Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 4, pp. 781–798
doi: https://doi.org/10.15407/mag19.04.781

Quasiconformal Extensions and Inner Radius of Univalence by pre-Schwarzian Derivatives of Analytic and Harmonic Mappings

Zhenyong Hu, Jinhua Fan, and Xiaoyuan Wang

In this paper, we study the criterion for univalence, quasiconformal extensions and inner radius of univalence for locally univalent analytic and harmonic mappings in the complex plane. For locally univalent analytic functions in the unit disk, we give a sufficient condition for univalence and quasiconformal extensions by pre-Schwarzian derivatives, which generalizes Becker's result. For strongly spirallike domains, we consider the quasiconformal extension and obtain the lower bounds of the inner radius of univalence by pre-Schwarzian derivatives and Schwarzian derivatives. Furthermore, for harmonic mappings in a simply connected domain Ω, we prove that Ω is a quasidisk if and only if the inner radius of univalence of the domain Ω by pre-Schwarzian derivatives of harmonic mappings is positive, and we obtain a general sufficient condition for univalence and quasiconformal extensions.

Key words: quasiconformal extension, quasidisk, inner radius of univalence, strongly spirallike function, harmonic mapping

Mathematical Subject Classification 2020: 30C62, 30C45, 30C55, 31A05

1. Introduction

1.1. Quasiconformal extensions and inner radius of univalence by pre-Schwarzian derivatives of locally univalent analytic functions. Let $\mathbf{D}$ be the unit disk in the complex plane $\mathbf{C}$ and $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the extended complex plane. We denote by $\mathbf{R}$ the real axis. Let ϕ be a locally univalent analytic function. The Schwarzian derivative of ϕ is defined as

$$S_\phi = (P_\phi)' - \frac{1}{2}P_\phi^2,$$

where $P_\phi = \phi''/\phi'$ denotes the pre-Schwarzian derivatives of ϕ. We call a complex plane domain with more than one boundary point a hyperbolic domain. Let Ω be a hyperbolic domain, the hyperbolic metric $\rho_\Omega(z)$ is induced by $\rho_\Omega(\varphi(z))|\varphi'(z)| = \rho_{\mathbf{D}}(z) = 1/(1-|z|^2)$, where $\varphi : \mathbf{D} \to \Omega$ is a covering mapping. The norms of Schwarzian derivatives and pre-Schwarzian derivatives of ϕ in Ω are defined as

$$\|S_\phi\|_\Omega = \sup_{z\in\Omega} |S_\phi(z)|\rho_\Omega^{-2}(z) \quad \text{and} \quad \|P_\phi\|_\Omega = \sup_{z\in\Omega} |P_\phi(z)|\rho_\Omega^{-1}(z).$$

A homeomorphism F on D is K-quasiconformal if F has locally L^2-derivatives and satisfies

$$|\partial F/\partial \bar{z}| \le k|\partial F/\partial z| \text{ a.e. } z \in D,$$

where $K = (1+k)/(1-k) \ge 1$. The Jordan curve is a quasicircle if it is an image of the unit circle under a quasiconformal self-mapping of $\widehat{\mathbf{C}}$. The domain bounded by a quasicircle is called a quasidisk (see [18]). Let C be a Jordan curve bounding the domains A_1 and A_2. A sense-reversing quasiconfromal mapping f of the plane which maps A_1 onto A_2 is a quasiconformal reflection in C if f keeps every point of C fixed (see [17]). It is well known that quasiconformal mappings have the continuous extension of mappings to the boundary of the domain. Apart from quasiconformal mappings, there are many studies related to the continuous extension of mappings to the boundary of the domain (see [10, 22–24, 26]).

Becker [3] proved that if

$$\|P_\phi\|_{\mathbf{D}} \le k < 1, \tag{1.1}$$

then ϕ is not only univalent in $\mathbf{D}$, but also has a continuous extension $\widetilde{\phi}$ to the closed unit disk $\overline{\mathbf{D}}$. Using the Löwner chain, Becker [3] proved that ϕ has a quasiconformal extension to $\widehat{\mathbf{C}}$ if (1.1) holds. Constructing different Löwner chains, one can establish different criteria of univalence and quasiconformal extension for analytic functions (see [11, 12]).

For locally univalent analytic functions in the unit disk, we give a sufficient condition for univalence and quasiconformal extensions by the pre-Schwarzian derivatives, which generalizes the criterion (1.1) as follows.

Theorem 1.1. *Let ϕ_1 and ϕ_2 be locally univalent analytic in $\mathbf{D}$, and α be a constant with $\alpha \in [0,1]$. If the principal branch of $(\frac{\phi_1'(z)}{\phi_2'(z)})^\alpha$ is intended,*

$$\alpha|P_{\phi_1}(z) - P_{\phi_2}(z)|\rho_{\mathbf{D}}^{-1}(z) + |1 - \phi_1'(z)^{1-\alpha}\phi_2'(z)^\alpha| \le k < 1, z \in \mathbf{D}, \tag{1.2}$$

and $\phi_1'(z)^{1-\alpha}\phi_2'(z)^\alpha$ is analytic in $\mathbf{D}$, then ϕ_1 is univalent in $\mathbf{D}$ and has a quasiconformal extension to $\widehat{\mathbf{C}}$.

Remark 1.2. When $\alpha = 1$ and $\phi_2 = z$, the criterion (1.2) corresponds to the criterion (1.1).

For some subclasses of univalent analytic functions in $\mathbf{D}$, the quasiconformal extensions of them have been studied, such as strongly starlike functions of order α, $\alpha \in (0,1)$ ([8,25,29,30]). We denote by S the class of all analytic and univalent functions f in $\mathbf{D}$ with $f(0) = f'(0) - 1 = 0$. We say that $f \in S_k$ $(0 \le k < 1)$ if $f \in S$ and f has a k-quasiconformal extension in $\widehat{\mathbf{C}}$. Let $S^*(\alpha)$ denote the class of functions consisting of strongly starlike functions of order α in $\mathbf{D}$, that is, of functions f which satisfy $f \in S$ and

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \frac{\pi\alpha}{2}, \quad z \in \mathbf{D}, \alpha \in (0,1).$$

Fait, Krzyż and Zygmunt [8] showed that if $f \in S^*(\alpha)$, then $f \in S_k$ with $k \leq \sin(\alpha\pi/2)$. Sugawa [30] proposed an open question whether $S^*(\alpha) \subset S_\alpha$ holds. By constructing examples, Shen [25] gave a negative answer to this question.

Suppose that $\beta \in (-\pi\alpha/2, \pi\alpha/2)$ and $\alpha \in (0,1)$. Let $S^\beta(\alpha)$ denote the class of functions consisting of strongly β-spirallike functions of order α in $\mathbf{D}$, that is, of functions f which satisfy $f \in S$ and

$$\left|\arg \frac{zf'(z)}{f(z)} - \beta\right| \leq \frac{\pi\alpha}{2}, \quad z \in \mathbf{D}, \alpha \in (0,1).$$

When $\beta = 0$, $S^\beta(\alpha) = S^*(\alpha)$. Sugawa [28] showed that if $f \in S^\beta(\alpha)$, then $f \in S_k$ with $k \leq \sin(\alpha\pi/2)$. Since $S^*(\alpha) \subset S_\alpha$ is not true for all $\alpha > 0$, it is natural to consider whether there is a similar conclusion for $S^\beta(\alpha)$ like $S^*(\alpha)$. In fact, by an example, we show that $S^\beta(\alpha) \subset S_{\alpha\cos(\beta/\alpha)}$ is not true for all $\alpha > 0$.

