Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 4, pp. 799–821
doi: https://doi.org/10.15407/mag19.03.799

# On Hilbert–Schmidt Frames for Operators and Riesz Bases

Jyoti and Lalit Kumar Vashisht

Stable analysis and reconstruction of vectors in closed subspaces of Hilbert spaces can be studied by Găvruta's type frame conditions which are related with the concept of atomic systems in separable Hilbert spaces. In this work, first we give Găvruta's type frame conditions for a class of Hilbert–Schmidt operators (in short, $\mathcal{C}_2$ class), where a bounded linear operator controls the lower frame condition. We discuss frame-preserving mappings for Hilbert–Schmidt frames for subspaces of a separable Hilbert space. We establish the existence of Hilbert–Schmidt frames for subspaces of the Hilbert–Schmidt class $\mathcal{C}_2$. It is shown that every separable Hilbert space admits a Hilbert–Schmidt frame with respect to a given separable Hilbert space. We obtain necessary and sufficient conditions for Găvruta's type frame conditions for sums of Hilbert–Schmidt frames for subspaces. Finally, we discuss Hilbert–Schmidt Riesz bases in separable Hilbert spaces.

*Key words:* frames, Hilbert–Schmidt frames, K-frames, perturbation

*Mathematical Subject Classification 2020:* 42C15, 42C30, 42C40, 43A32

## 1. Introduction

Frames for a separable Hilbert space $\mathcal{H}$ are redundant building blocks which provide a series representation, not necessarily unique, of each vector of the space $\mathcal{H}$. The concept of a *"Hilbert frame"* first appeared in the work of Duffin and Schaeffer [12] while studying some intense problems related to non-harmonic Fourier series. The mathematical definition of a frame is based on an operator inequality. Let $\mathcal{H}$ be a separable (infinite or finite-dimensional) complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We recall that a self-adjoint operator $T$ acting on $\mathcal{H}$ is said to be positive if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Let $S(\mathcal{H})$ be the family of self-adjoint operators acting on $\mathcal{H}$. If $U, T \in \mathcal{S}(\mathcal{H})$ and $U - T$ is positive, then we write $T \preceq U$. A countable sequence $\{x_k\}_{k \in \mathbb{I}}$ of members of $\mathcal{H}$ is called a *frame* (or *Hilbert frame*) for the space $\mathcal{H}$ if the map, called the *frame operator*, $\mho : \mathcal{H} \to \mathcal{H}$, given by

$$\mho : x \mapsto \sum_{k \in \mathbb{I}} \langle x, x_k \rangle x_k, \quad x \in \mathcal{H},$$

satisfies

$$m_o I_{\mathcal{H}} \preceq \mho \preceq M_o I_{\mathcal{H}} \tag{1.1}$$

for some positive constants $m_o$ and $M_o$; here $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Condition (1.1) is equivalent to

$$m_o\|x\|^2 \leq \|\{\langle x, x_k\rangle\}_{k\in\mathbb{I}}\|^2_{\ell^2(\mathbb{I})} \leq M_o\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

The scalars $m_o$ and $M_o$, obviously not unique, are known as *lower frame bound* and *upper frame bound*, and the supremum of lower frame bounds and infimum of upper frame bounds are called *optimal frame bounds*. The frame $\{x_k\}_{k\in\mathbb{I}}$ is *normalized tight* (or *Parseval*) if $m_o = M_o = 1$. If $\{x_k\}_{k\in\mathbb{I}}$ fulfills the right inequality in (1.1), then we say that $\{x_k\}_{k\in\mathbb{I}}$ is a *Bessel sequence* with *Bessel bound* $M_o$. The frame operator $\mho : \mathcal{H} \to \mathcal{H}$ of the frame $\{x_k\}_{k\in\mathbb{I}}$ is bounded, linear, positive and invertible on $\mathcal{H}$. Since $\mho^{-1}$ is a self-adjoint operator, the reconstruction formula is

$$x = \mho\mho^{-1}x = \sum_{k\in\mathbb{I}}\langle x, \mho^{-1}x_k\rangle x_k, \quad x \in \mathcal{H}.$$

The scalars $\langle x, \mho^{-1}x_k\rangle$ are known as *frame coefficients*. The frame condition (1.1), known as *frame inequality*, is a powerful tool in the study of the operator theory [2, 3, 6, 8, 17], iterated function systems [10, 26], quantum physics [16, 18] and many areas in both pure and engineering sciences. Detailed discussions about various types of frames and their applications can be found in the texts of Casazza and Kutyniok [5], Christensen [7], and Heil [15].

Frame properties of a given system in Hilbert (or Banach) spaces have been extensively studied in the last three decades. More precisely, flexibility and density of frame vectors are responsible for the development of frame theory in various directions, including pure mathematics and engineering science. In the direction of development of frames related to operators, Sun [24] introduced the concept of "generalized frame" ($g$-frame, in short) and "generalized Riesz basis" (*g-Riesz basis*, in short) in a separable complex Hilbert space. Generalized frames are the families of operators acting on a given Hilbert space with range in closed subspaces of another Hilbert space. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $\{\mathcal{K}_j\}_{j\in\mathbb{I}}$ be a sequence of closed subspaces of $\mathcal{K}$. For each $j \in \mathbb{I}$, let $\Lambda_j$ be a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}_j$. The collection $\Lambda \equiv \{\Lambda_j\}_{j\in\mathbb{I}}$ is called a *generalized frame* (in short, *g-frame*) for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{I}}$ if there exist positive constants $m_\Lambda$ and $M_\Lambda$ such that

$$m_\Lambda\|x\|^2 \leq \sum_{j\in\mathbb{I}}\|\Lambda_j x\|^2 \leq M_\Lambda\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

Sun [24] showed that generalized frames (respectively, generalized Riesz bases) share many useful properties with standard frames (respectively, standard Riesz bases). Furthermore, Sun [24] provided a link between generalized frames and another generalization of frames in the context of numerical analysis called *"stable space splittings"*. Generalized frames are also used in the atomic resolution of bounded linear operators, see [24] for technical details. In [25], Sun proved that generalized frames are stable under small perturbation.

Koo and Lim [20] studied the relations of Schatten $p$-class operators, the space of linear compact operators whose sequence of singular values is in $\ell^p$, to Bessel sequences and frame conditions. The authors of [21] studied von Neumann-Schatten $p$-frames in separable Banach spaces. They showed in [21] that generalized frames and $p$-frames are a class of von Neumann-Schatten $p$-frames. It is proved in [3, Lemma 3.1, Corollary 3.1] that Hilbert–Schmidt operators can be characterized by using frame conditions. Dual properties of Hilbert–Schmidt frames in separable Hilbert spaces can be found in [28]. It is necessary to mention the papers of Schatten [22] and Simon [23] for basic theory on Hilbert–Schmidt operators and their applications.

Găvruta, in [13], proposed a new concept of frames for the range of bounded linear operators under the name $K$-frame, where $K$ is a bounded linear operator acting on a separable Hilbert space. A countable family of vectors $\{x_k\}_{k\in\mathbb{I}}$ in $\mathcal{H}$ is called a *$K$-frame* for $\mathcal{H}$ if for some $\gamma_1, \gamma_2 \in (0,\infty)$,

$$\gamma_1\|K^*x\|^2 \le \sum_{k\in\mathbb{I}} |\langle x, x_k\rangle|^2 \le \gamma_2\|x\|^2 \tag{1.2}$$

holds for all $x \in \mathcal{H}$. One can observe that if $K$ is the identity operator on $\mathcal{H}$, then the $K$-frame $\{x_k\}_{k\in\mathbb{I}}$ turns out to be a standard Hilbert frame. The lower frame condition in a $K$-frame depends on $K$, which gives the linear expansion of each vector in the range of the operator $K$, see [13, Theorem 5] for technical details. Găvruta, in [13], relates this concept of $K$-frames with *"local atoms"* in closed subspaces of Hilbert spaces. Some differences between standard Hilbert frames and $K$-frames can be found in [13]. One of the major differences between standard Hilbert frames and $K$-frames is that the frame operator of a $K$-frame for a separable Hilbert space $\mathcal{H}$ may be not invertible on $\mathcal{H}$. It is invertible on its range $\mathrm{Ran}(K)$, whenever $\mathrm{Ran}(K)$ is closed in $\mathcal{H}$. Recent development of the study of $K$-frames in separable Hilbert spaces can be found in [8, 9, 27].

The present study concerns Găvruta's type frame conditions (1.2) for the Hilbert–Schmidt class of operators, where the lower frame condition is a function of a bounded linear operator. The paper is structured as follows. We start with a brief review about Hilbert–Schmidt class in Section 2. Hilbert–Schmidt frames for the range of operators under the name Θ-Hilbert–Schmidt frames are studied in Section 3. Proposition 3.6 and Theorem 3.7 give new Θ-Hilbert–Schmidt frames from a given Θ-Hilbert–Schmidt frame by composing frame elements with bounded operators. Theorem 3.12 gives the existence of Θ-Hilbert–Schmidt frames for the Hilbert–Schmidt class $\mathcal{C}_2$. In Theorem 3.14, we show that every separable Hilbert space admits a Θ-Hilbert–Schmidt frame with respect to a given separable Hilbert space. We also give the construction of Θ-Hilbert–Schmidt frames from ordinary Hilbert Θ-frames. In Section 5, we extend some results for the standard Riesz bases to the class of Hilbert–Schmidt operators.

