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On Circular Tractrices in $\mathbb{R}^3$

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Dedicated to the 80th anniversary of Professor Yuriy Aminov

We explore geometric properties of circular analogues of tractrices and pseudospheres in $\mathbb{R}^3$.

Key words: tractrix, circular tractrix, pseudosphere

Mathematical Subject Classification 2020: 53A04, 53A07

1. Introduction

In 2000, Yuriy Aminov and Antony Sym settled the question whether one can extend the classical theory of Bianchi–Backlund transformations of pseudo-spherical surfaces in $\mathbb{R}^3$ to the case of pseudo-spherical surfaces in $\mathbb{R}^4$, see [2]. This question, as well as its multi-dimensional generalizations, was addressed in a series of research papers [7–12], where it was shown to be rather non-trivial and actually it still remains widely open.

While studying the problem we (re)discovered a particular family of spatial curves in $\mathbb{R}^3$ called *circular tractrices*, which can be used for constructing novel examples of pseudo-spherical surfaces in $\mathbb{R}^4$ similar to the Beltrami and Dini surfaces in $\mathbb{R}^3$ [13, 15]. Independently of possible applications to the theory of pseudo-spherical surfaces, the circular tractrices themselves turn out to be of independent interest. The aim of this research note is to survey beautiful geometric properties of circular tractrices with particular emphasis on justifying the use of the terms *tractrix* and *circular*.

Let us introduce the principal hero of our story.

Consider the three-dimensional Euclidean space $\mathbb{R}^3$ endowed with Cartesian coordinates (x^1, x^2, x^3).

Definition 1.1. A *circular tractrix* is a curve in $\mathbb{R}^3$ represented by

$$\begin{cases} x^1 = \xi_1 \cos \dfrac{t}{R} + \xi_2 \sin \dfrac{t}{R}, \\ x^2 = -\xi_2 \cos \dfrac{t}{R} + \xi_1 \sin \dfrac{t}{R}, \qquad t \in \mathbb{R}, \\ x^3 = \xi_3, \end{cases} \tag{1.1}$$

where $R > 0$ is a fixed constant, and $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ are functions given explicitly by the following formulae depending on whether R is greater, equal or less than 1, respectively:

$$R > 1: \quad \xi_1 = \frac{(R - \frac{1}{R})\cosh\lambda t}{\frac{c_1}{R} + \cosh\lambda t}, \quad \xi_2 = \frac{\lambda\sinh\lambda t}{\frac{c_1}{R} + \cosh\lambda t}, \quad \xi_3 = \frac{\lambda c_2}{\frac{c_1}{R} + \cosh\lambda t}, \tag{1.2}$$

where $\lambda = \frac{\sqrt{R^2-1}}{R}$, and c_1, c_2 are arbitrary constants subject to $c_1^2 + c_2^2 = 1$;

$$R = 1: \quad \xi_1 = \frac{2}{c_1 + t^2}, \quad \xi_2 = \frac{2t}{c_1 + t^2}, \quad \xi_3 = \frac{c_2}{c_1 + t^2}, \tag{1.3}$$

where c_1, c_2 are arbitrary constants subject to $4(c_1 - 1) = c_2^2$;

$$R < 1: \quad \xi_1 = \frac{(R - \frac{1}{R})\cos\lambda t}{\frac{c_1}{R} + \cos\lambda t}, \quad \xi_2 = -\frac{\lambda\sin\lambda t}{\frac{c_1}{R} + \cos\lambda t}, \quad \xi_3 = \frac{\lambda c_2}{\frac{c_1}{R} + \cos\lambda t}, \tag{1.4}$$

where $\lambda = \frac{\sqrt{1-R^2}}{R}$, and c_1, c_2 are arbitrary constants subject to $c_1^2 - c_2^2 = 1$.

Thus, up to rigid motions in $\mathbb{R}^3$, we have a two-parameter family of circular tractrices: one parameter is the positive constant R and another parameter is encoded in the constants c_1, c_2 related by one relation.

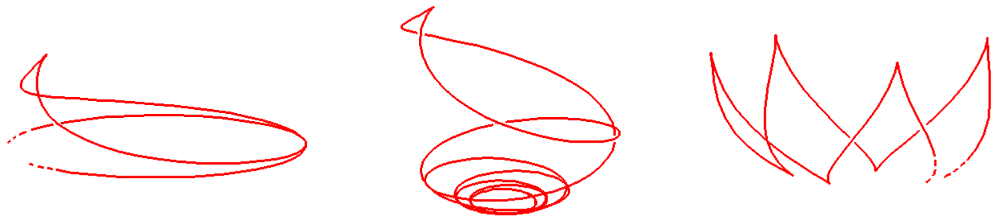

Fig. 1.1: Typical examples of circular tractrices with $R > 1$, $R = 1$, $R < 1$.

Qualitative properties of circular tractrices with R greater, equal or less than 1 differ greatly. These three cases will be discussed separately in Sections 2–4 below.

2. Circular tractrices with $R > 1$

Fix $R > 1$, chose arbitrary c_1, c_2 satisfying $c_1^2 + c_2^2 = 1$, and consider the corresponding circular tractrix γ represented by (1.1), (1.2). Denote by $x = f(t)$ the position vector of γ. List the elementary geometrical properties of γ.

1) The circular tractrix γ is symmetric with respect to the plane $x^2 = 0$:

$$f^1(-t) = f^1(t), \quad f^2(-t) = -f^2(t), \quad f^3(-t) = f^3(t). \tag{2.1}$$

This is the unique symmetry of γ.

