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Estimates for Diameter and Width for the Isoperimetrix in Minkowski Geometry

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The following estimates are obtained for the diameter $D_B(I)$ and the width $\Delta_B(I)$ of isoperimetrix in Minkowski space M^n

$$\frac{4v_{n-1}}{nv_n} \leqslant \Delta_B(I) \leqslant D_B(I) \leqslant \frac{4v_{n-1}}{v_n},$$

where v_n is a volume of the unit ball in n-dimensional Euclidean space R^n. The first inequality turns into equality for a bicone, the last inequality turns into equality for a cube.

Key words: isoperimetrix, Minkowski geometry, convex body, diameter, width.

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Main Part

Let B be a convex compact central-symmetric body with non-empty interior in n-dimensional affine space A^n ($n \geqslant 2$) and point o be the center of symmetry for B. For the point $x \in A^n$, $x \neq o$, we consider a ray from o through x. Suppose x_0 is the point of intersection of the ray and boundary of B.

Put $g(\bar{x}) = \frac{\bar{x}}{\bar{x_0}}$, $g(\bar{o}) = 0$, where $\bar{x}$ is a position-vector of point x. Function $g(\bar{x})$ is called Minkowski distance function [1, p. 26].

By using a distance function, G. Minkowski defined the distance $\rho_B(x, y)$ between the points x and y by

$$\rho_B(x, y) = g(\bar{y} - \bar{x}).$$

G. Minkowski proved that $\rho_B(x, y)$ is a metric on A^n [2, p. 114].

Suppose A^n is affine space with the Minkowski metric ρ_B which is defined by using B . Such a space is called an n-dimensional Minkowski space M^n. The body B is called a normalizing one for M^n [2, p. 114].

The distance from o to the points of B and only for these points is less or equal to one, and the distance from o to the boundary of B is equal to one. The body B is also called a unit ball of M^n.

Consider a coordinate system in M^n choosing o as an origin. Using a positively definite symmetric bilinear form we determine a scalar product on M^n. Minkowski space with such a scalar product is called a space with the auxiliary Euclidean metric. In this paper by the convex body in M^n we mean a convex compact set in M^n. An n-dimensional volume $V_B(A)$ for a convex body A in M^n is defined by

$$V_B(A) = \frac{V(A)}{V(B)} v_n. \tag{1}$$

Here $V(A)$ and $V(B)$ are n-dimensional volumes of A and B with respect to the auxiliary Euclidean metric, v_n is the volume of a unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$ [2, p. 278].

It follows from (1) that $V_B(B) = v_n$. If the auxiliary Euclidean metric satisfies

$$V(B) = v_n, \tag{2}$$

then the Euclidean volume V is the volume V_B of Minkowski space, i.e., $V(A) = V_B(A)$ for each convex body A in M^n.

Suppose that M^n is a Minkowski space with the auxiliary Euclidean metric and the metric satisfies (2), $\bar{u} \in \Omega$ is a unit vector, Ω is a unit sphere with respect to this metric centered at o. Denote by $\overline{T(\bar{u})}$ a closed half-space in M^n that contains o and is bounded by the hyperplane $T(\bar{u})$. Here $T(\bar{u})$ is perpendicular to vector $\bar{u}$ and is at the Euclidean distance

$$\frac{v_{n-1}}{V_{n-1}(B \bigcap T_o(\bar{u}))}$$

from o. Here $T_o(\bar{u})$ is a hyperplane through o parallel to $T(\bar{u})$, V_{n-1} is the $(n-1)$-dimensional volume in the auxiliary Euclidean metric.

The body $I = \bigcap_{\bar{u} \in \Omega} \overline{T(\bar{u})}$ is said to be the isoperimetrix of M^n. The body I is a convex central-symmetric centered at o. The isoperimetrix depends only on the unit ball B of M^n and does not depend on the auxiliary Euclidean metric that satisfies (2) [2, p. 279].