## 2. Notation and basic results

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the set of positive integers, integers and complex numbers, respectively. The symbol $\mathbb{I}$ denotes a countable infinite indexing set. $\mathcal{H}$ and $\mathcal{K}$ denote real or complex separable Hilbert spaces. The space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. If $\mathcal{H}=\mathcal{K}$, then we write $\mathcal{B}(\mathcal{H},\mathcal{K})=\mathcal{B}(\mathcal{H})$. The Hilbert-adjoint of an operator $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$, denoted by $T^*$, is defined as $\langle Tx,y\rangle=\langle x,T^*y\rangle$, $x\in\mathcal{H}$, $y\in\mathcal{K}$, see [15, Definition 2.10]. The space of compact linear operators acting on $\mathcal{H}$ is denoted by $\mathcal{B}_o(\mathcal{H})$. $\mathrm{Ran}(U)$ denotes the range of an operator $U$. The singular values $s_n(T)$ of $T\in\mathcal{B}_o(\mathcal{H})$ are the eigenvalues of the positive operator $(T^*T)^{\frac{1}{2}}$. Further, the set of singular values $s_n(T)$ of $T$ counted with their multiplicity is at most countable, they are bigger than zero and can be arranged in a decreasing sequence as follows:

$$s_1(T)\geq s_2(T)\geq\cdots\geq 0 \quad\text{and}\quad s_n(T)\to 0 \text{ as } n\to\infty.$$

The Hilbert–Schmidt class, denoted by $\mathcal{C}_2$, is defined as

$$\mathcal{C}_2:=\left\{T\in\mathcal{B}_o(\mathcal{H}):\sum_i s_i^2(T)<\infty\right\}.$$

The space $\mathcal{C}_2$ is a Banach space with respect to the norm $\|.\|_2$ defined as

$$\|T\|_2=\Big(\sum_i s_i^2(T)\Big)^{\frac{1}{2}}=\Big(\mathbf{trace}\,(T^*T)\Big)^{\frac{1}{2}},\ T\in\mathcal{C}_2,$$

where **trace** is the trace functional defined by $\mathbf{trace}\,(T)=\sum_{i\in\mathbb{N}}\langle Te_i,e_i\rangle$ for any orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}$. Note that $\|T\|_2=\big(\sum_{i\in\mathbb{N}}\|Te_i\|^2\big)^{\frac{1}{2}}$ for $T\in\mathcal{C}_2$ . The space $\mathcal{C}_2$ is a Hilbert space with respect to the inner product $[\cdot,\cdot]_{\mathbf{tr}}$ defined as

$$[T,S]_{\mathbf{tr}}=\mathbf{trace}\,(S^*T),\quad T,S\in\mathcal{C}_2.$$

The class $\mathcal{C}_2$ is a 2-sided $*$-closed ideal in $\mathcal{B}(\mathcal{H})$. To be precise, $T^*\in\mathcal{C}_2$ with $\|T^*\|_2=\|T\|_2$, and $TU, UT\in\mathcal{C}_2$ with $\|TU\|_2,\|UT\|_2\leq\|U\|\|T\|_2$ for $T\in\mathcal{C}_2$ and $U\in\mathcal{B}(\mathcal{H})$. Moreover, if $U\in\mathcal{B}(\mathcal{H})$ is unitary, then $\|UT\|_2=\|TU\|_2=\|T\|_2$.

The space $\bigoplus\mathcal{C}_2$, defined as

$$\bigoplus\mathcal{C}_2=\bigoplus_{i\in\mathbb{I}}\mathcal{C}_2:=\left\{\{A_i\}_{i\in\mathbb{I}}\subset\mathcal{C}_2:\Big(\sum_{i\in\mathbb{I}}\|A_i\|_2^2\Big)^{\frac{1}{2}}<\infty\right\},$$

is a Hilbert space with respect to the inner product given by

$$\Big\langle\{A_i\}_{i\in\mathbb{I}},\{B_i\}_{i\in\mathbb{I}}\Big\rangle_{\bigoplus\mathcal{C}_2}=\sum_{i\in\mathbb{I}}[A_i,B_i]_{\mathbf{tr}}.$$

It is easy to see that if $\mathcal{H} = \mathbb{C}$, then $\mathcal{B}(\mathcal{H}) = \mathcal{C}_2 = \mathbb{C}$ and $\bigoplus \mathcal{C}_2 = \ell^2(\mathbb{I})$.

For $x, y \in \mathcal{H}$, the operator $x \otimes y : \mathcal{H} \to \mathcal{H}$ is defined as

$$(x \otimes y)(u) = \langle u, y \rangle x, \;\; u \in \mathcal{H}.$$

It can be easily verified that $x \otimes y \in \mathcal{B}(\mathcal{H})$ with $\|x \otimes y\| = \|x\|\|y\|$. For any $x, y, u, v \in \mathcal{H}$ and for any $U \in \mathcal{B}(\mathcal{H})$, the following properties hold:

$$\begin{aligned}
(x \otimes y)(u \otimes v) &= \langle u, y \rangle (x \otimes v); \\
\textbf{trace}\,(x \otimes y) &= \langle x, y \rangle; \\
(x \otimes y)^* &= y \otimes x; \\
U(x \otimes y) &= Ux \otimes y; \\
(x \otimes y)U &= x \otimes U^* y.
\end{aligned}$$

Notice that $x \otimes y \in \mathcal{C}_2$ with $\|x \otimes y\|_2 = \|x\|\|y\|$ for $x, y \in \mathcal{H}$. Indeed, for any $x$, $y \in \mathcal{H}$, we have

$$\begin{aligned}
\|x \otimes y\|_2^2 &= \textbf{trace}\big((x \otimes y)^*(x \otimes y)\big) = \textbf{trace}\big((y \otimes x)(x \otimes y)\big) \\
&= \textbf{trace}\big(\langle x, x \rangle (y \otimes y)\big) = \langle x, x \rangle \textbf{trace}\,(y \otimes y) = \langle x, x \rangle \langle y, y \rangle = \|x\|^2\|y\|^2.
\end{aligned}$$

We conclude this section by recording the following result which will be used later.

**Theorem 2.1** ([11]). *Let $L_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. Then the following statements are equivalent:*

(i) $\operatorname{Ran}(L_1) \subset \operatorname{Ran}(L_2)$*;*

(ii) *there exists a $\lambda > 0$ such that $L_1 L_1^* \preceq \lambda^2 L_2 L_2^*$;*

(iii) *there exists $M \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $L_1 = L_2 M$.*

## 3. Hilbert–Schmidt frames for operators

We begin this section with the following definition.

**Definition 3.1.** Let $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$ and $\Theta \in \mathcal{B}(\mathcal{H})$. A countable collection of operators $\{T_i\}_{i \in \mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ is called a *$\Theta$-Hilbert–Schmidt frame* (*$\Theta$-HS frame*, in short) for $\mathcal{H}$ with respect to $\mathcal{K}$ if there exist positive scalars $\alpha_o$, $\beta_o$ such that

$$\alpha_o \|\Theta^* x\|^2 \leq \sum_{i \in \mathbb{I}} \|T_i x\|_2^2 \leq \beta_o \|x\|^2 \tag{3.1}$$

holds for all $x \in \mathcal{H}$.

If only the upper inequality in (3.1) is satisfied, then we say that $\{T_i\}_{i \in \mathbb{I}}$ is a *Hilbert–Schmidt Bessel sequence* or simply *$HS$ Bessel sequence* with *Bessel bound* (or *upper frame bound*) $\beta_o$. If the lower inequality in (3.1) holds, then $\alpha_o$ will be called a *lower $\Theta$-$HS$ frame bound*.

Let $\{T_i\}_{i\in\mathbb{I}}$ be an $HS$ Bessel sequence for $\mathcal{H}$ with Bessel bound $\beta_o$. The operator $\mathfrak{V} : \bigoplus \mathcal{C}_2 \to \mathcal{H}$, defined by

$$\mathfrak{V} : \{A_i\}_{i\in\mathbb{I}} \mapsto \sum_{i\in\mathbb{I}} T_i^* A_i, \quad \{A_i\}_{i\in\mathbb{I}} \in \bigoplus \mathcal{C}_2,$$

is called the *pre-frame operator* (or *synthesis operator*) associated with $\{T_i\}_{i\in\mathbb{I}}$. The *analysis operator* $\mathfrak{V}^* : \mathcal{H} \to \bigoplus \mathcal{C}_2$ is given by

$$\mathfrak{V}^* : x \mapsto \{T_i x\}_{i\in\mathbb{I}}, \quad x \in \mathcal{H}.$$

Note that both the synthesis and the analysis operators are linear and bounded. The composition $\Omega = \mathfrak{V}\mathfrak{V}^* : \mathcal{H} \to \mathcal{H}$ is called the *frame operator* associated with $\{T_i\}_{i\in\mathbb{I}}$ and it is given by

$$\Omega : x \mapsto \sum_{i\in\mathbb{I}} T_i^* T_i x, \quad x \in \mathcal{H}.$$

If $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$, then $\Omega$ is a linear and bounded operator, but it may not be invertible on $\mathcal{H}$. It is invertible on $\mathrm{Ran}(\Omega)$ whenever the range $\mathrm{Ran}(\Omega) \subset \mathcal{H}$ is closed.

*Remark* 3.2. If $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$, then inequality (3.1) can be written as

$$\alpha_o \Theta\Theta^* \preceq \Omega \preceq \beta_o I_{\mathcal{H}}.$$

Regarding the existence of $\Theta$-$HS$ frames in Hilbert spaces, we have the following example.

*Example* 3.3.

a) Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis for the Hilbert space $\ell^2(\mathbb{N})$. For any $i \in \mathbb{N}$, define $T_i : \ell^2(\mathbb{N}) \to \mathbb{C}$ as

$$T_i x = \langle x, e_{i+1}\rangle, \ x \in \ell^2(\mathbb{N}).$$

Then $T_i \in \mathcal{B}(\ell^2(\mathbb{N}), \mathbb{C})$ for all $i \in \mathbb{N}$. Let $\Theta$ be the right shift operator acting on $\ell^2(\mathbb{N})$. Then, for every $x \in \ell^2(\mathbb{N})$, we have

$$\|\Theta^* x\|^2 = \sum_{i\in\mathbb{N}} |\langle x, e_{i+1}\rangle|^2 = \sum_{i\in\mathbb{N}} \|T_i x\|_2^2 \leq \|x\|^2.$$

Hence, $\{T_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame for $\ell^2(\mathbb{N})$ with respect to the unitary space $\mathbb{C}$.

b) Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal basis for the space $\ell^2(\mathbb{Z})$. Let $U$ be the left shift operator acting on $\ell^2(\mathbb{Z})$, which is defined as

$$U : \{\xi_i\}_{i\in\mathbb{Z}} \mapsto \{\xi_{i+1}\}_{i\in\mathbb{Z}}, \ \{\xi_i\}_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Then $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is unitary. For $\mathcal{C}_2 \subset \mathcal{B}(\ell^2(\mathbb{Z}))$, let $\Theta : \mathcal{C}_2 \to \mathcal{C}_2$ be defined as

$$\Theta : T \mapsto UT,\ T \in \mathcal{C}_2.$$

Then $\Theta \in \mathcal{B}(\mathcal{C}_2)$. Now, for any $T,\ S \in \mathcal{C}_2$, we compute

$$\begin{aligned}[\Theta^* T, S]_{\mathbf{tr}} = [T, \Theta S]_{\mathbf{tr}} = [T, US]_{\mathbf{tr}} &= \mathbf{trace}\left((US)^* T\right) \\ &= \mathbf{trace}\left(S^* U^* T\right) = [U^* T, S]_{\mathbf{tr}},\end{aligned}$$

which entails

$$\Theta^* T = U^* T, \quad T \in \mathcal{C}_2.$$

Define $T_i : \mathcal{C}_2 \to \mathcal{C}_2$ as

$$T_i : T \mapsto \begin{cases} U^* T e_i \otimes e_i,\ i \leq 1, \\ U^* T e_{i-1} \otimes e_{i-1},\ i > 1 \end{cases}$$

for all $T \in \mathcal{C}_2$. Then $\{T_i\}_{i\in\mathbb{Z}} \subset \mathcal{B}(\mathcal{C}_2)$. For any $T \in \mathcal{C}_2$, we compute

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \|T_i T\|_2^2 &= \sum_{i\leq 1} \|U^* T e_i \otimes e_i\|_2^2 + \sum_{i>1} \|U^* T e_{i-1} \otimes e_{i-1}\|_2^2 \\ &\leq 2 \sum_{i\in\mathbb{Z}} \|U^* T e_i \otimes e_i\|_2^2 = 2 \sum_{i\in\mathbb{Z}} \|U^* T e_i\|^2 \|e_i\|^2 \\ &= 2 \sum_{i\in\mathbb{Z}} \|U^* T e_i\|^2 = 2\|U^* T\|_2^2 = 2\|T\|_2^2.\end{aligned}$$

This gives the upper frame condition for the $\Theta$-$HS$ frame. For the lower $\Theta$-$HS$ frame bounded in a similar way we can show that

$$\sum_{i\in\mathbb{Z}} \|T_i T\|_2^2 \geq \sum_{i\in\mathbb{Z}} \|U^* T e_i \otimes e_i\|_2^2 = \|U^* T\|_2^2 = \|\Theta^* T\|_2^2 \text{ for all } T \in \mathcal{C}_2.$$

Hence, $\{T_i\}_{i\in\mathbb{Z}}$ is a $\Theta$-$HS$ frame for $\mathcal{C}_2$ with respect to $\ell^2(\mathbb{Z})$ with $\Theta$-$HS$ frame bounds 1 and 2.

*Remark* 3.4. If $\Theta = I_{\mathcal{H}}$, the identity operator on $\mathcal{H}$, then $\Theta$-$HS$ frame is the standard $HS$ frame for the Hilbert space $\mathcal{H}$ with respect to $\mathcal{K}$. A $\Theta$-$HS$ frame, in general, is not a $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. This is clear by Example 3.3(a)) wherein $\{T_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame but not an $HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$ as for $x = e_1 \in \ell^2(\mathbb{N})$, we have

$$\sum_{i\in\mathbb{N}} \|T_i x\|_2^2 = \sum_{i\in\mathbb{N}} |\langle x, e_{i+1}\rangle|^2 = 0.$$

On the contrary, an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ is always a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ for $\Theta \in \mathcal{B}(\mathcal{H})$. In fact, this is a particular case of the following result which is an application of Douglas' majorization Theorem (see Theorem 2.1).

**Theorem 3.5.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-HS frame for $\mathcal{H}$ with respect to $\mathcal{K}$ and let $\Xi \in \mathcal{B}(\mathcal{H})$ with* $\operatorname{Ran}(\Xi) \subset \operatorname{Ran}(\Theta)$*. Then $\{T_i\}_{i\in\mathbb{I}}$ is a $\Xi$-HS frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

Aldroubi [1] characterized frame-preserving mappings, that is, mappings that transform frames of $\mathcal{H}$ into other frames of $\mathcal{H}$. Some necessary and sufficient conditions satisfied by frame-preserving operators on the underlying space can be found in [1]. Aldroubi gave the class of operators on $\ell^2(\mathbb{N})$ that generates all the frames of a Hilbert space $\mathcal{H}$. Inspired by Aldroubi's type frame-preserving mappings, we discuss $HS$ frames under the action of operators in separable Hilbert spaces. Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ and $\Theta \in \mathcal{B}(\mathcal{H})$. Then $\{T_i\Theta^*\}_{i\in\mathbb{I}}$, in general, need not be an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. For example, for any $i \in \mathbb{N}$, let $T_i \in \mathcal{B}(\ell^2(\mathbb{N}), \mathbb{C})$ be given by

$$T_i x = \langle x, e_i\rangle, \quad x \in \ell^2(\mathbb{N}),$$

where $\{e_i\}_{i\in\mathbb{N}}$ denotes the canonical orthonormal basis for $\ell^2(\mathbb{N})$. Then, for the right shift operator $\Theta \in \mathcal{B}(\ell^2(\mathbb{N}))$, the sequence $\{T_i\Theta^*\}_{i\in\mathbb{N}}$ is not an $HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$ as for $x = e_1 \in \ell^2(\mathbb{N})$, we have

$$\sum_{i\in\mathbb{N}} \|T_i\Theta^* x\|_2^2 = \sum_{i\in\mathbb{N}} |\langle x, e_{i+1}\rangle|^2 = 0.$$

However, it is easy to check that $\{T_i\Theta^*\}_{i\in\mathbb{N}}$ becomes an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $\{T_i\}_{i\in\mathbb{N}}$ is an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if $\Theta \in \mathcal{B}(\mathcal{H})$ is surjective. So, we have seen that $\{T_i\Theta^*\}_{i\in\mathbb{I}}$ need not be an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ where $\Theta \in \mathcal{B}(\mathcal{H})$. But $\{T_i\Theta^*\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ for every $\Theta \in \mathcal{B}(\mathcal{H})$, whenever $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ . In fact, this is a particular case of the following result.

**Proposition 3.6.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ and $E \in \mathcal{B}(\mathcal{H})$. Then $\{T_iE^*\}_{i\in\mathbb{I}}$ is a $E\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

The following result shows that a $\Theta$-$HS$ frame for a Hilbert space $\mathcal{H}$ with respect to $\mathcal{K}$ can be retained by composing its elements with a bounded below operator.

**Theorem 3.7.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Then:*

a) *$\{ET_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if $E \in \mathcal{B}(\mathcal{C}_2)$ is a bounded below operator.*

b) *$\{T_iE\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if $E \in \mathcal{B}(\mathcal{H})$ is a bounded below operator such that $E\Theta^* = \Theta^*E$.*

*Proof.* Let $\alpha$, $\beta$ be $\Theta$-$HS$ frame bounds for $\{T_i\}_{i\in\mathbb{I}}$. If $E \in \mathcal{B}(\mathcal{C}_2)$ is bounded below, then there exists a positive real number $M$ such that

$$\|EA\|_2 \geq M\|A\|_2 \quad \text{for all } A \in \mathcal{C}_2.$$

Therefore, for any $x \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} \|ET_ix\|_2^2 \geq \sum_{i\in\mathbb{I}} M^2\|T_ix\|_2^2 \geq M^2\alpha\|\Theta^*x\|^2.$$

Analogously,

$$\sum_{i\in\mathbb{I}} \|ET_ix\|_2^2 \leq \sum_{i\in\mathbb{I}} \|E\|^2\|T_ix\|_2^2 \leq \|E\|^2\beta\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

This proves a). Condition b) can be proved in a similar way. □

The concept of orthonormal bases plays a significant role in the characterization of frames for subspaces. This leads to the definition of the Hilbert–Schmidt orthonormal basis for a Hilbert space.

**Definition 3.8.** Let $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$. A sequence $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ is called a *Hilbert–Schmidt orthonormal basis* (*HS orthonormal basis*, in short) for $\mathcal{H}$ with respect to $\mathcal{K}$ if:

a) $\sum_{i\in\mathbb{I}} \|T_ix\|_2^2 = \|x\|^2$ for all $x \in \mathcal{H}$.

b) $\{T_i\}_{i\in\mathbb{I}}$ is $HS$ orthonormal, i.e., $\left\langle T_i^*T, T_j^*S\right\rangle = \delta_{i,j}[T,S]_{\mathbf{tr}}$ for all $T, S \in \mathcal{C}_2$.

*Remark* 3.9. If $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ is an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathcal{K}$, then every element $x$ of $\mathcal{H}$ can be expressed as

$$x = \sum_{i\in\mathbb{I}} T_i^*T_ix.$$

Regarding the existence of $HS$ orthonormal bases, we have the following example.

*Example* 3.10.

(I) Let $\{e_i\}_{i\in\mathbb{I}}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. For each $i \in \mathbb{I}$, define $T_i : \mathcal{H} \to \mathbb{C}$ as

$$T_ix = \langle x, e_i\rangle, \; x \in \mathcal{H}.$$

Then $T_i \in \mathcal{B}(\mathcal{H}, \mathbb{C})$ for each $i \in \mathbb{I}$, and $T_i^* : \mathbb{C} \to \mathcal{H}$ is given by

$$T_i^*\alpha = \alpha e_i, \; \alpha \in \mathbb{C}.$$

Therefore, we have:

a) $\sum_{i\in\mathbb{I}} \|T_ix\|_2^2 = \sum_{i\in\mathbb{I}} |\langle x, e_i\rangle|^2 = \|x\|^2$, $x \in \mathcal{H}$.

b) $\langle T_i^*\alpha, T_j^*\beta\rangle = \langle \alpha e_i, \beta e_j\rangle = \alpha\overline{\beta}\langle e_i, e_j\rangle = \delta_{i,j}\langle \alpha, \beta\rangle$ for all $\alpha, \beta \in \mathbb{C}$.

Hence, $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathbb{C}$.

