the operator $H(t)$ is self-adjoint (for each $t \in J$). A self-adjoint operator $H(t) \in \mathrm{L}(X)$ ($t \in J$) is called *positive definite* if there exists a constant $c > 0$ such that $(H(t)x, x) \geq c\|x\|^2$ for all t, x. A self-adjoint operator function $H : J \to \mathrm{L}(X)$ is called *positive definite* if the operator $H(t)$ is positive definite (see, e.g., [12, Definition 2.2]).

Theorem 6.1. *Let* $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^m)$ *and* $\lambda A + B$ *be a singular pencil of operators such that its regular block* $\lambda A_r + B_r$ *from* (2.17) *has the index not higher than 1. Assume that condition* 1 *of Theorem* 3.2 *holds and condition* 2 *of Theorem* 3.2 *or condition* 2 *of Theorem* 3.4 *holds. Let the following conditions be satisfied*:

3. *There exists a number* $R > 0$*, a function* $V \in C^1([t_+, \infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ *positive on* $[t_+, \infty) \times D_{s_1} \times D_{p_1}$*, where* $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid \|(x_{s_1}, x_{p_1})\| > R\}$*, and functions* $U_j \in C([0, \infty))$*,* $j = 0, 1, 2$*, such that* $U_0(r)$

is non-decreasing and $U_0(r) \to +\infty$ *as* $r \to +\infty$, $U_1(r)$ *is increasing,* $U_2(r) > 0$ *for* $r > 0$, *and for all* $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, *for which* $x_{s_2} \in D_{s_2}$ *and* $\|(x_{s_1}, x_{p_1})\| > R$, *the inequality*

$$U_0(\|(x_{s_1}, x_{p_1})\|) \le V(t, x_{s_1}, x_{p_1}) \le U_1(\|(x_{s_1}, x_{p_1})\|)$$

and one of the following inequalities (where $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1})$ *has the form* (2.47)) *hold:*

(a) $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \le -U_2\big(\|(x_{s_1}, x_{p_1})\|\big)$;

(b) $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \le -U_2\big(((x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1}))_H\big)$, *where* $H \in C([t_+, \infty), \mathrm{L}(X_{s_1} \times X_1))$ *is a positive definite self-adjoint operator function such that* $H(t)\big|_{X_{s_1}} : X_{s_1} \to X_{s_1} \times \{0\}$ *and* $H(t)\big|_{X_1} : X_1 \to \{0\} \times X_1$ *for any fixed* t, *and* $\sup_{t\in[t_+,\infty)} \|H(t)\| < \infty$;

(c) $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \le -\alpha V(t, x_{s_1}, x_{p_1})$, *where* $\alpha > 0$ *is some constant.*

4. *There exist constants* $\beta > 0$, $T > t_+$ *such that* $\|Q_2 f(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2})\| \le \beta \|(x_{s_1}, x_{p_1})\|$ *for all* $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_T$ *where* $x_{s_2} \in D_{s_2}$.
5. $\|F_2 f(t, x)\| < \infty$ *for all* $(t, x) \in L_{t_+}$ *such that* $S_2 x \in D_{s_2}$ *and* $\|x\| \le C < \infty$ (*C is an arbitrary constant*).

Then, for the initial points $(t_0, x_0) \in L_{t_+}$ *where* $S_2 x_0 \in D_{s_2}$ *and any function* $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ *satisfying the relations* $\phi_{s_2}(t_0) = S_2 x_0$ *and* $\sup_{t\in[t_0,\infty)} \|\phi_{s_2}(t)\| < \infty$ *the equation* (2.1), *where* $S_2 x = \phi_{s_2}(t)$, *is uniformly dissipative (uniformly ultimately bounded); when* $\operatorname{rank}(\lambda A + B) = n < m$, *the component* $S_2 x$ *is absent.*

Remark 6.2. If condition 3 of Theorem 6.1 holds, then condition 3 of Theorem 3.2 holds.

The proof of theorem 6.1. It follows from the conditions of the present theorem and Remark 6.2 that the conditions of Theorem 3.2 (or Theorem 3.4) are satisfied. Hence, there exists a unique global solution $x(t)$ of the IVP (2.1), (2.2) for each consistent initial point (t_0, x_0) with $S_2 x_0 \in D_{s_2}$ and some chosen function $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ with the initial value $S_2 x_0$ which defines the component $S_2 x(t) = \phi_{s_2}(t)$. As shown in the proof of Theorem 3.2, the solution can be represented as $x(t) = \omega_{s_1}(t) + \phi_{s_2}(t) + \omega_{p_1}(t) + q(t, \omega_{s_1}(t), \omega_{p_1}(t))$. By virtue of the conditions of the present theorem, it is assumed that $\sup_{t\in[t_0,\infty)} \|\phi_{s_2}(t)\| = \gamma < \infty$. In a similar way as in the proof of the theorem [12, Theorem 4.3], using condition 3 and the proof of the theorem [26, Theorem 10.4] and its corollary, we obtain that solutions of (3.9) are uniformly dissipative, i.e., there exists a number $M > 0$ and, for each solution $\omega(t) = (\omega_{s_1}(t)^{\mathrm{T}}, x_{p_1}(t)^{\mathrm{T}})^{\mathrm{T}}$ satisfying the initial condition (3.10), there exists a number $\tau_1 = \tau_1(x_0) \ge t_0$ such that $\|\omega(t)\| < M$ for each $t \ge t_0 + \tau_1$. Recall that the function $u(t) = q(t, \omega_{s_1}(t), \omega_{p_1}(t))$ satisfies the equality (4.2). Therefore, according to condition 4, there exists a constant $\beta_0 > 0$ and a number $\tau_2 = \tau_2(x_0) > t_0$ such that $\|u(t)\| \le \beta_0 \|\omega(t)\| < \beta_0 M$ for all $t \ge \tau_2$. Hence, for each solution with the initial values t_0, x_0 there exists a number $\tau = \tau(x_0) \geq t_0$ such that $\|x(t)\| < (2+\beta_0)M + \gamma = k$ for all $t \in [t_0 + \tau, \infty)$, where the constant $k > 0$ does not depend on t_0, x_0. Consequently, the DAE (2.1) is uniformly dissipative, and condition 5 ensures the correctness of the equality (2.46). □

7. Replacement of some conditions of the theorems by weaker ones

This section shows how we can weaken some requirements of Theorems 3.2, 3.4 and, as a consequence, some requirements of the other theorems as well.

Let Z and W be n-dimensional linear spaces. A system of n pairwise disjoint projectors $\{\Theta_i\}_{i=1}^n$ ($\Theta_l\,\Theta_j = \delta_{lj}\,\Theta_l$; the projectors are one-dimensional), where $\Theta_i \in \mathrm{L}(Z)$, such that their sum is the identity operator $I_Z = \sum_{i=1}^n \Theta_i$ in Z is called an *additive resolution (or decomposition) of the identity* in Z (cf. [8, 24]). Notice that an additive resolution of the identity in Z generates the direct decomposition $Z = Z_1 \dot{+} \cdots \dot{+} Z_n$ where $Z_i = \Theta_i\, Z$, $i = 1, \dots, n$. An operator function $\Phi\colon D \to \mathrm{L}(W, Z)$, where $D \subset W$, is called *basis invertible* on an interval $J \subset D$ (or on a convex hull $J = \mathrm{Conv}\{w_1, w_2\}$ of w_1, $w_2 \in D$) if for some additive resolution (decomposition) of the identity $\{\Theta_i\}_{i=1}^n$ in Z and for each set of elements $\{w^k\}_{k=1}^n \subset J$ the operator $\Lambda = \sum_{i=1}^n \Theta_i \Phi(w^i) \in \mathrm{L}(W, Z)$ has the inverse $\Lambda^{-1} \in \mathrm{L}(Z, W)$ (cf. [8, 24]). This definition in terms of matrices is given in [10, p. 176]. Note that the property of basis invertibility does not depend on the choice of an additive resolution of the identity in Z. Obviously, it follows from the basis invertibility of the mapping Φ on an interval J that for each $w^* \in J$ the operator $\Phi(w^*) \in \mathrm{L}(W, Z)$ is invertible. The converse statement does not hold true, except for the case when W, Z are one-dimensional spaces (see [8, Example 1]).

Theorem 7.1. *Theorem 3.2 remains valid if conditions 1 and 2 are replaced by the following:*

1. *For any fixed $t \in [t_+, \infty)$, $x_{s_1} \in X_{s_1}$, $x_{s_2} \in D_{s_2}$, where $D_{s_2} \subset X_{s_2}$ is a some set, and $x_{p_1} \in X_1$, there exists $x_{p_2} \in X_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$.*
2. *There exists the partial derivative $\frac{\partial}{\partial x} f \in C([t_+, \infty) \times \mathbb{R}^n, \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m))$. For any fixed t_*, $x_*^i = x_{s_1}^* + x_{s_2}^* + x_{p_1}^* + x_{p_2}^i$ such that $(t_*, x_*^i) \in L_{t_+}$ and $x_{s_2}^* \in D_{s_2}$, the operator function $\Phi_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(x_{p_2})$ defined by*

$$\Phi_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*} \colon X_2 \to \mathrm{L}(X_2, Y_2),$$
$$\Phi_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(x_{p_2}) = \left[\frac{\partial Q_2 f}{\partial x}(t, x_{s_1}^* + x_{s_2}^* + x_{p_1}^* + x_{p_2}) - B\right] P_2, \tag{7.1}$$

   is basis invertible on $[x_{p_2}^1, x_{p_2}^2]$.

Remark 7.2. Note that $\Phi_{t_*, x_{s_1}^*, x_{s_2}^*, x_{p_1}^*}(x_{p_2}^*) = \Phi_{t_*, x_*}$, where $x_* = x_{s_1}^* + x_{s_2}^* + x_{p_1}^* + x_{p_2}^*$ and Φ_{t_*, x_*} is the operator defined by (3.2), for any fixed $x_{p_2}^* \in X_2$. In addition, if the space X_2 is one-dimensional, then condition 2 of Theorem 7.1 is equivalent to condition 2 of Theorem 3.2.

