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Periodic Gibbs Measures for Three-State Hard-Core Models in the Case Wand

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We consider fertile three-state Hard-Core (HC) models with the activity parameter $\lambda > 0$ on a Cayley tree. It is known that there exist four types of such models: wrench, wand, hinge, and pipe. These models arise as simple examples of loss networks with nearest-neighbor exclusion. In the case wand on a Cayley tree of order $k \geq 2$, exact critical values $\lambda > 0$ are found for which two-periodic Gibbs measures are not unique. Moreover, we study the extremality of the existing two-periodic Gibbs measures on a Cayley tree of order two.

Key words: Cayley tree, configuration, fertile Hard-core model, Gibbs measure, critical temperature, extreme measure

Mathematical Subject Classification 2020: 82B26, 60K35

1. Introduction

Description of all limit Gibbs measures for a given Hamiltonian is one of the main problems of the theory of Gibbs measures. It is known that each Gibbs measure is associated with one phase of the physical system. Therefore, in the theory of Gibbs measures, one of the important problems is the existence of a phase transition, i.e., when the physical system changes its state when the temperature changes. This occurs when the Gibbs measure is not unique. In this case, the temperature at which the state of the physical system changes is usually called the critical temperature. Moreover, it is known that for continuous Hamiltonians (see [5]) the Gibbs measures form a non-empty convex compact set in the space of all probability measures endowed with the weak topology (see, e.g., [8, Chapter 7]). The set of the Gibbs measures on $\mathbb{Z}^d$ is the convex hull of the set of all limit Gibbs measures (see [4]).

In this connection, it is particularly interesting to describe all the extreme points of this convex set, i.e., the extreme Gibbs measures.

The definition of the Gibbs measure and other concepts related to Gibbs measure theory can be found, for example, in [8, 20, 21, 25]. Although there are many works devoted to studying Gibbs measures, a complete description of all limit Gibbs measures has not yet been obtained for any of the models on Cayley trees.

Hard constraints arise in fields as diverse as combinatorics, statistical mechanics, and telecommunications. In particular, the hard-core model arises in the study of random independent sets of a graph [2,6], the study of gas molecules on a lattice [1], and in the analysis of multi-casting in telecommunication networks [9,19].

Mazel and Suhov introduced and studied the HC model on the d-dimensional lattice $\mathbb{Z}^d$ [18]. In [3], fertile HC models were identified as those that correspond to graphs of the hinge, pipe, wand and wrench types. The Gibbs measures for HC models with three states on the Cayley tree of order $k \geq 1$ were studied in [3,13,16,22,24,26]. In particular, in [13,24], in the "wand" case, a full description of translation-invariant Gibbs measures (TIGM) is given on the Cayley tree of orders two and three, respectively. Also in this case, the existence of at least three TIMGs on a Cayley tree of arbitrary order is proved in [22]. Moreover, in [22], the areas of the (non) extremality of TIMG on the Cayley tree of order $k = 2$ were found. Work [12] is devoted to the study of translation-invariant and periodic Gibbs measures for three-state HC models with an external field. Translation-invariant and periodic Gibbs measures in "hinge", "pipe", and "wrench" cases were studied in [13,16,22,24]. In the "wand" case, periodic measures have not yet been studied. See Chap. 7 in [21] for other HC model properties and their generalizations on a Cayley tree.

In this paper, we study periodic Gibbs measures for a fertile three-state HC model in the case of a "wand" on a homogeneous Cayley tree. In this case, on a Cayley tree of arbitrary order under certain conditions, the translation invariance of the $G_k^{(2)}$-periodic Gibbs measures is proved. In addition, on the Cayley tree of orders two and three under certain conditions an exact critical value λ_{cr} is found such that, for $\lambda \geq \lambda_{cr}$ there exists exactly one $G_k^{(2)}$-periodic Gibbs measure, which is translation-invariant, and for $0 < \lambda < \lambda_{cr}$ there are exactly three $G_k^{(2)}$-periodic Gibbs measures, one of which is translation-invariant and the other two are $G_k^{(2)}$-periodic (non translation-invariant). Also, under certain conditions, we find explicit value $\lambda_{cr}(k)$ such that for $0 < \lambda < \lambda_{cr}$ there exist no less than two $G_k^{(2)}$-periodic (non translation-invariant) Gibbs measures on a Cayley tree of order $k \geq 2$. Moreover, we check extremality of the $G_k^{(2)}$-periodic Gibbs measures existing on the Cayley tree of order two.

2. Preliminaries

The Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree, i.e., a connected graph without cycles such that exactly $k+1$ edges originate from each vertex. Let $\Im^k = (V, L, i)$, where V is the set of vertices $\Im^k$, L is the set of edges and i is the incidence function setting each edge $l \in L$ into correspondence with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called the *nearest neighbors*, denoted by $l = \langle x, y \rangle$.

For a fixed point $x^0 \in V$,

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{\langle x, y\rangle \in L \mid x, y \in V_n\},$$

where $d(x, y)$ is the distance between the vertices x and y on a Cayley tree, i.e., the number of edges of the shortest path connecting x and y.

Write $x \prec y$ if the path from x^0 to y goes through x. Call the vertex y a direct successor of x if $y \succ x$ and x, y are the nearest neighbors. Notice that in $\Im^k$ any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$ direct successors. Denote by $S(x)$ the set of direct successors of x, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, \ i = 1, 2, \ldots, k\}.$$

HC model. Let $\Phi = \{0, 1, 2\}$ and $\sigma \in \Omega = \Phi^V$ be a configuration on V, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. In this model, to each vertex x, one of the values $\sigma(x) \in \Phi = \{0, 1, 2\}$ is assigned. The values $\sigma(x) = 1, 2$ mean that the vertex x is 'occupied', and $\sigma(x) = 0$ means that x is 'vacant'. We let Ω denote the set of all configurations on V. Configurations in V_n and W_n can be defined in a similar way, with the set of all configurations in V_n and W_n denoted by Ω_{V_n} and Ω_{W_n}.