Theorem 1.3. *Let $\beta \in (-\pi\alpha/2, \pi\alpha/2)$ and $\alpha \in (0,1)$. Then there exists a strongly β-spirallike function of order α that can not be extended to a k-quasiconformal mapping with $k \leq \alpha\cos\frac{\beta}{\alpha}$ of $\widehat{\mathbf{C}}$.*

Remark 1.4. When $\beta = 0$, Theorem 1.3 further gives a negative answer to the question by Sugawa [30] whether $S^*(\alpha) \subset S_\alpha$ holds.

The inner radii of domains Ω by the pre-Schwarzian and Schwarzian derivatives play an important role in the characterization of quasidisks (see [2, 9, 20]). Recall that the inner radii of domains Ω by the pre-Schwarzian and Schwarzian derivatives are defined by

$$\sigma_1(\Omega) = \sup\{c \geq 0 : \|P_\phi\|_\Omega \leq c \Rightarrow \phi \text{ is univalent in } \Omega\},$$
$$\sigma(\Omega) = \sup\{c \geq 0 : \|S_\phi\|_\Omega \leq c \Rightarrow \phi \text{ is univalent in } \Omega\}.$$

Martio–Sarvas [20] and Astala–Gehring [2] proved the following result.

Theorem A ([2, 20])**.** *A domail Ω is a quasidisk iff $\sigma_1(\Omega) > 0$.*

Astala and Gehring [2] also showed a criterion of quasiconformal extension for locally univalent analytic functions.

Theorem B ([2])**.** *Let Ω be a quasidisk in $\widehat{\mathbf{C}}$. If a locally univalent analytic function ϕ in Ω satisfies $\|P_\phi\|_\Omega \leq b < \sigma_1(\Omega)$, then ϕ admits a quasiconformal extension to $\widehat{\mathbf{C}}$.*

Using the inner radii of domains Ω by the Schwarzian derivatives, Ahlfors [1] and Gehring [9] gave similar results to Theorem A and Theorem B. In [1], Ahlfors gave a lower bound of $\sigma(\Omega)$ under $\partial\Omega$ that admits a continuously differentable quasiconformal reflection. Sugawa [29] improved Ahlfors' result by removing the assumption of the continuously differentable quasiconformal reflection. Cheng [5] considered the inner radius of univalence by the pre-Schwarzian derivatives.

Theorem C ([5, 29]). *Let Ω be a quasidisk and $\partial\Omega$ admit a continuously differentable quasiconformal reflection λ. Then*

$$\sigma(\Omega) \geq 2 \operatorname*{ess\,inf}_{z\in\Omega} \frac{|\lambda_{\bar{z}}(z)| - |\lambda_z(z)|}{|\lambda(z) - z|^2 \rho_\Omega^2(z)}, \tag{1.3}$$

$$\sigma_1(\Omega) \geq \operatorname*{ess\,inf}_{z\in\Omega} \frac{|\lambda_{\bar{z}}(z)| - |\lambda_z(z)|}{|\lambda(z) - z| \rho_\Omega(z)}. \tag{1.4}$$

The inequalities (1.3) and (1.4) are due to Sugawa [27] and Cheng [5], respectively.

Applying (1.3) and (1.4), one can get the explicit lower bounds for $\sigma(\Omega)$ and $\sigma_1(\Omega)$ of special domains such as a unit disk, an upper half plane and an angular domain [5, 17]. Sugawa [27] and Cheng [5] considered the inner radius of univalence of strongly starlike domains of order α (a domain Ω is a strongly starlike domain of order α if $f : \mathbf{D} \to \Omega$ satisfies $f(0) = 0$ and $|\arg(zf'(z)/f(z))| \leq \pi\alpha/2, z \in \mathbf{D}$).

Theorem D ([5, 27]). *A strongly starlike domain Ω of order α satisfies*

$$\sigma(\Omega) \geq \frac{2}{M(\alpha)^2} \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}, \tag{1.5}$$

$$\sigma_1(\Omega) \geq \frac{1}{M(\alpha)} \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}, \tag{1.6}$$

where

$$M(\alpha) = \exp\left[\int_0^1 \left\{\left(\frac{1+t}{1-t}\right)^\alpha - 1\right\} \frac{dt}{t}\right] = \frac{1}{4} \exp\left\{-\frac{\Gamma'((1-\alpha)/2)}{\Gamma((1-\alpha)/2)} - \gamma\right\},$$

Γ is the Euler gamma function and $\gamma = 0.5772\ldots$ is the Euler constant.

The inequalities (1.5) and (1.6) are due to Sugawa [27] and Cheng [5], respectively.

We say that the domain Ω is a strongly β-spirallike domain of order α if $f : \mathbf{D} \to \Omega$ satisfies $f(0) = 0$ and $|\arg((zf'(z)/f(z)) - \beta)| \leq \pi\alpha/2, z \in \mathbf{D}$. For strongly spirallike domains of order α, we consider and obtain the lower bounds of the inner radius of univalence by the pre-Schwarzian derivatives and Schwarzian derivatives.

Theorem 1.5. *Let $\beta \in (-\pi\alpha/2, \pi\alpha/2)$ and $\alpha \in (0, 1)$. A strongly β-spirallike domain Ω of order α satisfies*

$$\sigma(\Omega) \geq \frac{2}{L^2(\beta,\alpha)} \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)},$$

$$\sigma_1(\Omega) \geq \frac{1}{L(\beta,\alpha)} \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)},$$

where

$$L(\beta, \alpha) = \sup_{\zeta\in\mathbf{D}} \frac{|1 - u_{\beta,\alpha}^2(\zeta)|}{(1 - |\zeta|^2)|u'_{\beta,\alpha}(\zeta)|},$$

$$u_{\beta,\alpha}(\zeta) = \zeta \exp\left[\int_1^\zeta \left\{\left(\frac{1+te^{i2\beta/\alpha}}{1-t}\right)^\alpha - 1\right\}\frac{dt}{t}\right]. \tag{1.7}$$

In particular,

$$L(0,\alpha) = M(\alpha) = \exp\left[\int_0^1 \left\{\left(\frac{1+t}{1-t}\right)^\alpha - 1\right\}\frac{dt}{t}\right].$$

Remark 1.6. When $\beta = 0$, we refer to [27, Lemma 2] and obtain $L(0,\alpha) = M(\alpha)$, where $M(\alpha)$ is defined in Theorem D. It follows that Theorem 1.5 corresponds to Theorem D when $\beta = 0$.

1.2. Quasiconformal extensions and inner radius of univalence by the pre-Schwarzian derivatives of harmonic mappings. It is well known that complex-valued harmonic mappings are generalizations of analytic functions and have been researched widely (see [6]). Recall that a C^2 complex-valued function f in a simply connected domain Ω is harmonic if $\Delta f = 4f_{z\overline{z}} = 0$. Such f has a canonical representation $f = h + \overline{g}$ in Ω, where h and g are analytic in Ω. Lewy [19] proved that a harmonic mapping f is locally univalent if and only if its Jacobian $J_f \neq 0$. If $J_f > 0$, then f is sense-preserving. Let $\omega = g'/h'$ be the second complex dilatation of $f = h + \overline{g}$. Hernández and Martín [15] proposed a definition of the pre-Schwarzian derivatives Pf for all sense-preserving harmonic mappings as

$$Pf = \frac{\partial}{\partial z}\log J_f = P_h - \frac{\overline{\omega}\omega'}{1-|\omega|^2}.$$

For details about the pre-Schwarzian derivatives of harmonic mappings, we refer to [4, 13–16].

Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in a simply connected domain Ω with the second complex diliatation ω. Similarly to the definition of the inner radius of univalence of the simply connected domain Ω by the pre-Schwarzian derivatives, we define the inner radius of univalence of the simply connected domain Ω by the pre-Schwarzian derivatives of harmonic mappings as

$$\sigma_H(\Omega) = \sup\left\{c \geq 0 : \|Pf\|_\Omega \leq c \Rightarrow f \text{ is univalent in } \Omega\right\},$$

where

$$\|Pf\|_\Omega = \sup_{z\in\Omega}\left\{\left(|Pf| + \frac{|\omega'(z)|}{1-|\omega(z)|^2}\right)\rho_\Omega^{-1}(z)\right\}.$$

Noting that every locally univalent analytic function is harmonic, we have $\sigma_H(\Omega) \leq \sigma_1(\Omega)$. However, it is worth considering whether $\sigma_H(\Omega) = \sigma_1(\Omega)$ holds. We will give an affirmative answer to this question in Theorem 1.7.

Using the pre-Schwarzian derivatives of harmonic mappings, Hernández and Martín [14] obtained a harmonic mapping version of the criterion (1.1).

Theorem E ([14])**.** *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in $\mathbf{D}$ with $\|\omega\|_\infty < 1$. If $\|Pf\|_{\mathbf{D}} \leq k < 1$, then f has a quasiconformal extension to $\widehat{\mathbf{C}}$.*

In light of Theorems A, B, and E, it makes sense to ask whether there might be the corresponding version of harmonic mappings in a quasidisk. Motivated by this, for harmonic mappings in a simply connected domain Ω, we prove that Ω is a quasidisk if and only if the inner radius of univalence of the domain Ω by the pre-Schwarzian derivatives of harmonic mappings is positive, and we obtain a general sufficient condition for univalence and quasiconformal extensions. Now we state our results as follows.

Theorem 1.7. *A domain Ω is a quasidisk iff $\sigma_H(\Omega) = \sigma_1(\Omega) > 0$.*

Theorem 1.8. *Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping in a quasidisk Ω with $\|\omega\|_\infty < 1$. If*

$$\|Pf\|_\Omega \leq b < \sigma_H(\Omega), \tag{1.8}$$

then f is univalent in Ω and admits a quasiconformal extension to $\widehat{\mathbf{C}}$.

Remark 1.9. When $g = 0$, Theorem 1.7 and Theorem 1.8 correspond to Theorem A and Theorem B, respectively. When $\Omega = \mathbf{D}$, Theorem 1.8 corresponds to Theorem E.

In general, the norm of the pre-Schwarzian derivatives of harmonic mappings is defined as

$$\|\mathbf{P}f\|_\Omega = \sup_{z\in\Omega} |Pf|\rho_\Omega^{-1}(z).$$

However, instead of $\|Pf\|_\Omega$ by $\|\mathbf{P}f\|_\Omega$, we do not know whether Theorem E, Theorem 1.7 and Theorem 1.8 hold.

Efraimidis [7] got similar results to Theorem 1.7 and Theorem 1.8 by using harmonic mapping Schwarzian radius of injectivity of the simply connected domain.

2. A criterion for univalence and quasiconformal extensions for locally univalent analytic functions

In this section, we prove our results. By some lemmas, we prove Theorem 1.1. Let

$$f_t(z) = f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n \quad \text{on } \mathbf{D} \times [0,\infty).$$

The function $f(z,t)$ is said to be a Löwner chain if $f(z,t)$ is univalent analytic in $\mathbf{D}$ for any fixed $t \in [0,\infty)$ and $f_s(\mathbf{D}) \subseteq f_t(\mathbf{D})$ for $0 \leq s \leq t < \infty$. Pommerenke [21] proved the following result.

Lemma 2.1 ([21])**.** *Let $0 < r_0 \leq 1$ and $\mathbf{D}_{r_0} = \{z : |z| < r_0\}$. The function $f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ defined on $\mathbf{D} \times [0,\infty)$ is a Löwner chain if and only if the following two conditions hold:*

(i) *$f(z,t)$ is analytic in $z \in \mathbf{D}_{r_0}$, for each $t \in [0,\infty)$, absolutely continuous in t for each $z \in \mathbf{D}_{r_0}$ and satisfies $|f(z,t)| \leq K_0 e^t$ $(z \in \mathbf{D}_{r_0},\ t \in [0,\infty))$ for some positive constant K_0.*

(ii) *There exists a function* $p(z,t)$ *analytic in* $z \in \mathbf{D}$ *and measurable in* $t \in [0,\infty)$ *satisfying* $\operatorname{Re} p(z,t) > 0$ *such that*

$$\frac{\partial f(z,t)}{\partial t} = z\frac{\partial f(z,t)}{\partial z} p(z,t) \quad (z \in \mathbf{D}_{r_0}, \quad a.e.\ t \in [0,\infty)). \tag{2.1}$$

The following result is due to Becker [3].

Lemma 2.2 ([3]). *Suppose that* $f(z,t)$ *is a Löwner chain and*

$$\lambda(z,t) = \frac{p(z,t)-1}{p(z,t)+1}, \quad z \in \mathbf{D},\ t \geq 0,$$

where $p(z,t)$ *is given in Lemma 2.1. If* $|\lambda(z,t)| \leq k < 1$ *for all* $z \in \mathbf{D}$ *and* $t \geq 0$*, then* $f(z,t)$ *admits a continuous extension to* $\overline{\mathbf{D}}$ *for any* $t \geq 0$ *and the function* $F(z,z)$*, defined by the formula*

$$F(z,z) = \begin{cases} f(z,0), & |z| < 1, \\ f(z/|z|, \log|z|), & |z| \geq 1, \end{cases}$$

is a quasiconformal extension of $f(z,0)$ *to* $\widehat{\mathbf{C}}$.

Proof of Theorem 1.1. Without loss of generality, we suppose that

$$\phi_1(z) = z + a_2 z^2 + \cdots \quad \text{and} \quad \phi_2(z) = z + b_2 z^2 + \cdots .$$

Now we construct a function $\phi_1(z,t)$: $\mathbf{D} \times [0,+\infty) \to \mathbf{C}$ by

$$\phi_1(z,t) = \phi_1(e^{-t}z) + (e^t - e^{-t})z\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha}.$$

Calculations yield

$$\begin{aligned} \phi_1'(z,t) &= e^{-t}\phi_1'(e^{-t}z) + (e^t - e^{-t})\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha} \\ &\quad - \alpha(1-e^{-2t})z\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha}\left(\frac{\phi_2''(e^{-t}z)}{\phi_2'(e^{-t}z)} - \frac{\phi_1''(e^{-t}z)}{\phi_1'(e^{-t}z)}\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial \phi_1(z,t)}{\partial t} &= -ze^{-t}\phi_1'(e^{-t}z) + (e^t + e^{-t})z\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha} \\ &\quad + \alpha(1-e^{-2t})z^2\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha}\left(\frac{\phi_2''(e^{-t}z)}{\phi_2'(e^{-t}z)} - \frac{\phi_1''(e^{-t}z)}{\phi_1'(e^{-t}z)}\right). \end{aligned} \tag{2.3}$$

We first prove that $\phi_1(z,t)$ satisfies the conditions (i) in Lemma 2.1. Since $\frac{\phi_1'(0)}{\phi_2'(0)} = 1$, there exists a disk $\mathbf{D}_{r_1}$, $0 < r_1 \leq 1$, in which $\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)} \neq 0$ for all $t \geq 0$. Then we can choose a uniform branch of $\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha}$ analytic in $\mathbf{D}_{r_1}$. We fix the principal branch, it follows that

$$\phi_1(z,t) = \phi_1(e^{-t}z) + (e^t - e^{-t})z\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha} = e^t z + \cdots .$$