The proof of Theorem 7.1. The partial derivative of the mapping (3.6) with respect to x_{p_2} at the point $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2})$ has the form (3.8) and can be written as $W_{t_*,x_*} = \dfrac{\partial \Psi}{\partial x_{p_2}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^*_{p_2}) = B_2^{-1}\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x^*_{p_2}) \in \mathrm{L}(X_2)$, where $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$ and $\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}} \in C(X_2, \mathrm{L}(X_2, Y_2))$ is the operator function defined by (7.1). Define the operator function $W_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}} : X_2 \to \mathrm{L}(X_2)$,

$$W_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x_{p_2}) := \frac{\partial \Psi}{\partial x_{p_2}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x_{p_2}) = B_2^{-1}\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x_{p_2}),$$

where t_*, $x^*_{s_1}$, $x^*_{s_2}$ and $x^*_{p_1}$ are arbitrary fixed elements of $[t_+,\infty)$, X_{s_1}, X_{s_2} and X_1, respectively. Recall that the basis invertibility of the operator function $\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}} : X_2 \to \mathrm{L}(X_2, Y_2)$ on some interval J imply the invertibility of the operator $\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x^*_{p_2})$ for each (fixed) $x^*_{p_2} \in J$. Thus, it follows from condition 2 of the present theorem that for any fixed element $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}) \in L_{t_+}$ such that $x^*_{s_2} \in D_{s_2}$ the operator $W_{t_*,x_*} = W_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x^*_{p_2})$ (where $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$, and W_{t_*,x_*} was defined in (3.8)) has the inverse $W^{-1}_{t_*,x_*} = \big(\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x^*_{p_2})\big)^{-1} B_2 \in \mathrm{L}(X_2)$.

Let us prove that condition 1 of Theorem 3.2 holds. Due to condition 1 of the present theorem, for each (fixed) $t \in [t_+,\infty)$, $x_{s_1} \in X_{s_1}$, $x_{s_2} \in D_{s_2}$, $x_{p_1} \in X_1$ there exists $x_{p_2} \in X_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, and it is necessary to prove the uniqueness of such a x_{p_2} in order to show that condition 1 of Theorem 3.2 is satisfied.

Take an arbitrary fixed $t_* \in [t_+,\infty)$, $x^*_{s_1} \in X_{s_1}$, $x^*_{s_2} \in D_{s_2}$, $x^*_{p_1} \in X_1$ and $x^i_{p_2} \in X_2$, $i = 1, 2$, such that $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^i_{p_2}) \in L_{t_+}$, then $(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^i_{p_2})$ must satisfy (3.7), i.e., $\Psi(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^i_{p_2}) = 0$, $i = 1, 2$. Note that the projector P_2 restricted to X_2 is the identity operator in X_2. It follows from the basis invertibility of the operator function $\Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}$ on $[x^1_{p_2}, x^2_{p_2}]$ that for some additive resolution of the identity $\{\Theta_i\}_{i=1}^d$ in X_2 (where $d = \dim X_2$; $\sum_{i=1}^d \Theta_i = I_{X_2} = P_2\big|_{X_2}$) the operator

$$\Lambda = \sum_{i=1}^{d} \Theta_i W_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x_{p_2,i}) = B_2^{-1} \sum_{i=1}^{d} \Theta_i \Phi_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}(x_{p_2,i}) \tag{7.2}$$

is invertible for each set of the elements $\{x_{p_2,k}\}_{k=1}^d \subset [x^1_{p_2}, x^2_{p_2}]$. Hence, the operator function $W_{t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1}}$ is basis invertible on $[x^1_{p_2}, x^2_{p_2}]$. Using the additive resolution of the identity $\{\Theta_i\}_{i=1}^d$, we define the functions

$$\Psi_i := \Theta_i \Psi \colon [t_+,\infty) \times X_{s_1} \times X_{s_2} \times X_1 \times X_2 \to X_{2,i} = \Theta_i X_2, \qquad i = 1, \ldots, d.$$

Note that $X_{2,i}$, $i = 1, \ldots, d$, are one-dimensional spaces isomorphic to $\mathbb{R}$, and $X_2 = X_{2,1} \dot{+} \cdots \dot{+} X_{2,d}$. By the finite increment formula, there exist $x_{p_2,i} \in [x^1_{p_2}, x^2_{p_2}]$, $i = 1, ..., d$, such that

$$\Psi_i(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^2_{p_2}) - \Psi_i(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^1_{p_2})$$

$$= \frac{\partial \Psi_i}{\partial x_{p2}}(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x_{p_2,i})\big(x^2_{p_2} - x^1_{p_2}\big)$$
$$= \Theta_i W_{t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}}(x_{p_2,i})\big(x^2_{p_2} - x^1_{p_2}\big), \quad i = 1, \dots, d.$$

Since $\Psi(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}, x^i_{p_2}) = 0$, $i = 1, 2$, then, summing the obtained expressions over i, we obtain $\sum_{i=1}^{d} \Theta_i W_{t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}}(x_{p_2,i})\big(x^2_{p_2} - x^1_{p_2}\big) = \Lambda\big(x^2_{p_2} - x^1_{p_2}\big) = 0$, where Λ is defined in (7.2). Since the operator Λ^{-1} exists, then $x^2_{p_2} = x^1_{p_2}$. This holds for each point $(t_*, x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^i_{p_2}) \in L_{t_+}$, $i = 1, 2$, where $x^*_{s_2} \in D_{s_2}$, since these points were chosen arbitrarily. Thus, the proof of condition 1 of Theorem 3.2 is complete.

As in the proof of Theorem 3.2, take arbitrary fixed $t_* \in [t_+, \infty)$, $x^*_{s_1} \in X_{s_1}$, $x^*_{s_2} \in D_{s_2}$, $x^*_{p_1} \in X_1$. As proved above, there exists a unique $x^*_{p_2} \in X_2$ such that $(t_*, x_*) \in L_{t_+}$, where $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$, and for this (t_*, x_*) the operator $W_{t_*, x_*} = W_{t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}}(x^*_{p_2})$ has the inverse $W^{-1}_{t_*, x_*} \in \mathrm{L}(X_2)$.

The further proof coincides with the proof of Theorem 3.2. Generally, we proved above that conditions 1, 2 of Theorem 3.2 are satisfied, and the rest of the conditions of Theorem 3.2 are the same as in the present theorem. □

Corollary 7.3. *Theorems 4.1, 5.1, and 6.1 (which contain conditions 1, 2 of Theorem 3.2) remain valid if one requires that conditions 1 and 2 of Theorem 7.1 hold instead of condition 1 of Theorem 3.2 and instead of condition 2 of Theorem 3.2 or condition 2 of Theorem 3.4.*

Below we show how condition 3 of Theorem 3.2 can be weakened.

First, we consider an ODE

$$\frac{dx}{dt} = F(t, x), \tag{7.3}$$

where $t \in [t_+, \infty)$, $t_+ \geq 0$, $x \in W$ and W is an n-dimensional Euclidean space, and the function $F \in C([t_+, \infty) \times W, W)$ satisfies locally a Lipschitz condition with respect to x on $[t_+, \infty) \times W$, i.e., for each $(t_*, x_*) \in [t_+, \infty) \times W$ there exist open neighborhoods $U(t_*)$, $\widetilde{U}(x_*)$ of the points t_*, x_* and a constant $L \geq 0$ such that $\|F(t, x_1) - F(t, x_2)\| \leq L\|x_1 - x_2\|$ for any $t \in U(t_*)$, $x_1, x_2 \in \widetilde{U}(x_*)$. According to [19], a solution $x(t)$ of the ODE (7.3), which satisfies some initial condition $x(t_0) = x_0$, is called *defined in the future*, if it can be extended for all $t \geq t_0$, i.e., to the whole interval $[t_0, \infty)$, and hence this solution is global by the definition given in this paper. Thus, these definitions are equivalent. Consider the ODE (the ODE (7.3) with a truncation)

$$\frac{dx}{dt} = F_T(t, x), \tag{7.4}$$

where T is a parameter,

$$F_T(t, x) := \begin{cases} F(t, x), & t_+ \leq t \leq T \\ F(T, x), & t > T \end{cases}.$$

The function $F_T(t,x)$ is called the *truncation* of the function $F(t,x)$ over t, and it has the same properties as $F(t,x)$, i.e., $F_T(t,x)$ is continuous and locally satisfies a Lipschitz condition with respect to x on $[t_+,\infty)\times W$.

Below is the lemma proved in [7], which generalizes Theorem [19, Chapter IV, Theorem XIII] and will be used in the sequel.

