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On a Schrödinger–Kirchhoff Type Equation Involving the Fractional p-Laplacian without the Ambrosetti–Rabinowitz Condition

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In this paper, we consider the existence and multiplicity of many weak solutions for the following fractional Schrödinger–Kirchhoff type equation:

$$\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\times(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u = f(x,u)+h(x)\quad \text{in } \mathbb{R}^N,$$

where $N>sp$, $a,b>0$ are constants, λ is a parameter, $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0<s<1<p<\infty$, nonlinearity $f(x,u)$ and potential function $V(x)$ satisfy some suitable assumptions. Under those conditions, some new results are obtained for $\lambda>0$ large enough by applying the variation methods.

Key words: fractional p-Laplacian operator, fractional Sobolev space, Schrödinger–Kirchhoff type equation, Ambrosetti–Rabinowitz condition, variational methods

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1. Introduction

In this paper, we are concerned with a class of fractional p-Laplacian equations of Schrödinger–Kirchhoff type of the following form:

$$\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\times(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u = f(x,u)+h(x)\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N>sp$, $0<s<1<p<\infty$, $a,b>0$ are constants, λ is a parameter, $f\in C(\mathbb{R}^N,\mathbb{R})$, $V:\mathbb{R}^N\to\mathbb{R}$ is a potential function and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to normalization factors, can be defined as

$$(-\Delta)_p^s u(x)=\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\mathrm{d}y,$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, see [1–5]. In particular, $(-\Delta)_p^s$ becomes the fractional Laplacian $(-\Delta)^s$ as $p = 2$, and it is known that $(-\Delta)_p^s$ reduces to the standard p-Laplacian as $s \to 1^-$, see, for example, [6–8] and the references therein. Fractional p-Laplacian equations have gained importance because of their numerous applications in various fields such as phase transitions, turbulent flows, chaotic dynamics of classical conservative systems, finances, quantum mechanics, stratified materials, flame propagation, ultra-relativistic limits of quantum mechanics, minimal surfaces and water waves, as they are the typical outcome of stochastically stabilization of Lévy processes, see, for example, [9–14]. The body of literature on the fractional Sobolev space is quite large, we refer the reader to [15–17]. Recently, many authors have studied the existence of solutions for problems governed by the fractional p-Laplacian operator by using variational methods and critical point theory, see [18–25]. For example, in [22], the authors studied the following fractional p-Laplacian equations with perturbations:

$$(-\Delta)_p^s u + \lambda V(x)|u|^{p-2}u = f(x,u) - \mu g(x)|u|^{q-2}u, \quad u \in \mathbb{R}^N. \tag{1.2}$$

Basing on the variant fountain theorems, they obtained the existence of infinitely many solutions for equation (1.2), where $\mu,\ \lambda$ are two positive parameters $p \geq 2$, $N \geq 2$, $0 < s < 1$, the potentials $V,\ g\ : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. In particular, for $\lambda = 1,\ g(x) \equiv 0$, by using the Mountain pass theorem, the researchers in [26] established a nontrivial solution for equation (1.2) without a sign changing the potential $V(x)$ and by some appropriate assumptions on the nonlinearity $f(x,u)$. As for Kirchhoff problems, there are several works that deal with the existence and multiplicity of solutions. For example, in [24], the authors considered the following related problem:

$$\begin{aligned} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right) \times (-\Delta)_p^s u + V(x)|u|^{p-2}u& \\ = f(x,u) + g(x) \ \text{ in } \mathbb{R}^N&, \end{aligned} \tag{1.3}$$

where $0 < s < 1 < p < \infty$ and $ps < N$, the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the superlinear condition, that is, there exists $\theta \in [1, N/(N-sp))$ and $\mu > \theta p$ such that

$$\mu F(x,u) = \mu \int_0^u f(x,s)\mathrm{d}s \leq f(x,u)u, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.4}$$

and $V : \mathbb{R}^N \to \mathbb{R}^+$ is a potential function and $g : \mathbb{R}^N \to \mathbb{R}$ is a perturbation term. By using Ekeland's variational principle and the Mountain pass theorem, the authors obtained the existence and multiplicity of solutions for equation (1.3). More recently, for $M(t) = (a+bt)^{p-1}$ and $g \equiv 0$ in equation (1.3), the authors studied in [23] the following fractional p-Laplacian Schrödinger–Kirchhoff type equation with a parameter λ attached to the potential $V(x)$:

$$\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1} \times (-\Delta)_p^s u + \lambda V(x)|u|^{p-2}u$$

$$= f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $0 < s < 1$, $2 \leq p < \infty$, $a, b > 0$. With some appropriate assumptions on $V(x)$ and $f(x,u)$, in several main results they proved that equation (1.5) has infinitely many nontrivial solutions. Finally, when $\lambda \neq 1$ and when the nonlinearity $f(x,u)$ satisfies the Ambrosetti–Rabinowitz condition, that is, there exist $\mu > p^2$ such that

$$\mu F(x,u) \leq f(x,u)u, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.6}$$

and the potential $V(x)$ satisfies

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, where $V_0 > 0$ is a constant,

(V_2) there exists $r > 0$ such that $\lim\limits_{|y|\to\infty} \operatorname{meas}(\{x \in B_r(y) : V(x) \leq \omega\}) = 0$ for any $\omega > 0$, where $B_R(x)$ denotes the open ball of $\mathbb{R}^N$ centered at x and of radius $R > 0$,

the authors showed in [25] the existence and multiplicity of many nontrivial weak solutions when $\lambda > 0$ is sufficiently large. Recently, in [23], the authors treated problem (1.1) when the weight $V(x)$ changes the sign and $h(x) = 0$. Not using condition (1.6), they showed the existence and multiplicity of nontrivial solutions for λ large enough. The novelty of our work is in studying problem (1.1) when $h(x) \neq 0$ and eliminating the lemma used in [23] to verify the convergence of Palais–Smale sequence [23, Lemma 3.2]. Also, we added other main results that guarantee the existence of infinitely many solutions for problem (1.1) by using the Fountain theorem. For the potential $V(x)$, we impose the following hypotheses:

(V_1') $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$,

(V_2') there exists a constant $\omega > 0$ such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) \leq \omega\}) < \infty,$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Remark 1.1. The conditions like (V_1') and (V_2') were first introduced in [30], but $\inf_{x\in\mathbb{R}^N} V(x) > 0$ was required. In view of (V_1') and (V_2'), one can see that (V_1') is much weaker than (V_1) and (V_2), also the potential $V(x)$ is allowed to be sign-changing.

Inspired by [31,32], from condition (V_1'), there exists a constant $W_0 > 0$ such that $W(x) = V(x) + W_0 \geq 1$ for all $x \in \mathbb{R}^N$ and $g(x,u) = f(x,u) + W_0|u|^{p-2}u$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Hence, (1.1) is equivalent to the following equation:

$$\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1} \times (-\Delta)_p^s u + \lambda W(x)|u|^{p-2}u$$

$$= g(x,u) + h(x) \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

In this paper, for $\lambda > 0$ sufficiently large, we establish the existence and multiplicity of nontrivial weak solutions for (1.7) when the nonlinear term $g(x,u)$ does not satisfy condition (1.6). Observe that $W(x)$ satisfies the following conditions provided that conditions (V_1'), (V_2') hold:

(W_1) $W \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} W(x) \geq 1$ and

$$\operatorname{meas}(\{x \in \mathbb{R}^N : W(x) \leq d\}) < \infty, \quad d > 0;$$

(W_2) there exists $r > 0$ such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : W(x) \leq d\} \backslash (B_r \cap \{x \in \mathbb{R}^N : W(x) \leq d\})) = 0,$$

where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$.

After that we state the basic assumptions on the nonlinearity $g(x, u)$ in (1.7), which are weaker than those in the work [25]:

(H_1) $g \in C(\mathbb{R}^N \times \mathbb{R})$ and there exist constants $c_1,\ c_2 > 0$ and $q \in (p^2, p_s^*)$ such that

$$|g(x, u)| \leq c_1|u|^{p-1} + c_2|u|^{q-1}, \quad (x, u) \in \mathbb{R}^N \times \mathbb{R},$$

where p_s^* is the fractional Sobolev critical exponent defined by $p_s^* = \infty$ if $N \leq sp$ and $p_s^* = Np/(N - sp)$ if $N > sp$.

(H_2) $g(x, u) = o(|u|^{p-1})$ as $|u| \to 0$ uniformly in $x \in \mathbb{R}^N$.