(II) Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{C}_2 \subset \mathcal{B}(\mathcal{H})$. For $i \in \mathbb{I}$, define $P_i : \bigoplus \mathcal{C}_2 \to \mathcal{C}_2$ as

$$P_i : \{A_j\}_{j\in\mathbb{I}} \mapsto A_i, \quad \{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2.$$

Then, for $T \in \mathcal{C}_2$ and $\{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$, we compute

$$\begin{aligned}\Big\langle P_i^* T, \{A_j\}_{j\in\mathbb{I}}\Big\rangle_{\bigoplus \mathcal{C}_2} &= \Big[T, P_i\Big(\{A_j\}_{j\in\mathbb{I}}\Big)\Big]_{\mathbf{tr}} = [T, A_i]_{\mathbf{tr}} \\ &= \Big\langle (0, \cdots, 0, \underbrace{T}_{i^{\text{th}} \text{ place}}, 0, \cdots, 0), \{A_j\}_{j\in\mathbb{I}}\Big\rangle_{\bigoplus \mathcal{C}_2}.\end{aligned}$$

Thus, for each $i \in \mathbb{I}$, $P_i^* : \mathcal{C}_2 \to \bigoplus \mathcal{C}_2$ is given by

$$P_i^* T = \big(0, \cdots, 0, \underbrace{T}_{i^{\text{th}} \text{ place}}, 0, \cdots, 0\big), \quad T \in \mathcal{C}_2.$$

For any $\{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$ and for any $T, S \in \mathcal{C}_2$, we have

$$\sum_{i\in\mathbb{I}} \Big\| P_i\Big(\{A_j\}_{j\in\mathbb{I}}\Big)\Big\|_2^2 = \sum_{i\in\mathbb{I}} \|A_i\|_2^2 = \Big\|\{A_j\}_{j\in\mathbb{I}}\Big\|_{\bigoplus \mathcal{C}_2}^2,$$

and

$$\begin{aligned}&\Big\langle P_i^* T, P_j^* S\Big\rangle_{\bigoplus \mathcal{C}_2} \\ &= \Big\langle (0, \cdots, 0, \underbrace{T}_{i^{\text{th}} \text{ place}}, 0, \cdots, 0), (0, \cdots, 0, \underbrace{S}_{j^{\text{th}} \text{ place}}, 0, \cdots, 0)\Big\rangle_{\bigoplus \mathcal{C}_2} \\ &= \delta_{i,j}[T, S]_{\mathbf{tr}}.\end{aligned}$$

Hence, $\{P_i\}_{i\in\mathbb{I}}$ is an $HS$ orthonormal basis for $\bigoplus \mathcal{C}_2$ with respect to $\mathcal{H}$.

In [13, Theorem 4], Găvruta characterized $K$-frames using linear bounded operators and orthonormal bases. The following theorem generalizes this result to $\Theta$-$HS$ frames for separable Hilbert spaces. For completeness we include its proof. Using this result, we can construct a $\Theta$-$HS$ frame for a separable Hilbert space for any linear bounded operator $\Theta$ acting on $\mathcal{C}_2$, see Theorem 3.12.

**Theorem 3.11.** *Let $\Theta \in \mathcal{B}(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$. Let $\{P_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\bigoplus \mathcal{C}_2, \mathcal{C}_2)$ be an $HS$ orthonormal basis for the space $\bigoplus \mathcal{C}_2$ with respect to $\mathcal{K}$. Then $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if and only if there exists an operator $\Xi \in \mathcal{B}(\bigoplus \mathcal{C}_2, \mathcal{H})$ such that*

a) *$T_i = P_i \Xi^*$, $i \in \mathbb{I}$.*

b) *$\eta \Theta\Theta^* \preceq \Xi\Xi^*$ for some positive scalar $\eta > 0$.*

*Proof.* Suppose first that $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\alpha$, $\beta$. Define a linear operator $R : \mathcal{H} \to \bigoplus \mathcal{C}_2$ by

$$Rx = \sum_{i\in\mathbb{I}} P_i^* T_i x, \quad x \in \mathcal{H}.$$

Then $R$ is bounded. Indeed, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\|Rx\|^2_{\bigoplus \mathcal{C}_2} &= \Big\| \sum_{i\in\mathbb{I}} P_i^* T_i x \Big\|^2_{\bigoplus \mathcal{C}_2} \\ &= \Big\langle \sum_{i\in\mathbb{I}} P_i^* T_i x, \sum_{j\in\mathbb{I}} P_j^* T_j x \Big\rangle_{\bigoplus \mathcal{C}_2} = \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} \Big\langle P_i^* T_i x, P_j^* T_j x \Big\rangle_{\bigoplus \mathcal{C}_2} \\ &= \sum_{i\in\mathbb{I}} [T_i x, T_i x]_{\mathbf{tr}} = \sum_{i\in\mathbb{I}} \|T_i x\|_2^2 \le \beta \|x\|^2.\end{aligned}$$

Now, for any $i \in \mathbb{I}$ and $T \in \mathcal{C}_2$, we have

$$\begin{aligned}\langle R^* P_i^* T, x\rangle = \langle P_i^* T, Rx\rangle_{\bigoplus \mathcal{C}_2} &= \Big\langle P_i^* T, \sum_{j\in\mathbb{I}} P_j^* T_j x \Big\rangle_{\bigoplus \mathcal{C}_2} = \sum_{j\in\mathbb{I}} \langle P_i^* T, P_j^* T_j x\rangle \\ &= [T, T_i x]_{\mathbf{tr}} = \langle T_i^* T, x\rangle \quad \text{for all } x \in \mathcal{H}.\end{aligned}$$

This gives $R^* P_i^* = T_i^*$, $i \in \mathbb{I}$. Thus, a) is proved, where $\Xi = R^*$. Next we prove condition b). For any $x \in \mathcal{H}$, we have

$$\alpha \|\Theta^* x\|^2 \le \sum_{i\in\mathbb{I}} \|T_i x\|_2^2 = \|Rx\|^2_{\bigoplus \mathcal{C}_2}, \ x \in \mathcal{H}.$$

This can be expressed as $\alpha \Theta\Theta^* \preceq R^* R$, which gives b).

Conversely, suppose a) and b) hold for some $\eta > 0$. Then, for any $x \in \mathcal{H}$ and $\{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$, we have

$$\begin{aligned}\big\langle \Xi^* x, \{A_j\}_{j\in\mathbb{I}} \big\rangle_{\bigoplus \mathcal{C}_2} &= \big\langle x, \Xi(\{A_j\}_{j\in\mathbb{I}}) \big\rangle = \Big\langle x, \Xi\Big( \sum_{i\in\mathbb{I}} P_i^* P_i(\{A_j\}_{j\in\mathbb{I}}) \Big) \Big\rangle \\ &= \Big\langle x, \sum_{i\in\mathbb{I}} \Xi P_i^* P_i(\{A_j\}_{j\in\mathbb{I}}) \Big\rangle = \Big\langle x, \sum_{i\in\mathbb{I}} T_i^* P_i(\{A_j\}_{j\in\mathbb{I}}) \Big\rangle \\ &= \sum_{i\in\mathbb{I}} [T_i x, P_i(\{A_j\}_{j\in\mathbb{I}})]_{\mathbf{tr}} = \Big\langle \sum_{i\in\mathbb{I}} P_i^* T_i x, \{A_j\}_{j\in\mathbb{I}} \Big\rangle_{\bigoplus \mathcal{C}_2}.\end{aligned}$$

This implies that

$$\Xi^* x = \sum_{i\in\mathbb{I}} P_i^* T_i x, \quad x \in \mathcal{H}.$$

Therefore, for any $x \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} \|T_i x\|_2^2 = \Big\| \sum_{i\in\mathbb{I}} P_i^* T_i x \Big\|^2_{\bigoplus \mathcal{C}_2} = \|\Xi^* x\|^2_{\bigoplus \mathcal{C}_2} \le \|\Xi^*\|^2 \|x\|^2.$$

This gives the upper frame condition. For the lower $\Theta$-$HS$ frame bound, by condition b), we have

$$\eta \|\Theta^* x\|^2 \le \|\Xi^* x\|^2_{\bigoplus \mathcal{C}_2} = \sum_{i\in\mathbb{I}} \|T_i x\|_2^2, \quad x \in \mathcal{H}.$$

Hence, $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\eta$, $\|\Xi\|^2$. □

As an application of Theorem 3.11, we obtain the existence of $\Theta$-$HS$ frames for $\mathcal{C}_2$ with respect to a given separable Hilbert space.

**Theorem 3.12.** *Let $\mathcal{H}$ be a separable Hilbert space. Then, for every bounded linear operator $\Theta$ acting on $\mathcal{C}_2$, there exists a $\Theta$-$HS$ frame for $\mathcal{C}_2$ with respect to $\mathcal{H}$.*

*Proof.* Let $\{x_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{H}$ with frame bounds $\alpha$, $\beta$, where $x_i \neq 0$ for all $i \in \mathbb{I}$. Let $\{e_i\}_{i\in\mathbb{I}}$ be an orthonormal basis for $\mathcal{H}$. For each $i \in \mathbb{I}$, define $T_i : \mathcal{C}_2 \to \mathcal{C}_2$ by

$$T_i : T \mapsto \Theta^* T x_i \otimes \frac{x_i}{\|x_i\|}, \ T \in \mathcal{C}_2.$$

Then $T_i \in \mathcal{B}(\mathcal{C}_2)$ for all $i \in \mathbb{I}$. Indeed, for any $T \in \mathcal{C}_2$, we have

$$\|T_i T\|_2 = \left\|\Theta^* T x_i \otimes \frac{x_i}{\|x_i\|}\right\|_2 = \|\Theta^* T x_i\| \left\|\frac{x_i}{\|x_i\|}\right\| \leq \|\Theta^* T\|_2 \|x_i\| \leq \|\Theta^*\| \|T\|_2 \|x_i\|.$$

Moreover, $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ Bessel sequence as

$$\begin{aligned}\sum_{i\in\mathbb{I}} \|T_i(T)\|_2^2 &= \sum_{i\in\mathbb{I}} \|\Theta^* T x_i\|^2 = \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} |\langle \Theta^* T x_i, e_j\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} |\langle x_i, (\Theta^* T)^* e_j\rangle|^2 \leq \beta \sum_{j\in\mathbb{I}} \|(\Theta^* T)^* e_j\|^2 \\ &= \beta \|(\Theta^* T)^*\|_2^2 = \beta \|\Theta^* T\|_2^2 \leq \beta \|\Theta^*\|^2 \|T\|_2^2 \text{ for all } \quad T \in \mathcal{C}_2.\end{aligned}$$