Lemma 7.4 (cf. [7, Lemma 3.1])**.** *Let there exist a function* $V\in C^1([t_+,\infty)\times D^c,\mathbb{R})$ *positive on* $[t_+,\infty)\times D^c$*, where* D^c *is the complement of some bounded set* $D\subset W$ *containing* 0 *(the origin). Let for each number* $T>0$ *there exist a set* $D_T\supset D$ *and a function* $\chi_T\in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ *such that:*

1. $V(t,x)\to\infty$ *as* $\|x\|\to\infty$ *uniformly in* t *on each finite interval* $[a,b)\subset[t_+,\infty)$;
2. $V'_{(7.4)}(t,x)=\dfrac{\partial V}{\partial t}(t,x)+\dfrac{\partial V}{\partial x}(t,x)F_T(t,x)\le\chi_T(t,V(t,x))$ *for all* $t\in[t_+,\infty)$, $x\in D_T^c$ ($V'_{(7.4)}$ *is the derivative of* V *along the trajectories of* (7.4));
3. *the differential inequality* $dv/dt\le\chi_T(t,v)$ $(t\in[t_+,\infty))$ *does not have positive solutions with finite escape time.*

Then every solution of the ODE (7.3) *is global* (*defined in the future*)*.*

Proof. The proof is carried out in the same way as the proof of the lemma [7, Lemma 3.1]. □

We return to the consideration of the DAE (2.1). Recall that it is equivalent to the system (2.43)–(2.46). Introduce the truncation of the function $f(t,x)$ over t:

$$f_T(t,x):=\begin{cases}f(t,x), & t_+\le t\le T\\ f(T,x), & t>T\end{cases},\quad T\ge t_+ \text{ is a parameter.}$$

Then the truncation of the function $\Upsilon(t,x)$ (see (2.48)) over t has the form

$$\begin{aligned}\Upsilon_T(t,x)&:=\begin{pmatrix}\mathcal{A}_{\rm gen}^{(-1)}\big(F_1f_T(t,x)-\mathcal{B}_{\rm gen}x_{s_1}-\mathcal{B}_{\rm und}x_{s_2}\big)\\ \mathcal{A}_1^{(-1)}\big(Q_1f_T(t,x)-\mathcal{B}_1x_{p_1}\big)\end{pmatrix}\\ &=\begin{cases}\Upsilon(t,x), & t_+\le t\le T\\ \Upsilon(T,x), & t>T\end{cases}.\end{aligned}\tag{7.5}$$

It consists of the right-hand sides of the equations (2.43), (2.44) with a truncation:

$$\frac{dx_{s_1}}{dt}=\mathcal{A}_{\rm gen}^{(-1)}[F_1f_T(t,x)-\mathcal{B}_{\rm gen}x_{s_1}-\mathcal{B}_{\rm und}x_{s_2}],\tag{7.6}$$
$$\frac{dx_{p_1}}{dt}=\mathcal{A}_1^{(-1)}[Q_1f_T(t,x)-\mathcal{B}_1x_{p_1}].\tag{7.7}$$

The derivative of a function $V\in C^1([t_+,\infty)\times D_{s_1}\times D_{p_1},\mathbb{R})$ ($D_{s_1}\times D_{p_1}\subset X_{s_1}\times X_1$ is an open set) along the trajectories of the system (7.6), (7.7) has the form

$$V'_{(7.6),(7.7)}(t,x_{s_1},x_{p_1})=\frac{\partial V}{\partial t}(t,x_{s_1},x_{p_1})+\frac{\partial V}{\partial(x_{s_1},x_{p_1})}(t,x_{s_1},x_{p_1})\Upsilon_T(t,x)$$

$$= \frac{\partial V}{\partial t}(t, x_{s_1}, x_{p_1})$$
$$+ \frac{\partial V}{\partial x_{s_1}}(t, x_{s_1}, x_{p_1}) \left[\mathcal{A}_{\text{gen}}^{(-1)}\big(F_1 f_T(t,x) - \mathcal{B}_{\text{gen}} x_{s_1} - \mathcal{B}_{\text{und}} x_{s_2}\big)\right]$$
$$+ \frac{\partial V}{\partial x_{p_1}}(t, x_{s_1}, x_{p_1}) \left[\mathcal{A}_1^{(-1)}\big(Q_1 f_T(t,x) - \mathcal{B}_1 x_{p_1}\big)\right]. \tag{7.8}$$

Theorem 7.5. *Theorems 3.2 and 3.4 remain valid if condition 3 of Theorem 3.2 (which must also hold for Theorem 3.4) is replaced by the following:*

3. *There exists a function $V \in C^1([t_+,\infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ positive on $[t_+,\infty) \times D_{s_1} \times D_{p_1}$, where $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid \|(x_{s_1}, x_{p_1})\| > R\}$ and $R > 0$ is some number, and for each number $T > 0$ there exists a number $R_T \geq R$ and a function $\chi_T \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:*
   (a) *$V(t, x_{s_1}, x_{p_1}) \to \infty$ as $\|(x_{s_1}, x_{p_1})\| \to \infty$ uniformly in t on each finite interval $[a, b) \subset [t_+,\infty)$;*
   (b) *for all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1}, x_{p_1})\| \geq R_T$, the inequality*
   $$V'_{(7.6),(7.7)}(t, x_{s_1}, x_{p_1}) \leq \chi_T\big(t, V(t, x_{s_1}, x_{p_1})\big), \tag{7.9}$$
   where $V'_{(7.6),(7.7)}(t, x_{s_1}, x_{p_1})$ has the form (7.8), holds;
   (c) *the differential inequality $dv/dt \leq \chi_T(t, v)$ $(t \in [t_+,\infty))$ does not have positive solutions with finite escape time.*

Proof. The proof coincides with the proof of Theorem 3.2 (or 3.4), except for the part where the existence of a global solution of (3.9), i.e., $\frac{d\omega}{dt} = \widetilde{\Upsilon}(t,\omega)$, is proved. Let us prove this part using the conditions of the present theorem.

As shown in the proof of Theorem 3.2 (as well as Theorem 3.4), there exists the unique solution $\omega = \omega(t)$ of the IVP (3.9), (3.10) on the maximal interval of existence $[t_0, \beta)$. Recall that $(t_0, x_0) \in L_{t_+}$, where $S_2 x_0 \in D_{s_2}$, is an arbitrarily chosen initial point and that $\phi_{s_2} \in C([t_0,\infty), D_{s_2})$ is an arbitrarily chosen function with the initial value $\phi_{s_2}(t_0) = S_2 x_0$.

Consider the ODE (3.9) with a truncation, that is,

$$\frac{d}{dt}\omega = \widetilde{\Upsilon}_T(t,\omega), \tag{7.10}$$

where T is a parameter,

$$\omega = \begin{pmatrix} x_{s_1} \\ x_{p_1} \end{pmatrix}, \quad \widetilde{\Upsilon}_T(t,\omega) := \begin{cases} \widetilde{\Upsilon}(t,\omega), & t_0 \leq t \leq T \\ \widetilde{\Upsilon}(T,\omega), & t > T \end{cases}.$$

Note that

$$\widetilde{\Upsilon}_T(t,\omega) := \begin{cases} \widetilde{\Upsilon}(t,\omega) = \Upsilon(t, x_{s_1} + \phi_{s_2}(t) + x_{p_1} + q(t, x_{s_1}, x_{p_1})), & t_0 \leq t \leq T \\ \widetilde{\Upsilon}(T,\omega) = \Upsilon(T, x_{s_1} + \phi_{s_2}(T) + x_{p_1} + q(T, x_{s_1}, x_{p_1})), & t > T \end{cases}.$$

We choose a number $R > 0$ ($R < \infty$) and a function $V(t, x_{s_1}, x_{p_1})$ such that condition 3 of the theorem holds, and introduce the function $V(t, \omega) := V(t, x_{s_1}, x_{p_1})$. Due to condition 3, for each $T > 0$ there exists a number $R_T \geq R$ and a function $\chi_T \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$ such that the derivative of V along the trajectories of the equation (7.10) satisfies the inequality

$$V'_{(7.10)}(t, \omega) = \frac{\partial V}{\partial t}(t, \omega) + \frac{\partial V}{\partial \omega}(t, \omega)\widetilde{\Upsilon}_T(t, \omega) \leq \chi_T\big(t, V(t, \omega)\big) \tag{7.11}$$

for all $t \geq t_0$ and $\|\omega\| \geq R_T$. Since, by virtue of condition (c), the differential inequality $\dfrac{dv}{dt} \leq \chi_T(t, v)$ ($t \in [t_0, \infty)$) does not have positive solutions with finite escape time, then by Lemma 7.4 the solution $\omega(t) = (\omega_{s_1}(t)^{\mathrm{T}}, \omega_{p_1}(t)^{\mathrm{T}})^{\mathrm{T}}$ is global, i.e., exists on $[t_0, \infty)$. Thus, what was needed has been proved. □

8. On the choice of the functions χ and V when checking the conditions of proved theorems

The proved theorems contain conditions in a general form, and the main difficulty in applying the theorems lies in choosing suitable functions χ and V.

Choose the function $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$, which is present in Theorems 3.2, 3.4, 4.1, 5.1, and 7.1, in the form (2.5), that is,

$$\chi(t, v) = k(t)\, U(v),$$

where $k \in C([t_+, \infty), \mathbb{R})$ and $U \in C(0, \infty)$, then the conditions of the theorems take the following form:

- In Theorems 3.2, 3.4 and 7.1 on the global solvability all conditions remain unchanged, except for condition 3 which takes the form:

  3. There exists a number $R > 0$, a function $V \in C^1([t_+, \infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ positive on $[t_+, \infty) \times D_{s_1} \times D_{p_1}$, where $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid \|(x_{s_1}, x_{p_1})\| > R\}$, and functions $k \in C([t_+, \infty), \mathbb{R})$, $U \in C(0, \infty)$ such that condition (a) of Theorem 3.2 holds,

  $$\int_{v_0}^{\infty} \frac{dv}{U(v)} = \infty$$

  ($v_0 > 0$ is some number) and

  $$V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \leq k(t)\, U\big(V(t, x_{s_1}, x_{p_1})\big) \tag{8.1}$$

  for all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ for which $x_{s_2} \in D_{s_2}$, $\|(x_{s_1}, x_{p_1})\| > R$.

- In Theorem 4.1 on the Lagrange stability all conditions remain unchanged, except for condition 3 which takes the form:

  3. There exists a number $R > 0$, a function $V \in C^1([t_+, \infty) \times D_{s_1} \times D_{p_1}, \mathbb{R})$ positive on $[t_+, \infty) \times D_{s_1} \times D_{p_1}$, where $D_{s_1} \times D_{p_1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid$

$\|(x_{s_1}, x_{p_1})\| > R\}$, and functions $k \in C([t_+,\infty),\mathbb{R})$, $U \in C(0,\infty)$ such that condition (a) of Theorem 4.1 holds,

$$\int_{v_0}^{\infty} \frac{dv}{U(v)} = \infty, \qquad \int_{t_0}^{\infty} k(t)dt < \infty$$

($t_0 \ge t_+$, $v_0 > 0$ are some numbers) and the inequality (8.1) holds for all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ for which $x_{s_2} \in D_{s_2}$, $\|(x_{s_1}, x_{p_1})\| > R$.