(H_3) There exist $\mu > p^2$ and $r_0 > 0$ such that

$$\beta = \inf_{\substack{x\in\mathbb{R}^N \\ |u|=r_0}} G(x, u) > 0$$

and

$$\mu G(x, u) - g(x, u)u \leq C_0|u|^p, \quad x \in \mathbb{R}^N,\ |u| \geq r_0,$$

where $G(x, u) = \int_0^u g(x, t)\mathrm{d}t$ and $0 < C_0 < \dfrac{\beta(\mu - p)}{r_0^p}$.

(H_4) $\dfrac{G(x, u)}{|u|^{p^2}} \to \infty$ as $|u| \to \infty$ uniformly in $x \in \mathbb{R}^N$.

(H_5) There exist $r > 0$ and $C > 0$ such that

$$p^2G(x, u) - g(x, u)u \leq C|u|^p, \quad x \in \mathbb{R}^N,\ |u| \geq r.$$

(H_6) $g(x, -u) = -g(x, u)$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$.

(H_7) $h \in L^{p'}(\mathbb{R}^N)$, where $\dfrac{1}{p'} + \dfrac{1}{p} = 1$.

The main results of this paper are stated as follows.

Theorem 1.2. *Assume that (W_1), (W_2), (H_1)–(H_3), and (H_7) hold and suppose that $h \neq 0$. Then there exists a constant $\delta_0 > 0$ such that problem (1.7) has at least one nontrivial weak solution whenever $\lambda > 0$ is sufficiently large, provided that $\|h\|_{L^{p'}(\mathbb{R}^N)} \leq \delta_0$.*

Theorem 1.3. *Assume that (W_1), (W_2), (H_1), (H_2), (H_4), (H_5), and (H_7) hold and suppose that $h \neq 0$. Then there exists a constant $\delta_0 > 0$ such that problem (1.7) has at least one nontrivial weak solution whenever $\lambda > 0$ is sufficiently large, provided that $\|h\|_{L^{p'}(\mathbb{R}^N)} \leq \delta_0$.*

Theorem 1.4. *Assume that* (W_1), (W_2), (H_1), (H_3), (H_6), *and* (H_7) *hold and suppose that* $h \neq 0$. *Then there exists a constant* $\delta_1 > 0$ *such that problem* (1.7) *has infinitely many nontrivial weak solutions whenever* $\lambda > 0$ *is sufficiently large, provided that* $\|h\|_{L^{p'}(\mathbb{R}^N)} \leq \delta_1$.

Theorem 1.5. *Assume that* (W_1), (W_2), (H_1), (H_4), *and* (H_7) *hold and suppose that* $h \neq 0$. *Then there exists a constant* $\delta_1 > 0$ *such that problem* (1.7) *has infinitely many nontrivial weak solutions whenever* $\lambda > 0$ *is sufficiently large, provided that* $\|h\|_{L^{p'}(\mathbb{R}^N)} \leq \delta_1$.

Theorem 1.6. *Assume that* (W_1), (W_2), (H_1)–(H_4), (H_6), *and* (H_7) *hold. Then problem* (1.7) *has infinitely many nontrivial weak solutions whenever* $\lambda > 0$ *is sufficiently large.*

Theorem 1.7. *Assume that* (W_1), (W_2), (H_1), (H_2), *and* (H_4)–(H_7) *hold. Then problem* (1.7) *has infinitely many nontrivial weak solutions whenever* $\lambda > 0$ *is sufficiently large.*

Remark 1.8. Condition (1.6) plays an important role in checking the compactness condition ((PS)-condition) of the functional energy. Moreover, there are functions which are superlinear at infinity, but do not satisfy (1.6). For example, the superlinear function $f(x,u)$,

$$f(x,t) = \xi(x)|t|^{p-2}t\ln(1+|t|),$$

where $0 < \inf_{x\in\mathbb{R}^N} \xi(x) \leq \sup_{x\in\mathbb{R}^N} \xi(x) < \infty$. Also, if we consider the following function:

$$g(x,t) = \begin{cases} |t|^{\theta-2}t(\theta\ln|t|+1), & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{1.8}$$

By a straightforward computation, we obtain

$$G(x,t) = \begin{cases} |t|^{\theta}\ln|t|, & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{1.9}$$

Hence, it is easy to verify that (1.8) satisfies (H_1), (H_2), (H_4)–(H_6) but it does not satisfy (1.6) and (H_3) for $\theta = p^2$. In addition, we can also verify that (1.8) satisfies (H_1)–(H_6) and (1.6) for $\theta = \mu > p^2$.

2. Preliminaries

In this section, we introduce some definitions and basic properties of the fractional Sobolev space that will be used in proving the main results.

Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is given by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y < \infty \right\},$$

which can be equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

where $\|u\|_{L^p(\mathbb{R}^N)}$ is the norm for the usual Lebesgue space $L^p(\mathbb{R}^N)$ denoted by

$$\|u\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

We define the fractional Sobolev space with potential $W(x)$ and the parameter λ, which is large enough, by

$$E_\lambda = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda W(x)|u|^p \mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_{E_\lambda} = \left(\iint_{\mathbb{R}^{2N}} a^{p-1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda W(x)|u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

In particular, we have the following result

Lemma 2.1. *$(E_\lambda, \|\cdot\|_{E_\lambda})$ is a uniformly convex Banach space.*

Proof. The proof is similar to that of [24, Lemma 10] and hence omitted. □

Lemma 2.2. *Assume that (W_1) holds. Then the embeddings $E_\lambda \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ are continuous for $\nu \in [p, p_s^*]$. In particular, there exists a constant $C_\nu > 0$ such that*

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq C_\nu \|u\|_{E_\lambda} \quad \text{for all } u \in E_\lambda. \tag{2.1}$$

Moreover, for any $R > 0$ and $\nu \in [1, p_s^*)$, the embedding $E_\lambda \hookrightarrow L^\nu(B_R(0))$ is compact.

Proof. The proof is similar to that of [24, Lemma 1] and hence omitted. □

Lemma 2.3. *Assume that (W_1) and (W_2) hold. Let $\nu \in [p, p_s^*)$ be a fixed exponent and let $\{u_n\}_n$ be a bounded sequence in E_λ.*
Then there exist $u \in E_\lambda \cap L^\nu(\mathbb{R}^N)$ such that up to a subsequence

$$u_n \to u \quad \text{strongly in } L^\nu(\mathbb{R}^N) \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Proof. The proof is similar to that of [24, Theorem 2.1] and hence omitted. □

Definition 2.4. A function $u \in E_\lambda$ is said to be a weak solution of (1.7) if for any $\varphi \in E_\lambda$ we have

$$\begin{aligned}\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\\ \times\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))\mathrm{d}x\mathrm{d}y\\ +\int_{\mathbb{R}^N}\lambda W(x)|u|^{p-2}u\varphi\mathrm{d}x=\int_{\mathbb{R}^N}g(x,u)\varphi\mathrm{d}x+\int_{\mathbb{R}^N}h(x)\varphi\mathrm{d}x.\end{aligned}$$

Let $I_{\lambda,W}: E_\lambda \to \mathbb{R}$ be the energy functional associated with (1.7) defined by

$$\begin{aligned}I_{\lambda,W}(u)=\frac{1}{bp^2}\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p}\\ +\frac{1}{p}\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x-\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x-\int_{\mathbb{R}^N}h(x)u\mathrm{d}x.\end{aligned}\tag{2.3}$$

Under the assumptions (W_1) and (H_1), the functional $I_{\lambda,W}$ is of class $C^1(E_\lambda,\mathbb{R})$, and

$$\begin{aligned}\langle I'_{\lambda,W}(u),\varphi\rangle=\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\\ \times\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))\mathrm{d}x\mathrm{d}y\\ +\int_{\mathbb{R}^N}\lambda W(x)|u|^{p-2}u\varphi\mathrm{d}x-\int_{\mathbb{R}^N}g(x,u)\varphi\mathrm{d}x-\int_{\mathbb{R}^N}h(x)\varphi\mathrm{d}x.\end{aligned}$$

Then it is clear that the critical points of $I_{\lambda,W}$ are weak solutions of problem (1.7). In the next, we shall use the Mountain pass theorem, Symmetric mountain pass theorem and Fountain theorem to prove our main results.

