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On the Compactness of One Class of Solutions for the Dirichlet Problem

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We consider the Dirichlet problem for the Beltrami equation in an arbitrary bounded simply connected domain in the complex plane $\mathbb{C}$. Namely, we study the class of all regular solutions of such a problem with a normalization condition and set-theoretic constraints on their complex characteristics. We have proved the compactness of this class in terms of prime ends for an arbitrary continuous function in the Dirichlet condition.

Key words: Beltrami equation, Dirichlet problem, prime ends, plane mappings with a finite and bounded distortion

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1. Introduction

Relatively recently, some authors have obtained the theorems on the compactness of the classes of solutions of the Beltrami equations with different types of restrictions on their complex characteristics, see, for example, [3, 4, 12–14]. In particular, the results of [12, 13] were used in the variational method (see [14]). As for this paper, it deals with the compactness of the classes of solutions of the Dirichlet problem for the Beltrami equation. We consider here the case of the Dirichlet problem in an arbitrary simply connected domain D in $\mathbb{C}$, which does not imply the presence of a "good" boundary. In addition, we consider the so-called set-theoretic constraints, i.e., when the complex characteristics of solutions belong to a fixed family of sets. Notice that some results close in context were obtained by us in [2, 23].

In what follows, a mapping $f : D \to \mathbb{C}$ is assumed to be *sense-preserving*, moreover, we assume that f has partial derivatives almost everywhere. Put $f_{\overline{z}} = (f_x + if_y)/2$ and $f_z = (f_x - if_y)/2$. The *complex dilatation* of f at $z \in D$ is defined as follows: $\mu(z) = \mu_f(z) = f_{\overline{z}}/f_z$ for $f_z \neq 0$ and $\mu(z) = 0$ otherwise. The *maximal dilatation* of f at z is the following function:

$$K_\mu(z) = K_{\mu_f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{1.1}$$

Notice that the Jacobian of f at $z \in D$ is calculated by the formula

$$J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2.$$

It is easy to see that $K_{\mu_f}(z) = \frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|}$ whenever partial derivatives of f exist at $z \in D$ and, in addition, $J(z, f) \neq 0$.

We will call the *Beltrami equation* the differential equation of the form

$$f_{\overline{z}} = \mu(z) f_z, \tag{1.2}$$

where $\mu = \mu(z)$ is a given function with $|\mu(z)| < 1$ a.e. The *regular solution* of (1.2) in $D \subset \mathbb{C}$ is a homeomorphism $f : D \to \mathbb{C}$ of the class $W^{1,1}_{\rm loc}(D)$ such that $J(z, f) \neq 0$ for almost all $z \in D$.

In the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we use the so-called *chordal metric* h defined by the equalities

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \ x \neq \infty, \ y \neq \infty, \quad h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}, \tag{1.3}$$

see, e.g., [27, Definition 12.1]. For a given set $E \subset \overline{\mathbb{R}^n}$, we put

$$h(E) := \sup_{x,y \in E} h(x,y). \tag{1.4}$$

The quantity $h(E)$ in (1.4) is called the *chordal diameter* of the set E. As usual, the family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{C}}$ is called *normal* if from each sequence $f_n \in \mathfrak{F}$, $n = 1, 2, \ldots$, one can choose a subsequence f_{n_k}, $k = 1, 2, \ldots$, converging to some mapping $f : D \to \overline{\mathbb{C}}$ locally uniformly with respect to the metric h. If, in addition, $f \in \mathfrak{F}$, the family $\mathfrak{F}$ is called *compact*.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$. Recall some definitions (see, for example, [7,8,10,11]). Let ω be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n-1$. A continuous mapping $\sigma\colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* in $\mathbb{R}^n$. A *surface* is an arbitrary $(n-1)$-dimensional surface σ in $\mathbb{R}^n$. The surface σ is called *a Jordan surface* if $\sigma(x) \neq \sigma(y)$ for $x \neq y$. In the following, we will use σ instead of $\sigma(\omega) \subset \mathbb{R}^n$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$ and $\partial\sigma$ instead of $\overline{\sigma(\omega)} \setminus \sigma(\omega)$. A Jordan surface $\sigma\colon \omega \to D$ is called a *cut* of D if σ separates D, that is, $D \setminus \sigma$ has more than one component, $\partial\sigma \cap D = \varnothing$ and $\partial\sigma \cap \partial D \neq \varnothing$.