Hence the function $\phi_1(z,t)$ is analytic in $\mathbf{D}_{r_1}$. By

$$e^{-t}\phi_1(z,t) = e^{-t}\phi_1(e^{-t}z) + (1-e^{-2t})z\left(\frac{\phi_1'(e^{-t}z)}{\phi_2'(e^{-t}z)}\right)^{\alpha} = z + \cdots,$$

we obtain

$$\lim_{t\to+\infty} e^{-t}\phi_1(z,t) = z,$$

locally uniformly in z-variable, which implies that $\{e^{-t}\phi_1(z,t)\}_{t\geq 0}$ is a normal family in $\mathbf{D}_{r_1}$ by Montel's theorem. Therefore there exists a positive constant K_0 such that $|\phi_1(z,t)| \leq K_0 e^t$ for all $z \in \mathbf{D}_{r_1}$ and $t \in [0,+\infty)$. From the analyticity of $\frac{\partial \phi_1(z,t)}{\partial t}$ for all fixed numbers $T > 0$ and r_2, $0 < r_2 < r_1$, and a constant K_1 such that

$$\left|\frac{\partial \phi_1(z,t)}{\partial t}\right| < K_1, \quad z \in \mathbf{D}_{r_2},\ t \in [0,T]$$

it follows that $\phi_1(z,t)$ is locally absolutely continuous in $t \in [0,+\infty)$, locally uniform with respect to $\mathbf{D}_{r_2}$.

Now we show that Lemma 2.1 (ii) holds. To prove that there exists a measurable $p(z,t)$ with respect to t such that $\operatorname{Re} p(z,t) > 0$ and equation (2.1) holds, we suppose that

$$\lambda(z,t) = \frac{\frac{\partial \phi_1(z,t)}{\partial t} - z\phi_1'(z,t)}{\frac{\partial \phi_1(z,t)}{\partial t} + z\phi_1'(z,t)}. \tag{2.4}$$

By (2.2)–(2.4), a short calculation yields that

$$\begin{aligned}\lambda(z,t) &= e^{-2t}(1-\phi_1'(e^{-t}z)^{1-\alpha}\phi_2'(e^{-t}z)^{\alpha}) \\ &\quad + z\alpha\frac{1-e^{-2t}}{e^t}\left(\frac{\phi_2''(e^{-t}z)}{\phi_2'(e^{-t}z)} - \frac{\phi_1''(e^{-t}z)}{\phi_1'(e^{-t}z)}\right).\end{aligned}$$

Notice that $|e^{-t}z|^2 < e^{-2t}$ for $z \in \mathbf{D}$. Using z to represent $e^{-t}z$, by (1.2), we have

$$\begin{aligned}|\lambda(z,t)| &\leq |1-\phi_1'(e^{-t}z)^{1-\alpha}\phi_2'(e^{-t}z)^{\alpha}| \\ &\quad + |\alpha z e^{-t}|(1-e^{-2t})\left|\frac{\phi_2''(e^{-t}z)}{\phi_2'(e^{-t}z)} - \frac{\phi_1''(e^{-t}z)}{\phi_1'(e^{-t}z)}\right| \\ &\leq |1-\phi_1'(z)^{1-\alpha}\phi_2'(z)^{\alpha}| + \alpha(1-|z|^2)\left|\frac{\phi_2''(z)}{\phi_2'(z)} - \frac{\phi_1''(z)}{\phi_1'(z)}\right| \leq k < 1.\end{aligned} \tag{2.5}$$

Combining (2.5) and

$$p(z,t) = \frac{\frac{\partial \phi_1(z,t)}{\partial t}}{z\phi_1'(z,t)} = \frac{1+\lambda(z,t)}{1-\lambda(z,t)},$$

we obtain $\operatorname{Re} p(z,t) > 0$. Moreover, $p(z,t)$ is analytic in $\mathbf{D}$ for $t \in [0,+\infty)$ since $|\lambda(z,t)| < 1$ and $\phi_1'(z)^{1-\alpha}\phi_2'(z)^{\alpha}$ is analytic in $\mathbf{D}$. Hence, $\phi_1(z,t)$ is a Löwner chain from above analysis. Thus, for each $t \geq 0$, the functions $\phi_1(z,t)$ are univalent in $\mathbf{D}$, it follows that $\phi_1(z)$ is univalent in $\mathbf{D}$. Furthermore, by Lemma 2.2, we yield that $\phi_1(z)$ admits a quasiconformal extension onto $\widehat{\mathbf{C}}$. □

3. On quasiconformal extensions and inner radius of univalence of strongly spirallike domains

For a univalent analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{in } \mathbf{D},$$

its Grunsky coefficients α_{mn} are determined from the expression

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m+n=1}^{\infty} \alpha_{mn} z^m \zeta^n.$$

Define the Grunsky functional $g(f)$ of $f \in S_k$ as

$$g(f) = \sup_{\|x\|=1} \left| \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn} x_m x_n \right|$$

(see [25]). Let $k(f) = \min\{k : f \in S_k\}$. To prove Theorem 1.3, we need the following result.

Lemma 3.1 ([25]). *Let $g(f)$ denote the Grunsky functional for $f \in S_k$. For any function $f \in S_k$ one has $|S_f(0)| \le 6g(f)$ and the equality holds if and only if the Schwarzian derivative of f is of the form*

$$S_f(z) = \frac{S_f(0)}{\left(1 + S_f(0) z^2/6\right)^2}, \tag{3.1}$$

and in this case, $k(f) = g(f) = |S_f(0)|/6$.

Proof of Theorem 1.3. For $\beta \in (-\pi\alpha/2, \pi\alpha/2)$ and $\alpha \in (0,1)$, let

$$f_{\alpha,\beta}(z) = z \exp\left[\int_0^z \left\{ \left(\frac{1 + t^2 e^{i2\beta/\alpha}}{1 - t^2} \right)^{\alpha} - 1 \right\} \frac{dt}{t} \right].$$

It is easy to calculate that

$$\frac{z f'_{\alpha,\beta}(z)}{f_{\alpha,\beta}(z)} = \left(\frac{1 + z^2 e^{2i\beta/\alpha}}{1 - z^2} \right)^{\alpha}. \tag{3.2}$$

It follows that $f_{\alpha,\beta}$ is a strongly β-spirallike function of order α. By [28, Theorem 1], we know that $f_{\alpha,\beta} \in S_k$, where $k \le \sin \frac{\pi\alpha}{2}$. By (3.2), we obtain

$$\log z + \log f'_{\alpha,\beta}(z) - \log f_{\alpha,\beta}(z) = \alpha \left[\log\left(1 + z^2 e^{2i\beta/\alpha}\right) - \log\left(1 - z^2\right) \right].$$

After differentiation of the above identity, we get

$$\frac{1}{z} + \frac{f''_{\alpha,\beta}(z)}{f'_{\alpha,\beta}(z)} - \frac{f'_{\alpha,\beta}(z)}{f_{\alpha,\beta}(z)} = \frac{2\alpha(1 + e^{2i\beta/\alpha})z}{(1 + z^2 e^{2i\beta/\alpha})(1 - z^2)}. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\frac{f''_{\alpha,\beta}(z)}{f'_{\alpha,\beta}(z)} = \frac{1}{z}\left[\left(\frac{1+z^2e^{2i\beta/\alpha}}{1-z^2}\right)^{\alpha} - 1\right] + \frac{2\alpha(1+e^{2i\beta/\alpha})z}{(1+z^2e^{2i\beta/\alpha})(1-z^2)}.$$