Theorem 2.5 ([27], Mountain pass theorem)**.** *Let X be a real Banach space and let $I \in C^1(X,\mathbb{R})$ satisfying (PS)-condition. Suppose $I(0)=0$ and*

(i) *there are constants $\rho,\alpha>0$ such that $I_{|_{\partial B_\rho(0)}} \geq \alpha$;*

(ii) *there is an $e \in X\backslash\overline{B_\rho}(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I(\gamma(s)),$$

where

$$\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)=e\}.$$

Theorem 2.6 ([27], Symmetric mountain pass theorem)**.** *Let X be an infinite dimensional Banach space and let $I \in C^1(X,\mathbb{R})$ be even, satisfying (PS)-condition, and $I(0)=0$. If $X=Y\oplus Z$, where Y is finite dimensional and I satisfies the conditions:*

(i) *there exist constants* $\rho,\ \alpha > 0$ *such that* $I_{|\partial B_\rho(0)\cap Z} \geq \alpha$*;*

(ii) *for any finite dimensional subspace* $\tilde{X} \subset X$*, there is* $R = R(\tilde{X}) > 0$ *such that* $I(u) \leq 0$ *on* $\tilde{X}\backslash B_R$*,*

then I *possesses an unbounded sequence of critical values.*

Theorem 2.7 ([28], Fountain theorem)**.** *Let* X *be a real Banach space, let an even functional* $I \in C^1(X,\mathbb{R})$ *satisfy the Palais–Smale condition* ($(PS)_c$*-condition for short) for every* $c > 0$*, and let there be* $k_0 > 0$ *such that for every* $k \geq k_0$ *there exists* $\rho_k > r_k > 0$ *such that the following properties hold:*

(i) $a_k = \max\limits_{\substack{u\in Y_k\\ \|u\|_X=\rho_k}} I(u) \leq 0$*;*

(ii) $b_k = \inf\limits_{\substack{u\in Z_k\\ \|u\|_X=\gamma_k}} I(u) \to \infty$ *as* $k \to \infty$*.*

Then I *has a sequence of critical points* $\{u_k\}$ *such that* $I(u_k) \to \infty$*.*

Definition 2.8. Let X be a Banach space with its dual X^* and $I \in C^1(X,\mathbb{R})$. For any $\{u_n\} \subset X$, $\{u_n\}$ has a convergent subsequence if $I(u_n)$ is bounded or $I(u_n) \to c$, $c \in \mathbb{R}$ and $I'(u_n) \to 0$ as $n \to \infty$. Then we say that $I(u)$ satisfies the Palais–Smale condition or the Palais–Smale condition at the level c ((PS)-condition or $(PS)_c$-condition for short).

3. Proof of main results

Lemma 3.1. *Assume that* (W_1)*,* (H_1)*, and* (H_3) *hold. Then* $I_{\lambda,W}$ *satisfies the* $(PS)_c$*-condition for large* $\lambda > 0$*.*

Proof. Let $\{u_n\} \subset E_\lambda$ such that

$$I_{\lambda,W}(u_n) \to c \quad\text{and}\quad I'_{\lambda,W}(u_n) \to 0 \quad\text{as}\quad n \to \infty.$$

We first prove that $\{u_n\}$ is a bounded sequence in E_λ for large $\lambda > 0$. We prove this by contrary arguments. Suppose, by contradiction, that $\{u_n\}$ is unbounded in E_λ. We may assume that $\|u_n\|_{E_\lambda} \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|_{E_\lambda}$, then $\{v_n\}$ is bounded in E_λ and $\|v_n\|_{E_\lambda} = 1$, also $\|v_n\|_{L^\nu(\mathbb{R}^N)} \leq C_\nu\|v_n\|_{E_\lambda} = C_\nu$ for all $\nu \in [p, p_s^*]$. For any $(x,z) \in \mathbb{R}^N \times \mathbb{R}$, set

$$k(t) = G(x, t^{-1}z)t^\mu, \quad t \in [1,\infty).$$

For $|z| > r_0$ and $t \in [1, r_0^{-1}z]$, by (H_3), we can get

$$\begin{aligned} k'(t) &= g(x,t^{-1}z)(-\frac{z}{t^2})t^\mu + \mu t^{\mu-1}G(x,t^{-1}z) \\ &= t^{\mu-1}\left[\mu G(x,t^{-1}z) - t^{-1}zg(x,t^{-1}z)\right] \leq C_0 t^{\mu-1-p}|z|^p. \end{aligned}$$

Therefore,

$$k(r_0^{-1}z) - k(1) = \int_1^{r_0^{-1}z} k'(s)\mathrm{d}s$$

$$\leq \int_1^{r_0^{-1}z} C_0 s^{\mu-1-p}|z|^p \mathrm{d}x = \frac{C_0|z|^\mu}{(\mu-p)r_0^{\mu-p}} - \frac{C_0|z|^p}{\mu-p}.$$

Consequently, we have

$$G(x,z) = k(1) \geq k(r_0^{-1}z) - \frac{C_0|z|^\mu}{(\mu-p)r_0^{\mu-p}} \geq \left(\frac{\beta}{r_0^\mu} - \frac{C_0}{(\mu-p)r_0^{\mu-p}}\right)|z|^\mu.$$

Noting that $C_0 < \dfrac{\beta(\mu-p)}{r_0^p}$, we have $\dfrac{\beta}{r_0^\mu} - \dfrac{C_0}{(\mu-p)r_0^{\mu-p}} > 0$. Since $\mu > p^2$, there exists a constant $p^2 < \theta < p_s^*$, and hence

$$\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^\theta} = \infty. \tag{3.1}$$

In particularly, we have

$$\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^{p^2}} = \infty. \tag{3.2}$$

Due to (H_1), we have

$$|g(x,u)| \leq c_1|u|^{p-1} + c_2|u|^{q-1} \tag{3.3}$$

and

$$|G(x,u)| \leq \frac{c_1}{p}|u|^p + \frac{c_2}{q}|u|^q. \tag{3.4}$$

By using (3.1) and (3.4), we know that for any $M > 0$ there is a constant $C(M) > 0$ such that

$$G(x,u) \geq M|u|^\theta - C(M)|u|^p \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.5}$$

Moreover, we have

$$\begin{aligned}\frac{I_{\lambda,W}(u_n)}{\|u_n\|_{L^\theta(\mathbb{R}^N)}} = \frac{1}{\|u_n\|_{L^\theta(\mathbb{R}^N)}} \frac{1}{bp^2}&\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p \\ &+ \frac{1}{p}\frac{1}{\|u_n\|_{L^\theta(\mathbb{R}^N)}}\int_{\mathbb{R}^N} \lambda W(x)|u_n|^p \mathrm{d}x \\ &- \int_{\mathbb{R}^N} \frac{G(x,u_n)}{\|u_n\|_{L^\theta(\mathbb{R}^N)}}\mathrm{d}x - \int_{\mathbb{R}^N} \frac{h(x)u_n}{\|u_n\|_{L^\theta(\mathbb{R}^N)}}\mathrm{d}x.\end{aligned}$$

Since $\theta > p^2$, we can deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x,u_n)}{\|u_n\|_{L^\theta(\mathbb{R}^N)}}\mathrm{d}x = 0.$$

Furthermore, as $\|v_n\|_{E_\lambda} = 1$, passing to a subsequence, there exists $v \in E_\lambda$ such that $v_n \rightharpoonup v$ in E_λ, $v_n \to v$ in $L^\nu(\mathbb{R}^N)$ for $p \leq \nu < p_s^*$ and $v_n(x) \to v(x)$ a.e. in

$\mathbb{R}^N$. Set $\Omega = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. If meas$(\Omega) > 0$, then $\int_\Omega |v|^\theta \mathrm{d}x > 0$. Thus, from (3.5), we have

$$\int_{\mathbb{R}^N} \frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^\theta}\mathrm{d}x \geq M\|v_n\|_{L^\theta(\mathbb{R}^N)}^\theta - C(M)\frac{\|v_n\|_{L^p(\mathbb{R}^N)}^p}{\|u_n\|_{E_\lambda}^{\theta-p}}.$$

Hence,

$$\begin{aligned}0 &= \liminf_{n\to\infty}\left(\int_{\mathbb{R}^N} \frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^\theta}\mathrm{d}x + C(M)\frac{\|v_n\|_{L^p(\mathbb{R}^N)}^p}{\|u_n\|_{E_\lambda}^{\theta-p}}\right)\\ &\geq \liminf_{n\to\infty} M\|v_n\|_{L^\theta(\mathbb{R}^N)}^\theta \geq M\int_\Omega |v|^\theta \mathrm{d}x > 0.\end{aligned}$$