2) The circular tractrix γ is piecewise regular. It has a unique singular point, a cusp, at $t = 0$. This follows immediately from the relation:

$$|f'| = \left| \frac{\lambda \sinh \lambda t}{\frac{c_1}{R} + \cosh \lambda t} \right|. \tag{2.2}$$

Notice that $|f'| = |\xi_2|$. Besides, by integrating (2.2), one can show that γ has infinite length.

3) As t tends to $\pm\infty$, the circular tractrix γ becomes asymptotically close to the circle C_∞ of radius $\sqrt{R^2-1}$ centered at the origin in the coordinate plane $x^3 = 0$. More exactly, if we introduce two vector-functions,

$$f_+(t) = \lambda \begin{pmatrix} \sqrt{R^2-1}\cos\frac{t}{R} + \sin\frac{t}{R} \\ \sqrt{R^2-1}\sin\frac{t}{R} - \cos\frac{t}{R} \\ 0 \end{pmatrix}, \quad f_-(t) = \lambda \begin{pmatrix} \sqrt{R^2-1}\cos\frac{t}{R} - \sin\frac{t}{R} \\ \sqrt{R^2-1}\sin\frac{t}{R} + \cos\frac{t}{R} \\ 0 \end{pmatrix}$$

which both represent the circle C_∞ in appropriate parameterizations, then we have

$$|f(t) - f_+(t)| < 2e^{-\lambda t}, \quad |f(t) - f_-(t)| < 2e^{\lambda t},$$

and hence

$$\lim_{t\to+\infty} |f(t) - f_+(t)| = 0, \quad \lim_{t\to-\infty} |f(t) - f_-(t)| = 0.$$

Evidently, similar asymptotic closeness at $t \to \pm\infty$ extends to the Frenet frames, curvatures and torsions of γ and C_∞, respectively.

Notice that the asymptotic circle C_∞ does not depend on the choice of c_1, c_2.

4) The position vector $x = f(t)$ of the circular tractrix γ satisfies the following relation:

$$f + \frac{1}{\xi_2} f' = \begin{pmatrix} R\cos\frac{t}{R} \\ R\sin\frac{t}{R} \\ 0 \end{pmatrix}. \tag{2.3}$$

This means that if one draws appropriately chosen unit segments tangent to γ, then the endpoints of these segments sweep out the circle C of radius R centered at the origin in the coordinate plane $x^3 = 0$, and t is an arc length for C. Thus, the circular tractrix γ is related to the circle C in the same manner as the classical *linear* tractix is related to its asymptotic straight line, c.f. [5, p.8]. In terms of the general theory of tractrices, the circle C is *the directrix* for the circular tractrix γ in question, c.f. [4,14]. Notice that the circle C does not depend on the choice of c_1, c_2.

5) The circular tractrix γ has non-vanishing torsion for any choice of c_1, c_2 except two particular cases, $c_1 = 1$, $c_2 = 0$ and $c_1 = -1$, $c_2 = 0$, where γ belongs to the coordinate plane $x^3 = 0$ and represents the well-known planar circular tractrices with $R > 1$, see Fig. 2.1, c.f. [4, 14].

6) The position vector $x = f(t)$ of the circular tractrix γ satisfies the following relation:

$$\lim_{R\to+\infty} \begin{pmatrix} f^1(t) - R \\ f^2(t) \\ f^3(t) \end{pmatrix} = \begin{pmatrix} -c_1\frac{1}{\cosh t}, \\ t - \tanh t, \\ c_2\frac{1}{\cosh t}. \end{pmatrix}. \tag{2.4}$$

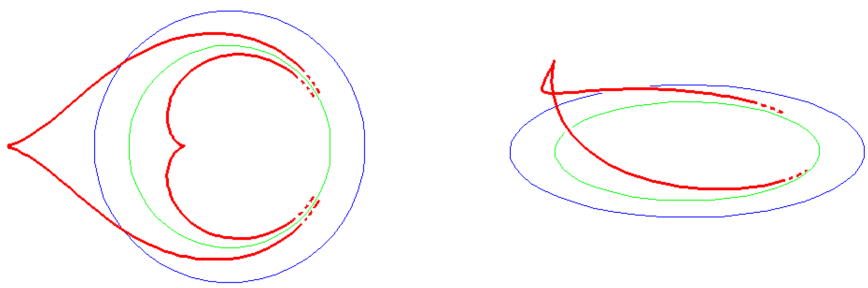

Fig. 2.1: Left: The planar circular tractrices (red), circle C (blue) and circle C_∞ (green) viewed from the top. Right: A non-planar circular tractrix (red) and the same circles C (blue) and C_∞ (green).

This means that if we use a shift along the x^1-axis at the distance R so that the circle C passes through the origin, then at the limit $R \to +\infty$ the circle C transforms into the x^2-axis, the circular tractrix γ transforms into the well-known linear tractrix situated so that its asymptotic straight line is the x^2-axis, and the parameters c_1, c_2 satisfying $c_1^2 + c_2^2 = 1$ describe the rotation of the linear tractrix around the x^2-axis in $\mathbb{R}^3$, c.f. [16, p.7].

Next, the constant $R > 1$ being fixed, we set $c_1 = \cos\alpha$, $c_2 = \sin\alpha$ and allow $\alpha \in S^1$ to be varied. Then we obtain a one-parameter family of circular tractrices which sweep out a two-dimensional surface F. This surface is represented by the position vector $x = f(t, \alpha)$ given by (1.1), (1.2) with $c_1 = \cos\alpha$, $c_2 = \sin\alpha$. We will call F *a circular pseudosphere*, see Fig. 2.2.