A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in D is called *a chain* if:

(i) the set σ_{m+1} is contained in exactly one component d_m of the set $D \setminus \sigma_m$, wherein $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$;

(ii) $\bigcap_{m=1}^{\infty} d_m = \varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent* if for each $m = 1, 2, \ldots$ the domain d_m contains all the domains d'_k, except for a finite number, and for each $k = 1, 2, \ldots$ the domain d'_k also contains all domains d_m, except for a finite number.

The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{R}^n$, then the set $I(K) = \bigcap_{m=1}^{\infty} \overline{d_m}$ is called *the impression of the end* K. Throughout the paper, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma\colon [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for every $t \in [a, b]$. In what follows, M denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$, see [27]. Following [18], we say that the end K is *a prime end* if K contains a chain of cuts $\{\sigma_m\}$ such that $\lim_{m\to\infty} M(\Gamma(C,\sigma_m,D)) = 0$ for some continuum C in D. Further, the following notations are used: the set of prime ends, corresponding to the domain D, is denoted by E_D, and the completion of the domain D by its prime ends is denoted by $\overline{D}_P$.

Consider the definition which goes back to Näkki [18], see also [10,11]. We say that the boundary of the domain D in $\mathbb{R}^n$ is *locally quasiconformal* if each point $x_0 \in \partial D$ has a neighborhood U in $\mathbb{R}^n$, which can be mapped by a quasiconformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ with the coordinate hyperplane.

For the sets $A, B \subset \mathbb{R}^n$, we set, as usual,

$$\operatorname{diam} A = \sup_{x,y\in A} |x-y|, \quad \operatorname{dist}(A,B) = \inf_{x\in A, y\in B} |x-y|.$$

Sometimes we also write $d(A)$ instead of $\operatorname{diam} A$ and $d(A,B)$ instead of $\operatorname{dist}(A,B)$ if no misunderstanding is possible. The sequence of cuts σ_m, $m = 1,2,\ldots$, is called *regular* if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \varnothing$ for $m \in \mathbb{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end K contains at least one regular chain, then K will be called *regular*. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Notice that a space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g : D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put

$$\rho(x,y) := |g^{-1}(x) - g^{-1}(y)|, \tag{1.5}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique and well-defined by [8, Theorem 2.1, Remark 2.1], cf. [18, Theorem 4.1]. It is easy to verify that ρ in (1.5) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map g with the indicated property.

We say that a sequence $x_m \in D$, $m = 1,2,\ldots$, converges to a prime end of $P \in E_D$ as $m \to \infty$, write $x_m \to P$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements x_m belong to d_k except for a finite number. Here d_k denotes a sequence of nested domains corresponding to the definition of the prime end P. It should be noticed that for a homeomorphism of a domain D onto D', the end of the domain D uniquely corresponds to some sequence of nested domains in the image under the mapping.

Consider the following Dirichlet problem:

$$f_{\overline{z}} = \mu(z) f_z, \tag{1.6}$$

$$\lim_{\zeta\to P} \operatorname{Re} f(\zeta) = \varphi(P), \quad P \in E_D, \tag{1.7}$$

where $\varphi : E_D \to \mathbb{R}$ is a prescribed continuous function. In what follows, we assume that D is a simply connected domain in $\mathbb{C}$. The solution of the problem

(1.6), (1.7) is called *regular* if one of two conditions is fulfilled: either $f(z) =$ const in D, or f is an open discrete $W^{1,1}_{\rm loc}(D)$-mapping such that $J(z, f) \neq 0$ for almost any $z \in D$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$. We say that a function $\psi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write $\psi \in FMO(x_0)$, if

$$\limsup_{\varepsilon\to 0} \frac{1}{\Omega_n\varepsilon^n} \int_{B(x_0,\varepsilon)} |\psi(x) - \overline{\psi}_\varepsilon|\, dm(x) < \infty, \tag{1.8}$$

where

$$\overline{\psi}_\varepsilon = \frac{1}{\Omega_n\varepsilon^n} \int_{B(x_0,\varepsilon)} \psi(x)\, dm(x),$$

and the number Ω_n denotes the volume of a unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, see [6]. Observe that, as known, $\Omega_n\varepsilon^n = m(B(x_0, \varepsilon))$, and that the situation $\overline{\psi_\varepsilon} \to \infty$ as $\varepsilon \to 0$ under the condition (1.8) is possible. We also say that a function $\psi : D \to \mathbb{R}$ has a finite mean oscillation in D, write $\psi \in FMO(D)$, or simply $\psi \in FMO$, if ψ has a finite mean oscillation at any point $x_0 \in D$.