We consider the set Φ as the set of vertices of a graph G. We use the graph G to define a G-admissible configuration as follows. A configuration σ is called a *G-admissible configuration* on the Cayley tree (in V_n or in W_n) if $\{\sigma(x), \sigma(y)\}$ is the edge of the graph G for any pair of nearest neighbors x, y in V (in V_n). We let Ω^G ($\Omega^G_{V_n}$) denote the set of G-admissible configurations.

The activity set [3] for a graph G is a function $\lambda : G \to R_+$ from the set G to the set of positive real numbers. The value λ_i of the function λ at the vertex $i \in \{0, 1, 2\}$ is called the vertex activity.

For given G and λ, we define the Hamiltonian of the $G-$HC model as

$$H^\lambda_G(\sigma) = \begin{cases} \sum\limits_{x \in V} \log \lambda_{\sigma(x)} & \text{if } \sigma \in \Omega^G, \\ +\infty & \text{if } \sigma \notin \Omega^G. \end{cases}$$

The union of configurations $\sigma_{n-1} \in \Phi^{V_{n-1}}$ and $\omega_n \in \Phi^{W_n}$ is determined by the following formula:

$$\sigma_{n-1} \vee \omega_n = \{\{\sigma_{n-1}(x), x \in V_{n-1}\}, \{\omega_n(y), y \in W_n\}\}.$$

Let $\mathbf{B}$ be the σ-algebra generated by cylindric subsets of Ω^G. For any arbitrary n we let $\mathbf{B}_{V_n} = \{\sigma \in \Omega^G \mid \sigma|_{V_n} = \sigma_n\}$, where $\sigma|_{V_n}$ is the restriction of σ to V_n and $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is an admissible configuration in V_n, denote subalgebra of $\mathbf{B}$.

Definition 2.1. For $\lambda > 0$, the HC model Gibbs measure is a probability measure μ on $(\Omega^G, \mathbf{B})$ such that for any n and $\sigma_n \in \Omega^G_{V_n}$, we have

$$\mu\{\sigma \in \Omega^G \mid \sigma|_{V_n} = \sigma_n\} = \int_{\Omega^G} \mu(d\omega) P_n(\sigma_n \mid \omega_{W_{n+1}}),$$

where

$$P_n(\sigma_n \mid \omega_{W_{n+1}}) = \frac{e^{-H_G^\lambda(\sigma_n)}}{Z_n(\lambda;\omega|_{W_{n+1}})}\mathbf{1}(\sigma_n \vee \omega|_{W_{n+1}} \in \Omega^G_{V_{n+1}}).$$

Here, $Z_n(\lambda;\omega|_{W_{n+1}})$ is the normalization multiplier with the boundary condition $\omega|_{W_{n+1}}$:

$$Z_n(\lambda;\omega|_{W_{n+1}}) = \sum_{\widetilde{\sigma}_n\in\Omega_{V_n}} e^{-H_G^\lambda(\widetilde{\sigma}_n)}\mathbf{1}(\widetilde{\sigma}_n \vee \omega|_{W_{n+1}} \in \Omega^G_{V_{n+1}}).$$

Definition 2.2 ([3])**.** A graph is said to be fertile if there is a set of activities λ such that the corresponding Hamiltonian has at least two translation-invariant Gibbs measures.

In this paper, we consider the case $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = \lambda$ and study periodic Gibbs measures in the case of fertile graph $G = \mathit{wand}$:

$$\mathit{wand}: \quad \{0,1\}\{0,2\}\{1,1\}\{2,2\}.$$

For $\sigma_n \in \Omega^G_{V_n}$, we let

$$\#\sigma_n = \sum_{x\in V_n} \mathbf{1}(\sigma_n(x) \geq 1)$$

denote the number of occupied vertices in V_n.

Let $z: \ x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in R^3_+$ be a vector-valued function on V. For $n = 1, 2, \ldots$ and $\lambda > 0$, we consider the probability measure $\mu^{(n)}$ on $\Omega^G_{V_n}$ defined as

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n}\lambda^{\#\sigma_n}\prod_{x\in W_n} z_{\sigma(x),x}, \tag{2.1}$$

where Z_n is a normalization factor,

$$Z_n = \sum_{\widetilde{\sigma}_n\in\Omega^G_{V_n}} \lambda^{\#\widetilde{\sigma}_n}\prod_{x\in W_n} z_{\widetilde{\sigma}(x),x}.$$

The probabilistic measure $\mu^{(n)}$ is said to be consistent if for all $n \geq 1$ and any $\sigma_{n-1} \in \Omega^G_{V_{n-1}}$:

$$\sum_{\omega_n\in\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1}\vee\omega_n)\mathbf{1}(\sigma_{n-1}\vee\omega_n \in \Omega^G_{V_n}) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.2}$$

In this case, there is a unique measure μ on $(\Omega^G, \mathbf{B})$ such that

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$$

for all n and any $\sigma_n \in \Omega^G_{V_n}$.

Definition 2.3. A measure μ defined by formula (2.1) with consistency condition (2.2) is called a splitting hard core Gibbs measure with activity $\lambda > 0$, corresponding to the function $z: \ x \in V \setminus \{x^0\} \mapsto z_x$.

It is known (see Chapter 12, [8]) that any extreme Gibbs measure is splitting Gibbs measure; therefore, for each given Hamiltonian on Cayley tree, the description of the set of all Gibbs measures is equivalent to the full description of the set of all extreme splitting Gibbs measures.

Let $L(G)$ be the set of edges of a graph G. We let $A \equiv A^G = \left(a_{ij}\right)_{i,j=0,1,2}$ denote the adjacency matrix of the graph G, i.e.,

$$a_{ij} \equiv a_{ij}^G = \begin{cases} 1 & \text{if } \{i,j\} \in L(G), \\ 0 & \text{if } \{i,j\} \notin L(G). \end{cases}$$

The following theorem presents a condition on z_x ensuring that the measure $\mu^{(n)}$ is consistent.