1*) The circular pseudosphere F is symmetric with respect to the coordinate plane $x^2 = 0$:

$$f^1(-t,\alpha) = f^1(t,\alpha), \quad f^2(-t,\alpha) = -f^2(t,\alpha), \quad f^3(-t,\alpha) = f^3(t,\alpha). \tag{2.5}$$

Thus, F consists of 2 symmetric parts sharing the common coordinate line $t = 0$ situated in the plane $x^2 = 0$.

Moreover, F is symmetric with respect to the coordinate plane $x^3 = 0$:

$$f^1(t,-\alpha) = f^1(t,\alpha), \quad f^2(t,-\alpha) = f^2(t,\alpha), \quad f^3(t,-\alpha) = -f^3(t,\alpha). \tag{2.6}$$

2*) The circular pseudosphere F is piecewise regular. Its singular set, a cuspidal edge, is a unit circle composed of the singular points $t = 0$ of the circular tractrices $\alpha = \mathit{const}$ sweeping out the surface F.

3*) The coordinate curves $t = \text{const}$ in F are circles whose radii are equal to $\frac{\sqrt{R^2-1}}{\sqrt{R^2\cosh^2\lambda t - 1}}$ and tend to 0 as $t \to \pm\infty$. This is verified easily by computing the curvature and torsion of the curves in question viewed as curves in $\mathbb{R}^3$.

4*) As t tends to $\pm\infty$, the circular pseudosphere F becomes asymptotically close to the circle C_∞ in the same manner as it was described above for the circular tractrices constituting F.

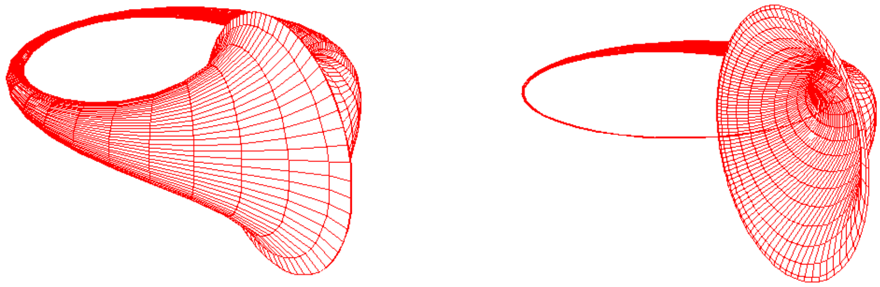

Fig. 2.2: A typical example of circular pseudosphere with $R > 1$: the complete surface (left) and its one half (right).

5*) The position vector $x = f(t, \alpha)$ of the circular pseudosphere F satisfies the following relation:

$$f + \frac{1}{\xi_2}\frac{\partial f}{\partial t} = \begin{pmatrix} R\cos\frac{t}{R} \\ R\sin\frac{t}{R} \\ 0 \end{pmatrix}. \tag{2.7}$$

This means that if one draws appropriately chosen unit segments tangent to coordinate t-curves of F, then the endpoints of these segments will sweep out the circle C. Thus, the circular pseudosphere F is related to the circle C in the same manner as the classical pseudosphere is related to its axis of rotation.

6*) The first fundamental form of F reads as follows:

$$g = \frac{R^2 - 1}{(\cos\alpha + R\cosh\lambda t)^2}\left(\sinh^2\lambda t\, dt^2 + d\alpha^2\right).$$

Hence, the coordinate curves in F, which are the circular tractrices $\alpha = const$ and the circles $t = const$, form an isothermic net on F. Evidently, this isothermic net on the circular tractrix F can be viewed as an analogue of the standard horocyclic net on the classical pseudosphere.

7*) Clearly, the circular pseudosphere F depends on R. For instance, it is situated inside the tube of unit radius around the circle C in $\mathbb{R}^3$. Hence, the greater R is, the larger the distance between F and the origin $O \in \mathbb{R}^3$ is, see Fig. 2.3.

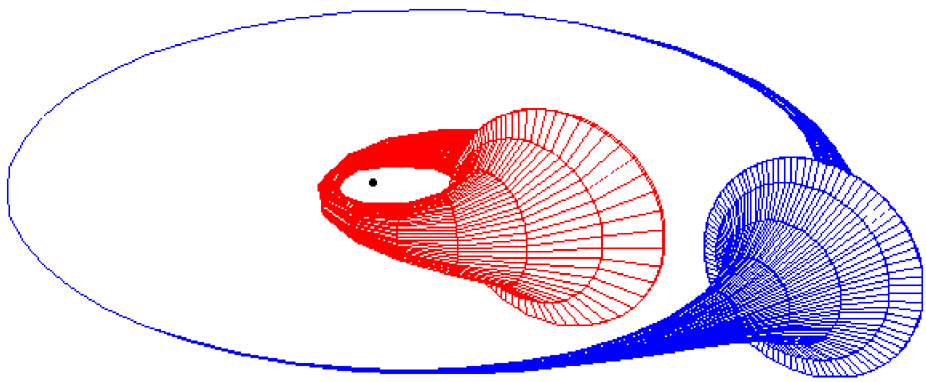

Fig. 2.3: Circular pseudospheres with $R = 1.25$ (red) and $R = 3.5$ (blue).