Calculations lead to

$$\begin{aligned} S_{f_{\alpha,\beta}}(z) &= \frac{2\alpha(1+e^{2i\beta/\alpha})\left(2e^{2i\beta/\alpha}z^4 - \alpha(1+e^{2i\beta/\alpha})z^2 + 2\right)}{(1+z^2e^{2i\beta/\alpha})^2(1-z^2)^2} \\ &\quad - \frac{1}{2z^2}\left[\left(\frac{1+z^2e^{2i\beta/\alpha}}{1-z^2}\right)^{2\alpha} - 1\right]. \end{aligned}$$

Then we get $S_{f_{\alpha,\beta}}(0) = 3\alpha(1+e^{2i\beta/\alpha})$. Notice that $S_{f_{\alpha,\beta}}(z)$ is not of the form (3.1). Hence, using Lemma 3.1, we have

$$g\left(f_{\alpha,\beta}\right) > \frac{\left|S_{f_{\alpha,\beta}}(0)\right|}{6} = \alpha\cos\frac{\beta}{\alpha}.$$

This means that the proof is completed. □

Let $\beta \in (-\pi\alpha/2, \pi\alpha/2)$ and $\alpha \in (0,1)$. In [28], Sugawa constructed the quasiconformal reflection in the boundary of a strongly β-spirallike domain. Let Ω be a strongly β-spirallike domain of order α with respect to the origin and let $R_\beta(\theta) = \sup\{r > 0 : [0, P_\beta(r,\theta)]_\beta \subset \Omega\}$ be its radius function, where $\theta \in \mathbb{R}$ and $P_\beta(r,\theta) = re^{i(\theta+\tan\beta\log r)} \in \Omega$. Then a quasiconformal reflection λ in $\partial\Omega$ is given by

$$\begin{aligned} \lambda(P_\beta(r,\theta)) &= \frac{R_\beta(\theta)^2}{r}e^{i\left(\tan\beta\log\frac{R_\beta(\theta)^2}{r}+\theta\right)} \\ &= e^{\left((1+i\tan\beta)\log\frac{R_\beta(\theta)^2}{r}+i\theta\right)}, \quad \theta \in \mathbf{R},\ r \in (0,\infty). \end{aligned} \tag{3.4}$$

To prove Theorem 1.5, we need the following results obtained by Sugawa [28].

Lemma 3.2 ([28]). *Let β and α be real numbers with $|\beta| < \pi\alpha/2 < \pi/2$. For a domain Ω in $\mathbf{C}$ with $0 \in \Omega$, the following are equivalent:*

(1) *There exists a strongly β-spirallike function f of order α such that $\Omega = f(\mathbf{D})$.*
(2) *The radius function $R_\beta(\theta)$ of Ω with respect to β-spirals is bounded, absolutely continuous on $[0,2\pi]$ and satisfies*

$$\left|\frac{R'_\beta(\theta)}{R_\beta(\theta)} + \sin\beta\cos\beta\right| \le \cos^2\beta\tan(\pi\alpha/2) \tag{3.5}$$

for almost every θ.

(3) *$wU_{\beta,\alpha} \subset \Omega$ whenever $w \in \partial\Omega$; $wU_{\beta,\alpha} \subset \Omega$ whenever $w \in \Omega$, where*

$$U_{\beta,\alpha} = \Big\{ \exp((1 + i \tan \beta)t + i\theta) : \theta \in [0, 2\pi), \\ t < \max\{- \cos^2 \beta(\tan(\pi\alpha/2) + \tan \beta)\theta, \\ - \cos^2 \beta(\tan(\pi\alpha/2) - \tan \beta)(2\pi - \theta)\}\Big\} \cup \{0\}.$$

Lemma 3.3 ([28])**.** *The function defined by (1.7) maps* **D** *conformally onto $U_{\beta,\alpha}$ in such a way that $u_{\beta,\alpha}(0) = u_{\beta,\alpha}(1) - 1 = 0$.*

We also need the following result.

Lemma 3.4. *Let $\beta \in (-\pi/2, \pi/2)$ and $\alpha \in (0, 1)$. The function $\varphi(x) := \sqrt{x^2 + \cos^4 \beta} - x$ is decreasing and*

$$\varphi(x) \geq \varphi(\cos^2 \beta \tan(\pi\alpha/2)) \\ = \cos^2 \beta \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}, \quad x \in [0, \cos^2 \beta \tan(\pi\alpha/2)]. \tag{3.6}$$

Proof. Let $\varphi(x) = \sqrt{x^2 + \cos^4 \beta} - x$. It is easy to see that

$$\varphi'(x) = \frac{x}{\sqrt{x^2 + \cos^4 \beta}} - 1 < 0, \quad x \in [0, \cos^2 \beta \tan(\pi\alpha/2)].$$

It follows that (3.6) holds. The proof is completed. □

Proof of Theorem 1.5. According to Theorem C, our proof consists of four steps.

Step 1. We estimate λ_z and $\lambda_{\bar{z}}$. Let $w = \lambda(z)$, where $\lambda(z)$ is defined by (3.4). We use the logarithmic coordinates $Z = X+iY = \log z$, $z = re^{i(\theta+\tan \beta \log r)}$, $W = U + iV = \log w$. Using (3.4) and the relation $\theta = Y - X \tan \beta$, we have

$$W = 2 \log R_\beta(\theta)(Y - X \tan \beta) - X + i[(Y - X \tan \beta) \\ + (2 \log R_\beta(\theta)(Y - X \tan \beta) - X) \tan \beta].$$

Short computations lead to the following:

$$W_X = -2(1 + i \tan \beta)\frac{R'_\beta(\theta)}{R_\beta(\theta)} \tan \beta - (1 + 2i \tan \beta),$$

$$W_Y = 2(1 + i \tan \beta)\frac{R'_\beta(\theta)}{R_\beta(\theta)} + i,$$

$$W_Z = \frac{W_X - iW_Y}{2} = -i\left(\frac{R'_\beta(\theta)}{R_\beta(\theta)}(1 + \tan^2 \beta) + \tan \beta\right),$$

$$W_{\bar{Z}} = \frac{W_X + iW_Y}{2} = ie^{2i\beta}\left(\frac{R'_\beta(\theta)}{R_\beta(\theta)} + \sin \beta \cos \beta + i \cos^2 \beta\right)(1 + \tan^2 \beta).$$

Then combining (3.5) and Lemma 3.4, we get that

$$\begin{aligned}
|\lambda_{\bar z}|-|\lambda_z| &= |W_{\bar z}\lambda|-|W_z\lambda| = \left|W_{\bar Z}\frac{1}{r}\lambda\right|-\left|W_Z\frac{1}{r}\lambda\right| \\
&= \frac{R_\beta^2(\theta)}{r^2}(1+\tan^2\beta)\left[\sqrt{\left|\frac{R'_\beta(\theta)}{R_\beta(\theta)}+\sin\beta\cos\beta\right|^2+\cos^4\beta}\right. \\
&\qquad\left. -\left|\frac{R'_\beta(\theta)}{R_\beta(\theta)}+\sin\beta\cos\beta\right|\right] \\
&\geq \frac{R_\beta^2(\theta)}{r^2}(1+\tan^2\beta)\cos^2\beta\frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)} \\
&= \frac{R_\beta^2(\theta)}{r^2}\frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)}. \qquad (3.7)
\end{aligned}$$

Step 2. We calculate

$$\begin{aligned}
|\lambda(z)-z| &= \left|\frac{R_\beta^2(\theta)}{r}e^{i\tan\beta\log\frac{R_\beta^2(\theta)}{r}}-re^{i\tan\beta\log r}\right| \\
&= \left|\frac{1}{r}\left(r^2-R_\beta^2(\theta)e^{i2\tan\beta\log\frac{R_\beta(\theta)}{r}}\right)\right| \qquad (3.8)
\end{aligned}$$

for all $z=re^{i(\theta+\tan\beta\log r)}\in\Omega$.