Theorem 2.4 ([24]). *The probability measures $\mu^{(n)}$, $n = 1, 2, \ldots$, defined by formula (2.1) are consistent if and only if the following relations hold for any $x \in V$:*

$$z'_{1,x} = \lambda \prod_{y \in S(x)} \frac{a_{10} + a_{11} z'_{1,y} + a_{12} z'_{2,y}}{a_{00} + a_{01} z'_{1,y} + a_{02} z'_{2,y}}, \tag{2.3}$$

$$z'_{2,x} = \lambda \prod_{y \in S(x)} \frac{a_{20} + a_{21} z'_{1,y} + a_{22} z'_{2,y}}{a_{00} + a_{01} z'_{1,y} + a_{02} z'_{2,y}}, \tag{2.4}$$

where $z'_{i,x} = \lambda z_{i,x}/z_{0,x}$, $i = 1, 2$.

In (2.3), (2.4), we assume that $z_{0,x} \equiv 1$ and $z_{i,x} = z'_{i,x} > 0$ for $i = 1, 2$. Then, by Theorem 2.4, there exists a unique G-HC Gibbs measure μ if and only if for any functions $z : x \in V \longmapsto z_x = (z_{1,x}, z_{2,x})$ the equality holds:

$$z_{i,x} = \lambda \prod_{y \in S(x)} \frac{a_{i0} + a_{i1} z_{1,y} + a_{i2} z_{2,y}}{a_{00} + a_{01} z_{1,y} + a_{02} z_{2,y}}, \quad i = 1, 2. \tag{2.5}$$

It is known that we have one-to-one correspondence between the set V of vertices of a Cayley tree of order $k \geq 1$ and the group G_k that is the free product of $k+1$ cyclic groups of second order with the corresponding generators $a_1, a_2, \ldots, a_{k+1}$ (see [7]).

Let $G_k/\widehat{G}_k = \{H_1, ..., H_r\}$ be the quotient group, where $\widehat{G}_k$ is a normal subgroup of index $r \geq 1$.

Definition 2.5. The set of vectors $z = \{z_x, x \in G_k\}$ is said to be $\widehat{G}_k$- periodic if $z_{yx} = z_x$ for all $\forall x \in G_k, y \in \widehat{G}_k$. G_k-periodic sets are said to be translation-invariant.

Definition 2.6. A measure μ is said to be $\widehat{G}_k$-periodic if it corresponds to the $\widehat{G}_k$-periodic set of vectors z.

For TIGM in the case $G = wand$, the following facts are known:

- In the case $k=2$ $(k=3)$ for $\lambda \leq 1$ $\left(\lambda \leq \frac{4}{27}\right)$, there is a unique TIGM ν_0 and for $\lambda > 1$ $(\lambda > \frac{4}{27})$ there are exactly three TIGMs ν_0, ν_1, ν_2 (see [13, 24]).
- In the case $k > 3$ for $\lambda \leq \lambda_{cr}$, there is a unique TIGM and for $\lambda > \lambda_{cr}$ there are at least three TIGMs, where $\lambda_{cr} = \frac{1}{k-1} \cdot \left(\frac{2}{k}\right)^k$ (see [22]).
- In the case $k=2$, the measure ν_0 for $0 < \lambda < \lambda_0$ and the measures ν_1, ν_2 for $1 < \lambda < \lambda_1$ are extreme and the measure ν_0 for $\lambda > \lambda_0$ is not extreme, where $\lambda_0 \approx 2.287572$, $\lambda_1 \approx 1.303094$ (see [22]).

3. Periodic splitting Gibbs measures in the case $G = wand$

In the case $G = wand$, we write (2.5) in the following form:

$$
\begin{aligned}
h_{1,x} &= \ln\lambda + \sum_{y\in S(x)} \ln \frac{1+e^{h_{1,y}}}{e^{h_{1,y}}+e^{h_{2,y}}},\\
h_{2,x} &= \ln\lambda + \sum_{y\in S(x)} \ln \frac{1+e^{h_{2,y}}}{e^{h_{1,y}}+e^{h_{2,y}}},
\end{aligned}
\tag{3.1}
$$

where $h_{i,x} = \ln z_{i,x}$, $i = 1, 2$. We study periodic solutions of system (3.1).

Let the function $F(\cdot) : h = (h_1, h_2) \longmapsto F(h) = (F_1(h), F_2(h))$ be given by

$$
F_1(h) = \ln \frac{1+e^{h_1}}{e^{h_1}+e^{h_2}}, \quad F_2(h) = \ln \frac{1+e^{h_2}}{e^{h_1}+e^{h_2}}.
$$

Proposition 3.1. *The function F is injective.*

Proof. Necessity. Let $F(h) = F(l)$. Then $F_1(h) = F_1(l)$, $F_2(h) = F_2(l)$, where $h = (h_1, h_2)$, $l = (l_1, l_2)$. From these equalities we obtain the following system of equations:

$$
\begin{aligned}
(1-z_2)(z_1-t_1) + (1+z_1)(z_2-t_2) &= 0,\\
(1+z_2)(z_1-t_1) + (1-z_1)(z_2-t_2) &= 0.
\end{aligned}
\tag{3.2}
$$

Here, $z_i = e^{h_i}, t_i = e^{l_i}, i = 1, 2$. It is easy to see that the determinant of system (3.2) is nonzero: $\Delta = -2(z_1+z_2) \neq 0$. Therefore, the system (3.2) has a unique solution $z_1 = t_1$, $z_2 = t_2$. □

Let $G_k^{(2)}$ be the subgroup of G_k consisting the words of even length.