Let $M(z) \subset \mathbb{D}$, $z \in \mathbb{C}$ be a system of sets (that is, for each $z_0 \in \mathbb{C}$ the symbol $M(z_0)$ denotes some set in $\mathbb{D}$). Denote by $\mathfrak{M}_M$ the set of all complex measurable functions $\mu : \mathbb{C} \to \mathbb{D}$ such that $\mu(z) \in M(z)$ for almost all $z \in \mathbb{C}$.

Set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A set $A \subset \mathbb{D}$ is called *invariantly convex* if the set $g(A)$ is convex for any fractional-linear automorphism g of the unit disk, see, e.g., [21].

We fix a point $z_0 \in D$ and a function $\varphi : E_D \to \mathbb{R}$. Let $M(z) \subset \mathbb{D}$, $z \in D$, be some system of sets. Let $\mathfrak{F}_{\varphi,M,z_0}(D)$ be the class of all regular solutions $f : D \to \mathbb{C}$ of the Dirichlet problem (1.6), (1.7), which satisfy the condition $\operatorname{Im} f(z_0) = 0$ with $\mu \in \mathfrak{M}_M$. We define a function $Q_M(z)$ by the relation

$$Q_M(z) = \frac{1 + q_M(z)}{1 - q_M(z)}, \quad q_M(z) = \sup_{\nu\in M(z)} |\nu|, \tag{1.9}$$

and we consider that $Q_M(z) \equiv 1$ for $z \in \mathbb{C}\backslash D$. The following statement generalizes [3, Theorem 2] for the case of arbitrary simply connected Jordan domains.

Theorem 1.1. *Let D be a simply connected domain in $\mathbb{C}$, and let $\varphi : E_D \to \mathbb{R}$ be a continuous function in (1.7). Let $M(z)$, $z \in D$, be a family of invariantly convex compact sets, and let Q_M be integrable in D and satisfies at least one of the following conditions: either $Q_M \in FMO(\overline{D})$, or*

$$\int_0^{\delta_0} \frac{dt}{tq_{M_{x_0}}(t)} = \infty \tag{1.10}$$

for any $x_0 \in \overline{D}$ and some $\delta_0 = \delta(x_0) > 0$, where

$$q_{M_{x_0}}(t) = \frac{1}{2\pi} \int_0^{2\pi} Q_M(x_0 + te^{i\theta})\, d\theta.$$

Then the family $\mathfrak{F}_{\varphi,M,z_0}(D)$ is compact.

2. Preliminaries

Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of an extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad S_i = S(x_0, r_i), \quad i = 1, 2,$$
$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}. \tag{2.1}$$

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$, and let $p \geqslant 1$. By following [16, Chap. 7.6], a mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point* $x_0 \in \overline{D} \setminus \{\infty\}$ if the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int_{A \cap D} Q(x)\, \eta^n(|x - x_0|)\, dm(x) \tag{2.2}$$

holds for all $0 < r_1 < r_2 < d_0 := \sup_{x \in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \tag{2.3}$$

The mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping in* $\overline{D} \setminus \{\infty\}$ if (2.2) holds for any $x_0 \in \overline{D} \setminus \{\infty\}$. This definition can also be applied to the point $x_0 = \infty$ by inversion: $\varphi(x) = \frac{x}{|x|^2}$, $\infty \mapsto 0$. In what follows, h denotes the so-called chordal metric defined by (1.3). The next important lemma follows from [20, Theorems 4.1 and 4.2].

Lemma 2.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $Q : D \to [1, \infty]$ be a Lebesgue measurable function. In addition, let f_k, $k = 1, 2, \ldots$ be a sequence of homeomorphisms of D into $\mathbb{R}^n$, which satisfy conditions (2.2), (2.3) at any point $x_0 \in D$ that converges to some mapping $f : D \to \overline{\mathbb{R}^n}$ locally uniformly in D with respect to the chordal metric h. Assume that the function Q satisfies at least one of two following conditions: either $Q \in FMO(D)$, or*

$$\int_0^{\delta_0} \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t)} = \infty \tag{2.4}$$

for any $x_0 \in D$ and some $\delta_0 = \delta(x_0) > 0$, where

$$q_{x_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0, t)} Q(x)\, d\mathcal{H}^{n-1},$$

$\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$. Then the mapping f is either a homeomorphism $f : D \to \mathbb{R}^n$, or a constant $c \in \overline{\mathbb{R}^n}$.