Step 3. We estimate $\rho_\Omega(z)$. For fixed $z=re^{i(\theta+\tan\beta\log r)}$ and $z_0=R_\beta(\theta)e^{i(\theta+\tan\beta\log R_\beta(\theta))}$. By Lemma 3.2, we have $N=z_0U_{\beta,\alpha}\subset\Omega$. Due to the monotonicity of the Poincaré metric, we get $\rho_\Omega(z)\leq\rho_N(z)$. Then $\rho_N(z)=\rho_{U_{\beta,\alpha}}(z/z_0)/|z_0|$, which implies that

$$\rho_\Omega(z)\leq\rho_N(z)=\rho_{U_{\beta,\alpha}}\left(\frac{r}{R_\beta(\theta)}e^{i\tan\beta\log\frac{r}{R_\beta(\theta)}}\right)/R_\beta(\theta). \qquad (3.9)$$

Since $u_{\beta,\alpha}(\zeta)$ defined by (1.7): $\mathbf{D}\to U_{\beta,\alpha}$ is conformal by Lemma 3.3, we have

$$\rho_{U_{\beta,\alpha}}(u_{\beta,\alpha}(\zeta))=\frac{1}{(1-|\zeta|^2)|u'_{\beta,\alpha}(\zeta)|},\zeta\in\mathbf{D}. \qquad (3.10)$$

It is easy to see that $u_{\beta,\alpha}(\zeta)$ maps ζ to $\frac{r}{R_\beta(\theta)}e^{i\tan\beta\log\frac{r}{R_\beta(\theta)}}$. However, $u_{0,\alpha}(\zeta)$ maps $(0,1)$ onto $(0,1)$.

Step 4. We calculate the lower bounds for $\sigma(\Omega)$ and $\sigma_1(\Omega)$. By (3.7)–(3.10), we have

$$\begin{aligned}
\sigma(\Omega) &\geq 2\operatorname*{ess\,inf}_{z\in\Omega}\frac{|\lambda_{\bar z}(z)|-|\lambda_z(z)|}{|\lambda(z)-z|^2\rho_\Omega^2(z)} \\
&\geq \frac{2\frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)}}{\left|1-\left(\frac{r}{R_\beta(\theta)}e^{i\tan\beta\log\frac{r}{R_\beta(\theta)}}\right)^2\right|^2\rho_{U_{\beta,\alpha}}^2\left(\frac{r}{R_\beta(\theta)}e^{i\tan\beta\log\frac{r}{R_\beta(\theta)}}\right)}
\end{aligned}$$

$$\geq \frac{2}{L^2(\beta,\alpha)}\frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)},$$

$$\sigma_1(\Omega) \geq \operatorname*{ess\,inf}_{z\in\Omega} \frac{|\lambda_{\bar z}(z)| - |\lambda_z(z)|}{|\lambda(z)-z|\rho_\Omega(z)} \geq \frac{1}{L(\beta,\alpha)}\frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)},$$

where

$$L(\beta,\alpha) = \sup_{\zeta\in\mathbf{D}} \frac{|1-u^2_{\beta,\alpha}(\zeta)|}{(1-|\zeta|^2)|u'_{\beta,\alpha}(\zeta)|}$$

and $u_{\beta,\alpha}(\zeta)$ is defined by (1.7). In particular, if $u_{0,\alpha}(\zeta)$ maps $(0,1)$ onto $(0,1)$, then

$$L(0,\alpha) = M(\alpha) = \exp\left[\int_0^1 \left\{\left(\frac{1+t}{1-t}\right)^\alpha - 1\right\}\frac{dt}{t}\right]$$

by [27, Lemma 2]. The proof is completed. □

When $\beta=0$, we denote $U_{\beta,\alpha}$ by $U_{0,\alpha}$. Sugawa [27] proved that $\sigma(U_{0,\alpha}) \leq 2(1-\alpha)^2$ for $\alpha\in(0,1)$. It is natural to consider the upper bound for $\sigma_1(U_{0,\alpha})$. In fact, we obtain the following result.

Theorem 3.5. *$\sigma_1(U_{0,\alpha}) \leq 2(1-\alpha)$ for $\alpha\in(0,1)$.*

Proof. We consider the analytic function $f_0 = \log(1-w)$ on the domain $\mathbf{C}\backslash[1,+\infty)$. Although f_0 is univalent, $f_0(U_{0,\alpha})$ has an outward pointing cusp. This means that $f_0(U_{0,\alpha})$ is not a quasidisk. Thus, by Theorem B, we get $\sigma_1(U_{0,\alpha}) \leq \|P_{f_0}\|_{U_{0,\alpha}}$. Now we will estimate $\|P_{f_0}\|_{U_{0,\alpha}}$. Since $U_{0,\alpha}\subset V = \{w : |\arg(1-w)| < (1-\alpha)\pi/2\}$, we have $\rho_V(w) \leq \rho_{U_{0,\alpha}}(w)$. Then $\|P_{f_0}\|_{U_{0,\alpha}} \leq \|P_{f_0}\|_V$. It should be noticed that $(1-w)^{\frac{1}{1-\alpha}}$ conformally maps V onto the right plane. Then, by the uniformization theorem, we deduce

$$\rho_V(w) = \frac{|1-w|^{\frac{\alpha}{1-\alpha}}}{2(1-\alpha)\mathrm{Re}\big[(1-w)^{\frac{1}{1-\alpha}}\big]}.$$

Therefore we have $\|P_{f_0}\|_V = \sup_{w\in V}|P_{f_0}|\rho_V(w)^{-1} = 2(1-\alpha)$, which implies that $\sigma_1(U_{0,\alpha}) \leq 2(1-\alpha)$ for $\alpha\in(0,1)$. The proof is completed. □

4. Inner univalence for a quasidisk involving harmonic mappings

In this section, we prove Theorems 1.7 and 1.8.

Proof of Theorem 1.7. It is obvious that if $\sigma_H(\Omega) = \sigma_1(\Omega) > 0$, by using Theorem A, we know that Ω is a quasidisk. Hence, we only need to prove that if Ω is a quasidisk, then $\sigma_1(\Omega) = \sigma_H(\Omega) > 0$. For $a\in\mathbf{D}$, we consider the functions $f_a = f + a\overline{f}$. It is easy to see that the dilatation of f_a satisfies

$$\omega_a = \frac{a+\omega}{1+\bar a\omega} \quad\text{and}\quad \frac{|z\omega_a'|}{1-|\omega_a|^2} = \frac{|z\omega'|}{1-|\omega|^2}.$$

Furthermore, by $Pf_a = Pf$ and the triangle inequality, we can get that

$$|P_{h+ag}(z)| \leq |Pf(z)| + \frac{|\omega'(z)|}{1-|\omega(z)|^2}.$$

Therefore,

$$|P_{h+ag}(z)|\lambda_\Omega^{-1}(z) \leq \left(|Pf(z)| + \frac{|\omega'(z)|}{1-|\omega(z)|^2}\right)\lambda_\Omega^{-1}(z), \quad z \in \Omega.$$