Theorem 3.2. *Let H be a normal subgroup of finite index in G_k. Then for HC model each H-periodic splitting Gibbs measure is either $G_k^{(2)}$-periodic or translation-invariant.*

Proof. The proof is similar to that of Theorem 2 from [16] using the result of Proposition 3.1. □

Remark 3.3. The analogies of Theorem 3.2 can be proved for a wide class of hard constraint models.

By Theorem 3.2, there are only $G_k^{(2)}$-periodic Gibbs measures, and for them from (3.1) we obtain the following system of equations:

$$t_1 = \lambda\left(\frac{1+z_1}{z_1+z_2}\right)^k, \qquad t_2 = \lambda\left(\frac{1+z_2}{z_1+z_2}\right)^k,$$
$$z_1 = \lambda\left(\frac{1+t_1}{t_1+t_2}\right)^k, \qquad z_2 = \lambda\left(\frac{1+t_2}{t_1+t_2}\right)^k. \tag{3.3}$$

We consider the map $W: R^4 \to R^4$ defined as

$$t_1^{'} = \lambda\left(\frac{1+z_1}{z_1+z_2}\right)^k, \qquad t_2^{'} = \lambda\left(\frac{1+z_2}{z_1+z_2}\right)^k,$$
$$z_1^{'} = \lambda\left(\frac{1+t_1}{t_1+t_2}\right)^k, \qquad z_2^{'} = \lambda\left(\frac{1+t_2}{t_1+t_2}\right)^k. \tag{3.4}$$

We note that the system (3.3) is the equation $z = W(z)$. Therefore, solving the system (3.3) is equivalent to finding fixed points of the map $z^{'} = W(z)$.

Lemma 3.4. *The following sets are invariant under the map W:*

$$I_1 = \{(t_1, t_2, z_1, z_2) \in R^4 : t_1 = t_2 = z_1 = z_2\},$$
$$I_2 = \{(t_1, t_2, z_1, z_2) \in R^4 : t_1 = t_2,\ z_1 = z_2\},$$
$$I_3 = \{(t_1, t_2, z_1, z_2) \in R^4 : t_1 = z_1,\ t_2 = z_2\},$$
$$I_4 = \{(t_1, t_2, z_1, z_2) \in R^4 : t_1 = z_2,\ t_2 = z_1\}.$$

Proof. The proof is similar to the proof of Lemma 2 from [23]. □

Remark 3.5. It is difficult to solve system (3.3) in the general case, so we will solve it on invariant sets I_i, $i = 1, 2, 3, 4$. Notice that there may be other invariant sets.

Theorem 3.6. *For HC model in the case $G =$ wand the following statements are true:*

1. *For $k \geq 2$, $\lambda > 0$ on I_1 and I_4, each $G_k^{(2)}$-periodic splitting Gibbs measure is translation- invariant. Moreover, this measure coincides with the unique translation-invariant Gibbs measure ν_0.*
2. *For $k \geq 2$, $\lambda > 0$ on I_3 each $G_k^{(2)}$-periodic splitting Gibbs measure is translation-invariant and this measure is not unique.*
3. *Let $k = 2$ and $\lambda_{cr} = 1$. Then on I_2 for $\lambda \geq \lambda_{cr}$ there is exactly one $G_k^{(2)}$-periodic splitting Gibbs measure which coincides with the unique TIGM ν_0, and for $0 < \lambda < \lambda_{cr}$ there are exactly three $G_k^{(2)}$-periodic splitting Gibbs measures ν_0, μ_1, μ_2, where μ_1, μ_2 are non translation-invariant.*
4. *Let $k = 3$ and $\lambda_{cr} = \frac{128}{27}$. Then on I_2 for $\lambda \geq \lambda_{cr}$ there is exactly one $G_k^{(2)}$-periodic splitting Gibbs measure which is translation-invariant and for $0 < \lambda < \lambda_{cr}$ there are exactly three $G_k^{(2)}$-periodic splitting Gibbs measures, one of which is translation-invariant and the other two are non translation-invariant.*

Proof. 1. *The case* I_1 is obvious. *The case* I_4. In this case, the system of equations (3.3) has the form

$$z_1 = \lambda \left(\frac{1+z_2}{z_1+z_2} \right)^k, \quad z_2 = \lambda \left(\frac{1+z_1}{z_1+z_2} \right)^k. \tag{3.5}$$

It suffices to show that the system of functional equations (3.5) has only roots of the form $z_1 = z_2$ for any $z_1 > 0$, $z_2 > 0$, $\lambda > 0$ and $k \geq 2$. Introducing the notation $\sqrt[k]{z_1} = x$, $\sqrt[k]{z_2} = y$, we rewrite the system of equations (3.5):

$$x = \sqrt[k]{\lambda} \left(\frac{1+y^k}{x^k+y^k} \right), \quad y = \sqrt[k]{\lambda} \left(\frac{1+x^k}{x^k+y^k} \right).$$

In the last system of equations, subtract the second from the first equation

$$(x-y)(x^k + y^k + \sqrt[k]{\lambda}(x^{k-1} + x^{k-2}y + ... + y^{k-1})) = 0.$$

Hence, $x = y$, i.e., $(t_1, t_2, z_1, z_2) \in I_1$. So $G_k^{(2)}$-periodic Gibbs measure is translation-invariant and this measure is unique.

2. *The case* I_3. In this case, we obtain the system of equations for the TIGM which was studied in [13, 22, 24].