This is also a contradiction. Thus, meas$(\Omega) = 0$, and as a result $v(x) = 0$ a.e. in $\mathbb{R}^N$. Therefore, by (W_1), we have

$$\begin{aligned}\|v_n\|_{L^p(\mathbb{R}^N)}^p &= \int_{W(x)\geq d} |v_n|^p \mathrm{d}x + \int_{W(x)<d} |v_n|^p \mathrm{d}x\\ &\leq \frac{1}{\lambda d}\|v_n\|_{E_\lambda}^p + o(1) \leq \frac{2}{\lambda d},\end{aligned}$$

for large n. From (H_1) and (H_3), we know that there is a constant $c > 0$ such that

$$\mu G(x,u) - ug(x,u) \leq c|u|^p \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Consequently, we have

$$\begin{aligned}0 \leftarrow &\frac{1}{\|u_n\|_{E_\lambda}^p}\left[\mu I_{\lambda,W}(u_n) - \langle I'_{\lambda,W}(u_n), u_n\rangle\right]\\
= &\frac{1}{\|u_n\|_{E_\lambda}^p}\Bigg[\frac{\mu}{bp^2}\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p + \frac{\mu}{p}\int_{\mathbb{R}^N} \lambda W(x)|u_n|^p \mathrm{d}x\\
&- \mu\int_{\mathbb{R}^N} G(x,u_n)\mathrm{d}x - \mu\int_{\mathbb{R}^N} h(x)u_n \mathrm{d}x\\
&- \left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&- \int_{\mathbb{R}^N} \lambda W(x)|u_n|^p\mathrm{d}x + \int_{\mathbb{R}^N} u_n g(x,u_n)\mathrm{d}x + \int_{\mathbb{R}^N} h(x)u_n\mathrm{d}x\Bigg]\\
= &\frac{1}{\|u_n\|_{E_\lambda}^p}\Bigg[\frac{a\mu}{bp^2}\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\\
&+ \frac{\mu-p^2}{p^2}\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&+ \frac{\mu-p}{p}\int_{\mathbb{R}^N} \lambda W(x)|u_n|^p\mathrm{d}x + \int_{\mathbb{R}^N} (g(x,u_n)u_n - \mu G(x,u_n))\mathrm{d}x\\
&- (\mu-1)\int_{\mathbb{R}^N} h(x)u_n\mathrm{d}x\Bigg]\end{aligned}$$

$$\geq \frac{1}{\|u_n\|_{E_\lambda}^p}\left[\frac{\mu-p^2}{p^2}a^{p-1}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right.$$
$$\left.+\frac{\mu-p}{p}\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x - c\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x - (\mu-1)C_p\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\|u_n\|_{E_\lambda}\right]$$
$$\geq \frac{\mu-p^2}{p^2} - c\int_{\mathbb{R}^N}|v_n|^p\mathrm{d}x - (\mu-1)C_p\frac{\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}}{\|u_n\|_{E_\lambda}^{p-1}} \to \frac{\mu-p^2}{p^2} - \frac{c}{\lambda d},$$

as $n\to\infty$. Taking $\lambda>0$ to be so large that the term $\dfrac{\mu-p^2}{p^2}-\dfrac{c}{\lambda d}$ is positive, we get a contradiction. Hence $\{u_n\}$ is bounded in E_λ for large λ. Going, if necessary to a subsequence, we may assume that

$$\begin{aligned}
&u_n \rightharpoonup u, && \text{weakly in } E_\lambda,\\
&u_n \to u, && \text{strongly a.e. in } \mathbb{R}^N,\\
&u_n \to u, && \text{strongly a.e. in } L^\nu(\mathbb{R}^N),\ p\leq\nu<p_s^*,\\
&|u_n|\leq l, && \text{a.e in } \mathbb{R}^N \text{ and for some } l\in L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N),
\end{aligned}\tag{3.6}$$

and

$$\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \to \varrho_1\geq 0 \quad\text{and}\quad \int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\to\varrho_2\geq 0.$$

We will prove the following equalities:

$$\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = \varrho_1 \quad\text{and}\quad \int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x = \varrho_2.$$

Let $\varphi\in E_\lambda$ be fixed and denote by Υ_φ the linear functional on E_λ defined by

$$\Upsilon_\varphi(v) = \iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}(v(x)-v(y))\mathrm{d}x\mathrm{d}y. \tag{3.7}$$

We set

$$[\varphi]_{s,p}^p = \iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y, \tag{3.8}$$

for all $v\in E_\lambda$. By Hölder's inequality and the definition of Υ_φ, we have

$$\langle I'_{\lambda,W}(u_n) - I'_{\lambda,W}(u), u_n-u\rangle = (a+b[u_n]_{s,p}^p)^{p-1}\Upsilon_{u_n}(u_n-u)$$
$$-(a+b[u]_{s,p}^p)^{p-1}\Upsilon_u(u_n-u) + \int_{\mathbb{R}^N}\lambda W(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x$$
$$\geq (a+b[u_n]_{s,p}^p)^{p-1}[u_n]_{s,p}^p - (a+b[u_n]_{s,p}^p)^{p-1}\left([u_n]_{s,p}^p\right)^{\frac{p-1}{p}}\left([u]_{s,p}^p\right)^{\frac{1}{p}}$$
$$+(a+b[u]_{s,p}^p)^{p-1}[u]_{s,p}^p - (a+b[u]_{s,p}^p)^{p-1}\left([u]_{s,p}^p\right)^{\frac{p-1}{p}}\left([u_n]_{s,p}^p\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x-\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
&+\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x-\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
&-\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x\\
&=(a+b[u_n]_{s,p}^p)^{p-1}\left([u_n]_{s,p}^p\right)^{\frac{p-1}{p}}\left[\left([u_n]_{s,p}^p\right)^{\frac{1}{p}}-\left([u]_{s,p}^p\right)^{\frac{1}{p}}\right]\\
&+(a+b[u]_{s,p}^p)^{p-1}\left([u]_{s,p}^p\right)^{\frac{p-1}{p}}\left[\left([u]_{s,p}^p\right)^{\frac{1}{p}}-\left([u_n]_{s,p}^p\right)^{\frac{1}{p}}\right]\\
&+\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}}-\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\\
&+\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}-\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\\
&-\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x\\
&=\left[\left([u_n]_{s,p}^p\right)^{\frac{1}{p}}-\left([u]_{s,p}^p\right)^{\frac{1}{p}}\right]\left[(a+b[u_n]_{s,p}^p)^{p-1}\left([u_n]_{s,p}^p\right)^{\frac{p-1}{p}}\right.\\
&\left.-(a+b[u]_{s,p}^p)^{p-1}\left([u]_{s,p}^p\right)^{\frac{p-1}{p}}\right]\\
&+\left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}}-\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}-\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right]\\
&-\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

Since $u_n \rightharpoonup u$ in E_λ and $I'_{\lambda,W}(u_n)\to 0$ as $n\to\infty$ in $(E_\lambda)^*$, one has

$$\langle I'_{\lambda,W}(u_n)-I'_{\lambda,W}(u),u_n-u\rangle\to 0\quad\text{as } n\to\infty.$$

Thus, by (H_1) and Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x\\
&\quad\le\int_{\mathbb{R}^N}|c_1(|u_n|^{p-1}+|u|^{p-1})+c_2(|u_n|^{q-1}+|u|^{q-1})||u_n-u|\mathrm{d}x\\
&\quad\le c_1(\|u_n\|_{L^p(\mathbb{R}^N)}^{p-1}+\|u\|_{L^p(\mathbb{R}^N)}^{p-1})\|u_n-u\|_{L^p(\mathbb{R}^N)}\\
&\qquad+c_2(\|u_n\|_{L^q(\mathbb{R}^N)}^{q-1}+\|u\|_{L^q(\mathbb{R}^N)}^{q-1})\|u_n-u\|_{L^q(\mathbb{R}^N)}.
\end{aligned}
$$

This latter implies by (3.6) that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x=0. \qquad (3.10)$$

By the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$ and Fatou's lemma, we have

$$[u]_{s,p}^p \le \liminf_{n\to+\infty}[u_n]_{s,p}^p = \varrho_1, \tag{3.11}$$

$$\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x \le \liminf_{n\to+\infty}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x = \varrho_2. \tag{3.12}$$