According to the definition of $\sigma_1(\Omega)$ and $\sigma_1(\Omega) > 0$, we know that if

$$|P_{h+ag}(z)|\lambda_\Omega^{-1}(z) \leq \left(|Pf(z)| + \frac{|\omega'(z)|}{1-|\omega(z)|^2}\right)\lambda_\Omega^{-1}(z) \leq \sigma_1(\Omega), \quad z \in \Omega,$$

then we deduce that $h_a = h + ag$ is univalent in Ω for all $|a| < 1$. By Huritz's theorem, for $\tau \in \partial\mathbf{D}$, $h_\tau = h + \tau g$ is either univalent or constant. Notice that any sense-preserving harmonic mapping satisfies $|g'(0)| < |h'(0)|$. That is to say, h_τ must be univalent in Ω, where $\tau \in \partial\mathbf{D}$. In conclusion, we obtain that $h_a = h + ag$ $(|a| \leq 1)$ is univalent in Ω. Furthermore, using the proof method from [14], we can easily prove that f is univalent in Ω. By the definition of $\sigma_H(\Omega)$, we get that $\sigma_H(\Omega) \geq \sigma_1(\Omega)$. Combining the fact that $\sigma_H(\Omega) \leq \sigma_1(\Omega)$, we have $\sigma_H(\Omega) = \sigma_1(\Omega)$. We complete the proof. □

Proof of Theorem 1.8. Let $\varphi : \mathbf{D} \to \Omega$ be a Riemann map of Ω and $\Omega_r = \varphi(\{|z| < r\})$ for $r < 1$. We set $\gamma_r = \partial\Omega_r$. Ω is a K_0-quasidisk which means that γ_r is a K_0-quasicircle. Considering the dilations $\Gamma_r = f(\gamma_r)$ for $r < 1$, we will prove that $f(\gamma_r)$ is K-quasiconformal and K does not depend on r. Once we prove this, we can consider λ_r and Λ_r to be K_0 and K-quaisconformal reflections across γ_r and Γ_r. Also, we construct

$$\tilde{f}_r(z) = \begin{cases} \tilde{f}(z), & z \in \overline{\Omega_r}, \\ \Lambda_r \circ f \circ \lambda_r(z), & z \in \widehat{\mathbb{C}} \backslash \overline{\Omega_r}, \end{cases}$$

which is a quasiconformal mapping in $\widehat{\mathbf{C}}$. Let $r \to 1$. Using Theorem 5.3 from [18], we will complete the proof. Our proof consists of four steps.

Step 1. Using the same method as in the proof of Theorem 1.7, we can also prove that $h(\Omega)$ is a quasidisk. Hence, by [1], for $\xi_i \in \partial\Omega_r$ $(i = 1, 2, 3, 4)$, we get

$$\big|(h(r\xi_1), h(r\xi_2), h(r\xi_3), h(r\xi_4))\big| = \left|\frac{h(r\xi_1) - h(r\xi_3)}{h(r\xi_1) - h(r\xi_4)} \frac{h(r\xi_2) - h(r\xi_4)}{h(r\xi_2) - h(r\xi_3)}\right| \leq M$$

for $0 < r < 1$, where M is a positive constant.

Step 2. We will prove that $h_a = h + ag$ is univalent in Ω for all $|a| < \delta$ $(\delta > 1)$. In Theorem 1.7, we have proved that $h_a = h + ag$ is univalent in Ω for all $|a| \leq 1$. Thus we only need to prove that $h_a = h + ag$ is univalent in Ω for all $1 < |a| < \delta$. Notice that

$$\frac{f_a''}{f_a'} = \left(\frac{h''}{h'} + \frac{a\omega'}{1+a\omega}\right) = Pf + \frac{\omega'}{1-|\omega|^2}\frac{\overline{\omega}+a}{1+a\omega}. \tag{4.1}$$

By (1.8), we see that

$$\left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right| \le b < \sigma_H(\Omega) \tag{4.2}$$

for all $z \in \Omega$. According to (1.8), (4.1) and (4.2), it follows that

$$\begin{aligned}
|P_{h_a}|\rho_\Omega^{-1}(z) &\le |Pf|\rho_\Omega^{-1}(z) + \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\frac{\overline{\omega}+a}{1+a\omega}\right| \\
&= |Pf|\rho_\Omega^{-1}(z) + \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right|\left|\frac{\overline{\omega}+a}{1+a\omega}\right| \\
&\le |Pf|\rho_\Omega^{-1}(z) + \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right|\frac{|a|-\|\omega\|_\infty}{1-\|\omega\|_\infty|a|} \\
&< b - \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right| + \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right|\frac{|a|-\|\omega\|_\infty}{1-\|\omega\|_\infty|a|} \\
&= b + \left|\frac{\omega'}{\rho_\Omega(z)(1-|\omega|^2)}\right|\left(\frac{|a|-\|\omega\|_\infty}{1-\|\omega\|_\infty|a|} - 1\right) \\
&\le b\frac{|a|-\|\omega\|_\infty}{1-\|\omega\|_\infty|a|} < \sigma_H(\Omega) = \sigma_1(\Omega)
\end{aligned}$$

if

$$|a| < \delta = \frac{\sigma_1(\Omega)+b\|\omega\|_\infty}{b+\sigma_1(\Omega)\|\omega\|_\infty}.$$

It is easy to see that

$$\frac{1}{\|\omega\|_\infty} \ge \frac{\sigma_1(\Omega)+b\|\omega\|_\infty}{b+\sigma_1(\Omega)\|\omega\|_\infty} > 1.$$

Hence, by Theorem B, we conclude that $f_a = h + ag$ is univalent in Ω for all

$$1 < |a| < \frac{\sigma_1(\Omega)+b\|\omega\|_\infty}{b+\sigma_1(\Omega)\|\omega\|_\infty} \le \frac{1}{\|\omega\|_\infty}.$$

Step 3. We estimate the upper bound of $\left|\frac{g(\alpha)-g(\beta)}{h(\alpha)-h(\beta)}\right|$ for $\alpha, \beta \in \overline{\Omega}$ if we suppose that h and g are analytic in $\overline{\Omega}$. Notice that h is univalent in $\overline{\Omega}$ by Step 1. For fixing $\beta \in \overline{\Omega}$, we define a function

$$\varphi_\beta(\alpha) = \begin{cases} \dfrac{g(\alpha)-g(\beta)}{h(\alpha)-h(\beta)}, & \alpha \ne \beta, \\ \omega(\alpha), & \alpha = \beta, \end{cases}$$

where $\alpha \in \overline{\Omega}$ and ω represents the second complex dilatation of f. Obviously, $\varphi_\beta(\alpha)$ is continuous for $\alpha \in \overline{\Omega}$. It follows that there exists $\alpha_0 \in \overline{\Omega}$ such that $\sup_{\alpha\in\overline{\Omega}} |\varphi_\beta(\alpha)| = |\varphi_\beta(\alpha_0)|$.

If $\alpha_0 = \beta$, then $\sup_{\in\overline{\Omega}} |\varphi_\beta(\alpha)| = |\varphi_\beta(\alpha_0)| = |\omega(\alpha_0)| \le \|\omega\|_\infty$. By $\delta \le \frac{1}{\|\omega\|_\infty}$, it follows that $\sup_{\alpha\in\overline{\Omega}} |\varphi_\beta(\alpha)| = |\varphi_\beta(\alpha_0)| \le \frac{1}{\delta}$. If $\alpha_0 \ne \beta$, here we suppose that $\sup_{\alpha\in\overline{\Omega}} |\varphi_\beta(\alpha)| > \frac{1}{\delta}$, then there exist $\varepsilon > 0$ and $|\mu| = 1$ such that

$$\frac{g(\alpha_1)-g(\beta)}{h(\alpha_1)-h(\beta)} = \frac{1+\varepsilon}{\delta\mu},$$

where $\alpha_1 \in \overline{\Omega}$. Therefore, we deduce that $h - \frac{\mu\delta}{1+\varepsilon}g$ is not univalent in $\overline{\Omega}$, which contradicts Step 2. So, if $\alpha_0 \neq \beta$, we have $\sup_{\alpha\in\overline{\Omega}} |\varphi_\beta(\alpha)| = |\varphi_\beta(\alpha_0)| \leq \frac{1}{\delta}$. It follows that

$$|\frac{g(\alpha) - g(\beta)}{h(\alpha) - h(\beta)}| \leq \frac{1}{\delta}$$

for all $\alpha, \beta \in \overline{\Omega}$.