3. *The case* I_2 *and* $k = 2$. In this case, we have $z_1 = z_2 = z$ and $t_1 = t_2 = t$. Then the system of equations (3.3) has the form

$$z = \lambda \left(\frac{1+t}{2t} \right)^k, \quad t = \lambda \left(\frac{1+z}{2z} \right)^k. \tag{3.6}$$

Let $k = 2$. Introducing the notation $\sqrt{z} = x$, $\sqrt{t} = y$ we rewrite the system of equations (3.6):

$$x = \sqrt{\lambda} \frac{1+y^2}{2y^2}, \quad y = \sqrt{\lambda} \frac{1+x^2}{2x^2}. \tag{3.7}$$

The system (3.7) leads to the following equation:

$$\lambda(1+x^2)^2 - 2x(1+x^2)^2\sqrt{\lambda} + 4x^4 = 0.$$

We regard the last equation as a quadratic equation for variable $\sqrt{\lambda} = a$ whose solutions have the following forms:

$$a_1 = \frac{2x}{1+x^2}, \quad a_2 = \frac{2x^3}{1+x^2}.$$

Notice that for any value $\lambda > 0$, the equation

$$a_2 = \sqrt{\lambda} = \frac{2x^3}{1+x^2} \tag{3.8}$$

has a unique solution which corresponds to the unique TIGM for the HC model in the case $G = wand$ (see [22] formula (3.15)).

Now, from the expression for $a_1 = \sqrt{\lambda_1} \equiv \sqrt{\lambda}$, we get the quadratic equation

$$\sqrt{\lambda}x^2 - 2x + \sqrt{\lambda} = 0, \tag{3.9}$$

Notice that equation (3.9) has solutions for $0 < \lambda \leq 1$, more exactly, for $\lambda = 1$ it has a unique solution of the form $x_1 = x_2 = 1$ which is also a solution to (3.8) for this value of λ, and for $0 < \lambda < 1$ it has two positive solutions:

$$x_1 = \frac{1+\sqrt{1-\lambda}}{\sqrt{\lambda}}, \quad x_2 = \frac{\sqrt{\lambda}}{1+\sqrt{1-\lambda}}.$$

From the second equation in (3.7), we obtain

$$y_1 = \frac{\sqrt{\lambda}}{1+\sqrt{1-\lambda}}, \quad y_2 = \frac{1+\sqrt{1-\lambda}}{\sqrt{\lambda}}.$$

So, for the system of equations (3.6), we have solutions of the form $(x_1^2, y_1^2) = (z, t)$ and $(x_2^2, y_2^2) = (t, z)$, where

$$z = \frac{(1+\sqrt{1-\lambda})^2}{\lambda}, \quad t = \frac{\lambda}{(1+\sqrt{1-\lambda})^2}. \tag{3.10}$$

Thus, the solutions (z, t) and (t, z) of system of equations (3.6) correspond to the $G_k^{(2)}$-periodic Gibbs measures μ_1 and μ_2 which are different from translation-invariant.

4. *The case I_2 and $k = 3$.* In this case, introducing the notation $\sqrt[3]{z} = x$, $\sqrt[3]{t} = y$, we rewrite the system of equations (3.6):

$$x = \sqrt[3]{\lambda}\frac{1+y^3}{2y^3}, \quad y = \sqrt[3]{\lambda}\frac{1+x^3}{2x^3}. \tag{3.11}$$

From the system of equations (3.11) we have

$$x = f(y), \quad y = f(x), \tag{3.12}$$

where

$$f(x) = \sqrt[3]{\lambda}\frac{1+x^3}{2x^3}.$$

It is easy to check that the equation $f(x) = x$ has a unique positive solution for any $\lambda > 0$, which corresponds to the unique TIGM.

Moreover, roots of the equation $f(x) = x$ are clearly roots of the equation $f(f(x)) = x$. To find the roots of the equation $f(f(x)) = x$ that differ from the roots of $f(x) = x$, we must therefore consider the equation

$$\frac{x - f(f(x))}{x - f(x)} = 0. \tag{3.13}$$

Equation (3.13) is equivalent to the equation

$$b^2(1+x^3)^2 - 2bx(1+x^3) - 4x^5 = 0,$$

where $b = \sqrt[3]{\lambda}$. We consider the last equation as a quadratic equation for b. It has one positive solution

$$b = \frac{x + x\sqrt{1+4x^3}}{1+x^3} > 0.$$

We consider next equation

$$b = \sqrt[3]{\lambda} = \frac{x + x\sqrt{1+4x^3}}{1+x^3} = \varphi(x). \tag{3.14}$$

It is easy to see that the function $\varphi(x)$ increases for $0 < x \leq \sqrt[3]{2}$ and decreases for $x \geq \sqrt[3]{2}$, i.e., $x_{max} = \sqrt[3]{2}$ and $\varphi(x_{max}) = \frac{4\sqrt[3]{2}}{3} = b = \sqrt[3]{\lambda_{cr}}$ (see Fig. 3.1, a)).

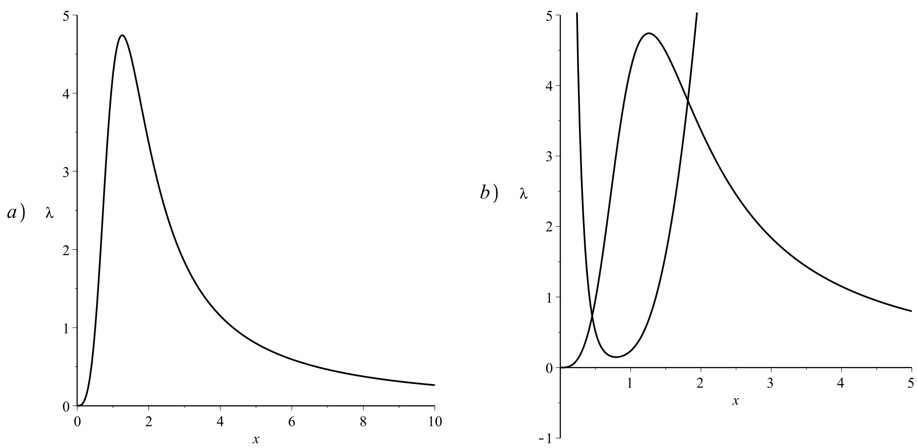

Fig. 3.1: a) Graph of the function $\varphi^3(x)$. b) Graph of the function $\varphi^3(x)$ (upper curve) and graph of the function $\psi^3(x)$ (lower curve).