It was shown in [2, p. 280] that for each $\bar{u} \in \Omega$ half-space $\overline{T(\bar{u})}$ is a support half-space for I in M^n. Hence the support function $h_I(\bar{u})$, $\bar{u} \in \Omega$ of the isoperimetrix in M^n is

$$h_I(\bar{u}) = \frac{v_{n-1}}{V_{n-1}(B \bigcap T_o(\bar{u}))}. \tag{3}$$

It was shown in [2, p. 282] that the area of surface $S_B(A)$ of the convex body A in M^n can be written as

$$S_B(A) = nV_1(A, I).$$

Here $V_1(A, I)$ denotes the first mixed volumes of A and I in the auxiliary Euclidean metric that satisfies (2). Therefore the solution of isoperimetric problem within a set of convex bodies in M^n is a body positively homothetic to the isoperimetrix I [2, p. 282]. In general, the body that is positively homothetic to the unit ball B does not give a solution to isoperimetric problem in M^n.

Let A be a convex body with nonempty interior in M^n. For each support hyperplane T_A for the body A we consider another support hyperplane T'_A, parallel to T_A. A set of points $Q(T_A) = \overline{T_A} \cap \overline{T'_A}$ is called the support layer corresponding to T_A. Here $\overline{T_A}$ denotes a closed support half-space for body A, bounded by T_A. The width of the support layer $Q(T_A)$ equals to $2q(Q(T_A), B)$. Here $q(Q(T_A), B)$ is the capacity coefficient, i.e., the maximum value of α such that the translation of the body αB may be contained in the layer $Q(T_A)$. If there is an auxiliary Euclidean metric on M^n we shall denote by $Q_A(\bar{u})$ the support layer $Q(T_A)$, where $\bar{u} \in \Omega$ is a unit vector orthogonal to the hyperplanes T_A and T'_A.

Theorem 1. *Suppose $\bar{u} \in \Omega$ is a unit vector in an auxiliary Euclidean metric on M^n, $Q_A(\bar{u})$ is a support layer for a body A that is bounded by the support hyperplanes orthogonal to $\bar{u}$. Then for the width of the support layer $Q_A(\bar{u})$ we have the following equality*

$$2q(Q(T_A), B) = 2\frac{h_A(\bar{u}) + h_A(-\bar{u})}{h_B(\bar{u}) + h_B(-\bar{u})}, \tag{4}$$

where $h_A(\bar{u})$ is a support function of the body A in this auxiliary metric.

R e m a r k 1. In the Euclidean case the numerator of (4) $h_A(\bar{u}) + h_A(\bar{u})$ is equal to the width of the body A in direction $\bar{u}$ [1, p. 62]. The maximum value of the numerator when $\bar{u} \in \Omega$ is said to be a diameter D of the body A in R^n, and the minimum value is to be the width Δ [1, p. 62]. It was shown that diameter of A in R^n coincides with the maximum of the distance between two points of A.

Since the numerator and the denominator of (4) are continuous functions of $\bar{u} \in \Omega$ and $h_A(\bar{u}) + h_A(\bar{-u}) > 0$, there exist the maximum and minimum of the right-hand (4). The maximum of the right-hand side of (4) on Ω is called the diameter and is denoted by $D_B(A)$, and the minimum of the right-hand side of (4) on Ω, denoted by $\Delta_B(A)$, is called the width of A in M^n.

It was shown that $D_B(A)$ coincides with the maximum of the distance between points of A in M^n [3, p. 220].

R e m a r k 2. In the case when A is a central symmetric convex body and o is a common symmetry center for A and B we have $h_A(\bar{u}) = h_A(-\bar{u})$, $h_B(\bar{u}) = h_B(-\bar{u})$ and (4) may be rewritten in the following form

$$2q(Q(T_A), B) = 2\frac{h_A(\bar{u})}{h_B(\bar{u})}.$$

Since B and I are central-symmetric convex bodies and o is their common center of symmetry, in any auxiliary Euclidean metric we have

$$2q(Q(T_I), B) = 2\frac{h_I(\bar{u})}{h_B(\bar{u})}. \tag{5}$$

Theorem 2. *Suppose $\bar{u} \in \Omega$ is a unit vector in an auxiliary Euclidean metric on M^n. Then for the width $2h_I(\bar{u})/h_B(\bar{u})$ of the support layer $Q_I(\bar{u})$ of the isoperimetrix I in M^n we have the following estimates*

$$\frac{4v_{n-1}}{nv_n} \leqslant 2\frac{h_I(\bar{u})}{h_B(\bar{u})} \leqslant \frac{4v_{n-1}}{v_n}. \tag{6}$$

These estimates are exact. For example, the left equality holds when B is a bicone in M^n and the vector $\bar{u}$ is orthogonal to the common base of the cones. The right equality holds when B is a cube in M^n and the vector $\bar{u}$ is orthogonal to the face of the cube.