Notice that $f(s) = (a+bs)^{p-1}s^{\frac{p-1}{p}}$ is a non-decreasing function for $s \ge 0$. Then we can get

$$[(\varrho_1)^{\frac{1}{p}} - ([u]_{s,p}^p)^{\frac{1}{p}}][(a+b\varrho_1)^{p-1}(\varrho_1)^{\frac{p-1}{p}} - (a+b[u]_{s,p}^p)^{p-1}([u]_{s,p}^p)^{\frac{p-1}{p}}] \ge 0 \tag{3.13}$$

and

$$\left[(\varrho_2)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\left[(\varrho_2)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right] \ge 0. \tag{3.14}$$

Now, in view of $\langle I'_{\lambda,W}(u_n) - I'_{\lambda,W}(u), u_n - u\rangle \to 0$ as $n \to \infty$ and (3.9) – (3.11), we have

$$\begin{aligned}
0 \ge \liminf_{n\to\infty}\Bigg\{ &\left[([u_n]_{s,p}^p)^{\frac{1}{p}} - ([u]_{s,p}^p)^{\frac{1}{p}}\right][(a+b[u_n]_{s,p}^p)^{p-1}([u_n]_{s,p}^p)^{\frac{p-1}{p}} \\
&- (a+b[u]_{s,p}^p)^{p-1}([u]_{s,p}^p)^{\frac{p-1}{p}}] \\
&+ \left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right] \\
&\times \left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right] \\
&- \int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x - \int_{\mathbb{R}^N}h(x)(u_n-u)\mathrm{d}x\Bigg\} \\
\ge \lim_{n\to\infty}\Big\{&[([u_n]_{s,p}^p)^{\frac{1}{p}} - ([u]_{s,p}^p)^{\frac{1}{p}}][(a+b[u_n]_{s,p}^p)^{p-1}([u_n]_{s,p}^p)^{\frac{p-1}{p}} \\
&- (a+b[u]_{s,p}^p)^{p-1}([u]_{s,p}^p)^{\frac{p-1}{p}}]\Big\} \\
&+ \lim_{n\to\infty}\Bigg\{\left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right] \\
&\times \left[\left(\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right]\Bigg\} \\
&- \lim_{n\to+\infty}\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))(u_n-u)\mathrm{d}x \\
\ge &\left[(\varrho_1)^{\frac{1}{p}} - ([u]_{s,p}^p)^{\frac{1}{p}}\right]\left[(a+b\varrho_1)^{p-1}(\varrho_1)^{\frac{p-1}{p}} - (a+b[u]_{s,p}^p)^{p-1}([u]_{s,p}^p)^{\frac{p-1}{p}}\right] \\
&+ \left[(\varrho_2)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]
\end{aligned}$$

$$\times\left[(\varrho_2)^{\frac{p-1}{p}}-\left(\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right]. \tag{3.15}$$

Finally, by (3.13) − (3.15), it comes that

$$\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y=\varrho_1 \quad \text{and} \quad \int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x=\varrho_2.$$

Therefore, we get $\|u_n\|_{E_\lambda}\to\|u\|_{E_\lambda}$. Since E_λ is a reflexive Banach space (see Lemma 2.1), it is isomorphic to a locally uniformly convex space. So, the weak convergence and convergence in norm imply the strong convergence. This completes the proof. □

Proof of Theorem 1.2. It is clear that $I_{\lambda,W}(0)=0$ and $I_{\lambda,W}\in C^1(E_\lambda,\mathbb{R})$ satisfies the $(PS)_c$-condition (see Lemma 3.1). For any $0<\varepsilon<\frac{1}{C_p^p}$ (C_p appears in (2.1)), combining hypotheses (H_1) and (H_2), there is a constant $C(\varepsilon)>0$ such that

$$|G(x,u)|\leq\frac{\varepsilon}{p}|u|^p+\frac{C(\varepsilon)}{q}|u|^q.$$

Thus, for small $\rho>0$, we can get

$$\begin{aligned}
I_{\lambda,W}(u)&=\frac{1}{bp^2}\left(a+b[u]_{s,p}^p\right)^p+\frac{1}{p}\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x-\int_{\mathbb{R}^N}h(x)u\mathrm{d}x\\
&\geq\frac{a^{p-1}}{p}[u]_{s,p}^p+\frac{1}{p}\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x-\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x-\int_{\mathbb{R}^N}h(x)u\mathrm{d}x\\
&\geq\frac{1}{p}\|u\|_{E_\lambda}^p-\varepsilon\frac{C_p^p}{p}\|u\|_{E_\lambda}^p-C(\varepsilon)\frac{C_q^q}{q}\|u\|_{E_\lambda}^q-C_p\|h\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{E_\lambda}\\
&=\|u\|_{E_\lambda}\left[\left(\frac{1}{p}-\varepsilon\frac{C_p^p}{p}\right)\|u\|_{E_\lambda}^{p-1}-C(\varepsilon)\frac{C_q^q}{q}\|u\|_{E_\lambda}^{q-1}-C_p\|h\|_{L^{p'}(\mathbb{R}^N)}\right].
\end{aligned}$$

Take $\varepsilon=\frac{1}{2C_p^p}$ and set $\eta_1(t)=\frac{1}{2p}t^{p-1}-C\frac{C_q^q}{q}t^{q-1}$ for all $t\in\mathbb{R}_*^+$.
Taking into consideration that $q>p>1$, we can deduce that there exists $\rho_0>0$ such that

$$\max_{t\in\mathbb{R}_*^+}\eta_1(t)=\eta_1(\rho_0)>0.$$

Taking $\delta_0=\frac{\eta_1(\rho_0)}{2C_p}$, and for all $u\in\overline{B_{\rho_0}}$, where $B_{\rho_0}=\{u\in E_\lambda:\|u\|_{E_\lambda}<\rho_0\}$, we obtain that

$$I_{\lambda,W_{|\partial B_{\rho_0}}}\geq\alpha=\rho_0\frac{\eta_1(\rho_0)}{2}>0.$$

Next, let us consider

$$I_{\lambda,W}(tu)=\frac{1}{bp^2}\left(a+bt^p[u]_{s,p}^p\right)^p+\frac{1}{p}\int_{\mathbb{R}^N}t^p\lambda W(x)|u|^p\mathrm{d}x$$

$$-\int_{\mathbb{R}^N} G(x,tu)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)tu\mathrm{d}x \quad \text{for all } t \in \mathbb{R}.$$

Set $0 \neq u \in E_\lambda$. By (H_1) and (H_4), we know that for any $\kappa > \dfrac{b^{p-1}([u]_{s,p}^p)^p}{p^2 \int_{\mathbb{R}^N} |u|^{p^2}\mathrm{d}x}$ there is a constant $C_\kappa > 0$ such that

$$|G(x,u)| \geq \kappa |u|^{p^2} - C_\kappa |u|^p \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Then we have

$$\begin{aligned}
I_{\lambda,W}(tu) &= \frac{1}{bp^2}\left(a + bt^p[u]_{s,p}^p\right)^p + \frac{1}{p}\int_{\mathbb{R}^N} t^p \lambda W(x)|u|^p\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} G(x,tu)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)tu\mathrm{d}x \\
&\leq \frac{1}{bp^2}\left(a + bt^p[u]_{s,p}^p\right)^p + \frac{t^p}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x + C_\kappa t^p \int_{\mathbb{R}^N} |u|^p\mathrm{d}x \\
&\quad - \kappa t^{p^2}\int_{\mathbb{R}^N} |u|^{p^2}\mathrm{d}x - t\int_{\mathbb{R}^N} h(x)u\mathrm{d}x \to -\infty \quad \text{as } t \to \infty.
\end{aligned}$$

Hence, there is a point $e \in E_\lambda \backslash \overline{B_\rho}$ such that $I_{\lambda,W}(e) \leq 0$. By Theorem 2.5, $I_{\lambda,W}$ possesses a critical value $c \geq \alpha > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I_{\lambda,W}(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \ \gamma(1) = e\}.$$

Therefore, there is $u \in E_\lambda$ such that $I_{\lambda,W}(u) = c$ and $I'_{\lambda,W}(u) = 0$, i.e., problem (1.7) has a nontrivial weak solution in E_λ. □

Proof of Theorem 1.3. From the proof of Theorem 1.2, we know that there exist constants $\rho > 0$ and $\alpha > 0$ such that $I_{\lambda,W_{|\partial B_\rho}} \geq \alpha > 0$ and there is a point $e \in E_\lambda \backslash \overline{B_\rho}$ such that $I_{\lambda,W}(e) \leq 0$. Next, we prove that $I_{\lambda,W}$ satisfies the $(PS)_c$-condition for large λ. Let $\{u_n\} \subset E_\lambda$ such that

$$I_{\lambda,W}(u_n) \to c \quad \text{and} \quad I'_{\lambda,W}(u_n) \to 0 \quad \text{as } n \to \infty.$$