Therefore, the pre-frame operator $\mathfrak{V}$ associated with $\{T_i\}_{i\in\mathbb{I}}$ is a well-defined linear bounded operator from $\bigoplus \mathcal{C}_2$ to $\mathcal{C}_2$. Considering the $HS$ orthonormal basis $\{\mathcal{P}_i\}_{i\in\mathbb{I}}$ for $\bigoplus \mathcal{C}_2$ with respect to $\mathcal{H}$ given in Example 3.10 ((II)), we get

$$\mathcal{P}_i \mathfrak{V}^* S = \mathcal{P}_i(\{T_j S\}_{j\in\mathbb{I}}) = T_i S, \ S \in \mathcal{C}_2.$$

That is, condition a) of Theorem 3.11 is satisfied for $\Xi = \mathfrak{V}$. Now, for any $T \in \mathcal{C}_2$, we compute

$$\begin{aligned}\alpha \|\Theta^* T\|_2^2 = \alpha \|(\Theta^* T)^*\|_2^2 &= \alpha \sum_{j\in\mathbb{I}} \|(\Theta^* T)^* e_j\|^2 \\ &\leq \sum_{j\in\mathbb{I}} \sum_{i\in\mathbb{I}} |\langle x_i, (\Theta^* T)^* e_j\rangle|^2 = \sum_{j\in\mathbb{I}} \sum_{i\in\mathbb{I}} |\langle \Theta^* T x_i, e_j\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} \|\Theta^* T x_i\|^2 = \sum_{i\in\mathbb{I}} \|T_i T\|_2^2 = \sum_{i\in\mathbb{I}} \|\mathcal{P}_i \Xi^* T\|_2^2 = \|\Xi^* T\|_{\bigoplus \mathcal{C}_2}^2.\end{aligned}$$

Thus, b) of Theorem 3.11 is satisfied for $\eta = \alpha$. Hence, $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{C}_2$ with respect to $\mathcal{H}$. □

*Remark* 3.13. Lemma 3.4 of [21] can be obtained from Theorem 3.12.

The next result gives necessary and sufficient conditions for $\Theta$-$HS$ frames in terms of a series associated with $HS$ Bessel sequences. This is an adaption of [13, Theorem 3]. As an application of the following result, the existence of $\Theta$-$HS$ frames in a separable Hilbert space with respect to a given Hilbert space is given in Theorem 3.16.

**Theorem 3.14.** *Let $\Theta \in \mathcal{B}(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be an $HS$ Bessel sequence in $\mathcal{H}$. The following statements are equivalent:*

a) *$\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

b) *There exists an $HS$ Bessel sequence $\{A_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ such that*

$$\Theta x = \sum_{i\in\mathbb{I}} T_i^* A_i x \text{ for every } x \in \mathcal{H}.$$

c) *There exists $\lambda > 0$ such that for every $x \in \mathcal{H}$ there exists a sequence $\{S_i\}_{i\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$ such that*

$$\Theta x = \sum_{i\in\mathbb{I}} T_i^* S_i \quad \text{and} \quad \sum_{i\in\mathbb{I}} \|S_i\|_2^2 \leq \lambda \|x\|^2.$$

*Remark* 3.15. The $HS$ Bessel sequence $\{A_i\}_{i\in\mathbb{I}}$ in part (b)) of Theorem 3.14 is known as a dual $\Theta$-$HS$ Bessel sequence. It can be easily verified that $\{A_i\}_{i\in\mathbb{I}}$ is a $\Theta^*$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. In fact, if $\beta_o$ is an upper $HS$ frame bound of $\{T_i\}_{i\in\mathbb{I}}$, then

$$\begin{aligned} \|\Theta x\|^2 &= \langle \Theta x, \Theta x\rangle \\ &= \left\langle \sum_{i\in\mathbb{I}} T_i^* A_i x, \Theta x \right\rangle = \sum_{i\in\mathbb{I}} [A_i x, T_i \Theta x]_{\mathbf{tr}} \leq \sum_{i\in\mathbb{I}} \|A_i x\|_2 \|T_i \Theta x\|_2 \\ &\leq \left(\sum_{i\in\mathbb{I}} \|A_i x\|_2^2\right)^{\frac{1}{2}} \left(\sum_{i\in\mathbb{I}} \|T_i \Theta x\|_2^2\right)^{\frac{1}{2}} \leq \left(\sum_{i\in\mathbb{I}} \|A_i x\|_2^2\right)^{\frac{1}{2}} \sqrt{\beta_o} \|\Theta x\|, \end{aligned}$$

which implies

$$\frac{1}{\beta_o} \|\Theta x\|^2 \leq \sum_{i\in\mathbb{I}} \|A_i x\|_2^2, \quad x \in \mathcal{H}. \tag{3.2}$$

As an application of Theorem 3.14, the following result shows that every separable Hilbert space admits a $\Theta$-$HS$ frame with respect to a given separable Hilbert space.

**Theorem 3.16.** *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, $\mathcal{C}_2 \subset \mathcal{B}(\mathcal{K})$ and $\Theta \in \mathcal{B}(\mathcal{H})$. Then there exists a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

*Proof.* Let $\{x_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{H}$ with frame bounds $\alpha$, $\beta$ and frame operator $\Omega$. Let $\{e_i\}_{i\in\mathbb{I}} \subset \mathcal{K}$ be an orthonormal set. For each $i \in \mathbb{I}$, define $T_i : \mathcal{H} \to \mathcal{C}_2$ by

$$T_i x = \langle x, \Theta x_i\rangle e_i \otimes e_i, \quad x \in \mathcal{H}.$$

Then, for any $x \in \mathcal{H}$, we have

$$\|T_i x\|_2 = \|\langle x, \Theta x_i\rangle e_i \otimes e_i\|_2 = \|\langle x, \Theta x_i\rangle e_i\|\|e_i\| = |\langle x, \Theta x_i\rangle| \leq \|\Theta x_i\|\|x\|.$$

Thus, $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ for all $i \in \mathbb{I}$. Moreover, $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ Bessel sequence as

$$\sum_{i\in\mathbb{I}} \|T_i x\|_2^2 = \sum_{i\in\mathbb{I}} |\langle x, \Theta x_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle \Theta^* x, x_i\rangle|^2 \leq \beta\|\Theta^*\|^2\|x\|^2, \quad x \in \mathcal{H}.$$

For any $T \in \mathcal{C}_2$ and for any $y \in \mathcal{H}$, we compute

$$\begin{aligned}\langle T_i^* T, y\rangle &= [T, T_i y]_{\mathbf{tr}} = [T, \langle y, \Theta x_i\rangle e_i \otimes e_i]_{\mathbf{tr}} = \langle \Theta x_i, y\rangle [T, e_i \otimes e_i]_{\mathbf{tr}} \\ &= \langle \Theta x_i, y\rangle \mathbf{trace}\left((e_i \otimes e_i)^* T\right) = \langle \Theta x_i, y\rangle \mathbf{trace}\left((e_i \otimes e_i) T\right) \\ &= \langle \Theta x_i, y\rangle \mathbf{trace}\left(e_i \otimes T^* e_i\right) = \langle \Theta x_i, y\rangle \langle e_i, T^* e_i\rangle = \Big\langle \langle T e_i, e_i\rangle \Theta x_i, y\Big\rangle,\end{aligned}$$

which entails

$$T_i^* T = \langle T e_i, e_i\rangle \Theta x_i, \quad i \in \mathbb{I}.$$

For $x \in \mathcal{H}$, we take $S_i = \langle x, \Omega^{-1} x_i\rangle e_i \otimes e_i$, $i \in \mathbb{I}$. Then $\{S_i\}_{i\in\mathbb{I}} \subset \bigoplus \mathcal{C}_2$ such that

$$\sum_{i\in\mathbb{I}} \|S_i\|_2^2 = \sum_{i\in\mathbb{I}} |\langle x, \Omega^{-1} x_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle \Omega x, x_i\rangle|^2 \leq \beta\|\Omega x\|^2 \leq \beta\|\Omega\|^2\|x\|^2,$$

and

$$\sum_{i\in\mathbb{I}} T_i^* S_i = \sum_{i\in\mathbb{I}} \left\langle \left(\langle x, \Omega^{-1} x_i\rangle e_i \otimes e_i\right) e_i, e_i\right\rangle \Theta x_i = \sum_{i\in\mathbb{I}} \langle x, \Omega^{-1} x_i\rangle \Theta x_i = \Theta x.$$

This gives condition c) of Theorem 3.14, where $\lambda = \beta\|\Omega\|^2$. Hence, by Theorem 3.14, $\{T_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. □

## 4. Sums of Θ-HS-frames

It is quite evident that the sum of two $\Theta$-$HS$ frames need not be a $\Theta$-$HS$ frame. However, the imposition of bounded belowness on the adjoint of $\Theta$ may lead to a condition under which the sum becomes a $\Theta$-$HS$ frame. Explicitly, we have

**Theorem 4.1.** *Let $\Theta \in \mathcal{B}(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{I}}, \{R_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be $\Theta$-$HS$ frames for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\alpha_1, \beta_1$ and $\alpha_2, \beta_2$, respectively. Let $\Theta^*$ be bounded below with a bound $m_o$ such that*

$$m_o^2(\alpha_1 + \alpha_2) > 2\sqrt{\beta_1\beta_2}. \tag{4.1}$$

*Then $\{T_i + R_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\left(\alpha_1 + \alpha_2 - \frac{2\sqrt{\beta_1\beta_2}}{m_o^2}\right)$ and $2(\beta_1 + \beta_2)$.*

*Proof.* For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} \|(T_i+R_i)x\|_2^2 &= \sum_{i\in\mathbb{I}} \|T_ix+R_ix\|_2^2 \geq \sum_{i\in\mathbb{I}} \Big(\|T_ix\|_2 - \|R_ix\|_2\Big)^2 \\ &= \sum_{i\in\mathbb{I}} \Big(\|T_ix\|_2^2 + \|R_ix\|_2^2 - 2\|T_ix\|_2\|R_ix\|_2\Big) \\ &\geq \sum_{i\in\mathbb{I}} \|T_ix\|_2^2 + \sum_{i\in\mathbb{I}} \|R_ix\|_2^2 - 2\Big(\sum_{i\in\mathbb{I}} \|T_ix\|_2^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in\mathbb{I}} \|R_ix\|_2^2\Big)^{\frac{1}{2}} \\ &\geq \alpha_1\|\Theta^*x\|^2 + \alpha_2\|\Theta^*x\|^2 - 2\sqrt{\beta_1\beta_2}\|x\|^2 \\ &\geq \left(\alpha_1+\alpha_2 - \frac{2\sqrt{\beta_1\beta_2}}{m_o^2}\right)\|\Theta^*x\|^2\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in\mathbb{I}} \|(T_i+R_i)x\|_2^2 &= \sum_{i\in\mathbb{I}} \|T_ix+R_ix\|_2^2 \\ &\leq 2\sum_{i\in\mathbb{I}} \|T_ix\|_2^2 + 2\sum_{i\in\mathbb{I}} \|R_ix\|_2^2 \leq 2(\beta_1+\beta_2)\|x\|^2.\end{aligned}$$