- In Theorem 5.1 on the Lagrange instability all conditions remain unchanged, except for condition 4 which takes the form:

4. There exists a function $V \in C^1([t_+,\infty) \times \Omega_1, \mathbb{R})$ positive on $[t_+,\infty) \times \Omega_1$ and functions $k \in C([t_+,\infty),\mathbb{R})$, $U \in C(0,\infty)$ such that

$$\int_{v_0}^{\infty} \frac{dv}{U(v)} < \infty, \qquad \int_{t_0}^{\infty} k(t)dt = \infty$$

($t_0 \ge t_+$, $v_0 > 0$ are some numbers) and

$$V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) \ge k(t)\, U\big(V(t, x_{s_1}, x_{p_1})\big)$$

for all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ for which $x_{s_2} \in D_{s_2}$, $(x_{s_1}, x_{p_1}) \in \Omega_1$.

Recall that $V'_{(2.43),(2.44)}$ has the form (2.47). The validity of the theorems with the above changes in the conditions follows directly from the remarks on differential inequalities given in Section 2.1.

Now, consider the scalar function V which is present in all theorems proved above and will be called a *Lyapunov type function*. Choose it in the form

$$V(t, x_{s_1}, x_{p_1}) = \big((x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\big)_H = \big(H(t)(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\big), \tag{8.2}$$

where $H \in C([t_+,\infty), \mathrm{L}(X_{s_1} \times X_1))$ is a positive definite self-adjoint operator function such that $H(t)\big|_{X_{s_1}} : X_{s_1} \to X_{s_1} \times \{0\}$ and $H(t)\big|_{X_1} : X_1 \to \{0\} \times X_1$ for any fixed t and (x_{s_1}, x_{p_1}) is a column vector. Due to the properties of the operator function H, the function (8.2) satisfies the conditions of Theorems 3.2, 3.4, 7.1, 4.1, 5.1 on the global solvability, Lagrange stability and instability, and if in addition $\sup_{t\in[t_+,\infty)} \|H(t)\| < \infty$, then the function (8.2) also satisfies the conditions of Theorem 6.1 on the uniform dissipativity, however, of course, the conditions on the derivative $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1})$ in these theorems need to be checked.

The conditions $H(t)\big|_{X_{s_1}} : X_{s_1} \to X_{s_1} \times \{0\}$ and $H(t)\big|_{X_1} : X_1 \to \{0\} \times X_1$ ($t \ge t_+$ is fixed) mean that the pair of subspaces $\{X_{s_1}, X_{s_1} \times \{0\}\}$ and the pair of subspaces $\{X_1, \{0\} \times X_1\}$ are invariant under the operator $H(t) \in \mathrm{L}(X_{s_1} \times X_1)$ (for each t) and it has the block structure

$$H(t) = \begin{pmatrix} H_{s_1}(t) & 0 \\ 0 & H_1(t) \end{pmatrix} : X_{s_1} \times X_1 \to X_{s_1} \times X_1, \tag{8.3}$$

where $H_{s_1} \in C([t_+,\infty), \mathrm{L}(X_{s_1}))$ and $H_1 \in C([t_+,\infty), \mathrm{L}(X_1))$ are positive definite self-adjoint operator functions. Note that if we identify $X_{s_1} \times \{0\}$ with X_{s_1}

and $\{0\} \times X_1$ with X_1, i.e., identify $X_{s_1} \times X_1 = X_{s_1} \times \{0\} \dot{+} \{0\} \times X_1$ with $X_{s_1} \dot{+} X_1$ as in Section 2.2, then $H(t)$ (t fixed) can be considered as the operator $H(t)\colon X_{s_1} \dot{+} X_1 \to X_{s_1} \dot{+} X_1$.

If $H(t) \equiv H \in \mathrm{L}(X_{s_1} \times X_1)$ is a time-invariant operator, then for all theorems it suffices to require it to be self-adjoint and positive and the pairs of subspaces $\{X_{s_1}, X_{s_1} \times \{0\}\}$ and $\{X_1, \{0\} \times X_1\}$ to be invariant under H. Then the function (8.2) takes the form $V(t, x_{s_1}, x_{p_1}) \equiv V(x_{s_1}, x_{p_1}) = \big(H(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\big)$ and satisfies the conditions of all theorems, except for the conditions on $V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1})$ which need to be checked.

For a function V of the form (8.2) the derivative (2.47) takes the form

$$\begin{aligned} V'_{(2.43),(2.44)}(t, x_{s_1}, x_{p_1}) &= \Big(\frac{d}{dt} H(t)(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\Big) \\ &+ 2\Big(H(t)(x_{s_1}, x_{p_1}), \Upsilon(t, x)\Big) \\ &= \Big(\frac{d}{dt} H(t)(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\Big) \\ &+ 2\Big(H_{s_1}(t) x_{s_1}, \Big[\mathcal{A}_{\mathrm{gen}}^{(-1)}\big(F_1 f(t, x) - \mathcal{B}_{\mathrm{gen}} x_{s_1} - \mathcal{B}_{\mathrm{und}} x_{s_2}\big)\Big]\Big) \\ &+ 2\Big(H_1(t) x_{p_1}, \Big[\mathcal{A}_1^{(-1)}\big(Q_1 f(t, x) - \mathcal{B}_1 x_{p_1}\big)\Big]\Big), \end{aligned}$$

where $H_{s_1}(t)$, $H_1(t)$ are operators defined in (8.3), and $\Upsilon(t, x)$ has the form (2.48).

9. Isothermal models of gas networks in the form of DAEs

9.1. A model of a gas flow for a single pipe (in the isothermal case). The mathematical model of the dynamics of a gas flow in a pipe in the case when the gas temperature is constant consists of the *isothermal Euler equations*, which we write in the form (see, e.g., [5, (ISO1), p. 38])

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{9.1}$$

$$\partial_t(\rho v) + \partial_x(p + \rho v^2) = -\varkappa\, \rho v |v| - g \rho\, s_{\mathrm{lope}}, \tag{9.2}$$

where $x \in [0, L]$, $L < \infty$ is the length of the pipe, and $t \in \mathcal{I} \subset [0, \infty)$, $\mathcal{I}$ is the time interval, and *the equation of state* for real gases for the constant temperature:

$$p = R_s T_0 \rho z(p). \tag{9.3}$$

Here $\rho = \rho(t, x)$ denotes the density, $v = v(t, x)$ is the velocity of the gas, $p = p(t, x)$ is the pressure, g is the gravitational acceleration, $\varkappa := 0.5 \lambda_{fr} D^{-1}$ where λ_{fr} is the pipe friction coefficient and D is the pipe diameter, $T_0 = \mathrm{const}$ is the gas temperature, R_s is the specific gas constant, $s_{\mathrm{lope}}(x) = \dfrac{dh}{dx}(x)$ (cf. [14]) denotes the slope $\dfrac{dh}{dx}(x)$ of the pipe, where $h = h(x)$ is the height profile of the pipe over ground, and $z = z(p)$ is the compressibility factor (see the description of the model in [5]). In particular, the compressibility factor $z(p) = 1 + \alpha p$, where α

is a certain constant (see, e.g., [5, p. 5]), is a good approximation for pressures up to 70 bar which is used by the American Gas Association. If this compressibility factor is used, then $p = R_s T_0 \rho/(1 - \alpha R_s T_0 \rho)$ that is one of the commonly used equation of state in the isothermal case.

When modelling the dynamics of a gas flow, the assumption $(\rho v^2)_x = 0$ (i.e., we assume that this term is negligibly small) can be used (see, e.g., [16]) in order to simplify the model, then we obtain the gas dynamics equations in the form (9.1) and

$$\partial_t(\rho v) + \partial_x p = -\varkappa\, \rho v|v| - g\rho\, s_{\mathrm{lope}} \tag{9.4}$$

(see [5, (ISO2), p.12]; the similar system is used in [16]) with the same equation of state (9.3). The equations (9.1), (9.4) are often referred to as a semilinear model of the gas flow dynamics [5, 16].

In [5, p.26] and [16, p.2,3], q denotes a mass flow and it is defined as $q = S\rho v$, where S is the cross-sectional area of a pipe. We denote by $q := \rho v$ a mass flow by the cross-sectional area equal to 1, in order not to introduce additional notation, and assume that the total mass flow is $\tilde{q} = qS$. We can assume that $|q| = \rho|v|$ (cf. [5, 16]). Also, assume that $s_{\mathrm{lope}}(x) \equiv \sin\theta$ where the parameter θ denotes the angle of the pipe slope (cf. [5,16]). Then the system of the isothermal Euler equations (9.1), (9.4) and the gas state equation (9.3) takes the form

$$\partial_t \rho + \partial_x q = 0, \tag{9.5}$$

$$\partial_t q + \partial_x p + \rho\, g \sin\theta = -\varkappa\, q|q|\rho^{-1}, \tag{9.6}$$

$$p = R_s T_0 \rho z(p). \tag{9.7}$$

Suppose that a pipe was previously divided into parts of a short length through the introduction of artificial nodes and L is the length of such a part (subpipe). In Section 9.2 we consider the model of a gas network, where a gas flow in each pipe is described by a system of the type (9.5)–(9.7), and show that it also can be represented as the DAE (2.1). Thus, the pipe of the original length can be considered as a gas network consists of pipes of a short length, and the model obtained in Section 9.2 (generally, this model describes a gas network including pipes, valves, regulators and compressors) can be applied for the description of this network. We discretize the equations (9.5), (9.6) (for the pipe of the length L) in the phase variable (in space) and obtain the spatially discretized equations

$$\frac{d\rho_r}{dt} + \frac{q_r - q_l}{L} = 0, \tag{9.8}$$

$$\frac{dq_l}{dt} + \frac{p_r - p_l}{L} + \rho_r\, g \sin\theta = -\varkappa\, \frac{q_l|q_l|}{\rho_r}, \tag{9.9}$$

$$p_r = R_s T_0 \rho_r z(p_r). \tag{9.10}$$

where $q_r(t) := q(t, L)$, $p_r(t) := p(t, L)$, $\rho_r(t) := \rho(t, L)$ and $q_l(t) := q(t, 0)$, $p_l(t) := p(t, 0)$. If we represent the pipe as a graph consisting of an edge and two vertices (nodes), define the vertices as the left and right nodes and fix the edge orientation from the left node to the right node, then $q_r(t)$, $p_r(t)$ and $\rho_r(t)$ are defined at the right end of pipe and $q_l(t)$, $p_l(t)$ are defined at the left end of pipe. For a gas network, the spatial discretization is performed on each pipe. Here we use a scheme similar to the topology-adapted discretization scheme from [2, 15].