8*) Whatever the value of $R > 1$ is, the complete area of the circular pseudosphere F is equal to 4π, and it is the same as the area of the classical pseudosphere, c.f. [17]. In view of the previous item in the list, this is really an astonishing fact. Its proof is based simply on calculations of corresponding integrals:

$$\begin{aligned}\mathrm{Area}(F) &= 2\int_0^{+\infty}\int_0^{2\pi}\sqrt{\det g}\,d\alpha\,dt\\ &= 2\int_0^{+\infty}\int_0^{2\pi}\frac{(R^2-1)\sinh\lambda t}{(\cos\alpha + R\cosh\lambda t)^2}\,d\alpha\,dt = 4\pi.\end{aligned}$$

9*) The circular pseudosphere F has self-intersections, which all are situated in the coordinate plane $x^2 = 0$. By cutting F with this plane, we decompose F into a countable set of pieces without self-intersections, $F = \bigcup_{j=1}^{\infty} F_j$. Every piece F_j encloses a well-defined body, $F_j = \partial B_j$. Then we define "the volume" enclosed by F as the total sum of the volumes of B_j, $j \geq 1$, i.e., $\mathrm{Vol}(F) := \sum_{j=1}^{\infty}\mathrm{Vol}(B_j)$. Actually, $\mathrm{Vol}(F)$ is the volume of a body enclosed by F, which is counted with multiplicities in view of self-intersections of F.

To find $\mathrm{Vol}(F)$, one needs to parameterize every body B_j. For instance, F_j is equipped with the coordinates t, α and foliated by circles $t = const$. Consequently, B_j is foliated by two-dimensional discs. Introducing polar coordinates l, α in every disc, we get a parameterization by t, α l for B_j. And then the volume of B_j is calculated via appropriate integrals in terms of t, α l, which appears to be quite cumbersome. However the result turns out to be absolutely surprising:

$$\mathrm{Vol}(F) = \frac{2}{3}\pi, \tag{2.8}$$

Thus, whatever the value of $R > 1$ is, the complete "volume" enclosed by the circular pseudosphere F is equal to $\frac{2}{3}\pi$, and it is the same as the volume enclosed by the classical pseudosphere, c.f. [17].

10*) If we make a shift along the x^1-axis at the distance R so that the circle C passes through the origin, then at the limit $R \to +\infty$ the circle C transforms into the x^2-axis, the circular pseudosphere F transforms into the classical pseudosphere whose axis of rotation is the x^2-axis, and α describes the corresponding angle parameter on the pseudosphere, c.f. [16, p.7], [17]. Therefore, the pseudosphere appears as the limit surface at $R \to \infty$ in the one-parameter family of circular pseudospheres with $R > 1$ under consideration.

Thus, the circular tractrices and circular pseudospheres with $R > 1$ inherit fundamental properties of the classical linear tractrix and pseudosphere, respectively, and hence can be naturally treated as their circular analogs.

3. Circular tractrices with $R = 1$

Next, fix $R = 1$, chose arbitrary c_1, c_2 satisfying $4(c_1 - 1) = c_2^2$, and consider the corresponding circular tractrix γ represented by (1.1), (1.3). Let $x = f(t)$ stand for the position vector of γ. List the elementary geometrical properties of γ.

1) γ is symmetric with respect to the plane $x^2 = 0$, its position-vector satisfies (2.1). This is the unique symmetry of γ.
2) γ is piecewise regular. It has a unique singular point, a cusp, at $t = 0$. This follows immediately from the relation:

$$|f'| = \left| \frac{2t}{t^2 + c_1} \right|.$$

As a consequence, γ has infinite length. Notice that $|f'| = |\xi_2|$ remains true.
3) As t tends to $\pm\infty$, the circular tractrix γ becomes asymptotically close to the origin point $O \in \mathbb{R}^3$. More exactly, we have

$$|f(t)| < \frac{2}{|t|},$$

and hence $|f(t)| \to 0$ as $t \to \pm\infty$. The origin point O can be viewed as a degenerate asymptotic circle C_∞ whose radius $\sqrt{R^2 - 1}$ becomes equal to 0 at $R = 1$. Notice that this asymptotic behavior of γ does not depend on the choice of c_1, c_2.
4) The position vector $x = f(t)$ of the circular tractrix γ satisfies the same relation as (2.3) but with $R = 1$. This still means that if one draws appropriately chosen unit segments tangent to γ, then the endpoints of these segments will sweep out the circle C of unit radius centered at the origin in the coordinate plane $x^3 = 0$, and t is an arc length of this circle. Thus, the circular tractrix γ is related to the unit circle C in the same manner as the linear tractix is related to its asymptotic straight line, i.e., the unit circle C is the directrix for the circular tractrix γ in question, c.f. [5, p.8], [4,14]. Notice that the circle C does not depend on the choice of c_1, c_2.
5) The circular tractrix γ has non-vanishing torsion for any choice of c_1, c_2 except one particular case, $c_1 = 1$, $c_2 = 0$, where γ belongs to the coordinate plane $x^3 = 0$ and describes the well-known planar circular tractrix with $R = 1$, see Fig. 3.1, c.f. [4, 14].