Hence, $\{T_i+R_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with the required frame bounds. □

*Remark* 4.2. Condition (4.1) in Theorem 4.1 is not necessary. Indeed, consider the sequence $\{T_i\}_{i\in\mathbb{Z}}$ given in part *b*) of Example 3.3 which is a $\Theta$-$HS$ frame for $\mathcal{C}_2$ with respect to $\ell^2(\mathbb{Z})$ with $\Theta$-$HS$ frame bounds $\alpha_1 = 1$ and $\beta_1 = 2$. Note that $\Theta^*$ is a bounded below operator with lower bound $m_o = 1$. Clearly, $\{T_i+T_i\}_{i\in\mathbb{Z}}$ is a $\Theta$-$HS$ frame $\mathcal{C}_2$ with respect to $\ell^2(\mathbb{Z})$, but condition (4.1) gives $2 = m_o^2(\alpha_1 + \alpha_2) > 2\sqrt{\beta_1\beta_2} = 4$, which is absurd.

The following example illustrates Theorem 4.1.

*Example* 4.3. Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis for the Hilbert space $\ell^2(\mathbb{N})$. For any $i \in \mathbb{N}$, define $R_i$, $T_i : \ell^2(\mathbb{N}) \to \mathbb{C}$ as

$$R_ix = \langle x, e_i\rangle \quad \text{and} \quad T_ix = \begin{cases} \left\langle x, \frac{e_1}{2\sqrt{2}}\right\rangle, & i = 1, \\ \left\langle x, \frac{e_{i-1}}{2\sqrt{2}}\right\rangle, & i \geq 2, \ x \in \ell^2(\mathbb{N}). \end{cases}$$

Then $T_i, R_i \in \mathcal{B}(\ell^2(\mathbb{N}), \mathbb{C})$ for all $i \in \mathbb{N}$. Let $\Theta$ be the left shift operator acting on $\ell^2(\mathbb{N})$. Then, for every $x \in \ell^2(\mathbb{N})$, we have

$$\frac{1}{8}\|\Theta^*x\|^2 = \frac{1}{8}\|x\|^2 \leq \sum_{i\in\mathbb{N}} \|T_ix\|_2^2 = \frac{1}{8}|\langle x, e_1\rangle|^2 + \frac{1}{8}\sum_{i\geq 2} |\langle x, e_{i-1}\rangle|^2 \leq \frac{1}{4}\|x\|^2.$$

Therefore, $\{T_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$ with bounds $\alpha_1 = \frac{1}{8}$, $\beta_1 = \frac{1}{4}$. Similarly, $\{R_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$ with bounds $\alpha_2 = \beta_2 = 1$.

Since

$$\frac{9}{8} = m_o^2(\alpha_1 + \alpha_2) > 2\sqrt{\beta_1\beta_2} = 1,$$

by Theorem 4.1, the sum $\{T_i + R_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$.

Recall that the perturbation theory related to operators is one of the important branches in analysis, which has been studied extensively, see [19] for technical details. In the frame theory, it is important that frame conditions should be stable under perturbation. The following result gives sufficient conditions under which the sum of a given $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with perturbed family of operators in $\mathcal{C}_2$ constitutes a $\Theta$-$HS$ frame for the underlying Hilbert space $\mathcal{H}$ with respect to $\mathcal{K}$. Here, the family of operators in the $\mathcal{C}_2$ space is perturbed by a fixed element of the space $\mathcal{B}(\mathcal{H}, \mathcal{C}_2)$. The proof is based on a straightforward modification of the proof of Theorem 4.1.

**Theorem 4.4.** *For $\Theta \in \mathcal{B}(\mathcal{H})$, let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\alpha_1, \beta_1$. Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ and $\{C_i\}_{i\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$. If $\Theta^*$ is bounded below with a bound $m_o$ such that*

$$\alpha_1 m_o^2 > 2\sqrt{\beta_1}\|T\|\sqrt{\sum_{i\in\mathbb{I}} \|C_i\|_2^2},$$

*then the perturbed sum $\{T_i + TC_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with frame bounds $\left(\alpha_1 - \frac{2\sqrt{\beta_1}}{m_o^2}\|T\|\sqrt{\sum_{i\in\mathbb{I}}\|C_i\|_2^2}\right)$ and $2\left(\beta_1 + \|T\|^2\sum_{i\in\mathbb{I}}\|C_i\|_2^2\right)$.*

Next, we use the concept of orthogonality of $HS$ Bessel sequences for the sum of two $\Theta$-$HS$ frames to be a $\Theta$-$HS$ frame. The concept of orthogonal Bessel sequences in separable Hilbert spaces was introduced by Han and Larson in [14] and further used by Bhatt, Johnson, and Weber [4] to study a vector-valued discrete wavelet transform. The orthogonality of $HS$ Bessel sequences can be defined in the same way.

**Definition 4.5.** Two HS-Bessel sequences $\{T_i\}_{i\in\mathbb{I}}$ and $\{R_i\}_{i\in\mathbb{I}}$ are said to be orthogonal if for all $x \in \mathcal{H}$,

$$\mathfrak{V}_1\mathfrak{V}_2^* x = 0,$$

where $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are the pre-frame operators associated with $\{T_i\}_{i\in\mathbb{I}}$ and $\{R_i\}_{i\in\mathbb{I}}$.

We now show that the sum of two orthogonal $\Theta$-$HS$ frames is always a $\Theta$-$HS$ frame. In fact, we will prove a more general result which shows that the sum of a $\Theta$-$HS$ frame with an orthogonal $HS$ Bessel sequence gives a $\Theta$-$HS$ frame.

**Theorem 4.6.** *Let $\Theta \in \mathcal{B}(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Let $\{R_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be an $HS$ Bessel sequence orthogonal to $\{T_i\}_{i\in\mathbb{I}}$. Then $\{T_i + R_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

*Proof.* Suppose that $\alpha_1, \beta_1$ are the $\Theta$-$HS$ frame bounds of $\{T_i\}_{i\in\mathbb{I}}$ and $\beta_o$ is an $HS$ Bessel bound for $\{R_i\}_{i\in\mathbb{I}}$. For any $x \in \mathcal{H}$, we compute

$$\begin{aligned}\sum_{i\in\mathbb{I}}\|(T_i+R_i)x\|_2^2 &= \sum_{i\in\mathbb{I}}[(T_i+R_i)x,(T_i+R_i)x]_{\mathbf{tr}}\\ &= \sum_{i\in\mathbb{I}}[T_ix,T_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[T_ix,R_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[R_ix,T_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[R_ix,R_ix]_{\mathbf{tr}}\\ &= \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 + \left\langle x, \sum_{i\in\mathbb{I}} T_i^*R_ix\right\rangle + \left\langle \sum_{i\in\mathbb{I}} T_i^*R_ix, x\right\rangle + \sum_{i\in\mathbb{I}}\|R_ix\|_2^2\\ &= \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 + \sum_{i\in\mathbb{I}}\|R_ix\|_2^2 \geq \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 \geq \alpha_1\|\Theta^*x\|^2.\end{aligned}$$

Similarly, $\sum_{i\in\mathbb{I}}\|(T_i+R_i)x\|_2^2 \leq (\beta_1+\beta_o)\|x\|^2$, $x \in \mathcal{H}$. This concludes the proof. □

*Remark* 4.7. The condition of orthogonality in Theorem 4.6 is not a necessary condition. Indeed, consider $\Theta$-$HS$ frames $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$ given in Example 4.3. It can be easily verified that $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ are not orthogonal. Yet their sum $\{T_i+R_i\}_{i\in\mathbb{N}}$ is a $\Theta$-$HS$ frame for $\ell^2(\mathbb{N})$ with respect to $\mathbb{C}$.

Lastly, we will discuss the sum of a $\Theta$-$HS$ frame and its dual $\Theta$-$HS$ Bessel sequence. By Remark 3.15, it is clear that the sum of a $\Theta$-$HS$ frame with its dual $\Theta$-$HS$ Bessel sequence need not be a $\Theta$-$HS$ frame. In the following result, we show that this happens if $\Theta$ is chosen to be a positive operator.

**Theorem 4.8.** *Let $\Theta \in \mathcal{B}(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$ with dual $\Theta$-$HS$ Bessel sequence $\{A_i\}_{i\in\mathbb{I}}$. Then $\{T_i+A_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$, provided $\Theta$ is positive.*

*Proof.* Clearly, $\{T_i+A_i\}_{i\in\mathbb{I}}$ is an $HS$ Bessel sequence in $\mathcal{H}$. Suppose that $\alpha_o, \beta_o$ are $\Theta$-$HS$ frame bounds of $\{T_i\}_{i\in\mathbb{I}}$. Then, for any $x \in \mathcal{H}$, using (3.2), we compute

$$\begin{aligned}&\sum_{i\in\mathbb{I}}\|(T_i+A_i)x\|_2^2 = \sum_{i\in\mathbb{I}}[(T_i+A_i)x,(T_i+A_i)x]_{\mathbf{tr}}\\ &= \sum_{i\in\mathbb{I}}[T_ix,T_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[T_ix,A_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[A_ix,T_ix]_{\mathbf{tr}} + \sum_{i\in\mathbb{I}}[A_ix,A_ix]_{\mathbf{tr}}\\ &= \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 + \left\langle x, \sum_{i\in\mathbb{I}} T_i^*A_ix\right\rangle + \left\langle \sum_{i\in\mathbb{I}} T_i^*A_ix, x\right\rangle + \sum_{i\in\mathbb{I}}\|A_ix\|_2^2\\ &= \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 + \langle x,\Theta x\rangle + \langle \Theta x, x\rangle + \sum_{i\in\mathbb{I}}\|A_ix\|_2^2\\ &\geq \sum_{i\in\mathbb{I}}\|T_ix\|_2^2 + \sum_{i\in\mathbb{I}}\|A_ix\|_2^2 \geq \alpha_o\|\Theta^*x\|^2 + \frac{1}{\beta_o}\|\Theta x\|^2 = \left(\alpha_o + \frac{1}{\beta_o}\right)\|\Theta^*x\|^2.\end{aligned}$$

Hence, $\{T_i+A_i\}_{i\in\mathbb{I}}$ is a $\Theta$-$HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. The proof is complete. □

## 5. Hilbert–Schmidt Riesz bases

This section studies Hilbert–Schmidt Riesz bases in separable Hilbert spaces. We begin with the following definition.