Let I be a fixed set of indices and let D_i, $i \in I$, be some sequence of domains. Following [19, Sect. 2.4], we say that a family of domains $\{D_i\}_{i \in I}$ is *equi-uniform* if for any $r > 0$ there exists a number $\delta > 0$ such that the inequality

$$M(\Gamma(F^*, F, D_i)) \geqslant \delta \tag{2.5}$$

holds for any $i \in I$ and any continua $F, F^* \subset D$ such that $h(F) \geqslant r$ and $h(F^*) \geqslant r$. If D is a domain satisfying condition (2.5), then it is called *uniform*.

Given numbers $\delta > 0$, a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, a point $a \in D$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to \mathbb{R}^n$, $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$, denote by $\mathfrak{F}_{Q,a,\delta}(D)$ the family of all homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ satisfying (2.2), (2.3) in $\overline{D}$ such that $h(f(a), \partial f(D)) \geqslant \delta$, $h(\overline{\mathbb{R}^n} \setminus f(D)) \geqslant \delta$. The following statement holds (see [24, Theorem 2]).

Lemma 2.2. *Let D be regular, and let $D'_f = f(D)$ be bounded domains with locally quasiconformal boundary which are equi-uniform over all $f \in \mathfrak{F}_{Q,a,\delta}(D)$. If $Q \in FMO(D)$, or condition (2.4) holds, then any $f \in \mathfrak{F}_{Q,a,\delta}(D)$ has a continuous extension $\overline{f} : \overline{D}_P \to \overline{\mathbb{R}^n}$ and, in addition, the family $f \in \mathfrak{F}_{Q,a,\delta}(\overline{D})$ of all extended mappings $\overline{f} : \overline{D}_P \to \overline{\mathbb{R}^n}$ is equicontinuous in $\overline{D}_P$.*

We also need to formulate a similar statement for homeomorphisms inverse to (2.2). For this purpose, consider the following definition.

For domains $D \subset \mathbb{R}^n$ and $D' \subset \overline{\mathbb{R}^n}$, $n \geqslant 2$, points $a \in D$, $b \in D'$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ such that $Q(x) \equiv 0$ for $x \notin D$, we denote by $\mathfrak{G}_{a,b,Q}(D, D')$ the family of all homeomorphisms g of D' onto D such that the mapping $f = g^{-1}$ satisfies condition (2.2) in $\overline{D}$, while $f(a) = b$.

The boundary of a domain D is called *weakly flat at a point* x_0 if for every number $P > 0$ and every neighborhood U of this point there is a neighborhood V of the point x_0 such that $M(\Gamma(E, F, D)) > P$ for any continua E and F, satisfying conditions $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. The boundary of a domain D is called *weakly flat* if it is weakly flat at each of its points. The following statement holds (see, e.g., [25, Theorem 7.1]).

Lemma 2.3. *Assume that D is a regular domain and that D' has a weakly flat boundary, none of the components of which degenerates into a point. Then any $g \in \mathfrak{G}_{a,b,Q}(D, D')$ has a continuous extension $\overline{g} : \overline{D'} \to \overline{D}_P$, while $\overline{g}(\overline{D'}) = \overline{D}_P$ and, in addition, the family $\mathfrak{G}_{a,b,Q}(\overline{D}, \overline{D'})$ of all extended mappings $\overline{g} : \overline{D'} \to \overline{D}_P$ is equicontinuous in $\overline{D'}$.*

3. Proof of Theorem 1.1

In general, we will use the scheme of proving Theorem 1.2 in [23].