Let the functions q_r and p_l be given, that is, we consider the boundary conditions of the form

$$q(t,L) = q_r(t), \quad p(t,0) = p_l(t), \qquad t \in \mathcal{I}. \tag{9.11}$$

Then functions p_r, ρ_r and q_l need to be found.

We introduce the variable vector $x = (\rho_r, q_l, p_r)^{\mathrm{T}}$ (we denote it by x for convenience and comparison with further results, since the original variable x is already absent from the equations) and denote

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -\frac{1}{L} & 0 \\ g\sin\theta & 0 & \frac{1}{L} \\ 0 & 0 & 1 \end{pmatrix}, \; f(t,x) = \begin{pmatrix} -\frac{q_r(t)}{L} \\ \frac{p_l(t)}{L} - \frac{\varkappa q_l |q_l|}{\rho_r} \\ R_s T_0\, \rho_r z(p_r) \end{pmatrix}. \tag{9.12}$$

Then the system (9.8)–(9.10) can be written in the vector form

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \qquad t \in \mathcal{I}, \tag{9.13}$$

where A, $B \in \mathbb{R}^{3\times 3}$ and $f \in C(\mathcal{I}\times\mathbb{R}^3, \mathbb{R}^3)$. The initial condition for (9.13) can be given as

$$x(t_0) = x_0, \qquad x_0 = (\rho_r^0, q_l^0, p_r^0)^{\mathrm{T}}. \tag{9.14}$$

where ρ_r^0 and p_r^0 have to satisfy (9.10) for $t = t_0$, i.e., $p_r^0 = R_s T_0 \rho_r^0 z(p_r^0)$.

In general, the DAE (9.13) is regular (since the pencil $\lambda A + B$ is regular), but if any of the input parameters (i.e., $q_r(t)$ or $p_l(t)$) is not specified, then the system (9.8)–(9.10) is underdetermined and the corresponding DAE is singular (nonregular). Also, if it is required to realize the evolution of some variable (i.e., p_r, or ρ_r, or q_l) such that it becomes equal to the prescribed function, then this system is overdetermined and the corresponding DAE is singular.

9.2. A model of a gas network (in the isothermal case). Now, consider the mathematical model of a gas network, where a gas flow in each pipe is described by a system of the type (9.5), (9.6), (9.7). In addition to pipes, the gas network also includes valves, regulators and compressors.

Following [5], [16], we describe a gas network as oriented connected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes a set of nodes (vertices), $\mathcal{E}$ denotes a set of edges, and each edge joins two distinct nodes (i.e., there are no self-loops). We fix the orientation of edge $e \in \mathcal{E}$, denoting its endpoints by v_l and v_r and assuming that the edge is oriented from the left node v_l to the right node v_r. Note that the orientation of the edge may not coincide with the direction of a gas flow. We collect all nodes with a fixed pressure in $\mathcal{V}_{\text{pset}}$ and refer to them as *pressure nodes* [2, 16]. Fixed pressure means the existence of a time-dependent function chosen in advance, which yields the respective pressure value at each point in time. All other nodes we collect in

$\mathcal{V}_{\rm qset}$. Accordingly, $\mathcal{V} = \mathcal{V}_{\rm pset} \cup \mathcal{V}_{\rm qset}$. We denote the sets of edges corresponding to the pipes, valves and regulating elements (regulators and compressors) by $\mathcal{E}_{pip}$, $\mathcal{E}_{\rm val}$ and $\mathcal{E}_{\rm reg}$, respectively. Thus, $\mathcal{E} = \mathcal{E}_{\rm pip} \cup \mathcal{E}_{\rm val} \cup \mathcal{E}_{\rm reg}$.

First, introduce the vector p of the pressures of nodes $u \in \mathcal{V}_{\rm pset}$, and the vectors $q_{{\rm pip},r}$, $q_{{\rm pip},l}$, $q_{\rm val}$ and $q_{\rm reg}$ of flows at the right ends of pipes, at the left ends of pipes, through valves and through regulating elements, respectively.

As mentioned above, at the pressure nodes $u \in \mathcal{V}_{\rm pset}$, the pressure function $p^{\rm set}(t) = (\dots, p_u^{\rm set}(t), \dots)^{\rm T}_{u\in\mathcal{V}_{\rm pset}}$ is given. At the nodes $u \in \mathcal{V}_{{\rm qset}=\mathcal{V}\setminus\mathcal{V}_{\rm pset}}$ (which include junction, demand and source nodes), the function $q^{\rm set}(t) = (\dots, q_u^{\rm set}(t), \dots)^{\rm T}_{u\in\mathcal{V}_{\rm qset}}$ specifying the relationships between the flows $q_{{\rm pip},r}$, $q_{{\rm pip},l}$, $q_{\rm val}$, $q_{\rm reg}$ in a Kirchhoff-type flow balance equation (see (9.19) below) is given.

The mathematical model of a gas network consisting of pipes, valves, regulators and compressors after applying spatial discretization (more precisely, a topologically adaptive discretization of the isothermal Euler equations for pipes and pipelines [2, 15]) has the form [16, (9), p. 7]:

$$A^{\rm T}_{{\rm pip},r}\frac{d}{dt}\phi(p) + D_q(q_{{\rm pip},r} - q_{{\rm pip},l}) = 0, \tag{9.15}$$

$$\frac{d}{dt}q_{{\rm pip},l} + D_p(A^{\rm T}_{{\rm pip},r} + A^{\rm T}_{{\rm pip},l})p = -f_{\rm pip}(p, q_{{\rm pip},l}, t), \tag{9.16}$$

$$D_{\rm val}\frac{d}{dt}q_{\rm val} = -f_{\rm val}(p, q_{\rm val}, t), \tag{9.17}$$

$$D_{\rm reg}\frac{d}{dt}q_{\rm reg} = f_{\rm reg}(p, q_{\rm reg}, t), \tag{9.18}$$

$$A_{{\rm pip},l}q_{{\rm pip},l} + A_{\rm val}q_{\rm val} + A_{\rm reg}q_{\rm reg} + A_{{\rm pip},r}q_{{\rm pip},r} = q^{\rm set}(t), \tag{9.19}$$

$$0 = f_{pb}(p), \tag{9.20}$$

$$0 = f_{qb}(q_{{\rm pip},l}, q_{{\rm pip},r}, q_{\rm val}, q_{\rm reg}), \tag{9.21}$$

where

$$A_{{\rm pip},l} := \big(a_{ij}^{{\rm pip},l}\big)_{\substack{i=1,\dots,|\mathcal{V}_{\rm qset}|,\\ j=1,\dots,|\mathcal{E}_{\rm pip}|}}, \qquad A_{{\rm pip},r} := \big(a_{ij}^{{\rm pip},r}\big)_{\substack{i=1,\dots,|\mathcal{V}_{\rm qset}|,\\ j=1,\dots,|\mathcal{E}_{\rm pip}|}},$$

$$A_{\rm val} := \big(a_{ij}^{\rm val}\big)_{\substack{i=1,\dots,|\mathcal{V}_{\rm qset}|,\\ j=1,\dots,|\mathcal{E}_{\rm val}|}}, \qquad A_{\rm reg} := \big(a_{ij}^{\rm reg}\big)_{\substack{i=1,\dots,|\mathcal{V}_{\rm qset}|,\\ j=1,\dots,|\mathcal{E}_{\rm reg}|}}$$

are constant incidence matrices with the entries presented in [16, Section 3.1, p. 4],

$$D_q := {\rm diag}\{\dots, \frac{\kappa_e}{L_e}, \dots\}_{e\in\mathcal{E}_{\rm pip}}, \qquad D_p := {\rm diag}\{\dots, \frac{S_e}{L_e}, \dots\}_{e\in\mathcal{E}_{\rm pip}},$$

$$D_{\rm val} := {\rm diag}\{\dots, \mu_e, \dots\}_{e\in\mathcal{E}_{\rm val}}, \qquad D_{\rm reg} := {\rm diag}\{\dots, \mu_e, \dots\}_{e\in\mathcal{E}_{\rm reg}}$$

are constant diagonal matrices, where $\mu_e \geq 0$, $\kappa_e = R_sT_0/S_e$ (as above, $T_0 =$ const is the temperature and R_s is the specific gas constant), S_e and L_e are the cross-sectional area and the length of pipe e, respectively. Here p, $q_{{\rm pip},r}$, $q_{{\rm pip},l}$, $q_{\rm val}$ and $q_{\rm reg}$ are unknown and the remaining functions and parameters are given. The functions $f_{\rm pip}(p, q_{{\rm pip},l}, t)$, $f_{\rm val}(p, q_{\rm val}, t)$ and $f_{\rm reg}(p, q_{\rm reg}, t)$ are specified in [16, (4),(5),(8), p.5,6,7]; $f_{pb}(p)$ and $f_{qb}(q_{\text{pip},l}, q_{\text{pip},r}, q_{\text{val}}, q_{\text{reg}})$ are given continuous functions (see [16] for details).