Theorem 3. *For the width $\Delta_B(I)$ and the diameter $D_B(I)$ of the isoperimetrix I in M^n the following estimates hold*

$$\frac{4v_{n-1}}{nv_n} \leqslant \Delta_B(I) \leqslant D_B(I) \leqslant \frac{4v_{n-1}}{v_n}. \tag{7}$$

For example, a left-hand equality holds when B is a bicone in M^n and a right-hand equality holds when B is a cube in M^n.

R e m a r k 3. In [4, p. 14] the estimate $D_B(I) \leqslant d$ is proved, where d depends on n and does not depend on B. But this inequality is not exact.

P r o o f o f T h e o r e m 1. Let T_A and T'_A be different parallel support hyperplanes for the body A, T_B and T'_B be different support hyperplanes for the body B that are parallel to T_A and co-directional to the hyperplanes T_A and T'_A respectively. Note that T_A is co-directional to T_B if at least one of the support half-spaces $\overline{T}_A$ or $\overline{T}_B$ lies in another one. Let o be the center of symmetry for B and belong to a bisecting hyperplane of hyperplanes T_A and T'_A. This can be done by a parallel translation of B. Consider $b \in T_B \cap B$. Draw a ray l from

o through b. Denote $a = l \cap T_A$. Then the body tB, where $t = \overline{oa}/\overline{ob}$, has the hyperplanes T_A and T'_A as support hyperplanes. Therefore $t = q(Q(T_A), B)$ and the width of the support layer A which corresponds to T_A is equal to $2t$. Now, let l_1 be a ray from o that intersects T_B, $b_1 = l_1 \cap T_B$, $a_1 = l_1 \cap T_A$. Then the triangles obb_1 and oaa_1 are similar. Hence, we get $\overline{oa_1}/\overline{ob_1} = t$. Thus, for each ray l_1 satisfying $l_1 \cap T_B \neq \oslash$ the ratio $2\overline{oa_1}/\overline{ob_1}$ equals to the width of the support layer $Q(T_A)$.

Now, let there be an auxiliary Euclidean metric on M^n and $Q(T_A) = Q_A(\bar{u})$, where $\bar{u}$ is a unit normal vector to T_A. Draw a ray l_1 from o which is orthogonal to the hyperplane T_A and co-directional to $\bar{u}$. Since the support function $h_A(\bar{u})$ is equal to the distance from o to T_A, we get

$$t = \frac{|\overline{oa_1}|}{|\overline{ob_1}|} = \frac{h_A(\bar{u})}{h_B(\bar{u})}.$$

The point o is the center of symmetry of the body B, moreover o belongs to the bisecting hyperplane of the layer $Q_A(\bar{u})$. Hence $h_A(\bar{u}) = h_A(-\bar{u})$, $h_B(\bar{u}) = h_B(-\bar{u})$. Thus the width of the support layer $Q_A(\bar{u})$ corresponding to $\bar{u}$ is equal to

$$2q(Q_A(\bar{u}), B) = 2\frac{h_A(\bar{u})}{h_B(\bar{u})} = 2\frac{h_A(\bar{u}) + h_A(-\bar{u})}{h_B(\bar{u}) + h_B(-\bar{u})}.$$

If we choose a new origin in the auxiliary Euclidean metric, then the values $h_A(\bar{u}) + h_A(-\bar{u})$, $h_B(\bar{u}) + h_B(-\bar{u})$ will not change. This completes the proof of Theorem 1. ∎

To prove Theorem 2 we need two lemmas. In these lemmas we use arbitrary auxiliary Euclidean metric on M^n.

Lemma 1. *If T_0 is a hyperplane in M^n containing the origin, then T_0 divides the unit ball B into parts of equal volumes. Sections of B by hyperplanes that are parallel to T_0 and are at the same distance from T_0 have equal $(n-1)$-dimensional volumes.*

Proof of Lemma 1. It follows from central symmetry of B. ∎

Let turn from the base of auxiliary Euclidean metric to the orthonormal base this Euclidean metric. Suppose that coordinate system $ox_1x_2 \dots x_n$ created by the orthonormal base, the hyperplane T_0 coincide with hyperplane $x_n = 0$, segment $[-b; b]$, $b > 0$, is a projection of the body B on the axis ox_n.