I. Let $f_m \in \mathfrak{F}_{\varphi,M,z_0}(D)$, $m = 1, 2, \ldots$. By Stoilow's factorization theorem (see, e.g., [26, 5(III).V]), a mapping f_m has a representation

$$f_m = \varphi_m \circ g_m, \tag{3.1}$$

where g_m is a homeomorphism, and φ_m is an analytic function. By Lemma 1 in [22], the mapping g_m belongs to the Sobolev class $W^{1,1}_{\rm loc}(D)$ and has a finite distortion. Moreover, by [1, (1), Chap. I],

$$f_{m_z}=\varphi_{m_z}(g_m(z))g_{m_z}, \quad f_{m_{\overline{z}}}=\varphi_{m_z}(g_m(z))g_{m_{\overline{z}}} \tag{3.2}$$

for almost all $z\in D$. Therefore, by the relation (3.2), $J(z,g_m)\neq 0$ for almost all $z\in D$, in addition, $K_{\mu_{f_m}}(z)=K_{\mu_{g_m}}(z)$.

II. We prove that $\partial g_m(D)$ contains at least two points. Suppose the contrary. Then either $g_m(D)=\mathbb{C}$, or $g_m(D)=\mathbb{C}\setminus\{a\}$, where $a\in\mathbb{C}$. Consider first the case $g_m(D)=\mathbb{C}$. By Picard's theorem, $\varphi_m(g_m(D))$ is the whole plane, except, perhaps, one point $\omega_0\in\mathbb{C}$. On the other hand, for every $m=1,2,\ldots$ the function $u_m(z):={\rm Re}\,f_m(z)={\rm Re}(\varphi_m(g_m(z)))$ is continuous on the compact set $\overline{D}$ under condition (1.7) by the continuity of $\varphi: E_D\rightarrow\mathbb{R}$. Therefore, there exists $C_m>0$ such that $|{\rm Re}\,f_m(z)|\leqslant C_m$ for any $z\in D$, but this contradicts the fact that $\varphi_m(g_m(D))$ contains all points of the complex plane except, perhaps, one point. The situation $g_m(D)=\mathbb{C}\setminus\{a\}$, $a\in\mathbb{C}$, is also impossible since the domain $g_m(D)$ must be simply connected in $\mathbb{C}$ as a homeomorphic image of the simply connected domain D.

Therefore, the boundary of the domain $g_m(D)$ contains at least two points. Then, according to Riemann's mapping theorem, we can transform the domain $g_m(D)$ onto the unit disk $\mathbb{D}$ using the conformal mapping ψ_m. Let $z_0\in D$ be a point from the condition of the theorem. By using an auxiliary conformal mapping

$$\widetilde{\psi}_m(z)=\frac{z-(\psi_m\circ g_m)(z_0)}{1-z\overline{(\psi_m\circ g_m)(z_0)}}$$

of the unit disk onto itself, we may consider that $(\psi_m\circ g_m)(z_0)=0$. Now, by (3.1), we obtain that

$$f_m=\varphi_m\circ g_m=\varphi_m\circ\psi_m^{-1}\circ\psi_m\circ g_m=F_m\circ G_m,\quad m=1,2,\ldots,$$

where $F_m:=\varphi_m\circ\psi_m^{-1}$, $F_m:\mathbb{D}\rightarrow\mathbb{C}$, and $G_m=\psi_m\circ g_m$. Obviously, a function F_m is analytic, and G_m is a regular Sobolev homeomorphism in D. In particular, ${\rm Im}\,F_m(0)=0$ for any $m\in\mathbb{N}$.

III. Observe that

$$\int\limits_D K_{\mu_{G_m}}(z)\,dm(z)\leqslant\int\limits_D Q_M(z)\,dm(z)<\infty, \tag{3.3}$$

because the condition $\mu(z)\in M(z)$ implies that $K_{\mu_{G_m}}(z)\leqslant Q_M(z)$ holds for almost any $z\in D$, where $Q_M(z)$ does not depend on $m=1,2,\ldots$ and is integrable by the assumption.

IV. We prove that each map G_m, $m=1,2,\ldots$, has a continuous extension to E_D, in addition, the family of extended maps $\overline{G}_m$, $m=1,2,\ldots$, is equicontinuous in $\overline{D}_P$. Indeed, as proved in item **III**, $K_{\mu_{G_m}}\in L^1(D)$. By [9, Theorem 3] (see also [15, Theorem 3.1]), each G_m, $m=1,2,\ldots$, is a ring Q-homeomorphism in $\overline{D}$ for $Q=K_{\mu_{G_m}}(z)$, where μ is defined in (1.6), and K_μ can be calculated by the formula (1.1). Notice that the unit disk $\mathbb{D}$ is a uniform domain as a finitely connected flat domain at its boundary with a finite number of boundary components (see, for example, [17, Theorem 6.2 and Corollary 6.8]). Then the desirable conclusion is a statement of Lemma 2.2.