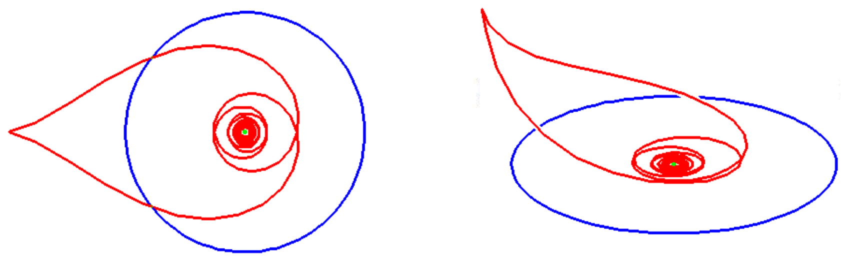

Fig. 3.1: Left: The planar circular tractrix (red) and the unit circle C (blue) viewed from the top. Right: A non-planar circular tractrix (red) and the unit circle C (blue).

Next, the constant $R = 1$ being fixed, we set $c_1 = 1 + \alpha^2$, $c_2 = 2\alpha$ and allow $\alpha \in \mathbb{R}^1$ to be varied. Then we obtain a one-parameter family of circular tractrices, which sweep out a two-dimensional surface F. This surface, also called a *circular pseudosphere*, is represented by the position vector $x = f(t, \alpha)$ given by (1.1), (1.3) with $c_1 = 1 + \alpha^2$, $c_2 = 2\alpha$, see Fig. 3.2.

Let us list the fundamental geometric properties of F.

1*) The circular pseudosphere F is symmetric with respect to the coordinate plane $x^2 = 0$, its position vector satisfies (2.5). Thus, F consists of two symmetric parts sharing the common coordinate line $t = 0$ situated in the plane $x^2 = 0$. Besides, F is symmetric with respect to the coordinate plane $x^3 = 0$, its position vector satisfies (2.6).

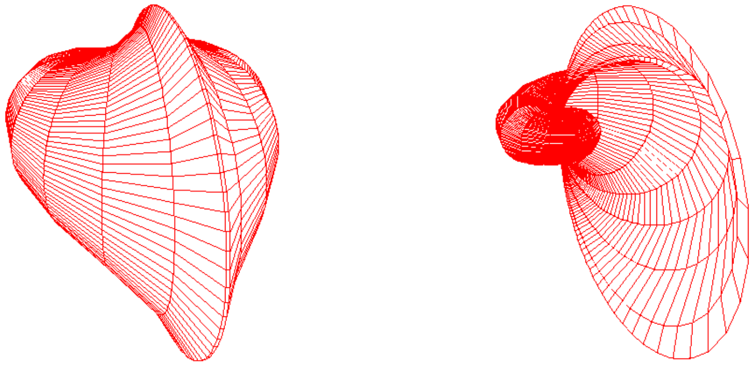

Fig. 3.2: The circular pseudosphere with $R = 1$: complete surface (left) and its one half (right).

2*) The circular pseudosphere F is piecewise regular. Its singular set, a cuspidal edge, is a unit circle composed of the singular points $t = 0$ of the circular tractrices sweeping out the surface F.

3*) The coordinate curves $t = const$ in F are circles of radius $\frac{1}{\sqrt{t^2+1}}$. All of them pass through the origin point $O \in \mathbb{R}^3$ which corresponds to the limit value $\alpha \to \pm\infty$ and has to be treated as removed from F.

4*) As t tends to $\pm\infty$, the circular pseudosphere F becomes asymptotically close to the origin $O \in \mathbb{R}^3$.

5*) The position vector $x = f(t, \alpha)$ of the circular pseudosphere F satisfies the same relation as in (2.7) but with $R = 1$. Hence, if one draws appropriately chosen unit segments tangent to coordinate t-curves of F, then the endpoints of these segments will sweep out the unit circle C. Moreover, t is still an arc length of this circle. Thus, the circular pseudosphere F in question is related to the unit circle C in the same manner as the classical pseudosphere is related to its axis of rotation.

6*) The first fundamental form of F reads as follows:

$$g = \frac{4}{(1 + \alpha^2 + t^2)^2} \left(t^2\, dt^2 + d\alpha^2\right).$$

Hence, the coordinate curves in F, circular tractrices $\alpha = const$ and circles $t = const$, form an isothermic net on F. Possibly, this isothermic net on the circular tractrix F could be viewed as an analogue of the standard horocyclic net on the classical pseudosphere.

7*) The complete area of the circular pseudosphere F is still equal to 4π, and it is the same as the area of the classical pseudosphere. The proof is based on calculations of corresponding integrals:

$$\mathrm{Area}(F) = 2\int_0^{+\infty}\int_{-\infty}^{+\infty}\sqrt{\det g}\,d\alpha\,dt = 2\int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{4t}{(1+\alpha^2+t^2)^2}\,d\alpha\,dt = 4\pi.$$

8*) The circular pseudosphere F has self-intersections. We can define the "volume" enclosed by F in the same manner as in the previous section for the case $R > 1$. Then we get the same formula (2.8). Therefore, the complete "volume" enclosed by the circular pseudosphere F is still equal to $\frac{2}{3}\pi$, and it is the same as the volume enclosed by the classical pseudosphere.

Thus, similarly to the case $R > 1$, the circular tractrices and circular pseudosphere with $R = 1$ inherit fundamental properties of the classical linear tractrix and pseudosphere, respectively, and hence can also be treated as their circular analogs.

Notice that the case $R = 1$ discussed in this section can be viewed as the limit for the case $R > 1$ considered in the previous section. Particularly, formulae (1.3) arise as the limit version of (1.2) as $R \to 1$ under appropriate scalings of involved parameters. Besides, qualitative geometric properties of circular tractrices and circular pseudospheres with $R > 1$ hold in the limit case $R = 1$.

