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On the Relative Decay of Unbounded Semigroups on the Domain of the Generator

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We study the asymptotics of C_0-semigroups on the domain of the generator. In particular, we analyze the behavior of $\|T(t)(A-\lambda I)^{-1}\|$ as time goes to infinity. Our results extend some existing results to the case when the intersection of the spectrum of the generator with the imaginary axis is non-empty. We also give a constructive example of a class of unbounded C_0-semigroups with pure imaginary point spectrum for which our theorem is applicable.

Key words: C_0-semigroups, Asymptotic behavior

Mathematical Subject Classification 2020: 47D06

1. Introduction and background

Consider the abstract Cauchy problem (ACP) given in a Banach space X:

$$\begin{cases} \dot{x}(t) = Ax(t), \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Before formulating our results, we recall some basic concepts concerning (1.1). A is a linear operator on the space X with domain $D(A)$ and solutions of such equations are closely related to the concept of strongly continuous semigroups, also called C_0-semigroups.

Definition 1.1 ([1, I, Definition 1.1.1]). A family $T = \{T(t)\}_{t\geq 0}$ of bounded linear operators acting on a Banach space X is called a C_0-semigroup if the following three properties are satisfied:

- $T(0) = I$, the identity operator on X;
- $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$;
- $\lim\limits_{t\downarrow 0}\|T(t)x - x\| = 0$ for all $x \in X$.

Clearly, the maps $t \to T(t)x$ are continuous for $t \geq 0$ for all $x \in X$. The generator of T is the linear operator A with domain $D(A)$ defined by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0}\frac{1}{t}(T(t)x - x) \text{ exists} \right\},$$

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$$Ax = \lim_{t\downarrow 0}\frac{1}{t}(T(t)x - x), \quad x \in D(A).$$

One can easily see that for $x \in D(A)$, one has

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \quad x \in D(A),$$

which means that $t \to T(t)x$ is a classical solution of the (1.1) with the initial condition $x \in D(A)$. An interesting and key for applications property of dynamical systems is their asymptotic behavior. In this paper, we will deal with the asymptotic behavior of families of semigroups $\{T(t)\}_{t\geq 0}$ in the sense of the behavior of the norm $\|T(t)\|$ as $t \to \infty$. A semigroup $\{T(t)\}_{t\geq 0}$ is called *uniformly stable* if $\|T(t)\| \to 0$ as $t \to \infty$, and *strongly stable* if $\|T(t)x\| \to 0$ as $t \to \infty$ for all x in the Banach space X. A critical quantity for asymptotic behavior which characterizes the semigroup is its *growth bound* $\omega_0(T)$, which gives a restraint on how much the norm of the semigroup $T = \{T(t)\}_{t\geq 0}$ grows or decays with time.

Definition 1.2 ([2, IV, Definition 2.1])**.** For a strongly continuous semigroup $T = \{T(t)\}_{t\geq 0}$, we call

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \ \forall t \geq 0 \ \ \|T(t)\| \leq M_\omega e^{\omega t}\}$$

its growth bound.

It is well-known that $e^{\omega_0 t} \leq \|T(t)\|$. Now we will state a crucial theorem concerning strong stability of C_0-semigroups. It was first proved for the case of a bounded generator by the first author and Shirman in 1982 [3] and later extended to the unbounded case independently in [4, 5].

Theorem 1.3. *Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X and let*

$$\sigma(A) \cap (i\mathbb{R}) \quad \text{be at most countable,}$$

then the semigroup $\{T(t)\}_{t\geq 0}$ is strongly asymptotically stable, i.e.,

$$\lim_{t\to+\infty} \|T(t)x\| = 0 \quad \text{for all } x \in X,$$

if and only if the adjoint operator A^* has no purely imaginary eigenvalues.

The asymptotic behavior of semigroups and their orbits has been a subject of an intense study for the last few decades, see, e.g. [6–10]. It follows from Theorem 1.3 that, for a bounded semigroup, if $\sigma(A)$ (the spectrum of the generator) is contained in the open left-half plane $\{z \in \mathbb{C} : \Re(z) < 0\}$, the semigroup is strongly stable. As a consequence of the the uniform boundedness principle, if the growth bound $\omega_0(T) = 0$, this stability cannot be uniform. However, due to the works [11, 12], we have the following theorem:

Theorem 1.4. *Let $T = \{T(t)\}_{t\geq 0}$ be a bounded C_0-semigroup acting on a Banach space X, and let A be its generator. Then $\|T(t)A^{-1}\| \to 0$ as $t \to +\infty$ if and only if the intersection of the spectrum of the generator A with the imaginary axis is empty.*

The above means that for a bounded semigroup T for which

$$\sigma(A) \subset \{z \in \mathbb{C} : \Re(z) < 0\}, \tag{1.2}$$

the orbits starting in the domain of the generator are dominated uniformly up to the multiplication by a constant by a decaying function $f(t) = \|T(t)A^{-1}\|$, i.e. $\|T(t)x\| \leq f(t)C_x$ for all $x \in D(A)$, where $C_x = \|Ax\|$. This can be easily seen by writing $\|T(t)x\| = \|T(t)A^{-1}Ax\|$. With this being the case, following [12], we call the semigroup *semi-uniformly stable*. The semi-uniform stability may occur even for unbounded semigroups (see [13] for example). For the case of unbounded semigroups, it was shown in [14] that the condition (1.2) remains necessary for $\|T(t)A^{-1}\| \to 0$. We note here that the sufficiency for C_0-semigroups of contractions has been proved independently in [15]. The mentioned results for bounded semigroups were later extended, keeping $\omega_0(T) = 0$, to the unbounded case in [16] to obtain

$$\lim_{t\to+\infty} \frac{1}{f(t)}\|T(t)A^{-1}\| = 0 \tag{1.3}$$

for a class of so-called weight functions ($f(s+t) \leq f(s)f(t)$) dominating the semigroup norm and satisfying some additional assumptions. The proof of (1.3) is based on results from [17] which required that Fourier transforms of functions converging to $e^{-\lambda t}$, $\lambda \notin \sigma(A)$, vanish on open neighborhoods of $\sigma(A) \cap (i\mathbb{R})$, thus requiring $\sigma(A) \cap (i\mathbb{R}) = \varnothing$.