We need to prove that $\{u_n\}$ possesses a convergent subsequence. In the same way as in Lemma 3.1, it is sufficient to prove that $\{u_n\}$ is bounded in E_λ. If not, we may assume that $\|u_n\|_{E_\lambda} \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|_{E_\lambda}$, then $\|v_n\|_{E_\lambda} = 1$ and $\|v_n\|_{L^\nu(\mathbb{R}^N)} \leq C_\nu \|v_n\|_{E_\lambda} = C_\nu$ for $\nu \in [p, p_s^*]$. Since $\|v_n\|_{E_\lambda} = 1$, passing to a subsequence, there is $v \in E_\lambda$ such that $v_n \rightharpoonup v$ in E_λ, $v_n \to v$ in $L^\nu(\mathbb{R}^N)$ for $p \leq \nu < p_s^*$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Set $\Omega = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. If meas$(\Omega) > 0$, then $\int_\Omega |v|^\theta \mathrm{d}x > 0$. From (H_1) and (H_4), we know that for any $M > \dfrac{b^{p-1}}{p^2 a^{p(p-1)} \int_\Omega |v|^{p^2}\mathrm{d}x}$ there is a constant $C(M) > 0$ such that

$$G(x,u) \geq M|u|^{p^2} - C(M)|u|^{p^2}. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}\frac{I_{\lambda,W}(u_n)}{\|u_n\|_{E_\lambda}^{p^2}} &= \frac{1}{\|u_n\|_{E_\lambda}^{p^2}}\frac{1}{bp^2}\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p \\ &+ \frac{1}{p\|u_n\|_{E_\lambda}^{p^2}}\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x \\ &- \int_{\mathbb{R}^N}\frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^{p^2}}\mathrm{d}x - \int_{\mathbb{R}^N}h(x)\frac{u_n}{\|u_n\|_{E_\lambda}^{p^2}}\mathrm{d}x.\end{aligned}$$

Since $p^2 > p$, we deduce that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^{p^2}}\mathrm{d}x \le \frac{1}{p^2}\frac{b^{p-1}}{a^{p(p-1)}}.$$

Moreover, from (3.16), it follows that

$$\int_{\mathbb{R}^N}\frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^{p^2}}\mathrm{d}x \ge M\|v_n\|_{L^{p^2}(\mathbb{R}^N)}^{p^2} - C(M)\frac{\|v_n\|_{L^p(\mathbb{R}^N)}^p}{\|u_n\|_{E_\lambda}^{p^2-p}},$$

which, by the last two inequalities, means that

$$\frac{1}{p^2}\frac{b^{p-1}}{a^{p(p-1)}} \ge \liminf_{n\to\infty}\left(\int_{\mathbb{R}^N}\frac{G(x,u_n)}{\|u_n\|_{E_\lambda}^{p^2}}\mathrm{d}x + C(M)\frac{\|v_n\|_{L^p(\mathbb{R}^N)}^p}{\|u_n\|_{E_\lambda}^{p^2-p}}\right) \ge M\int_\Omega |v|^{p^2}\mathrm{d}x.$$

This is a contradiction. Hence, $\mathrm{meas}(\Omega) = 0$ and, as a result, $v(x) = 0$ a.e. in $\mathbb{R}^N$. Then, by (W_1), we have

$$\|v_n\|_{L^p(\mathbb{R}^N)}^p = \int_{W(x)\ge d}|v_n|^p\mathrm{d}x + \int_{W(x)<d}|v_n|^p\mathrm{d}x \le \frac{1}{\lambda d}\|v_n\|_{E_\lambda}^p + o(1) \le \frac{2}{\lambda d},$$

for large n. By combining (H_1) and (H_5), we can find a constant $C > 0$ such that

$$p^2G(x,u) - g(x,u)u \le C|u|^p \quad \text{for all } (x,u)\in\mathbb{R}^N\times\mathbb{R}.$$

Notice that $p \ge 2$. By a calculation, for large n, we get

$$\begin{aligned}0 &\leftarrow \frac{1}{\|u_n\|_{E_\lambda}^p}\left[p^2 I_{\lambda,W}(u_n) - \langle I'_{\lambda,W}(u_n), u_n\rangle\right] \\ &= \frac{1}{\|u_n\|_{E_\lambda}^p}\left[\frac{1}{b}\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p\right. \\ &+ p\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x - p^2\int_{\mathbb{R}^N}G(x,u_n)\mathrm{d}x - p^2\int_{\mathbb{R}^N}h(x)u_n\mathrm{d}x \\ &- \left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1} \\ &\times\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} u_n g(x,u_n)\mathrm{d}x+\int_{\mathbb{R}^N} h(x)u_n\mathrm{d}x\Bigg] \\
&=\frac{1}{\|u_n\|_{E_\lambda}^p}\left[\frac{a}{b}\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\right. \\
&+(p-1)\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x \\
&\left.+\int_{\mathbb{R}^N}(u_n g(x,u_n)-p^2G(x,u_n))\mathrm{d}x-(p^2-1)\int_{\mathbb{R}^N} h(x)u_n\mathrm{d}x\right] \\
&\geq \frac{1}{\|u_n\|_{E_\lambda}^p}\left[(p-1)a^{p-1}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right. \\
&\left.+(p-1)\int_{\mathbb{R}^N}\lambda W(x)|u_n|^p\mathrm{d}x+C\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x-(p^2-1)C_p\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\|u_n\|_{E_\lambda}\right] \\
&\geq (p-1)-C\int_{\mathbb{R}^N}|v_n|^p\mathrm{d}x-(p^2-1)C_p\frac{\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}}{\|u_n\|_{E_\lambda}^{p-1}}\to(p-1)-\frac{C}{\lambda d}\ \text{as } n\to\infty.
\end{aligned}
$$

Letting $\lambda > 0$ be so large that the term $p - 1 - \dfrac{C}{\lambda d}$ is positive, we get a contradiction. Hence, $\{u_n\}$ is bounded in E_λ for large λ. Therefore, $I_{\lambda,W}$ possesses a critical value $c \geq \alpha$ by Theorem 2.5, i.e., problem (1.7) has a nontrivial weak solution in E_λ. This completes the proof. □

Proof of Theorem 1.4. Let $\{e_j\}$ be a total orthonormal basis of $L^2(B_r)$ (B_r appears in (V_2)) and define $X_j = \mathbb{R}e_j$ for $j \in \mathbb{N}$,

$$Y_k = \oplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \oplus_{j=k+1}^{\infty} X_j \quad \text{for } k \in \mathbb{N}.$$

Set

$$E_\lambda(B_r) = \left\{ u \in W^{s,p}(B_r)\ :\ \int_{B_r} \lambda W(x)|u|^p \mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_{E_\lambda(B_r)} = \left(\iint_{B_r\times B_r} a^{p-1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{B_r}\lambda W(x)|u|^p\mathrm{d}x \right)^{\frac{1}{p}}. \quad □$$

Lemma 3.2. *Assume that* (W_1) *holds. Then, for* $p < \theta < p_s^*$,

$$\beta_k = \sup_{\substack{u\in Z_k \\ \|u\|_{E_\lambda(B_r)}=1}} \|u\|_{L^\theta(B_r)} \to 0 \quad \textit{as} \quad k \to \infty.$$

Proof. From Lemma 2.2, $E_\lambda(B_r) \hookrightarrow L^s(B_r)$ is compact for $1 \leq s < p_s^*$. Indeed, it is clear that $\beta_{k+1} \leq \beta_k < \infty$, so $\beta_k \to \beta \geq 0$ as $k \to \infty$. For every $k \geq 1$, there exists $u_k \in Z_k$ such that $\|u\|_{E_\lambda(B_r)} = 1$ and $\|u\|_{L^s(B_r)} > \dfrac{\beta_k}{2}$. By the definition of Z_k, $u_k \rightharpoonup 0$ in $E_\lambda(B_r)$. Then this implies that $u_k \to 0$ in $L^s(B_r)$ and, as a result, $\beta = 0$. This completes the proof. □

In view of Lemma 3.2, we can choose an integer $k \geq 1$ such that

$$\int_{B_r} |u|^p \mathrm{d}x \leq \frac{1}{2c_1}\left(\iint_{B_r\times B_r} a^{p-1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{B_r} \lambda W(x)|u|^p\mathrm{d}x\right) \tag{3.17}$$

for all $u \in Z_k \cap E_\lambda(B_r)$, where c_1 appears in condition (H_1). Take

$$\psi(x) = \begin{cases} 1, & x > r, \\ 0, & x \leq r \end{cases}$$

and set

$$Y = \{(1-\psi)u : u \in E_\lambda, (1-\psi)u \in Y_k\} \tag{3.18}$$

and

$$Z = \{(1-\psi)u : u \in E_\lambda, (1-\psi)u \in Z_k\} + \{\psi v : v \in E_\lambda\}. \tag{3.19}$$

Hence, Y and Z are subspaces of E_λ, and $E_\lambda = Y \oplus Z$.