4. Circular tractrices with $0 < R < 1$

Finally, fix $0 < R < 1$, chose arbitrary c_1, c_2 satisfying $c_1^2 - c_2^2 = 1$, and consider the corresponding circular tractrix γ represented by (1.1), (1.4). Let $x = f(t)$ still stand for the position vector of γ. List the elementary geometrical properties of γ.

1) γ is symmetric with respect to the plane $x^2 = 0$, its position vector satisfies (2.1). Moreover, γ is periodic in the following sense:

$$\begin{aligned} f^1(t+T) &= f^1(t)\cos\varphi + f^2(t)\sin\varphi,\\ f^2(t+T) &= -f^1(t)\sin\varphi + f^2(t)\cos\varphi,\\ f^3(t+T) &= f^3(t), \end{aligned}$$

where $T = \frac{2\pi}{\lambda}$ and $\varphi = \frac{2\pi}{\sqrt{1-R^2}}$. Thus, γ is invariant under rotations around the x^3-axis at the angles $n\varphi$, $n \in \mathbb{Z}$, in $\mathbb{R}^3$. Consequently, γ is symmetric with respect to any plane $x^1 \sin\frac{\varphi}{2}n + x^2\cos\frac{\varphi}{2}n = 0$, $n \in \mathbb{Z}$, in $\mathbb{R}^3$.

2) γ is piecewise regular. It has two rotationally invariant series of singular points: $t = \frac{2n\pi}{\lambda}$, $n \in \mathbb{Z}$, and $t = \frac{(2n+1)\pi}{\lambda}$, $n \in \mathbb{Z}$. This follows immediately from the relation:

$$|f'| = \left|\frac{\lambda\sin\lambda t}{\frac{c_1}{R} + \cos\lambda t}\right|.$$

Notice that $|f'| = |\xi_2|$ remains true.

3) Any interval of γ, which is situated between two consecutive singular points, will be called *a unit* of γ. Any pair of adjacent units represents a piece of γ, which is symmetric with respect to the two-dimensional plane containing the x^3-axis and the common endpoint of both units; it will be called *a petal* of γ, see Fig. 4.1. The complete circular tractrix γ is reconstructed by applying to any of its petals the rotations around the x^3-axis at the angles $n\varphi$, $n \in \mathbb{Z}$.

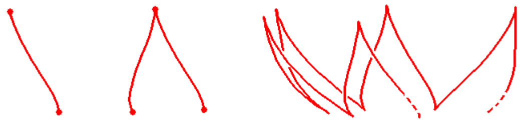

Fig. 4.1: A circular tractrix with $R < 1$ (right), its unit (left) and petal (center).

Particularly, γ is closed if and only if $\frac{\varphi}{\pi} \in \mathbb{Q}$, i.e., $\sqrt{1-R^2} \in \mathbb{Q}$. Otherwise, γ forms an everywhere dense subset in some rotationally invariant surface in $\mathbb{R}^3$.

Fig. 4.2: A closed circular tractrix with $R = \sqrt{1 - \left(\frac{3}{4}\right)^2}$ (left) and $R = \sqrt{1 - \left(\frac{4}{5}\right)^2}$ (center), and a non-closed circular tractrix with $R = \sqrt{1 - \frac{1}{e}}$ (right).

4) The length of a unit of γ is equal to $\operatorname{sign}(c_1) \ln \left| \frac{c_1+R}{c_1-R} \right|$. Hence it depends on R as well as on c_1. Notice that the length is finite, no matter what $0 < R < 1$ and $c_1 \in \mathbb{R}$ are. But it can tend to infinity as $R \to 1$, and this illustrates a quite complex behavior of the circular tractrix γ as R approaches 1 from below.
5) The position vector $x = f(t)$ of the circular tractrix γ satisfies the same relation as in (2.3) but with $0 < R < 1$. Once again, this means that if one draws appropriately chosen unit segments tangent to γ, then the endpoints of these segments will sweep out the circle C of radius R centered at the origin in the coordinate plane $x^3 = 0$, and t is an arc length of this circle. Thus, the circular tractrix γ is related to the circle C in the same manner as the linear tractix is related to its asymptotic straight line, i.e., the circle C is the directrix for the circular tractrix γ in question, c.f. [5, p.8] and [4, 14]. Notice that the circle C does not depend on the choice of c_1, c_2.
6) The circular tractrix γ has non-vanishing torsion for any choice of c_1, c_2 except two particular cases, $c_1 = 1$, $c_2 = 0$, and $c_1 = -1$, $c_2 = 0$, where γ belongs to the coordinate plane $x^3 = 0$ and represents the well-known planar circular tractrices with $0 < R < 1$, see Fig. 4.3, c.f. [4, 14].