Note that the elements of $\phi(p) = (\dots, \varphi(p_u), \dots)^{\mathrm{T}}_{u\in\mathcal{V}_{\text{qset}}}$ from (9.15) are expressed as $\varphi(p) = p/z(p)$, $p = p_u$, $u \in \mathcal{V}_{\text{qset}}$ (see [16, p. 2,5]), where the function $\varphi(p)$ can be also derived from the equation of state for real gases (in the isothermal case) $p = R_s T_0 \rho z(p)$ (9.3), i.e., $\varphi(p) = R_s T_0 \rho$. Thus, we introduce an additional variable $\varrho = \begin{pmatrix} \vdots \\ \rho_u \\ \vdots \end{pmatrix}_{u\in\mathcal{V}_{\text{qset}}}$, and instead of (9.15) we use the system

$$A^{\mathrm{T}}_{\text{pip},r} \frac{d}{dt}\varrho + D_q(q_{\text{pip},r} - q_{\text{pip},l}) = 0, \tag{9.22}$$

$$\varrho = \phi(p), \tag{9.23}$$

which is equivalent to (9.15), taking into account that $\kappa_e = R_s T_0/S_e$. Also, we rewrite the function $f_{\text{pip}}(p, q_{\text{pip},l}, t)$ (this function also includes $\phi(p)$ [16, p. 4,5]), without changing its notation, as $f_{\text{pip}}(\varrho, q_{\text{pip},l}, t)$. Then the equation (9.16) takes the form

$$\frac{d}{dt} q_{\text{pip},l} + D_p(A^{\mathrm{T}}_{\text{pip},r} + A^{\mathrm{T}}_{\text{pip},l})p = -f_{\text{pip}}(\varrho, q_{\text{pip},l}, t). \tag{9.24}$$

Finally, we obtain the differential-algebraic system (9.22), (9.24), (9.17), (9.18), (9.19), (9.23), (9.20) and (9.21). It is assumed that the resulting system with the spatially discretized equations satisfies conditions sufficient for its solution to approximate a solution of the original system sufficiently accurately.

The system (9.22), (9.24), (9.17), (9.18), (9.19), (9.23), (9.20), (9.21) can be written in the form of the singular (nonregular) DAE

$$\frac{d}{dt}[Ax] + Bx(t) = f(t, x), \qquad \text{where} \tag{9.25}$$

$$A = \begin{pmatrix} A^{\mathrm{T}}_{\text{pip},r} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{\text{val}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{\text{reg}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; f(t,x) = \begin{pmatrix} 0 \\ -f_{\text{pip}}(\varrho, q_{\text{pip},l}, t), \\ -f_{\text{val}}(p, q_{\text{val}}, t) \\ f_{\text{reg}}(p, q_{\text{reg}}, t), \\ q^{\text{set}}(t) \\ \phi(p) \\ f_{pb}(p) \\ f_{qb}(q_{\text{pip},l}, q_{\text{pip},r}, q_{\text{val}}, q_{\text{reg}}) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -D_q & 0 & 0 & D_q & 0 \\ 0 & 0 & 0 & 0 & 0 & D_p(A^{\mathrm{T}}_{\text{pip},r} + A^{\mathrm{T}}_{\text{pip},l}) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{\text{pip},l} & A_{\text{val}} & A_{\text{reg}} & A_{\text{pip},r} & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; x = \begin{pmatrix} \varrho \\ q_{\text{pip},l} \\ q_{\text{val}} \\ q_{\text{reg}} \\ q_{\text{pip},r} \\ p \end{pmatrix}.$$

The initial condition for the DAE (9.25) has the form

$$x(0) = x_0, \tag{9.26}$$

where $x_0 = (\varrho^0, q^0_{\mathrm{pip},l}, q^0_{\mathrm{val}}, q^0_{\mathrm{reg}}, q^0_{\mathrm{pip},r}, p^0)^{\mathrm{T}}$ is chosen so that the values t_0, x_0 satisfy the algebraic equations (9.19), (9.23), (9.20) and (9.21) (or satisfy the consistency condition defined in Remark 3.1).

In [16], the vector form of the DAE corresponding to the system (9.15)–(9.21) is slightly different from the above, but, in general, it is also a nonregular DAE in the sense that the number of unknowns is not equal to the number of equations. However, in [16], it is mentioned that with a proper choice of the directions of pipe and some additional conditions to the positions of regulators and valves (as described in, e.g., [15]), the resulting DAE system will have index 1 that means it will be a regular as well. A gas network model in the form of a nonregular DAE of the type (9.25) is also obtained in [1].

10. Analysis of a singular (nonregular) semilinear DAE with the characteristic pencil of the rank $\operatorname{rank}(\lambda A + B) < n, m$

In this section, we consider a simple example which demonstrates the application of the obtained results.

Consider the singular semilinear DAE (a DAE of the form (2.1))

$$\frac{d}{dt}[Ax] + Bx = f(t, x), \tag{10.1}$$

where $t \in [t_+, \infty)$ $(t_+ \geq 0)$, $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$, a function $f(t, x) = (f_1(t, x), f_2(t, x), f_3(t, x))^{\mathrm{T}} \in C([t_+, \infty) \times \mathbb{R}^3, \mathbb{R}^3)$ has the continuous partial derivative $\partial_x f$ on $[t_+, \infty) \times \mathbb{R}^3$, and A, $B \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ (here the terminology from Section 2.2 is used), $n = m = 3$, are the operators to which the matrices

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{pmatrix} \tag{10.2}$$

correspond with respect to the standard bases in $\mathbb{R}^n$, $n = 3$, and $\mathbb{R}^m$, $m = 3$. As usual, a basis in $\mathbb{R}^k$ is standard if the ith coordinate of the basis vector e_j $(j = 1, \dots, k)$ is equal to δ_{ij}. The pencil $\lambda A + B$ of the operators (10.2) is singular and its rank equals $\operatorname{rank}(\lambda A + B) = 2$.

Generally, in this section we consider the matrices corresponding to the operators (from $\mathbb{R}^3$ into $\mathbb{R}^3$) with respect to the standard bases in $\mathbb{R}^3$ (as well as we consider the coordinates of vectors with respect to the standard basis in $\mathbb{R}^3$), and if the bases are different, then this will be explicitly indicated.

The singular pencil (10.2) was studied in [9, Section 4.4].

In [9], it is shown that the subspaces from the decomposition (2.35) where $n = 3$, i.e., $\mathbb{R}^3 = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$, and from the decomposition (2.37) where $m = 3$, i.e., $\mathbb{R}^3 = Y_s \dot{+} Y_r = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_1 \dot{+} Y_2$, can be represented as $X_s = X_{s_1} \dot{+} X_{s_2} = \operatorname{Lin}\{s_i\}_{i=1}^2$, $X_{s_1} = \operatorname{Lin}\{s_1\}$, $X_{s_2} = \operatorname{Lin}\{s_2\}$, $Y_s =$

$Y_{s_1}\dot{+}Y_{s_2} = \mathrm{Lin}\{l_i\}_{i=1}^2$, $Y_{s_1} = \mathrm{Lin}\{l_1\}$, $Y_{s_2} = \mathrm{Lin}\{l_2\}$, $X_r = \mathrm{Lin}\{p\}$, $X_1 = \{0\}$, $X_2 = X_r$, $Y_r = \mathrm{Lin}\{q\}$, $Y_1 = \{0\}$, $Y_2 = Y_r$, where

$$s_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ s_2 = \begin{pmatrix}1\\0\\1\end{pmatrix},\ p = \begin{pmatrix}0\\1\\0\end{pmatrix},\ l_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ l_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ q = \begin{pmatrix}-1/2\\1/2\\1\end{pmatrix}, \tag{10.3}$$

and that the projection matrices corresponding to the projectors $S\colon \mathbb{R}^3 \to X_s$, $S = S_1 + S_2$, $S_i\colon \mathbb{R}^3 \to X_{s_i}$, $F\colon \mathbb{R}^3 \to Y_s$, $F = F_1 + F_2$, $F_i\colon \mathbb{R}^3 \to Y_{s_i}$, $P\colon \mathbb{R}^3 \to X_r$, $P = P_1 + P_2$, $P_i\colon \mathbb{R}^3 \to X_i$, $Q\colon \mathbb{R}^3 \to Y_r$, $Q = Q_1 + Q_2$, $Q_i\colon \mathbb{R}^3 \to Y_i$, $i = 1, 2$, which are defined in (2.13), (2.14) and (2.30), have the form

$$S_1 = \begin{pmatrix}1&0&-1\\0&0&0\\0&0&0\end{pmatrix},\qquad S_2 = \begin{pmatrix}0&0&1\\0&0&0\\0&0&1\end{pmatrix},\qquad S = \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix},$$
$$F_1 = \begin{pmatrix}1&0&1/2\\0&0&0\\0&0&0\end{pmatrix},\qquad F_2 = \begin{pmatrix}0&0&0\\0&1&-1/2\\0&0&0\end{pmatrix},\qquad F = \begin{pmatrix}1&0&1/2\\0&1&-1/2\\0&0&0\end{pmatrix},$$
$$P = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad P_1 = 0,\qquad P_2 = P,$$
$$Q = \begin{pmatrix}0&0&-1/2\\0&0&1/2\\0&0&1\end{pmatrix},\qquad Q_1 = 0,\qquad Q_2 = Q. \tag{10.4}$$

Note that if in $\mathbb{R}^3$, instead of the standard basis, we take the basis which is the union of the bases of the summands from the decomposition (2.35) of the space $\mathbb{R}^n$ where $n = 3$, i.e., we take the vectors s_1, s_2, p defined in (10.3), then the matrices corresponding to the projectors S, P, S_i, P_i, $i = 1, 2$, with respect to the basis s_1, s_2, p in $\mathbb{R}^3$ will have the simple form (and will be self-adjoint):

$$S_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad S_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad S = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},$$
$$P = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\qquad P_1 = 0,\qquad P_2 = P.$$

Similarly, if in $\mathbb{R}^3$, instead of the standard basis, we take the basis which is the union of the bases of the summands from the decomposition (2.37) of the space $\mathbb{R}^m$ where $m = 3$, i.e., we take l_1, l_2, q defined in (10.3), then the matrices corresponding to the projectors F, Q, F_i, Q_i, $i = 1, 2$, with respect to the basis l_1, l_2, q in $\mathbb{R}^3$ will have a simple form (and will be self-adjoint):

$$F_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad F_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad F = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_1 = 0, \qquad Q_2 = Q.$$

However, in this case, the operators A, B as the operators from $\mathbb{R}^n$ into $\mathbb{R}^m$, where $n = m = 3$, and in general the DAE (10.1), (10.2), must be considered with respect to the new bases s_1, s_2, p in $\mathbb{R}^n$ and l_1, l_2, q in $\mathbb{R}^m$ ($n = m = 3$). In what follows, we continue to use the standard bases in $\mathbb{R}^3$ ($\mathbb{R}^n$, $\mathbb{R}^m$, $n = m = 3$).