Thus, the equation $\lambda = \varphi^3(x)$ has two solutions if $0 < \lambda < \lambda_{cr} = \frac{128}{27}$, one solution if $\lambda = \lambda_{cr}$, and no solution if $\lambda > \lambda_{cr}$.

Notice that if $\lambda = \lambda_{cr} = \frac{128}{27}$, then solution (3.11) has the form: $(\sqrt[3]{2}, \sqrt[3]{2})$, i.e., this solution corresponds to the TIGM which exists for any $\lambda > 0$, and measures corresponding to the two existing solutions for $0 < \lambda < \lambda_{cr}$ are $G_k^{(2)}$-periodic different from translation-invariant. □

Remark 3.7. In [13], TIGMs were investigated for the HC model in the case $G = wand$ and a similar method was applied which was used in the proof of part 4 of Theorem 3.6. For TIGMs, equation (3.14) is given by

$$\sqrt[3]{\lambda} = \frac{x^2(x^3-3) + \sqrt{x^4(x^3+3)^2 + 4x}}{2x(x^3+1)} = \psi(x). \tag{3.15}$$

The equation shows that the functions $\varphi^3(x)$ and $\psi^3(x)$ differ and they intersect at two points: $x_1 \approx 0.4531316267$, $x_2 \approx 1.813976199$, i.e., measures corresponding to these solutions are TIGMs (see Fig. 3.1, b)).

The case I_2 and $k \geq 2$. In this case, we rewrite the system of equations (3.6):

$$z = h(t), \quad t = h(z), \tag{3.16}$$

where $h(x) = \alpha \left(1 + \frac{1}{x}\right)^k, \alpha = \frac{\lambda}{2^k}$.

The following lemma is known.

Lemma 3.8 ([10]). *Let $f : [0,1] \to [0,1]$ be a continuous function with a fixed point $\xi \in (0,1)$. We assume that f is differentiable at ξ and $f'(\xi) < -1$. Then there exist points x_0 and x_1, $0 \leq x_0 < \xi < x_1 \leq 1$, such that $f(x_0) = x_1$ and $f(x_1) = x_0$.*

Theorem 3.9. *Let $k \geq 2$ and $\lambda_{cr} = 2^k(k-1)\left(\frac{k-1}{k}\right)^k$. Then for HC model in the case $G =$ wand on I_2 for $0 < \lambda < \lambda_{cr}$ there exist at least three $G_k^{(2)}$-periodic splitting Gibbs measures, one of which is translation-invariant and the other two are non translation-invariant.*

Proof. Since $h(x) > \alpha$, we have $z > \alpha$ and $t > \alpha$. The function $h(x)$ is decreasing and $h_{\max} = h(\alpha) = \alpha\left(1 + \frac{1}{\alpha}\right)^k = \beta$. Moreover, the function $h(x)$ is continuous and differentiable in $[\alpha, \beta]$. It follows from the above argument that the equation $h(x) = x$ has a unique solution $x = \xi$.

We rewrite the equality $h(\xi) = \xi$ as follows,

$$\alpha \left(1 + \frac{1}{\xi}\right)^{k-1} = \frac{\xi^2}{1+\xi}.$$

Using the last equality, from the inequality

$$h'(\xi) = -\frac{\alpha k}{\xi^2}\left(1 + \frac{1}{\xi}\right)^{k-1} = -\frac{k}{1+\xi} < -1,$$

we get $\xi < k - 1$.

Since $\xi \in (\alpha, \beta)$ is a fixed point of the function h, we have

$$\lambda = 2^k \xi \left(\frac{\xi}{1+\xi}\right)^k = \phi(\xi).$$

Notice that $\phi'(\xi) > 0$, i.e., the function $\phi(\xi)$ is increasing. Hence, for $\xi < k - 1$, we have $\phi(\xi) < \phi(k-1)$, i.e.,

$$\lambda_{\max} = \lambda_{\text{cr}} = 2^k(k-1)\left(\frac{k-1}{k}\right)^k.$$

Consequently, by Lemma 3.6, if $\lambda < \lambda_{\text{cr}}$, then the system (3.16) has three solutions (ξ, ξ), (z_0, t_0) and (t_0, z_0). □

4. Extremality of periodic splitting Gibbs measures in the case $G = wand$

We have $G_k^{(2)}$-periodic splitting Gibbs measures μ_1 and μ_2 for $k = 2$. To study their (non) extremality we use the methods from [11,14,15,17] for TIGMs. For each translation-invariant measure we consider a tree-indexed Markov chain with states $\{0, 1, 2\}$, i.e., suppose we are given a Cayley tree with set vertices V, a probability measure ν, and a probability transition matrix $\mathbb{P} = (P_{ij})$ on $\{0, 1, 2\}$. Using transition probabilities given the value of its parent, regardless of everything else, we can construct a tree, indexed by a Markov chain $X : V \to \{0, 1, 2\}$ by choosing $X(x^0)$ according to ν and choosing $X(v)$, for each vertex $v \neq x^0$.

Since translation-invariant measures are obtained for $(t_1, t_2) = (z_1, z_2)$, the matrix $\mathbb{P}$ depends only on $z_1(= t_1)$ and $z_2(= t_2)$, more precisely,

$$\mathbb{P} = \begin{pmatrix} 0 & \frac{z_1}{z_1+z_2} & \frac{z_2}{z_1+z_2} \\ \frac{1}{1+z_1} & \frac{z_1}{1+z_1} & 0 \\ \frac{1}{1+z_2} & 0 & \frac{z_2}{1+z_2} \end{pmatrix}$$