Lemma 2. *$V_{n-1}(B \cap (x_n = a)) \leqslant V_{n-1}(B \cap (x_n = 0))$, $0 \leqslant a \leqslant b$. Here V_{n-1} denotes $(n-1)$-dimensional volume in an auxiliary Euclidean metric.*

P r o o f o f L e m m a 2. Let $\tilde{B}$ be a result of Schwarz symmetrization [2, p. 224] with respect to the axis ox_n. Then the projection of $\tilde{B}$ on ox_n is the segment [-b;b] and section $\tilde{B} \cap (x_n = c)$, $-b \leqslant c \leqslant b$ is the $(n-1)$-dimensional ball with the center in point c on ox_n. This section has the $(n-1)$-dimensional volume $V_{n-1}(\tilde{B} \cap (x_n = c)) = V_{n-1}(B \cap (x_n = c))$. It follows from Lemma 1 that the radii of the balls $\tilde{B} \cap (x_n = a)$ and $\tilde{B} \cap (x_n = -a)$ are equal. Since the body $\tilde{B}$ is convex [2, p. 227], the radius of the ball $\tilde{B} \cap (x_n = 0)$ is not less than the radius of the ball $\tilde{B} \cap (x_n = a)$. Thus, $V_{n-1}(B \cap (x_n = a)) = V_{n-1}(\tilde{B} \cap (x_n = a)) \leqslant V_{n-1}(\tilde{B} \cap (x_n = 0)) = V_{n-1}(B \cap (x_n = 0))$, where $0 \leqslant a \leqslant b$. ■

Later on we consider an auxiliary Euclidean metrics on M^n that satisfies condition (2).

P r o o f o f T h e o r e m 2. Without loss of generality, we can assume that the orthonormal system $ox_1x_2 \ldots x_n$ has vector $\bar{u} \in \Omega$ as a direction vector for the axis ox_n. Let $h_B(\bar{u}) = b$. It follows from (3) that

$$h_I(\bar{u}) = \frac{v_{n-1}}{V_{n-1}(B \cap (x_n = 0))}.$$

Then the width of the support layer $Q_I(\bar{u})$ for the isoperimetrix I can be written in the following form:

$$2\frac{h_I(\bar{u})}{h_B(\bar{u})} = \frac{2v_{n-1}}{bV_{n-1}(B \cap (x_n = 0))}. \tag{8}$$

Let us find an upper and lower bounds for the denominator of the right-hand side of (8) that depend only on n. To do this we consider bodies B_1, $\tilde{B}_1$, K, and Π. Here $B_1 = B \cap (x_n \geq 0)$, $\tilde{B}_1 = \tilde{B} \cap (x_n \geq 0)$, K is a ball cone with the vertex in the point b on ox_n and the base $\tilde{B}_1 \cap (x_n = 0)$. The base is a ball with the center in o. Denote by Π a right ball cylinder with the base $\tilde{B}_1 \cap (x_n = 0)$ that has the segment $[0; b]$ as a height.

The segment $[-b; b]$ is a projection of the body B on the axis ox_n, the segment $[0; b]$ is a projection of the bodies B_1, $\tilde{B}_1$, K, and Π on the same axis. Consequently, the hyperplane $x_n = b$ is a support hyperplane with a unit outward normal vector $\bar{u}$ for the bodies B, B_1, $\tilde{B}_1$, K, and Π. Since the origin o belongs to all these bodies, we have

$$h_B(\bar{u}) = h_{B_1}(\bar{u}) = h_{\tilde{B}_1}(\bar{u}) = h_K(\bar{u}) = h_\Pi(\bar{u}) = b.$$

From the definition of B_1 it follows that $V_{n-1}(B \cap (x_n = 0)) = V_{n-1}(B_1 \cap (x_n = 0))$. By the definition of Schwarz symmetrization with respect to a line we obtain $V_{n-1}(B \cap (x_n = 0)) = V_{n-1}(B_1 \cap (x_n = 0)) = V_{n-1}(\tilde{B}_1 \cap (x_n = 0))$. From the definition of K and Π it follows that $V_{n-1}(\tilde{B}_1 \cap (x_n = 0)) = V_{n-1}(K \cap (x_n = 0)) = V_{n-1}(\Pi \cap (x_n = 0))$.