Lemma 3.3. *If the conditions (W_1), (W_2), and (H_1) hold, then there exist constants ρ, $\alpha > 0$ such that $I_{\lambda,W}|_{\partial B_\rho(0)\cap Z} \geq \alpha$ for large λ.*

Proof. By (W_2), (3.17) and (3.19), we have

$$\begin{aligned} \|u\|^p_{L^p(\mathbb{R}^N)} &= \int_{|x|<h} |u|^p\mathrm{d}x + \int_{|x|\geq h} |u|^p\mathrm{d}x \\ &\leq \frac{1}{2c_1}\|u\|^p_{E_\lambda(B_h)} + \frac{1}{\lambda d}\int_{\{x\in\mathbb{R}^N:\ W(x)\geq d\}} \lambda W(x)|u|^p\mathrm{d}x \\ &\leq \frac{1}{2c_1}\|u\|^p_{E_\lambda} + \frac{1}{\lambda d}\|u\|^p_{E_\lambda}, \end{aligned} \tag{3.20}$$

for all $u \in Z$. It follows from (2.1), (3.4) and (3.20) that

$$\begin{aligned} I_{\lambda,W}(u) &= \frac{1}{bp^2}\left(a + b[u]^p_{s,p}\right)^p + \frac{1}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x \\ &\qquad - \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x \\ &\geq \frac{a^{p-1}}{p}[u]^p_{s,p} + \frac{1}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x \\ &\geq \frac{1}{p}\|u\|^p_{E_\lambda} - \frac{c_1}{p}\|u\|^p_{L^p(\mathbb{R}^N)} - \frac{c_2}{q}\|u\|^q_{L^q(\mathbb{R}^N)} - C_p\|h\|_{L^{p'}(\mathbb{R}^\mathbb{N})}\|u\|_{E_\lambda} \\ &\geq \frac{1}{2p}\|u\|^p_{E_\lambda} - \frac{c_1}{d\lambda p}\|u\|^p_{E_\lambda} - \frac{c_2C_q^q}{q}\|u\|^q_{E_\lambda} - C_p\|h\|_{L^{p'}(\mathbb{R}^\mathbb{N})}\|u\|_{E_\lambda} \\ &= \|u\|_{E_\lambda}\left[\left(\frac{1}{2p} - \frac{c_1}{d\lambda p}\right)\|u\|^{p-1}_{E_\lambda} - \frac{c_2C_q^q}{q}\|u\|^{q-1}_{E_\lambda} - C_p\|h\|_{L^{p'}(\mathbb{R}^\mathbb{N})}\right]. \end{aligned}$$

For λ large enough such that $\lambda > \dfrac{2c_1}{d}$, let

$$\eta_2(t) = \left(\frac{1}{2p} - \frac{c_1}{d\lambda p}\right)t^{p-1} - \frac{c_2C_q^q}{q}t^{q-1} \quad \text{for all } t \in \mathbb{R}^+_*.$$

Since $q > p > 1$, we can conclude that there exists $\rho_1 > 0$ such that

$$\max_{t\in\mathbb{R}_*^+} \eta_2(t) = \eta_2(\rho_1) > 0.$$

Taking $\delta_1 = \dfrac{\eta_2(\rho_1)}{2C_p}$, we have $I_{\lambda,W}|_{\partial B_{\rho_1}(0)\cap Z} \geq \alpha = \rho_1 \dfrac{\eta_2(\rho_1)}{2} > 0$. □

Lemma 3.4. *Assume that* (H_1) *and* (H_4) *are satisfied. Then, for any finite dimensional subspace* $\tilde{E}_\lambda \subset E_\lambda$, *there is* $R = R(\tilde{E}_\lambda) > 0$ *such that*

$$I_{\lambda,W}(u) \leq 0, \quad u \in \widetilde{E}_\lambda, \ \|u\|_{E_\lambda} \geq R.$$

Proof. For any finite dimensional subspace $\tilde{E}_\lambda \subset E_\lambda$, by the equivalence of norms in the finite dimensional space, there is a constant $D > 0$ such that

$$\|u\|_{L^{p^2}(\mathbb{R}^N)} \geq D\|u\|_{E_\lambda}, \quad u \in \widetilde{E}_\lambda.$$

Basing on (H_1) and (H_4), for any $M > \dfrac{b^{p-1}}{a^{p(p-1)}p^2 D^{p^2}}$, we can find a constant $C(M) > 0$ such that

$$G(x,u) \geq M|u|^{p^2} - C(M)|u|^p \ \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, we have

$$\begin{aligned}
I_{\lambda,W}(u) &= \frac{1}{bp^2}\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p \mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x \\
&\leq \frac{1}{bp^2}\left(a + \frac{b}{a^{p-1}}\|u\|_{E_\lambda}^p\right)^p + \frac{1}{p}\|u\|_{E_\lambda}^p \\
&\quad + C(M)\|u\|_{L^p(\mathbb{R}^N)}^p - M\|u\|_{L^{p^2}(\mathbb{R}^N)}^{p^2} + \|h\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} \\
&\leq \frac{1}{bp^2}\left(a + \frac{b}{a^{p-1}}\|u\|_{E_\lambda}^p\right)^p + \left(\frac{1}{p} + C(M)C_p^p\right)\|u\|_{E_\lambda}^p \\
&\quad - MD^{p^2}\|u\|_{E_\lambda}^{p^2} + C_p\|h\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{E_\lambda},
\end{aligned}$$

for all $u \in \tilde{E}_\lambda$. It follows that there is a large $R > 0$ such that $I_{\lambda,W}(u) \leq 0$ on $\tilde{E}_\lambda \backslash B_R$. □

Let X be a reflexive and separable Banach space. Then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_j \mid j = 1,2,\dots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* \mid j = 1,2,\dots\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For the convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Let $B_k = \{u \in Y_k : \|u\|_X \leq \rho_k\}$, $N_k = \{u \in Z_k : \|u\|_X = \gamma_k\}$, where $\rho_k > \gamma_k > 0$.

Lemma 3.5. *Assume that $\nu \in [p, p_s^*)$ and let $\beta_\nu(k) = \sup_{u\in Z_k,\ \|u\|_{E_\lambda}=1} \|u\|_\nu$. Then $\beta_\nu(k) \to 0$ as $k \to \infty$.*

Proof. The proof of this lemma is similar to that of [29, Lemma 6] and we omit the details. □

To prove Theorem 1.6 and Theorem 1.7, we first prove the following two lemmas.