V. Observe that the inverse homeomorphisms G_m^{-1}, $m = 1, 2, \dots$, have a continuous extension $\overline{G}_m^{-1}$ to $\partial\mathbb{D}$ as mappings from $\overline{\mathbb{D}}$ into $\overline{D}_P$ and $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ is equicontinuous in $\overline{\mathbb{D}}$. Indeed, by the item **IV**, mappings G_m, $m = 1, 2, \dots$, are ring $K_{\mu_{G_m}}(z)$-homeomorphisms in D such that $G_m^{-1}(0) = z_0$ for any $m = 1, 2, \dots$. In this case, the possibility of a continuous extension of G_m^{-1} to $\partial\mathbb{D}$, and the equicontinuity of $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ as mappings $G_m^{-1} : \overline{\mathbb{D}} \to \overline{D}_P$ follows from Lemma 2.3.

VI. Since, as proved above, the family $\{G_m\}_{m=1}^{\infty}$ is equicontinuous in D, by the Arzela–Ascoli criterion, there exists an increasing subsequence of numbers m_k, $k = 1, 2, \dots$ such that G_{m_k} converges locally uniformly in D to some continuous mapping $G : D \to \overline{\mathbb{C}}$ as $k \to \infty$ (see, e.g., [27, Theorem 20.4]). By Lemma 2.1, either G is a homeomorphism with values in $\mathbb{R}^n$, or a constant in $\overline{\mathbb{R}^n}$. Let us prove that the second case is impossible. We apply the approach used in the proof of the second part of Theorem 21.9 in [27]. Suppose the contrary: let $G_{m_k}(x) \to c = \text{const}$ as $k \to \infty$. Since $G_{m_k}(z_0) = 0$ for all $k = 1, 2, \dots$, we have that $c = 0$. By the item **V**, the family of mappings G_m^{-1}, $m = 1, 2, \dots$, is equicontinuous in $\mathbb{D}$. Then

$$h(z, G_{m_k}^{-1}(0)) = h(G_{m_k}^{-1}(G_{m_k}(z)), G_{m_k}^{-1}(0)) \to 0$$

as $k \to \infty$, which is impossible because z is an arbitrary point of the domain D. The obtained contradiction refutes the assumption made above. Thus, $G : D \to \mathbb{C}$ is a homeomorphism.

VII. According to **V**, the family of mappings $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ is equicontinuous in $\overline{\mathbb{D}}$. By the Arzela-Ascoli criterion (see, e.g., [27, Theorem 20.4]), we may consider that $\overline{G}_{m_k}^{-1}(y)$, $k = 1, 2, \dots$, converges to some mapping $\widetilde{F} : \overline{\mathbb{D}} \to \overline{D}$ as $k \to \infty$ uniformly in $\overline{D}$. Let us prove that $\widetilde{F} = \overline{G}^{-1}$. For this purpose, we show that $G(D) = \mathbb{D}$. Fix $y \in \mathbb{D}$. Since $G_{m_k}(D) = \mathbb{D}$ for every $k = 1, 2, \dots$, we obtain that $G_{m_k}(x_k) = y$ for some $x_k \in D$. Since D is regular, the metric space $(\overline{D}_P, \rho)$ is compact. Thus, we may assume that $\rho(x_k, x_0) \to 0$ as $k \to \infty$, where $x_0 \in \overline{D}_P$. By the triangle inequality and the equicontinuity of $\{\overline{G}_m\}_{m=1}^{\infty}$ onto $\overline{D}_P$, see **IV**, we obtain that

$$|\overline{G}(x_0) - y| = |\overline{G}(x_0) - \overline{G}_{m_k}(x_k)| \leqslant |\overline{G}(x_0) - \overline{G}_{m_k}(x_0)| + |\overline{G}_{m_k}(x_0) - \overline{G}_{m_k}(x_k)| \to 0$$

as $k \to \infty$. Hence, $\overline{G}(x_0) = y$. Observe that $x_0 \in D$ because G is a homeomorphism. Since $y \in \mathbb{D}$ is arbitrary, the equality $G(D) = \mathbb{D}$ is proved. In this case, $G_{m_k}^{-1} \to G^{-1}$ locally uniformly in $\mathbb{D}$ as $k \to \infty$ (see, e.g., [20, Lemma 3.1]). Thus, $\widetilde{F}(y) = G^{-1}(y)$ for every $y \in \mathbb{D}$.