As shown in [9], the matrices (with respect to the standard bases in $\mathbb{R}^3$)

$$\mathcal{A}_r = 0, \quad \mathcal{A}_{\text{gen}} = A, \ \mathcal{B}_{\text{und}} = 0, \quad \mathcal{B}_r = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$\mathcal{B}_{\text{gen}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B}_{\text{ov}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_{\text{gen}}^{(-1)} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

correspond to the operators $\mathcal{A}_r$, $\mathcal{B}_r$, $\mathcal{A}_{\text{gen}}$, $\mathcal{B}_{\text{gen}}$, $\mathcal{B}_{\text{und}}$, $\mathcal{B}_{\text{ov}}$ introduced in (2.18), (2.20) and the operator $\mathcal{A}_{\text{gen}}^{(-1)}$ defined in Remark 2.3. Since $A_r = \mathcal{A}_r\big|_{X_r} = 0$ and $B_r = \mathcal{B}_r\big|_{X_r}$ has the inverse $B_r^{-1} \in \mathrm{L}(Y_r, X_r)$, then $\lambda A_r + B_r$ is a regular pencil of index 1.

The DAE (10.1), (10.2), is the vector form of the system

$$\frac{d}{dt}(x_1 - x_3) + x_1 - x_2 - x_3 = f_1(t, x), \tag{10.5}$$

$$x_1 + x_2 - x_3 = f_2(t, x), \tag{10.6}$$

$$2x_2 = f_3(t, x). \tag{10.7}$$

Note that a point (t, x) belongs to the manifold L_{t_+} (introduced in Remark 3.1) if and only if it satisfies the equations (2.41), (2.42) or the equations equivalent to them, e.g., (2.45), (2.46). It is readily verified that the equations (2.41) and (2.42) (as well as (2.45), (2.46)) are equivalent to the equations

$$x_2 = \frac{1}{2} f_3(t, x), \tag{10.8}$$

$$x_1 - x_3 = f_2(t, x) - 0.5 f_3(t, x) \tag{10.9}$$

respectively, which are the "algebraic part" of the DAE (10.1), (10.2), and are equivalent to the algebraic equations (10.6), (10.7). Also, notice that the ODE (2.44) (or (2.40)) is not present in the system (10.5)–(10.7) since the projector $Q_1 = 0$, and the ODE (2.43) (or (2.39)) is equivalent to

$$\frac{d}{dt}(x_1 - x_3) = -(x_1 - x_3) + f_1(t, x) + \frac{1}{2} f_3(t, x), \tag{10.10}$$

that is, the equation (10.5) into which (10.7) (or (10.8)) is substituted.

The components (projections) of a vector $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ represented as (2.36) have the form

$$x_{s_1} = S_1 x = (x_1 - x_3, 0, 0)^{\mathrm{T}}, \qquad x_{s_2} = S_2 x = (x_3, 0, x_3)^{\mathrm{T}},$$
$$x_{p_1} = P_1 x = 0, \qquad x_{p_2} = P_2 x = (0, x_2, 0)^{\mathrm{T}},$$

$x_s = x_{s_1} + x_{s_2}$, $x_r = x_{p_2}$, where S_i, P_i, $i = 1, 2$, were presented in (10.4). Obviously, $x_1 - x_3$, x_3, x_2 are the coordinates of the vector $x = (x_1, x_2, x_3)$ with respect to the basis s_1, s_2, p in $\mathbb{R}^3$, i.e., $x = (x_1 - x_3)s_1 + x_3 s_2 + x_2 p$, where s_1, s_2, p are the vectors defined in (10.3). Make the change of variables

$$w = x_1 - x_3, \quad \xi = x_3, \quad u = x_2, \tag{10.11}$$

then $x_{s_1} = w\,(1, 0, 0)^{\mathrm{T}}$, $x_{s_2} = \xi\,(1, 0, 1)^{\mathrm{T}}$, $x_{p_2} = u\,(0, 1, 0)^{\mathrm{T}}$.

Taking into account the new notations (10.11), we consider the function

$$\widetilde{f}(t, w, \xi, u) := f(t, w + \xi, u, \xi) = f(t, x_1, x_2, x_3) = f(t, x) \in C([t_+, \infty) \times \mathbb{R}^3, \mathbb{R}^3), \tag{10.12}$$

which, obviously, has the continuous partial derivative $\dfrac{\partial f}{\partial(w, \xi, u)}(t, w, \xi, u)$ for all $(t, w, \xi, u) \in [t_+, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. In the new notations the system of the equations (10.10), (10.8), (10.9) takes the form

$$\frac{d}{dt} w = -w + \widetilde{f}_1(t, w, \xi, u) + 0.5\widetilde{f}_3(t, w, \xi, u), \tag{10.13}$$
$$u = \frac{1}{2}\widetilde{f}_3(t, w, \xi, u), \tag{10.14}$$
$$w = \widetilde{f}_2(t, w, \xi, u) - 0.5\widetilde{f}_3(t, w, \xi, u). \tag{10.15}$$

Now we find the conditions under which there exists a global solution of the DAE (10.1), (10.2), and, accordingly, the system (10.5)–(10.7). To do this, we use Theorems 3.2 and 7.1, and in addition the remarks regarding the functions χ and V from Section 8.

It follows from the above that condition 1 of Theorem 3.2 is satisfied if:

(i) There exists a set $\widetilde{D}_{s_2} \subset \mathbb{R}$ such that for any fixed $t \in [t_+, \infty)$, $w \in \mathbb{R}$ and $\xi \in \widetilde{D}_{s_2}$ there exists a unique $u \in \mathbb{R}$ such that the equalities (10.14), (10.15) are satisfied.

The matrix corresponding to the operator $\widehat{\Phi}_{t_*, x_*}$ defined (for fixed t_*, x_*) by (3.4) has the form

$$\widehat{\Phi}_{t_*, x_*} = \begin{pmatrix} 0 & -\dfrac{1}{2}\dfrac{\partial f_3}{\partial x_2}(t_*, x_*) + 1 & 0 \\ 0 & \dfrac{1}{2}\dfrac{\partial f_3}{\partial x_2}(t_*, x_*) - 1 & 0 \\ 0 & \dfrac{\partial f_3}{\partial x_2}(t_*, x_*) - 2 & 0 \end{pmatrix},$$

and since the equality $\widehat{\Phi}_{t_*,x_*}x_{p_2} = 0$, $x_{p_2} \in X_2$, yields $x_{p_2} = 0$ if the relation $\frac{\partial f_3}{\partial x_2}(t_*, x_*) - 2 \neq 0$ holds, then the operator $\Phi_{t_*,x_*} = \widehat{\Phi}_{t_*,x_*}\big|_{X_2}$ (3.2) has the inverse $\Phi_{t_*,x_*}^{-1} \in \mathrm{L}(Y_2, X_2)$ if this relation is satisfied. Note that

$$\frac{\partial f_3}{\partial x_2}(t, x_1, x_2, x_3) = \frac{\partial f_3}{\partial x_2}(t, w+\xi, u, \xi) = \frac{\partial \widetilde{f}_3}{\partial u}(t, w, \xi, u).$$

Thus, condition 2 of Theorem 3.2 is satisfied if:

(ii) $\frac{\partial \widetilde{f}_3}{\partial u}(t_*, w_*, \xi_*, u_*) \neq 2$ for any fixed $t_* \in [t_+, \infty)$, $w_* \in \mathbb{R}$, $\xi_* \in \widetilde{D}_{s_2}$, $u_* \in \mathbb{R}$ satisfying the equalities (10.14), (10.15).

Also, since the space X_2 is one-dimensional, then condition 2 of Theorem 7.1 is satisfied if condition (ii) holds (see Remark 7.2 for explanation). Consequently, we can use condition 1 of Theorem 7.1 instead of more restrictive condition 1 of Theorem 3.2 and, accordingly, replace condition (i) by the following:

(i)′ There exists a set $\widetilde{D}_{s_2} \subset \mathbb{R}$ such that for any fixed $t \in [t_+, \infty)$, $w \in \mathbb{R}$ and $\xi \in \widetilde{D}_{s_2}$ there exists $u \in \mathbb{R}$ such that the equalities (10.14), (10.15) hold.