But, in the case of periodic measures, the matrix $\mathbb{P}$ depends on t_1, t_2, z_1 and z_2, where $t_1 \neq z_1$, $t_2 \neq z_2$ and (t_1, t_2, z_1, z_2) are the solutions of the system of equations (3.3). We consider the measures μ_1 and μ_2 corresponding to the set of $I_2 : t_1 = t_2 = t$, $z_1 = z_2 = z$. In addition, notice that $zt = 1$. Then the transitions probabilities matrix P_{il}, defined by the given periodic Gibbs measure μ_1 (respectively, μ_2) $\mathbb{P} \equiv \mathbb{P}_{z,t} = \mathbf{P}_{\mu_1}$ (respectively $\mathbb{P} \equiv \mathbb{P}_{t,z} = \mathbf{P}_{\mu_2}$), is the product of two transition probabilities matrices:

$$\mathbf{P}_{\mu_1} = \mathbb{P}_z\mathbb{P}_t = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{1+z} & \frac{z}{1+z} & 0 \\ \frac{1}{1+z} & 0 & \frac{z}{1+z} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{1+t} & \frac{t}{1+t} & 0 \\ \frac{1}{1+t} & 0 & \frac{t}{1+t} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{1+t} & \frac{t}{2(1+t)} & \frac{t}{2(1+t)} \\ \frac{z}{(1+z)(1+t)} & \frac{t+3}{2(1+z)(1+t)} & \frac{1}{2(1+z)} \\ \frac{z}{(1+z)(1+t)} & \frac{1}{2(1+z)} & \frac{t+3}{2(1+z)(1+t)} \end{pmatrix}. \tag{4.1}$$

Thus, the matrix $\mathbf{P}_{\mu_1}$ defines a Markov chain on the Cayley tree of order k^2, which consists of the vertices of the tree Γ^k in even places.

So, a sufficient condition (i.e., the Kesten-Stigum condition, see [11]) for non-extremality of a Gibbs measure μ_1 corresponding to the matrix $\mathbf{P}_{\mu_1}$ is that $k^2 s_2^2 > 1$, where s_2 is the second largest (in absolute value) eigenvalue of $\mathbf{P}_{\mu_1}$.

It is clear that the eigenvalues of this matrix are

$$s_1 = 1, \quad s_2 = s_3 = \frac{1}{z + t + 2}.$$

We have solutions of the form (3.10) for $k = 2$. By virtue of the symmetry of the solutions, the region of non-extremality of the measure μ_2 coincides with the region of non-extremality of the measure μ_1. Therefore, it is sufficient to check the condition of non-extremality of the measure μ_1 for $k = 2$. For this, we calculate $z + t$,

$$z + t = \frac{2(2-\lambda)}{\lambda}.$$

Then, from $4s_2^2 > 1$, we obtain $\lambda > 2$, but the measures μ_1 and μ_2 exist for $0 < \lambda < 1$. Hence, this measures should be extreme, which will be checked below.

Let us first give some necessary definitions from [17]. If from a Cayley tree Γ^k we remove an arbitrary edge $\langle x^0, x^1\rangle = l \in L$, then it is divided into two components $\Gamma^k_{x^0}$ and $\Gamma^k_{x^1}$, each called semi-infinite Cayley tree or Cayley subtree.

We consider the finite complete subtrees $\mathcal{T}$ that are the initial points of Cayley tree $\Gamma^k_{x^0}$. The boundary $\partial\mathcal{T}$ of the subtree $\mathcal{T}$ consists of the neighbors which are on $\Gamma^k_{x^0} \setminus \mathcal{T}$. We identify the subgraphs of $\mathcal{T}$ with their vertex sets and write $E(A)$ for the edges within a subset A and ∂A for the boundary of A.

In [17], the key ingredients are two quantities, κ and γ. Both are properties of the collection of Gibbs measures $\{\mu^\tau_\mathcal{T}\}$, where the boundary condition τ is fixed and $\mathcal{T}$ ranges over all initial finite complete subtrees of $\Gamma^k_{x^0}$. For a given subtree $\mathcal{T}$ of $\Gamma^k_{x^0}$ and a vertex $x \in \mathcal{T}$, we write $\mathcal{T}_x$ for the half-tree growing from the root x. When x is not the root of $\mathcal{T}$, let $\mu^s_{\mathcal{T}_x}$ denote the (finite-volume) Gibbs measure in which the parent of x has its spin fixed to s and the configuration on the bottom boundary $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x \setminus \{\text{parent of } x\}$) is specified by τ.

For two measures μ_1 and μ_2 on Ω, $\|\mu_1 - \mu_2\|_x$ denotes the variation distance between the projections of μ_1 and μ_2 onto the spin at x, i.e.,

$$\|\mu_1 - \mu_2\|_x = \frac{1}{2}\sum_{i=0}^{2} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x set to s.

Following [17], define

$$\kappa \equiv \kappa(\mu) = \frac{1}{2}\max_{i,j}\sum_{l=0}^{2} |P_{il} - P_{jl}|,$$

$$\gamma \equiv \gamma(\mu) = \sup_{A\subset\Gamma^k} \max \|\mu_A^{\eta^{y,s}} - \mu_A^{\eta^{y,s'}}\|_x,$$

where the maximum is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y, and all spins $s, s' \in \{0, 1, 2\}$.

It is known that a sufficient condition for extremality of the translation-invariant Gibbs measure μ is that $k\kappa(\mu)\gamma(\mu) < 1$, but for the considered $G_k^{(2)}$-periodic measures $\mu_i, i = 1, 2$ this condition is $k^2\kappa(\mu_i)\gamma(\mu_i) < 1$.

Using (4.1), we obtain

$$\kappa = \frac{z}{(z+1)^2}.$$

By virtue of the symmetry of the solutions, the region of extremality of the measure μ_2 coincides with the region of extremality of the measure μ_1. It is known from [22] that $\gamma = \kappa$. Hence, for extremality of the measure μ_1 (also for an measure μ_2) we obtain the inequality

$$k^2\kappa(\mu_1)\gamma(\mu_1) = \frac{4z^2}{(z+1)^4} < 1,$$

which is true for any values z, in particular, for a solution of the form (3.10) which exists for $0 < \lambda < 1$. Consequently, in the case $k = 2$, the condition of the extremality measures μ_1 and μ_2 is satisfied for any values $0 < \lambda < 1$, i.e., in an area of the existence of these measures.