Finally, since $\widetilde{F}(y) = G^{-1}(y)$ for any $y \in \mathbb{D}$ and, in addition, $\widetilde{F}$ has a continuous extension on $\partial\mathbb{D}$, due to the uniqueness of the limit at the boundary points, we obtain that $\widetilde{F}(y) = \overline{G}^{-1}(y)$ for $y \in \overline{\mathbb{D}}$. Therefore, we have proved that $\overline{G}_{m_k}^{-1} \to$

$\overline{G}^{-1}$ uniformly in $\overline{\mathbb{D}}$ as $k\to\infty$ with respect to the metrics ρ in $\overline{D}_P$.

VIII. By **VII**, for $y=e^{i\theta}\in\partial\mathbb{D}$,

$$\operatorname{Re} F_{m_k}(e^{i\theta})=\varphi\left(\overline{G}_{m_k}^{-1}(e^{i\theta})\right)\to\varphi\left(\overline{G}^{-1}(e^{i\theta})\right) \tag{3.4}$$

as $k\to\infty$ uniformly on $\theta\in[0,2\pi)$. Since, by the construction, $\operatorname{Im} F_{m_k}(0)=0$ for any $k=1,2,\ldots$, by the Schwartz formula (see, e.g., [5, Section 8.III.3]), the analytic function F_{m_k} is uniquely restored by its real part, namely,

$$F_{m_k}(y)=\frac{1}{2\pi i}\int\limits_{S(0,1)}\varphi\left(\overline{G}_{m_k}^{-1}(t)\right)\frac{t+y}{t-y}\,\frac{dt}{t}. \tag{3.5}$$

Set

$$F(y):=\frac{1}{2\pi i}\int\limits_{S(0,1)}\varphi\left(\overline{G}^{-1}(t)\right)\frac{t+y}{t-y}\,\frac{dt}{t}. \tag{3.6}$$

Let $K\subset\mathbb{D}$ be an arbitrary compact set, and let $y\in K$. By (3.5) and (3.6), we obtain that

$$|F_{m_k}(y)-F(y)|\leqslant\frac{1}{2\pi}\int\limits_{S(0,1)}\left|\varphi(\overline{G}_{m_k}^{-1}(t))-\varphi(\overline{G}^{-1}(t))\right|\left|\frac{t+y}{t-y}\right|\,|dt|. \tag{3.7}$$

Since K is compact, there is $0<R_0=R_0(K)<\infty$ such that $K\subset B(0,R_0)$. By the triangle inequality, $|t+y|\leqslant 1+R_0$ and $|t-y|\geqslant|t|-|y|\geqslant 1-R_0$ for $y\in K$ and any $t\in\mathbb{S}^1$. Thus,

$$\left|\frac{t+y}{t-y}\right|\leqslant\frac{1+R_0}{1-R_0}:=M=M(K).$$

Put $\varepsilon>0$. By (3.4), for a number $\varepsilon':=\frac{\varepsilon}{M}$ there is $N=N(\varepsilon,K)\in\mathbb{N}$ such that $\left|\varphi\left(\overline{G}_{m_k}^{-1}(t)\right)-\varphi\left(\overline{G}^{-1}(t)\right)\right|<\varepsilon'$ for any $k\geqslant N(\varepsilon)$ and $t\in\mathbb{S}^1$. Now, by (3.7),

$$|F_{m_k}(y)-F(y)|<\varepsilon,\quad k\geqslant N. \tag{3.8}$$

It follows from (3.8) that the sequence F_{m_k} converges to F as $k\to\infty$ in the unit disk locally uniformly. In particular, we obtain that $\operatorname{Im} F(0)=0$. Notice that F is an analytic function in $\mathbb{D}$ (see the remarks made at the end of item 8.III in [5]), and

$$\operatorname{Re} F(re^{i\psi})=\frac{1}{2\pi}\int\limits_0^{2\pi}\varphi\left(\overline{G}^{-1}(e^{i\theta})\right)\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,d\theta$$

for $z=re^{i\psi}$. By [5, Theorem 2.10.III.3],

$$\lim_{\zeta\to z}\operatorname{Re} F(\zeta)=\varphi(\overline{G}^{-1}(z)),\quad z\in\partial\mathbb{D}. \tag{3.9}$$