Recall that $X_1 = \{0\}$, the equation (2.44) is not present in the system (10.13)–(10.14), and the equation (2.43) is equivalent to (10.13). Thus, condition 3 of Theorem 3.2 (as well as Theorem 7.1) is fulfilled if:

(iii) There exists a number $R > 0$ (R can be sufficiently large), a function $\widetilde{V} \in C^1([t_+, \infty) \times \widetilde{D}_{s_1}, \mathbb{R})$ positive on $[t_+, \infty) \times \widetilde{D}_{s_1}$, where $\widetilde{D}_{s_1} = \{|w| > R\}$, and a function $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$ such that:

 (a) $\widetilde{V}(t, w) \to \infty$ as $|w| \to \infty$ uniformly in t on each finite interval $[a, b) \subset [t_+, \infty)$;

 (b) $\widetilde{V}'_{(10.13)}(t, w) = \frac{\partial \widetilde{V}}{\partial t}(t, w) + \frac{\partial \widetilde{V}}{\partial w}(t, w)\left[-w + \widetilde{f}_1(t, w, \xi, u) + \frac{1}{2}\widetilde{f}_3(t, w, \xi, u)\right] \leq \chi\big(t, \widetilde{V}(t, w)\big)$ for all $t \in [t_+, \infty)$, $w \in \mathbb{R}$, $\xi \in \widetilde{D}_{s_2}$, $u \in \mathbb{R}$ satisfying (10.14), (10.15) and $|w| > R$.

 (c) the differential inequality $\frac{dv}{dt} \leq \chi(t, v)$ $(t \in [t_+, \infty))$ does not have positive solutions with finite escape time.

Condition (iii) is given in the most general form, and if we take the function $\widetilde{V}$ of the type (8.2) and the function χ of the form (2.5), then we obtain a particular case of this condition, which is convenient for practical application. Namely, let $\widetilde{V}(t, w) = Hw^2$, where $H = \mathrm{const} > 0$, $w \in \mathbb{R}$, and $\chi(t, v) = k(t)\,U(v)$, where $k \in C([t_+, \infty), \mathbb{R})$ and $U \in C(0, \infty)$. Then $\widetilde{V}'_{(10.13)}(t, w) = -2H\,w^2 + 2H\,w\big[\widetilde{f}_1(t, w, \xi, u) + \frac{1}{2}\widetilde{f}_3(t, w, \xi, u)\big]$, and, taking into account the remarks from Section 8, condition (iii) is converted into the following one:

(iii)′ There exists a number $R > 0$ and functions $k \in C([t_+, \infty), \mathbb{R})$, $U \in C(0, \infty)$ such that

$$\int_{v_0}^{\infty} \frac{dv}{U(v)} = \infty$$

($v_0 > 0$ is a constant) and

$$-2H\,w^2 + 2H\,w\Big[\widetilde{f}_1(t,w,\xi,u) + \frac{1}{2}\widetilde{f}_3(t,w,\xi,u)\Big] \le k(t)\,U(Hw^2),$$

where $H > 0$ is some constant, for all $t \in [t_+,\infty)$, $w \in \mathbb{R}$, $\xi \in \widetilde{D}_{s_2}$, $u \in \mathbb{R}$ satisfying (10.14), (10.15) and $|w| > R$.

Finally, the following conclusions can be drawn.

Let conditions (i)′ and (ii), where the function $\widetilde{f}(t,w,\xi,u)$ is defined by (10.12), be fulfilled and let condition (iii) or (iii)′ hold, then by Theorem 7.1 (as well as by Theorem 3.2 if condition (i)′ is replaced by (i)) for each initial point $(t_0,x_0) \in [t_+,\infty) \times \mathbb{R}^3$, where $x_0 = (x_{0,1},x_{0,2},x_{0,3})^{\mathrm{T}}$, for which the equalities (10.8), (10.9) hold and $x_{0,3} \in \widetilde{D}_{s_2}$, the initial value problem (10.1), (10.2), $x(t_0) = x_0$ has a unique global solution $x(t)$ with the component $x_{s_2}(t) = S_2x(t) = \varphi_{s_2}(t)\,(1,0,1)^{\mathrm{T}}$, where $\varphi_{s_2} \in C([t_0,\infty),\widetilde{D}_{s_2})$ is some function with the initial value $\varphi_{s_2}(t_0) = x_{0,3}$.

Acknowledgments. The author is supported by the Alexander von Humboldt Foundation (the host institution: Friedrich-Alexander University of Erlangen-Nuremberg, Chair for Dynamics, Control, Machine Learning and Numerics).

References

[1] T.P. Azevedo-Perdicoúlis and G. Jank, *Modelling aspects of describing a gas network through a DAE system*, IFAC Proceedings Volumes **40** (2007), No. 20, 40–45.

[2] P. Benner, S. Grundel, C. Himpe, C.Huck, T. Streubel, and C. Tischendorf, *Gas Network Benchmark Models*, Applications of Differential-Algebraic Equations: Examples and Benchmarks. Differential-Algebraic Equations Forum (Eds. S. Campbell, A. Ilchmann, V. Mehrmann, and T. Reis), Springer, Cham, 2018, 171–197.

[3] V.F. Chistyakov and E.V. Chistyakova, *Application of the least squares method to solving linear differential-algebraic equations*, Numer. Analys. Appl. **6**(2013), 77–90.

[4] S.M. Chuiko, *On a reduction of the order in a differential-algebraic system*, J. Math. Sci. **235** (2018), No. 1, 2–14.

[5] P. Domschke, B. Hiller, J. Lang, V. Mehrmann, R. Morandin, and C. Tischendorf, *Gas Network Modeling: An Overview*, Technische Universität Darmstadt, Darmstadt, 2021.

[6] D.K. Faddeev, *Lectures on algebra*, Nauka, Moscow, 1984 (Russian).

[7] M.S. Filipkovskaya, *Continuation of solutions of semilinear differential-algebraic equations and applications in nonlinear radiotechnics*, Bull. of V. Karazin Kharkiv National University. Series Math. Model. Inform. Tech. Automat. Control Syst. **19** (2012), No. 1015, 306–319 (Russian).

[8] M.S. Filipkovska, *Lagrange stability and instability of irregular semilinear differential-algebraic equations and applications*, Ukrainian Math. J. **70** (2018), No. 6, 947–979.

[9] M.S. Filipkovska (Filipkovskaya), *A block form of a singular pencil of operators and a method of obtaining it*, Visnyk of V.N. Karazin Kharkiv National University. Ser. "Mathematics, Applied Mathematics and Mechanics" **89** (2019), 33–58 (Russian). Available from: https://doi.org/10.26565/2221-5646-2019-89-04

[10] M.S. Filipkovska, *Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits*, J. Math. Phys. Anal. Geom. **14** (2018), No. 2, 169–196.

[11] M. Filipkovskaya, *Global solvability of singular semilinear differential equations and applications to nonlinear radio engineering*, Chall. Modern Technology. **6** (2015), No. 1, 3–13.

[12] M. Filipkovska (Filipkovskaya), *Existence, boundedness and stability of solutions of time-varying semilinear differential-algebraic equations*, Global and Stochastic Analysis **7** (2020), No. 2, 169–195.

[13] F.R. Gantmacher, *The theory of matrices, Vol. I, II*, Amer. Math. Soc., Providence, RI, 2000.

[14] M. Gugat and S. Ulbrich, *Lipschitz solutions of initial boundary value problems for balance laws*, Math. Models Methods Appl. Sci. **28** (2018), No. 5, 921–951.

[15] C. Huck, *Perturbation analysis and numerical discretisation of hyperbolic partial differential algebraic equations describing flow networks*, Dissertation, Humboldt Universität zu Berlin, 2018.

[16] T. Kreimeier, H. Sauter, S.T. Streubel, C. Tischendorf, and A. Walther, *Solving Least-Squares Collocated Differential Algebraic Equations by Successive Abs-Linear Minimization – A Case Study on Gas Network Simulation*, Humboldt-Universität zu Berlin, preprint, 2022.

[17] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations: Analysis and Numerical Solution*, European Mathematical Society, Zurich, 2006.

[18] R. Riaza, *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific, Hackensack, NJ, 2008.

[19] J. La Salle and S. Lefschetz, *Stability by Liapunov's direct method with applications*, Academic Press, New York, 1961.

[20] L. Schwartz, *Analyse Mathématique*, I. Hermann, Paris, 1967 (French).

[21] L. Schwartz, *Analyse Mathématique*, II, Hermann, Paris, 1967 (French).

[22] A.G. Rutkas, *Cauchy problem for the equation $Ax'(t)+Bx(t) = f(t)$*, Differ. Uravn. **11** (1975), No. 11, 1996–2010 (Russian).

[23] A.G. Rutkas, *Solvability of semilinear differential equations with singularity*, Ukrainian Math. J. **60** (2008), 262–276.

[24] Rutkas A.G. and Filipkovskaya (Filipkovska) M.S. *Extension of solutions of one class of differential-algebraic equations*. J. Comput. Appl. Math. **1** (2013), 135–145 (Russian).

[25] L.A. Vlasenko, *Evolution Models with Implicit and Degenerate Differential Equations*, System Technologies, Dnipropetrovsk, Ukraine, 2006 (Russian).

[26] T. Yoshizawa, *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, Tokyo, 1966.

Received September 1, 2023, revised December 6, 2023.

Maria Filipkovska,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany,

E-mail: maria.filipkovska@fau.de, filipkovskaya@ilt.kharkov.ua

Якісний аналіз нерегулярних диференціально-алгебраїчних рівнянь та динаміка газових мереж

Maria Filipkovska

Одержано умови існування, єдиності та обмеженості глобальних розв'язків, а також граничної обмеженості розв'язків, та умови руйнування розв'язків нерегулярних напівлінійних диференціально-алгебраїчних рівнянь. Розглянуто приклад, який демонструє застосування одержаних результатів. В якості застосувань наводяться ізотермічні моделі газових мереж.

Ключові слова: нерегулярне диференціально-алгебраїчне рівняння, вироджене диференціальне рівняння, сингулярний жмуток, глобальна розв'язність, обмеженість розв'язків, руйнування, дисипативність