Step 4. We will prove that

$$|(w_1, w_2, w_3, w_4)| = \left|\frac{w_1 - w_3}{w_1 - w_4}\frac{w_2 - w_4}{w_2 - w_3}\right|$$

is uniformly bounded, where $w_i = f_r(\xi_i) = h_i + \bar{g}_i \in \Gamma_r = f_r(\partial\Omega) = f(\partial\Omega_r)$ $(i = 1, 2, 3, 4)$. From Steps 1–3, for $\xi_i \in \partial\Omega_r$ $(i = 1, 2, 3, 4)$, we have

$$|(w_1, w_2, w_3, w_4)| \leq M\left(\frac{1 + 1/\delta}{1 - 1/\delta}\right)^2.$$

Letting $r \to 1$, and using Theorem 5.3 from [18], we complete the proof. □

Acknowledgments. We would like to thank the referee for his/her helpful comments and suggestions. This work is supported by Postgraduate Research & Practice Innovation Program of Jiangsu Province (No. KYCX21_0247). This work is also supported by National Natural Science Foundation of China (No. 11871215) and the Fundamental Research Funds for the Central University (No. 30922010804).

References

[1] L. Ahlfors, *Quasiconformal reflections,* Acta Math. **109** (1963), 291–301.

[2] K. Astala and F. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space,* J. d'Analyse Math. **46** (1986), 16–57.

[3] J. Becker, *Lownersche differentialgleichung und quasikonform fortsetzbare schlichet functionen,* J. Reine Angew. Math. **255** (1972), 23–43.

[4] S. Chen and S. Ponnusamy, *John disks and K-quasiconformal harmonic mappings,* J. Geom. Anal. **27** (2017), 1468–1488.

[5] T. Cheng and J. Chen, *On the inner radius of univalency by pre-Schwarzian derivative,* Sci. China Math. **50** (2007), 987–996.

[6] P. Duren, *Harmonic Mappings in the Plane,* Cambridge Univ. Press, 2004.

[7] I. Efraimidis, *Criteria for univalence and quasiconformal extension for harmonic mappings on plane domain,* Ann. Fenn. Math. **46** (2021), 1123–1134.

[8] M. Fait, J. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions,* Comment. Math. Helv. **51** (1976), 279–285.

[9] F. Gehring, *Univalent functions and the Schwarzian derivative,* Comment. Math. Helv. **52** (1977), 561–572.

[10] A. Golberg, R. Salimov, and E. Sevost'yanov, *Singularities of discrete open mappings with controlled p-module,* J. Anal. Math. **127** (2015), 303–328.

[11] I. Hotta, *Explicit quasiconformal extensions and Löwner chains,* Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), 108–111.

[12] I. Hotta, *Loewner theory for quasiconformal extensions: old and new,* Interdiscip. Inform. Sci. **25** (2019), 1–21.

[13] R. Hernández and M. Martín, *Stable geometric properties of analytic and harmonic functions,* Math. Proc. Cambridge Philos. Soc. **155** (2013), 343–359.

[14] R. Hernández and M. Martín, *Quasiconformal extensions of harmonic mappings in the plane,* Ann. Acad. Sci. Fenn. Math. **38** (2013), 617–630.

[15] R. Hernández and M. Martín, *Pre-Schwarzian and Schwarzian derivatives of harmonic mappings,* J. Geom. Anal. **25** (2015), 64–91.

[16] Z. Hu and J. Fan, *Criteria for univalency and quasiconformal extension for harmonic mappings,* Kodai Math. J. **44** (2021), 273–289.

[17] O. Lehto, *Univalent Functions and Teichmüller Space,* Graduate Texts in Mathematics, 109, Springer-Verlag, New York, 1987.

[18] O. Letho and K. Virtanen, *Quasiconformal Mappings in the Plane,* Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[19] H. Lewy, *On the non-vanishing of the Jacbian in certain one-to-one mappings,* Bull. Amer. Math. Soc. **429** (1936), 689–692.

[20] O. Martio and J. Sarvas, *Injectivity theorems in plane and space,* Ann. Acad. Sci. Fenn. Math. **4** (1978/79), 383–401.

[21] C. Pommerenke, *Univalent Functions,* Vandenhoeck Ruprecht, Göttingen, 1975.

[22] V. Ryazanov and S. Volkov, *On the boundary behavior of mappings in the class $W^{1,1}_{\mathrm{loc}}$ on Riemann surfaces,* Complex Anal. Operator Theory, **11** (2017), 1503–1520.

[23] E. Sevost'yanov, *On the boundary behavior of open discrete mappings with unbounded characteristic,* Ukrainian Math. J. **64** (2012), 979–984.

[24] E. Sevost'yanov, S. Skvortsov, and O. Dovhopiatyi, *On nonhomeomorphic mappings with the inverse Poletsky inequality,* J. Math. Sci. **252** (2021), 541–557.

[25] Y. Shen, *Counterexamples concerning quasiconformal extensions of strongly starlike functions,* Acta Math. Sinica Eng. Ser. **23** (2007), 1859–1868.

[26] E. Smolovaya, *Boundary behavior of ring Q-homeomorphisms in metric spaces,* Ukrainian Math. J. **62** (2010), 785–793.

[27] T. Sugawa, *Inner radius of univalence for a strongly starlike domain,* Monatsh. Math. **139** (2003), 61–68.

[28] T. Sugawa, *Quasiconformal extension of strongly spirallike functions,* Comput. Methods Funct. Theory **12** (2012), 19–30.

[29] T. Sugawa, *A remark on the Ahlfors-Lehto univalence criterion,* Ann. Acad. Sci. Fenn. Math. **27** (2002), 151–161.

[30] T. Sugawa, *On the norm of pre-Schwarzian derivatives of strongly starlike functions,* Ann. Univ. Marie Curie-Sklodowska, Section A, **52** (1998), 149–157.

Received December 23, 2021, revised February 20, 2022.

Zhenyong Hu,
School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, China,
E-mail: huzhenyongad@163.com

Jinhua Fan,
School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, China,
E-mail: jinhuafan@hotmail.com

Xiaoyuan Wang,
School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, China,
E-mail: mewangxiaoyuan@163.com

Квазіконформні продовження та внутрішній радіус унівалентності відносно перед-шварцових похідних аналітичного і гармонічного відображення

Zhenyong Hu, Jinhua Fan, and Xiaoyuan Wang

У роботі ми вивчаємо критерій унівалентності квазіконформних продовжень і внутрішній радіус унівалентності для локально унівалентних аналітичних і гармонічних відображень. Для локально унівалентних аналітичних функцій на одиничному диску ми надаємо достатні умови унівалентності і квазіконформні продовження відносно перед-шварцових похідних, які узагальнюють результат Беккера. Для сильно спіралеподібних областей ми розглядаємо квазіконформне продовження і одержуємо нижні оцінки внутрішніх радіусів унівалентності відносно перед-шварцових і шварцових похідних. Крім того, для гармонічних відображень в однозв'язній області Ω, ми доводимо, що Ω є квазідиском в тому і лише тому випадку, коли внутрішній радіус унівалентності області Ω відносно перед-шварцових похідних гармонічного відображення є додатним та одержуємо загальну достатню умову унівалентності і квазіконформні продовження.

Ключові слова: квазіконформне продовження, квазідиск, внутрішній радіус унівалентності, сильно спіралеподібна функція, гармонічне відображення