So, we have proved the following theorem.

Theorem 4.1. *Let $k = 2$. Then for the HC model in the case $G =$ wand $G_k^{(2)}$-periodic splitting Gibbs measures μ_1 and μ_2 are extreme for $0 < \lambda < 1$.*

Remark 4.2. Since in the case $k = 3$ we do not have an explicit form of the solution of the system of equations (3.11), it is very difficult to investigate (non) extremality of the corresponding periodic Gibbs measures. Therefore, this question is still open.

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References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic, London 1982.

[2] G.R. Brightwell and P. Winkler, *Hard constraints and the Bethe lattice: adventures at the interface of combinatorics and statistical physics*, Proceedings of the ICM 2002, **IIIi**, Higher Education Press, Beijing, 2002, 605–624.

[3] G. Brightwell and P. Winkler, *Graph homomorphisms and phase transitions*, J. Combin. Theory Ser.B. **77** (1999), 221–262.

[4] L. Coquille, *Examples of DLR states which are not weak limits of finite volume Gibbs measures with deterministic boundary conditions*, J. Stat. Phys. **159** (2015), 958–971.

[5] S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems. A Concrete Mathematical Introduction*, Cambridge University Press, Cambridge, 2018.

[6] D. Galvin and J. Kahn, *On phase transition in the hard-core model on $\mathbb{Z}^d$*. Comb. Prob. Comp. **13** (2004), 137–164.

[7] N.N. Ganikhodjaev and U.A. Rozikov, *Description of periodic extreme Gibbs measures of some lattice models on a Cayley tree*, Theor. Math. Phys. **111** (1997), 480–486.

[8] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter Studies in Mathematics, **9**, Walter de Gruyter, Berlin, 1988.

[9] F. Kelly, *Loss networks*, Ann. Appl. Probab. **1** (1991), 319–378.

[10] H. Kesten, *Quadratic transformations: a model for population growth*. I. Adv. Appl. Probab. **2** (1970), 1–82.

[11] H. Kesten and B.P. Stigum, *Additional limit theorem for indecomposable multi-dimensional Galton-Watson processes*, Ann. Math. Statist. **37** (1966), 1463–1481.

[12] S. Kissel, C. Külske, and U.A. Rozikov, *Hard-core and soft-core Widom-Rowlinson models on Cayley trees*, Jour. Stat. Mech.: Theory and Exper. **2019** (2019), P043204.

[13] R.M. Khakimov, *Translation-invariant Gibbs measures for fertile three-state "Hard Core" models on a Cayley tree*, Theor. Math. Phys. **183** (2015), 441–449.

[14] C. Külske and U.A. Rozikov, *Fuzzy transformations and extremality of Gibbs measures for the Potts model on a Cayley tree*, Random Structures Algorithms **50** (2017), 636–678.

[15] C. Külske and U.A. Rozikov, *Extremality of translation-invariant phases for a three-state SOS-model on the binary tree*. Jour. Stat. Phys. **160** (2015), 659–680.

[16] J.B. Martin, U.A. Rozikov, and Yu.M. Suhov, *A three state hard-core model on a Cayley tree*, J. Nonlin. Math. Phys. **12** (2005), 432–448.

[17] F. Martinelli, A. Sinclair, and D. Weitz, *Fast mixing for independent sets, coloring and other models on trees*, Random Structures Algorithms **31** (2007), 134–172.

[18] A.E. Mazel and Yu.M. Suhov, *Random surfaces with two-sided constraints: an application of the theory of dominant ground states*, J. Stat. Phys. **64** (1991), 111–134.

[19] P. Mitra, K. Ramanan, A. Sengupta, and I. Ziedins, *Markov random field models of multicasting in tree networks*, Adv. Appl. Probab.**34** (2002), 1–27.

[20] C.J. Preston, *Gibbs States on Countable Sets*, Cambridge Tracts Math., **68**, Cambridge Univ. Press, Cambridge, 1974.

[21] U.A. Rozikov, *Gibbs measures on Cayley trees*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[22] U.A. Rozikov and R.M. Khakimov, *Gibbs measures for the fertile three-state hard core models on a Cayley tree*, Queueing Systems **81** (2015), 49–69.

[23] U.A. Rozikov and R.M. Khakimov, *Periodic Gibbs measures for Potts model on the Cayley tree*, Theor. Math. Phys. **175** (2013), 699–709.

[24] U.A. Rozikov and Sh.A. Shoyusupov, *Fertile HC models with three states on a Cayley tree*, Theor. Math. Phys. **156** (2008), 1319–1330.

[25] Ya.G. Sinai, *Theory of Phase Transitions: Rigorous Results*, Nauka, Moscow, 1980 (Russian); English transl. Intl. Series Nat. Philos., **108**, Pergamon, Oxford, 1982.

[26] Yu.M. Suhov and U.A. Rozikov, *A hard-core model on a Cayley tree: an example of a loss network*, Queueing Systems **46** (2004), 197–212.

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Періодичні міри Гіббса для HC- моделей з трьома станами у випадку "палички"

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Ми розглядаємо фертильні (Hard-Core) HC-моделі з трьома станами з параметром активності $\lambda > 0$ на дереві Кейлі. Відомо, що існують чотири типи таких моделей: гайковий ключ, паличка, петля і труба. Ці моделі виникають як прості приклади втрат взаємодії з найближчим сусідом. У випадку "палички" на дереві Кейлі порядку $k \geq 2$ знайдено точні критичні значення $\lambda > 0$, для яких двоперіодичні міри Гіббса не є єдиними. Крім того, ми вивчаємо екстремальність існуючих двоперіодичних мір Гіббса на дереві Кейлі другого порядку.

Ключові слова: дерево Кейлі, конфігурація, фертильна модель Hard-core, міра Гіббса, критична температура, екстремальність міри