Next, the constant $0 < R < 1$ being fixed, we set $c_1 = \pm\cosh\alpha$, $c_2 = \sinh\alpha$ and allow $\alpha \in \mathbb{R}^1$ to be varied. Then we obtain a one-parameter two-component family of circular tractrices which sweep out a two-dimensional surface F. This surface is represented by the position vector $x = f(t, \alpha)$ given by (1.1),(1.4)

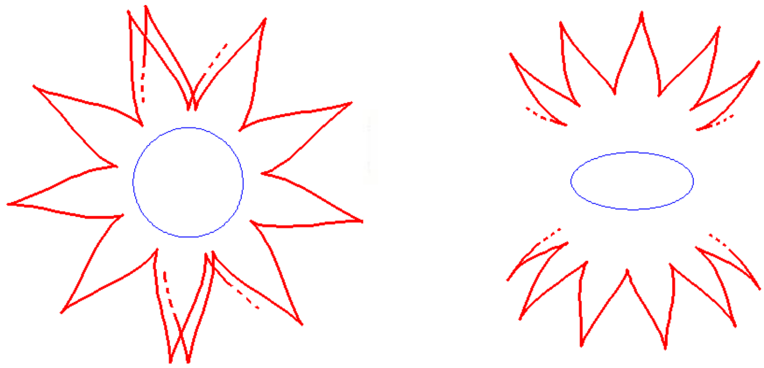

Fig. 4.3: Left: The planar circular tractrices (red) and the circle C (blue) viewed from the top. Right: The non-planar circular tractrices (red) and the circle C (blue).

with $c_1 = \pm\cosh\alpha$, $c_2 = \sinh\alpha$, it will be called *a circular pseudosphere* too. Notice that the surface F consists of two mutually congruent components which correspond to $c_1 = \cosh\alpha$ and $c_1 = -\cosh\alpha$.

Let us list the fundamental geometric properties of F.

1*) The circular pseudosphere F is symmetric with respect to the coordinate plane $x^2 = 0$, its position vector satisfies (2.5). Moreover, F is periodic in the following sense:

$$\begin{aligned} f^1(t+T,\alpha) &= f^1(t,\alpha)\cos\varphi + f^2(t,\alpha)\sin\varphi, \\ f^2(t+T,\alpha) &= -f^1(t,\alpha)\sin\varphi + f^2(t,\alpha)\cos\varphi, \\ f^3(t+T,\alpha) &= f^3(t,\alpha), \end{aligned}$$

where $T = \frac{2\pi}{\lambda}$ and $\varphi = \frac{2\pi}{\sqrt{1-R^2}}$. Thus, the surface F is invariant under rotations around the x^3-axis at the angles $n\varphi$, $n \in \mathbb{Z}$, in $\mathbb{R}^3$. Consequently, F is symmetric with respect to any plane $x^1 \sin\frac{\varphi}{2}n + x^2\cos\frac{\varphi}{2}n = 0$, $n \in \mathbb{Z}$, in $\mathbb{R}^3$. Besides, F is symmetric with respect to the coordinate plane $x^3 = 0$, its position vector satisfies (2.6).

2*) The circular pseudosphere F is piecewise regular. It has two rotationally invariant series of cuspidal edges, the coordinate curves $t = \frac{2n\pi}{\lambda}$, $n \in \mathbb{Z}$, and $t = \frac{(2n+1)\pi}{\lambda}$, $n \in \mathbb{Z}$, respectively, which are formed by singular points of circular tractrices $\alpha = const$ sweeping out the surface F.

3*) The coordinate curves $t = const$ in F are circles of radii $\frac{\sqrt{1-R^2}}{\sqrt{1-R^2\cos^2\lambda t}}$, this fact is easily verified by computing the curvature and torsion of the mentioned curves viewed as curves in $\mathbb{R}^3$. Particularly, singular edges of F are unit circles.

4*) All the coordinate circles $t = const$ of F pass through the points $O_1(0,0,-\sqrt{1-R^2})$ and $O_2(0,0,\sqrt{1-R^2})$, which correspond to the limit values $\alpha \to \pm\infty$ and hence have to be viewed as removed from F.

5*) Any piece of F, which is determined by $\frac{n\pi}{\lambda} \le t \le \frac{(n+1)\pi}{\lambda}$ for some $n \in \mathbb{Z}$ and hence situated between two consecutive singular edges, will be called a *unit* of F. Any pair of adjacent units represents a piece of F, which is symmetric with respect to the two-dimensional plane containing the x^3-axis and the singular coordinate circle shared by the units in question. This piece will be called *a petal* of F, see Fig. 4.4. The complete circular pseudosphere F is reconstructed by applying to any of its petals the rotations around the x^3-axis at the angles $n\varphi$, $n \in \mathbb{Z}$.

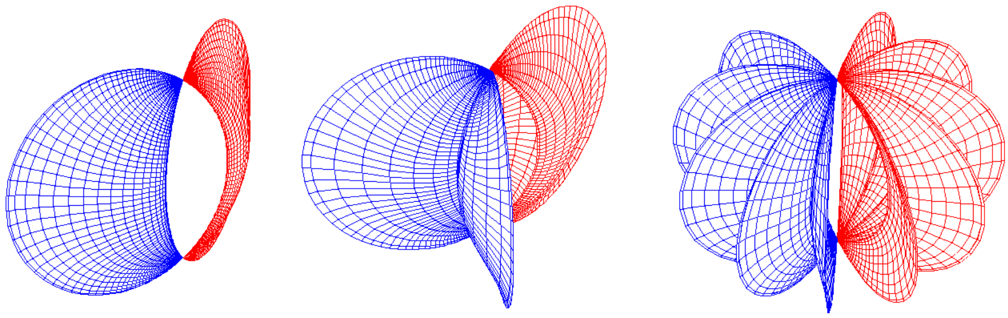

Fig. 4.4: A circular pseudosphere with $R < 1$: a unit (left), a petal (center), several petals (right). Different components are colored by red and blue.

6*) Any unit of F has two connected components, which correspond to the choice either $c_1 = \cosh\alpha$ or $c_1 = -\cosh\alpha$. Both components share the same pair of points $O_1(0, 0, -\sqrt{1-R^2})$ and $O_2(0, 0, \sqrt{1-R^2})$ viewed as removed from F. Moreover, these components are mutually congruent.