**Definition 5.1.** [21] Let $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$. A countable sequence $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ is called a *Hilbert–Schmidt Riesz basis* (*HS Riesz basis*, in short) for $\mathcal{H}$ with respect to $\mathcal{K}$ if:

1. $\{T_i\}_{i\in\mathbb{I}}$ is $HS$-complete, that is, $\{x \in \mathcal{H} : T_i(x) = 0 \text{ for every } i \in \mathbb{I}\} = \{0\}$.
2. There exist positive scalars $L_o \leq U_o < \infty$ such that

$$L_o\Big(\sum_{i\in\mathbb{J}} \|A_i\|_2^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_{i\in\mathbb{J}} T_i^* A_i\Big\| \leq U_o\Big(\sum_{i\in\mathbb{J}} \|A_i\|_2^2\Big)^{\frac{1}{2}}, \tag{5.1}$$

where $\mathbb{J} \subseteq \mathbb{I}$ is any finite set and $\{A_i\}_{i\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$.

*Remark* 5.2. One can observe that $\{T_i\}_{i\in\mathbb{I}}$ is $HS$-complete if and only if $\overline{\operatorname{span}}\{T_i^*(\mathcal{C}_2)\}_{i\in\mathbb{I}} = \mathcal{H}$. Indeed, let $\{T_i\}_{i\in\mathbb{I}}$ be $HS$-complete. Then, for $x \in \mathcal{H}$ satisfying $x \perp \operatorname{span}\{T_i^*(\mathcal{C}_2)\}_{i\in\mathbb{I}}$, we get

$$\|T_i(x)\|_2^2 = [T_i(x), T_i(x)]_{\mathbf{tr}} = \langle x, T_i^* T_i(x)\rangle = 0 \text{ for all } i \in \mathbb{I}.$$

This gives $T_i(x) = 0$ for every $i \in \mathbb{I}$. Thus, by the $HS$-completeness of $\{T_i\}_{i\in\mathbb{I}}$, we have $x = 0$. Hence, $\overline{\operatorname{span}}\{T_i^*(\mathcal{C}_2)\}_{i\in\mathbb{I}} = \mathcal{H}$. On the other hand, let $\overline{\operatorname{span}}\{T_i^*(\mathcal{C}_2)\}_{i\in\mathbb{I}} = \mathcal{H}$. Let $x \in \mathcal{H}$ be such that $T_i(x) = 0$ for every $i \in \mathbb{I}$. Then, for each $T \in \mathcal{C}_2$, $\langle x, T_i^*(T)\rangle = [T_i(x), T]_{\mathbf{tr}} = 0$. This implies that $x$ is orthogonal to $\operatorname{span}\{T_i^*(\mathcal{C}_2)\}_{i\in\mathbb{I}}$ and therefore $x$ is orthogonal to $\mathcal{H}$. This gives $x = 0$. Hence, $\{T_i\}_{i\in\mathbb{I}}$ is $HS$-complete.

The following is an example of an $HS$ Riesz basis for the Hilbert space $\bigoplus \mathcal{C}_2$.

*Example* 5.3. Let $\mathcal{C}_2 \subset \mathcal{B}(\mathcal{H})$. For each $i \in \mathbb{I}$, define $T_i : \bigoplus \mathcal{C}_2 \to \mathcal{C}_2$ as

$$T_i\Big(\{A_j\}_{j\in\mathbb{I}}\Big) = \begin{cases} 3A_1, & i = 1, \\ A_i, & i \geq 2. \end{cases}$$

Then, for $T \in \mathcal{C}_2$ and $\{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$, we have

$$\begin{aligned}\big\langle T_1^*(T), \{A_j\}_{j\in I}\big\rangle_{\bigoplus \mathcal{C}_2} &= \big[T, T_1\big(\{A_j\}_{j\in\mathbb{I}}\big)\big]_{\mathbf{tr}} \\ &= [T, 3A_1]_{\mathbf{tr}} = \Big\langle (3T, 0, \cdots, 0), \{A_j\}_{j\in\mathbb{I}}\Big\rangle_{\bigoplus \mathcal{C}_2};\end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned}\big\langle T_i^*(T), \{A_i\}_{j\in\mathbb{I}}\big\rangle_{\bigoplus \mathcal{C}_2} &= \big[T, T_i\big(\{A_j\}_{j\in\mathbb{I}}\big)\big]_{\mathbf{tr}} = [T, A_i]_{\mathbf{tr}} \\ &= \Big\langle (0, \ldots, 0, \underbrace{T}_{i^{\text{th}} \text{ place}}, 0, \ldots, 0), \{A_j\}_{j\in\mathbb{I}}\Big\rangle_{\bigoplus \mathcal{C}_2}.\end{aligned}$$

That is, $T_i^*: \mathcal{C}_2 \to \bigoplus \mathcal{C}_2$ is given by

$$T_i^*(T) = \begin{cases} (3T, 0, \cdots, 0), & i = 1, \\ (0, \ldots, 0, \underbrace{T}_{i^{\text{th}} \text{ place}}, 0, \ldots, 0), & i \geq 2. \end{cases}$$

Let $\{A_j\}_{j\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$ be such that $T_i\Big(\{A_j\}_{j\in\mathbb{I}}\Big) = 0$ for every $i \in \mathbb{I}$. Then $A_j = 0$ for every $j \in \mathbb{I}$, and hence $\{T_i\}_{i\in\mathbb{I}}$ is $HS$-complete. For a finite set $\mathbb{J} \subseteq \mathbb{I}$ and $\{A_i\}_{i\in\mathbb{I}} \in \bigoplus \mathcal{C}_2$, we have

$$\sum_{i\in\mathbb{J}} \|A_i\|_2^2 \leq \Bigg\| \sum_{i\in\mathbb{J}} T_i^* A_i \Bigg\|_{\bigoplus \mathcal{C}_2}^2 \leq 9 \sum_{i\in\mathbb{J}} \|A_i\|_2^2.$$

Therefore, $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ Riesz basis for $\bigoplus \mathcal{C}_2$ with respect to $\mathcal{H}$.

It is well-known that starting from an orthonormal basis for a separable Hilbert space $\mathcal{H}$, all orthonormal bases can be characterized in terms of unitary operators on $\mathcal{H}$, see [7]. In the direction of $HS$ orthonormal bases, let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathcal{K}$ and $U \in \mathcal{B}(\mathcal{H})$ be unitary. Then, for each $x \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} \|T_i U^* x\|_2^2 = \|U^*(x)\|^2 = \|x\|^2 \tag{5.2}$$

and

$$\big\langle (T_i U^*)^* T, (T_j U^*)^* S \big\rangle = \big\langle U T_i^* T, U T_j^* S \big\rangle = \big\langle T_i^* T, T_j^* S \big\rangle = \delta_{i,j} [T, S]_{\mathbf{tr}}.$$

Hence, $\{T_i U^*\}_{i\in\mathbb{I}}$ is an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathcal{K}$. In a similar way, we can show that if $\{T_i U_o^*\}_{i\in\mathbb{I}}$ is an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathcal{K}$ for some $U_o \in \mathcal{B}(\mathcal{H})$, then $U_o$ is unitary. Thus, $HS$ orthonormal bases for $\mathcal{H}$ with respect to $\mathcal{K}$ are precisely of the form $\{T_i U^*\}_{i\in\mathbb{I}}$.

The condition on the operator $U \in \mathcal{B}(\mathcal{H})$ in (5.2) is much weaker in the case of $HS$ frames. In fact, we have the following result which gives a bigger class of $HS$ frames constructed from a known $HS$ frame. For ordinary Hilbert frames, this result can be found in [7, Theorem 5.5.5].

**Proposition 5.4.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$. If $U \in \mathcal{B}(\mathcal{H})$ is surjective, then $\{T_i U^*\}_{i\in\mathbb{I}}$ is an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

The following result gives a characterization of the $HS$ Riesz basis in terms of the $HS$ orthonormal basis. This is an adaption of [7, Theorem 3.6.6].

**Theorem 5.5.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H}, \mathcal{C}_2)$ be a sequence and let $\{\Xi_i\}_{i\in\mathbb{I}}$ be an $HS$ orthonormal basis for $\mathcal{H}$ with respect to $\mathcal{K}$. Then $\{T_i\}_{i\in\mathbb{I}}$ is an $HS$ Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$ if and only if there is a bijective operator $U \in \mathcal{B}(\mathcal{H})$ such that $\{T_i\}_{i\in\mathbb{I}} = \{\Xi_i U^*\}_{i\in\mathbb{I}}$.*

**Corollary 5.6.** *Let $\{T_i\}_{i\in\mathbb{I}} \subset \mathcal{B}(\mathcal{H},\mathcal{C}_2)$ be an HS Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$. Then there exists an HS Riesz basis $\{\mathcal{F}_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\mathcal{K}$ such that*

$$x = \sum_{i\in\mathbb{I}} T_i^*\mathcal{F}_i(x), \quad x \in \mathcal{H}. \tag{5.3}$$

*Proof.* Let $\{T_i\}_{i\in\mathbb{I}}$ be of the form $\{\Xi_i U^*\}_{i\in I}$ for a bijective operator $U \in \mathcal{B}(\mathcal{H})$ and an $HS$ orthonormal basis $\{\Xi_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\mathcal{K}$. For $x \in \mathcal{H}$, we have

$$x = UU^{-1}(x) = U\Big(\sum_{i\in\mathbb{I}} \Xi_i^*\Xi_i(U^{-1}(x))\Big) = \sum_{i\in\mathbb{I}} T_i^*\Xi_i U^{-1}(x).$$