By assumption, the auxiliary Euclidean metric on M^n satisfies the condition $V(B) = v_n$. Then Lemma 1 implies $V(B_1) = v_n/2$. Since $\tilde{B}_1$ is the result of Schwarz symmetrization B with respect to the axis ox_n, we get $V(\tilde{B}_1) = V(B_1) = v_n/2$. The volume of the cone K equals

$$V(K) = \frac{1}{n} bV_{n-1}(\tilde{B}_1 \cap (x_n = 0)) = \frac{1}{n} bV_{n-1}(B \cap (x_n = 0)).$$

The volume of the cylinder Π equals

$$V(\Pi) = bV_{n-1}(\tilde{B}_1 \cap (x_n = 0)) = bV_{n-1}(B \cap (x_n = 0)).$$

The body $\tilde{B}_1$ contains the vertex of the cone K. The base of the cone K is the ball $\tilde{B}_1 \cap (x_n = 0)$ and this ball belongs to $\tilde{B}_1$. Hence, we have $K \subset \tilde{B}_1$. It follows from Lem. 2 that $\tilde{B}_1 \subset \Pi$. Then from the chain of inclusions

$$K \subset \tilde{B}_1 \subset \Pi$$

we get

$$V(K) \leqslant V(\tilde{B}_1) \leqslant V(\Pi)$$

or

$$\frac{1}{n} bV_{n-1}(B \cap (x_n = 0)) \leqslant v_n/2 \leqslant bV_{n-1}(B \cap (x_n = 0)).$$

The inequalities obtained above lead to the following estimates, depending only on n, for the denominator (8).

$$v_n/2 \leqslant bV_{n-1}(B \cap (x_n = 0)) \leqslant nv_n/2.$$

By using these estimates in (8) we get the estimates (6) for $2h_I(\bar{u})/h_B(\bar{u})$ in the statement of Th. 2. ■

R e m a r k 4. Let B be a bicone in M^n, o the center of B, T_0 a hyperplane that passes through o and has a common base of the cones. Consider the half-space where a unit vector $\bar{u}$ is orthogonal to T_0 and directed inside. Denote by b the height of the cone B_1 that lies in this half-space. Since $v_n/2 = bV_{n-1}(B \cap T_0)/n$, we get $h_B(\bar{u}) = b$, $h_I(\bar{u}) = v_{n-1}/V_{n-1}(B \bigcap T_o)$ and

$$\frac{h_I(\bar{u})}{h_B(\bar{u})} = \frac{2v_{n-1}}{nv_n}.$$

This implies that the left-hand estimate for $2h_I(\bar{u})/h_B(\bar{u})$ in the statement of Th. 2 is exact.

Let B be a cube in M^n and o the center of B. Denote by T_0 a hyperplane through o parallel to the face of B with the outward normal vector $\bar{u}$, let a be the edge of B. Since $a^n = v_n$, we obtain

$$h_B(\bar{u}) = a/2, \quad h_I(\bar{u}) = \frac{v_{n-1}}{a^{n-1}}, \quad \frac{h_I(\bar{u})}{h_B(\bar{u})} = \frac{2v_{n-1}}{v_n}.$$

Thus the right-hand estimate in the statement of Th. 2 is exact. ■

P r o o f o f T h e o r e m 3. It follows from Remark 1 and (5) that

$$D_B(I) = \max_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})}, \Delta_B(I) = \min_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})}.$$

Then using the inequalities (6) we obtain

$$\frac{4v_{n-1}}{nv_n} \leqslant \min_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})} \leqslant \max_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})} \leqslant \frac{4v_{n-1}}{v_n}.$$

These estimates and estimates (7) are equivalent. This completes the proof. ■

R e m a r k 5. Let B be a bicone from Remark 4, $\bar{u}$ be a unit vector that orthogonal to the common base of the cones. Then

$$2\frac{h_I(\bar{u})}{h_B(\bar{u})} = \frac{4v_{n-1}}{nv_n}.$$

Now, by (7), this quantity equals $\Delta_B(I)$. Thus, the left-hand estimate in (7) for $\Delta_B(I)$ is exact.

Let B be a cube from Remark 4, $\bar{u}$ an outward unit normal vector to one of the hyperplanes containing the face of the cube. Then

$$2\frac{h_I(\bar{u})}{h_B(\bar{u})} = \frac{4v_{n-1}}{v_n}.$$

Now, by (7), this quantity equals to $D_B(I)$. Thus, the right-hand estimate in (7) for $D_B(I)$ is exact.

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