7*) The position vector $x = f(t, \alpha)$ of the circular pseudosphere F satisfies the same relation as in (2.7) but with $0 < R < 1$. This still means that if one draws appropriately chosen unit segments tangent to the coordinate t-curves of F, then the endpoints of these segments will sweep out the circle C. Moreover, t is an arc length of this circle. Thus, the circular pseudosphere F is related to the circle C in the same manner as the classical pseudosphere is related to its axis of rotation.

8*) The first fundamental form of F reads as follows:

$$g = \frac{1-R^2}{(\pm\cosh\alpha + R\cos\lambda t)^2}\left(\sin^2\lambda t\, dt^2 + d\alpha^2\right).$$

Hence, the coordinate curves in F, which are the circular tractrices $\alpha = const$ and the circles $t = const$, form an isothermic net on F which can be viewed as an analogue of the standard horocyclic net on the classical pseudosphere.

9*) The complete area of a unit of F is equal to

$$\begin{aligned}\mathrm{Area}(F) &= 2\int_{-\infty}^{+\infty}\int_0^{\frac{\pi}{\lambda}} \sqrt{\det g}\, dt\, d\alpha = 2\int_{-\infty}^{+\infty}\int_0^{\frac{\pi}{\lambda}} \frac{(1-R^2)\sin\lambda t}{(\cosh\alpha + R\cos\lambda t)^2}\, dt\, d\alpha \\ &= 4\left(\arctan\sqrt{\frac{1+R}{1-R}} - \arctan\sqrt{\frac{1-R}{1+R}}\right).\end{aligned}$$

Therefore, the area is finite and depends on R. In this context, the case $0 < R < 1$ differs essentially from the case $R \geq 1$. Similar differences appear as well in the context of the volume enclosed by the circular pseudosphere.

5. Concluding remarks and questions

Remark 5.1. Notice that the Gauss curvature of circular pseudospheres is not constant negative, no matter what $R > 0$ is. In other words, the circular pseudospheres are not pseudospherical in the classical sense of this term. On the other hand, fix $R > 0$ and consider the corresponding one-parameter family of circular tractrices. All these tractrices have the same directrix, the circle C. By applying rotations along C in $\mathbb{R}^3$, we get a two-parameter family of circular tractrices with the same directrix C. The question is whether one can choose a one-parameter subfamily in this two-parameter family of circular tractrices so that the chosen circular tractrices sweep out a surface of constant negative curvature. If such a pseudo-spherical surface exists, then it can be treated as a circular analog of the well-known Dini surface.

Remark 5.2. It would be interesting to explore the extrinsic geometry of the circular pseudospheres. For instance, we claim that if one considers an arbitrary circular pseudosphere $F \subset \mathbb{R}^3$ parameterized by the coordinates (t,α) used above, then the coordinate lines are *lines of curvature* in F. Moreover, the asymptotic lines in F which live near singular edges of F turn out to traverse these edges tangentially. In this regard, the circular pseudospheres show resemblance with the classical pseudosphere too. Possibly, another surprising resemblances can be found in this context.

Remark 5.3. An arbitrary non-closed circular tractrix with $0 < R < 1$ form an everywhere dense subset in a rotationally invariant surface in $\mathbb{R}^3$. What can we say about that surface of revolution?

Remark 5.4. The circular tractrices can be characterized, with some exceptions, as the only tractrices in $\mathbb{R}^3$ whose directrices are circles. Namely, if we fix a circle $C \subset \mathbb{R}^3$ of radius R, then the tractrices in $\mathbb{R}^3$, whose directrix is C, are:

$R > 1$: the circular tractrices whose directix is C and, as an exception, their common asymptotic circle C_∞;

$R = 1$: the circular tractrices whose directix is C and, as a degenerate exception, the origin point $O \in R^3$;

$0 < R < 1$: the circular tractrices whose directix is C and, as a degenerate exception, the corresponding points O_1, $O_2 \in R^3$ that are situated at the same distance 1 relatively to *all* points of C.

Remark 5.5. An interesting problem, which seems to be quite non-trivial, is to find helical analogues for linear and circular tractrices in $\mathbb{R}^n$, $n \geq 3$. Namely, to describe explicitly the tractrices in $\mathbb{R}^n$ whose directrices are curves of constant curvatures.

Remark 5.6. In terms of the simple model of bicycle motion discussed in [3], [6], any circular tractix in $\mathbb{R}^3$ can be viewed as the rear track of a spatial bicycle of unit length whose front track is a circle. In this context, the explicit description of the circular tractrices explored in our research note could be used for illustrating deep mathematical ideas and statements from [3], [6] concerning tractrices in the frames of the modern theory of integrable systems.

Acknowledgments. This note is dedicated to the 80th anniversary of Professor Yuriy Aminov. The first author is eager to share his impression of Yu. Aminov's textbook "Differential Geometry and Topology of Curves" [1] which he bought by accident in the bookstore "Kobzar" (Kharkiv) in 1987: "It was a remarkable book as it made me admire the beauty of geometry and deeply influenced the choice of my further scientific interests." We sincerely hope, and it would be a great honor for us, if some results of this note could be included in a new edition of Yu. Aminov's textbook.

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Про циркулярні трактриси в R^3

V. Gorkavyy and A. Sirosh

Досліджуються властивості циркулярних аналогів трактрис і псевдосфер в R^3.

Ключові слова: трактриса, циркулярна трактриса, псевдосфера