Since $(U^{-1})^*$ is a bounded bijective operator on $\mathcal{H}$, the sequence $\{\mathcal{F}_i\}_{i\in\mathbb{I}} = \{\Xi_i U^{-1}\}_{i\in\mathbb{I}}$ is an $HS$ Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$ satisfying (5.3). □

*Remark* 5.7. The sequence $\{\mathcal{F}_i\}_{i\in\mathbb{I}}$ satisfying (5.3) is called the dual $HS$ Riesz basis of $\{T_i\}_{i\in\mathbb{I}}$. We have seen that the dual $HS$ Riesz basis of $\{T_i\}_{i\in\mathbb{I}} = \{\Xi_i U^*\}_{i\in\mathbb{I}}$ is $\{\mathcal{F}_i\}_{i\in\mathbb{I}} = \{\Xi_i U^{-1}\}_{i\in\mathbb{I}}$. Therefore, the dual $HS$ Riesz basis of $\{\mathcal{F}_i\}_{i\in\mathbb{I}}$ is given by

$$\Big\{\Xi_i\big((U^{-1})^*\big)^{-1}\Big\}_{i\in\mathbb{I}} = \{\Xi_i U^*\}_{i\in\mathbb{I}} = \{T_i\}_{i\in\mathbb{I}}.$$

Hence, for a pair of dual $HS$ Riesz bases $\{T_i\}_{i\in\mathbb{I}}$ and $\{\mathcal{F}_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$, we get

$$x = \sum_{i\in\mathbb{I}} T_i^*\mathcal{F}_i(x) = \sum_{i\in\mathbb{I}} \mathcal{F}_i^* T_i(x), \quad x \in \mathcal{H}.$$

*Remark* 5.8. It is clear by Theorem 5.5 that every $HS$ orthonormal basis for a Hilbert space $\mathcal{H}$ with respect to $\mathcal{K}$ is an $HS$ Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$. But an $HS$ Riesz basis need not be an $HS$ orthonormal basis. For example, consider the sequence $\{T_i\}_{i\in\mathbb{I}}$ in $\bigoplus \mathcal{C}_2$ given in Example 5.3. Then $\{T_i\}_{i\in\mathbb{I}}$ is not an $HS$ orthonormal basis since it is not an $HS$ orthonormal system as

$$\big\langle T_1^*(T), T_1^*(S)\big\rangle_{\bigoplus \mathcal{C}_2} = \big\langle (3T, 0, \cdots, 0), (3S, 0, \cdots, 0)\big\rangle_{\bigoplus \mathcal{C}_2} = 9[T,S]_{\mathbf{tr}} \neq [T,S]_{\mathbf{tr}}.$$

*Remark* 5.9. It is shown in [21] that every $HS$ Riesz basis for a Hilbert space $\mathcal{H}$ with respect to $\mathcal{K}$ is an $HS$ Bessel sequence in $\mathcal{H}$. Moreover, it satisfies the lower $HS$ frame inequality. In fact, let $\{T_i\}_{i\in\mathbb{I}}$ be an $HS$ Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$. By Theorem 5.5, there exists a bijective operator $U \in \mathcal{B}(\mathcal{H})$ and an $HS$ orthonormal basis $\{\Xi_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\mathcal{K}$ such that $\{T_i\}_{i\in\mathbb{I}} = \{\Xi_i U^*\}_{i\in\mathbb{I}}$. Then, for any $x$ in $\mathcal{H}$, we have

$$\|x\|^2 = \|(U^*)^{-1}U^*(x)\|^2 \leq \|(U^*)^{-1}\|^2 \sum_{i\in\mathbb{I}} \|\Xi_i U^*(x)\|_2^2 = \|(U^*)^{-1}\|^2 \sum_{i\in\mathbb{I}} \|T_i(x)\|_2^2,$$

which entails

$$\frac{1}{\|(U^*)^{-1}\|^2}\|x\|^2 \leq \sum_{i\in\mathbb{I}} \|T_i(x)\|_2^2, \quad x \in \mathcal{H}.$$

Hence, every $HS$ Riesz basis for $\mathcal{H}$ with respect to $\mathcal{K}$ is an $HS$ frame for $\mathcal{H}$ with respect to $\mathcal{K}$.

**Acknowledgments.** We are deeply indebted to the referee for critically reading the paper and for his/her effective comments that led to a considerable improvement in the paper.

The research of Jyoti is supported by the Council of Scientific and Industrial Research (CSIR), India (Grant No.: 09/045(1374)/2015-EMR-I). Lalit Kumar Vashisht is supported by the Faculty Research Programme (FRP) Grant-IoE, University of Delhi (Grant No.: Ref. No./IoE/2021/12/FRP).

## References

[1] A. Aldroubi, *Portraits of frames*, Proc. Amer. Math. Soc. 123 (1995), No. 6, 1661–1668.

[2] A. Aldroubi, C. Cabrelli, A. F. Cakmak, U. Molter, and A. Petrosyan, *Iterative actions of normal operators*, J. Funct. Anal. **272** (2017), No. 3, 1121–1146.

[3] P. Balazs, *Hilbert–Schmidt operators and frames–classification, best approximation by multipliers and algorithms*, Int. J. Wavelets Multiresolut. Inf. Process. **6** (2008), No. 2, 315–330.

[4] G. Bhatt, B.D. Johnson, and E. Weber, *Orthogonal wavelet frames and vector-valued wavelet transforms*, Appl. Comput. Harmon. Anal. **23** (2007), 215–234.

[5] P.G. Casazza and G. Kutyniok, *Finite frames: Theory and Applications*, Birkhäuser, New York, 2013.

[6] J. Cahill and P.G. Casazza, *The Paulsen problem in operator theory*, Oper. Matrices **7** (2013), No. 1, 117–130.

[7] O. Christensen, *An Introduction to Frames and Riesz Bases*, $2^{nd}$ ed., Birkhäuser, Boston, 2016.

[8] Deepshikha, L. K. Vashisht, and G. Verma, *Generalized weaving frames for operators in Hilbert spaces*, Results Math. **72** (2017), No. 3, 1369–1391.

[9] Deepshikha and L. K. Vashisht, *Weaving $K$-frames in Hilbert spaces*, Results Math. **73** (2018), No. 2, 81.

[10] D.-X. Ding, *Generalized continuous frames constructed by using an iterated function system*, J. Geom. Phys. **61** (2011), 1045–1050.

[11] R.G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), No. 2, 413–415.

[12] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[13] L. Găvruta, *Frames for operators*, Appl. Compu. Harmon. Anal. **32** (2012), 139–144.

[14] D. Han and D.R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc., **147**, Amer. Math. Soc., Providence, RI, 2000.

[15] C. Heil, *A Basis Theory Primer*, (Expanded ed.), Birkhäuser, New York, 2011.

[16] Jyoti and L.K. Vashisht, *$\mathcal{K}$-Matrix-valued wave packet frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$*, Math. Phys. Anal. Geom. **21** (2018), No. 3, 21.

[17] Jyoti, L.K. Vashisht, and G. Verma, *Operators related to the reconstruction property in Banach spaces*, Results Math. **74** (2019), No. 3, 125.

[18] Jyoti and L.K. Vashisht, *On matrix-valued wave packet frames in* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, Anal. Math. Phys. **10** (2020), No. 4, 66.

[19] T. Kato, *Perturbation Theory for Linear Operators*, $2^{nd}$ ed., Springer-Verlag, New York, 1976.

[20] Y.Y. Koo and J.K. Lim, *Schatten-class operators and frames*, Quaest. Math. **34** (2011), 203–211.

[21] G. Sadeghi and, A. Arefijamaal, *Von Neumann–Schatten frames in separable Banach spaces*, Mediterr. J. Math. **9** (2012), 525–535.

[22] R. Schatten, *Norm Ideals of Completely Continious Operators*, Springer, Berlin-Heidelberg, 1960.

[23] B. Simon, *Trace Ideals and their Applications*, $2^{nd}$ ed., Mathematical Surveys and Monographs, 120. Amer. Math. Soc., Providence, RI, 2005.

[24] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), No. 1, 437–452.

[25] W. Sun, *Stability of g-frames*, J. Math. Anal. Appl. **326** (2007), No. 2, 858–868.

[26] L.K. Vashisht and Deepshikha, *Weaving properties of generalized continuous frames generated by an iterated function system*, J. Geom. Phys. **110** (2016), 282–295.

[27] X. Xiao, Y. Zhu, and L. Găvruta, *Some properties of K-frames in Hilbert spaces*, Results Math. **63** (2013), No. 3-4, 1243–1255.

[28] W. Zhang, *Dual and approximately dual Hilbert–Schmidt frames in Hilbert spaces*, Results Math. **73** (2018), 4.

Received December 6, 2021, revised April 27, 2022.

Jyoti,
*Department of Mathematics, University of Delhi, Delhi-110007, India,*
E-mail: jyoti.sheoran3@gmail.com

Lalit Kumar Vashisht,
*Department of Mathematics, University of Delhi, Delhi-110007, India,*
E-mail: lalitkvashisht@gmail.com

## Щодо фреймів Гільберта–Шмідта для операторів і базисів Ріса

Jyoti and Lalit Kumar Vashisht

Стійкий аналіз і реконструкцію векторів у замкнених підпросторах гільбертових просторів можна вивчати за допомогою фреймових умов за типом Гурвіти, які пов'язані з поняттям атомарних систем у сепарабельних гільбертових просторах. У цій роботі спочатку ми надаємо фреймові умови за типом Гурвіти для класу операторів Гільберта–Шмідта (коротко, клас $\mathcal{C}_2$), де обмежений лінійний оператор контролює нижню фреймову умову. Ми обговорюємо відображення, що зберігають фрейм для фреймів Гільберта–Шмідта для підпросторів сепарабельного гільбертового простору. Встановлюємо існування фреймів Гільберта–Шмідта для підпросторів класу Гільберта–Шмідта $\mathcal{C}_2$. Показано, що кожен сепарабельний гільбертовий простір допускає фрейм Гільберта–Шмідта відносно даного сепарабельного гільбертового простору. Отримано необхідні та достатні умови для фреймових умов за типом Гурвіти для сум фреймів Гільберта–Шмідта для підпросторів. Нарешті, ми обговорюємо базиси Гільберта–Шмідта Ріса в сепарабельних гільбертових просторах.

*Ключові слова:* фрейми, фрейми Гільберта–Шмідта, K-фрейми, збурення