Lemma 3.6. *Assume that the assumptions (H_1) and (H_2) hold, then there exist constants r_k such that*

$$b_k = \inf_{\substack{u\in Z_k\\ \|u\|_{E_\lambda}=r_k}} I_{\lambda,W}(u) \to \infty \quad \text{as } k \to \infty. \tag{3.21}$$

Proof. Notice that from (H_1) and (H_2) it follows that there is a constant $C(\epsilon) > 0$ such that

$$|G(x,u)| \le \frac{\epsilon}{p}|u|^p + \frac{C(\epsilon)}{q}|u|^q. \tag{3.22}$$

Also notice that by Young's inequality we have

$$\begin{aligned}
a + b \iint_{\mathbb{R}^{2N}} &\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&= \left(a^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} + \left(\left(b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\right)^p \\
&= \frac{p-1}{p}\left(\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} \times a^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} \\
&\quad + \frac{1}{p}\left(\left(pb\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\right)^p \\
&\ge \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} a^{\frac{p-1}{p}}\left(pb\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \\
&\ge a^{\frac{p-1}{p}}\left(pb\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, we make the following conclusion:

$$\begin{aligned}
\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p& \\
\ge pba^{p-1}\iint_{\mathbb{R}^{2N}} &\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.23}$$

According to (3.22), (3.23) and Lemma 3.5, we obtain that

$$I_{\lambda,W}(u) = \frac{1}{bp^2}\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p + \frac{1}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} G(x,u(x))\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x \\
\geq\ & \frac{a^{p-1}}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \frac{1}{p}\int_{\mathbb{R}^N} \lambda W(x)|u|^p\mathrm{d}x \\
&-\int_{\mathbb{R}^N} G(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x \\
\geq\ & \frac{1}{p}\|u\|^p_{E_\lambda} - \frac{\epsilon}{p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \frac{C(\epsilon)}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x - \|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x \\
\geq\ & \frac{1}{p}\|u\|^p_{E_\lambda} - \left(\frac{\epsilon}{p} + \|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\right)\beta_p^p(k)\|u\|^p_{E_\lambda} - \frac{C(\epsilon)}{q}\beta_q^q(k)\|u\|^q_{E_\lambda}.
\end{aligned}
$$

Choose $r_k = \dfrac{1}{\beta_p(k)+\beta_q(k)}$. Then $r_k \to \infty$ as $k \to \infty$. For any $u \in Z_k$ with $\|u\|_{E_\lambda} = r_k$, we know

$$
\begin{aligned}
I_{\lambda,W}(u) &\geq \frac{1}{p}r_k^p - \left(\frac{\epsilon}{p} + \|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\right)\frac{\beta_p^p(k)}{|\beta_p(k)+\beta_q(k)|^p} - \frac{C(\epsilon)}{q}\frac{\beta_q^q(k)}{|\beta_p(k)+\beta_q(k)|^q} \\
&\geq \frac{1}{p}r_k^p - \left(\frac{\epsilon}{p} + \|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\right) - \frac{C(\epsilon)}{q} > 0.
\end{aligned}
$$

Therefore,

$$
b_k = \inf_{u\in Z_k,\ \|u\|_{E_\lambda}=r_k} I_{\lambda,W}(u) \to \infty \quad \text{as} \quad k\to\infty. \qquad \square
$$

Lemma 3.7. *Assume that the assumptions (H_1) and (H_4) hold, then there exist constants $\rho_k > 0$ such that*

$$
a_k = \max_{\substack{u\in Y_k \\ \|u\|_{E_\lambda}=\rho_k}} I_{\lambda,W}(u) \leq 0. \tag{3.24}
$$

Proof. By (H_1) and (H_4), there exist $\eta > \dfrac{2^{p-1}b^{p-1}}{C_k p^2}$, $C_\eta > 0$ such that

$$
|G(x,u)| \geq \eta|u|^{p^2} - C_\eta|u|^p \quad \text{for all } x\in\mathbb{R}^N \text{ and } u\in E_\lambda. \tag{3.25}
$$

Since all norms are equivalent on the finite dimensional Banach space Y_k, there exists a positive constant $C_k > 0$ such that $\|u\|_{L^{p^2}(\mathbb{R}^N)} \geq C_k\|u\|_{E_\lambda}$. Then, for $\|u\|_{E_\lambda} = \rho_k \geq 1$, from (3.25) and the inequality

$$
(x+y)^p \leq 2^{p-1}(x^p+y^p) \quad \text{for all} \ \ x,y\geq 0, \tag{3.26}
$$

we have

$$
\begin{aligned}
I_{\lambda,W}(u) = \frac{1}{bp^2}&\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^p \\
&+ \frac{1}{p}\int_{\mathbb{R}^N}\lambda W(x)|u|^p\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x - \int_{\mathbb{R}^N} h(x)u\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{2^{p-1}a^p}{bp^2} + \frac{2^{p-1}b^{p-1}}{p^2}\|u\|_{E_\lambda}^p + \frac{1}{p}\|u\|_{E_\lambda}^{p^2} - \eta \int_{\mathbb{R}^N} |u|^{p^2} \mathrm{d}x \\
&\quad + \left(C_\eta + C_p\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\right) \|u\|_{L^p(\mathbb{R}^N)}^p \\
&\leq \left(\frac{2^{p-1}b^{p-1}}{p^2} - \eta C_k\right) \rho_k^{p^2} + \frac{1}{p}\rho_k^p \\
&\quad + \left(C_\eta + C_p\|h\|_{L^{p'}(\mathbb{R}^{\mathbb{N}})}\right) \|u\|_{L^p(\mathbb{R}^N)}^p + \frac{2^{p-1}a^p}{bp^2}.
\end{aligned}$$

Therefore, since $p^2 > p > 1$, then there exists $\rho_k > 1$ large enough such that

$$a_k = \max_{u \in Y_k,\ \|u\|_{E_\lambda} = \rho_k} I_{\lambda,W}(u) \leq 0.$$

This completes the proof. □

Proof of Theorem 1.4. Take $X = E_\lambda$ and Y, Z defined in (3.18), (3.19). According to Lemma 3.1, Lemma 3.3 and Lemma 3.4 with (H_6) and by the fact that $I_{\lambda,W}(0) = 0$, we have that $I_{\lambda,W}$ satisfies all assumptions of Theorem 2.6. Hence, problem (1.7) has infinitely many nontrivial weak solutions, and thus the proof of Theorem 1.4 is completed. □

Proof of Theorem 1.5. Take $X = E_\lambda$ and Y, Z defined in (3.18), (3.19). From the proof of Theorem 1.3, especially in the passage where the $(PS)_c$-condition was checked, also with Theorem 1.4, we deduce that $I_{\lambda,W}$ satisfies all assumptions of Theorem 2.6. Therefore problem (1.7) has infinitely many nontrivial weak solutions, which completes the proof of Theorem 1.5. □

Proof of Theorem 1.6. Let $X = E_\lambda$ be a Banach space and let the conditions of Theorem 1.6 be verified. First, from Lemma 3.1, $I_{\lambda,W}$ satisfies the $(PS)_c$-condition. Moreover, we have $I_{\lambda,W}(0) = 0$ and, according to the condition (H_6), $I_{\lambda,W}$ is an even function. Finally, by Lemma 3.6 and Lemma 3.7, we deduce that $I_{\lambda,W}$ satisfies the conditions (i) and (ii) of Theorem 2.7. Therefore, $I_{\lambda,W}$ satisfies all conditions of Theorem 2.7 and we obtain that problem (1.7) has a sequence of solutions $\{u_k\}$ with unbounded energy. In conclusion, by Theorem 2.7, problem (1.7) has infinitely many nontrivial weak solutions. This completes the proof. □

Proof of Theorem 1.7. Let $X = E_\lambda$ be a Banach space and let the conditions of Theorem 1.7 be verified. First, to show that the energy functional $I_{\lambda,W}$ satisfies the $(PS)_c$-condition, we follow the same steps as in the proof of Theorem 1.3. Moreover, we have $I_{\lambda,W}(0) = 0$ and, according to the condition (H_6), $I_{\lambda,W}$ is an even function. Finally, by Lemma 3.6 and Lemma 3.7, we deduce that $I_{\lambda,W}$ satisfies the conditions (i) and (ii) of Theorem 2.7. Therefore, $I_{\lambda,W}$ satisfies all conditions of Theorem 2.7 and we obtain that problem (1.7) has a sequence of solutions $\{u_k\}$ with unbounded energy. In conclusion, by Theorem 2.7, problem (1.7) has infinitely many nontrivial weak solutions. This completes the proof. □

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Про рівняння типу Шредінгера–Кірхгофа з дробовим p-лапласіаном без умови Амброзетті–Рабіновица

Mohamed Bouabdallah, Omar Chakrone, and Mohammed Chehabi

У цій статті ми розглядаємо існування та множинність багатьох слабких розв'язків для наступного дробового рівняння типу Шредінгера–Кірхгофа:

$$\left(a+b\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1}\times(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u = f(x,u)+h(x) \quad \text{in } \mathbb{R}^N,$$

де $N > sp$, $a, b > 0$ — константи, λ — параметр, $(-\Delta)_p^s$ — дробовий p-оператор Лапласа з $0 < s < 1 < p < \infty$, нелінійність $f(x,u)$ і потенціальна функція $V(x)$ задовольняють деякі прийнятні припущення. За таких умов одержано деякі нові результати для достатньо великих $\lambda > 0$ шляхом застосування варіаційних методів.

Ключові слова: p-оператор Лапласа, дробовий простір Соболєва, рівняння типу Шредінгера–Кірхгофа, умова Амброзетті–Рабіновіца, варіаційні методи