Observe that F is either constant or open and discrete (see, e.g., [26, Chap. V, I.6 and II.5]). Thus, $f_{m_k}=F_{m_k}\circ G_{m_k}$ converges to $f=F\circ G$ locally uniformly as $k\to\infty$, where $f=F\circ G$ either is a constant or open and discrete. Moreover, by (3.9),

$$\lim_{\zeta\to P}\operatorname{Re} f(\zeta)=\lim_{\zeta\to P}\operatorname{Re} F(G(\zeta))=\varphi(G^{-1}(G(P)))=\varphi(P).$$

IX. Since, by **VI**, G is a homeomorphism, by [13, Lemma 1 and Theorem 1], G is a regular solution of equation (1.6) for some function $\mu : \mathbb{C} \to \mathbb{D}$. Since the set of points of the function F, where its Jacobian is zero, consist only of isolated points (see [26, Ch. V, 5.II and 6.II]), f is a regular solution of the Dirichlet problem (1.6), (1.7) whenever $F \not\equiv \text{const}$. It remains to show that $\mu \in \mathfrak{M}_M$. If $f(z) = c = \text{const}$ in D, due to the condition (1.7), we obtain that $f_n(z) = c$ in D, and $\mu_n(z) = 0 \in M(z)$ for almost any $z \in D$. In this case, $\mu(z) = 0$ for almost any $z \in D$, in particular, $\mu \in \mathfrak{M}_M$.

Let $f(z) \neq \text{const}$. As proved above, f is regular. Since $f_n(z)$ converge to $f(z)$ locally uniformly in D, in addition, the Jacobian of f does not vanish almost everywhere, by [13, Lemma 1], $\mu(z) \in \operatorname{inv}\operatorname{co} M_0(z)$ for almost any $z \in D$, where $\operatorname{inv}\operatorname{co} A$ denotes the invariant-convex hull of the set $A \subset \mathbb{C}$ (see, e.g., [21]), and $M_0(z)$ is a cluster set of $\mu_n(z)$, $n = 1, 2, \ldots$. Obviously, there is a set $D_0 \subset D$ such that $\mu_n(z) \in M(z)$ and $\mu(z) \in \operatorname{inv}\operatorname{co} M_0(z)$ for all $z \in D_0$ and any $n \in \mathbb{N}$, where $m(D \setminus D_0) = 0$. Fix $z_0 \in D_0$. Let $w_0 \in M_0(z_0)$. Then there is a subsequence of numbers n_k, $k = 1, 2, \ldots$, for which $\mu_{n_k}(z_0)$ converge as $k \to \infty$ and $\lim_{k\to\infty} \mu_{n_k}(z_0) = w_0$. Since, by the assumption, $\mu_{n_k}(z_0) \in M(z_0)$ for any $k = 1, 2, \ldots$, in addition, $M(z_0)$ is closed, we obtain that $w_0 \in M(z_0)$. Thus,

$$M_0(z_0) \subset M(z_0). \tag{3.10}$$

It follows from (3.10) that

$$\operatorname{inv}\operatorname{co} M_0(z_0) \subset M(z_0) \tag{3.11}$$

because $M(z_0)$ is invariant-convex. Thus,

$$\mu(z_0) \in \operatorname{inv}\operatorname{co} M_0(z_0) \subset M(z_0)$$

for almost any $z_0 \in D$, that is a desired conclusion. □

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Про компактність одного класу розв'язків задачі Діріхле

Evgeny Sevost'yanov and Oleksandr Dovhopiatyi

Ми розглядаємо задачу Діріхле для рівняння Бельтрамі у довільній обмеженій однозв'язній області комплексної площини $\mathbb{C}$. Само, вивчається клас усіх регулярних розв'язків цієї задачі з умовами нормування і теоретико-множинними обмеженнями на їх комплексну характеристику. Доведена компактність цього класу в термінах простих кінців за наявності довільної неперервної функції в умові Діріхле.

Ключові слова: рівняння Бельтрамі, прості кінці, плоскі відображення зі скінченним та обмеженим спотворенням