In this paper, we extend the result of [16] to the case $\sigma(A) \cap (i\mathbb{R}) \neq \varnothing$. Our main result is given in Section 2 (Theorem 2.1). In this theorem, we show that the property (1.3) (with A^{-1} replaced by the resolvent R_μ, $\mu \notin \sigma(A)$) holds for some class of operators with not necessarily empty set of pure imaginary points of spectrum. We require that the dominating function f satisfies the property

$$\limsup_{s\to+\infty} \frac{f(t+s)}{f(s)} = 1, \quad t \geq 0. \tag{1.4}$$

In the proof, we use different tools, which are based on some development of ideas given in [14]. It is worth noting that in [14] the author constructed a weight function dominating the norm of the semigroup and satisfying (1.4) for an arbitrary semigroup. The constructed function is monotonic and is similar to $\|T(t)\|$ in sense that $f(t_n) = \|T(t_n)\|$ holds for some unbounded sequence $t_n \in \mathbb{R}^+$. Moreover, for sufficiently regular semigroups, we show (see Corollary 2.2) that the assertion (1.3) takes the following form

$$\lim_{t\to+\infty} \frac{\|T(t)R_\mu\|}{\|T(t)\|} = 0 \quad \text{for any } \mu \notin \sigma(A), \tag{1.5}$$

where by R_μ we mean the resolvent of the semigroup generator at the point $\mu \notin \sigma(A)$.

The rest of the paper is organized as follows: Section 3 is devoted to the example of a family of semigroups with polynomial growth constructed in [23,24]. This operator has pure imaginary countable unbounded spectrum and satisfies condition (1.5), so the Theorem 2.1 is applicable. In the appendix (Section 4) we give calculations of exact asymptotics of $\frac{\|T(t)R_\mu\|}{\|T(t)\|}$ for the simplest case described in Section 3.

2. Main result

Now we will state the following theorem which is the main result of this work:

Theorem 2.1. *Let $T = \{T(t)\}_{t\geq 0}$ be a semigroup on a Banach space X, not necessarily bounded, with the growth bound $\omega_0(T) = 0$ and the generator A. Suppose $f(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive function such that*

$$\limsup_{s\to+\infty} \frac{f(t+s)}{f(s)} = 1, \quad t \geq 0, \tag{2.1}$$

$$\|T(t)\| \leq f(t), \quad t \geq 0. \tag{2.2}$$

Assume further that

(a) *for any $\lambda \in \sigma(A) \cap (i\mathbb{R})$, there exists a regular bounded curve Γ_λ enclosing λ, such that $\Gamma_\lambda \cap \sigma(A) = \varnothing$;*

(b) *for any $\lambda \in \sigma(A) \cap (i\mathbb{R})$,*

$$\lim_{t\to+\infty} \frac{\|T(t)P_{\Gamma_\lambda}\|}{f(t)} = 0, \tag{2.3}$$

where P_{Γ_λ} is the Riesz projection associated with the curve P_{Γ_λ}.

Then

$$\lim_{t\to+\infty} \frac{\|T(t)R_\mu\|}{f(t)} = 0, \tag{2.4}$$

for any fixed $\mu \notin \sigma(A)$.

Preliminary Remarks. The Theorem can be proved for an arbitrary $\omega_0 > -\infty$ by considering the shifted semigroup $\{e^{-\omega_0 t}T(t)\}_{t\geq 0}$. For this more general case, one has simply to change 1 to $e^{\omega_0 t}$ in (2.1) and consider the set $\sigma(A) \cap (\omega_0 + i\mathbb{R})$ instead of $\sigma(A) \cap (i\mathbb{R})$ in assumptions (a) and (b). For the bounded semigroups with $\omega_0 = 0$, the condition $\sigma(A) \cap (i\mathbb{R}) = \varnothing$ is sufficient and necessary (cf. Theorem 1.4) for (1.5) to hold. In the more general case of not necessarily bounded semigroups, this condition is no longer necessary as shown in Example 2.3, it is however sufficient since the conditions (a) and (b) are satisfied in a trivial way for any function satisfying (2.1) and (2.2). The *best candidate* for f would be the norm of the semigroup $\|T(t)\|$, but it may fail to satisfy (2.1). However, in the work [14], the author has constructed a function satisfying (2.1) and (2.2) which is similar to $\|T(t)\|$ in the sense that $f(t_k) = \|T(t_k)\|$ for some unbounded sequence $t_k \in \mathbb{R}^+$. This construction can be applied to any semigroup with $\omega_0(T) = 0$ and the constructed function is the minimal of all functions with concave downwards logarithm.

Proof of Theorem 2.1. We use here the construction of the special operator-valued semigroup introduced in [14]. We note here that a similar idea has already been used in [15, 16]. Let $\widetilde{X} \subset \mathcal{L}(X)$ (the space of bounded linear operators on X) be defined as

$$\widetilde{X} = \overline{\{DR_\mu,\ D \in \mathcal{L}(X)\}}, \quad \mu \notin \sigma(A),$$

for arbitrary $\mu \notin \sigma(A)$, where $\overline{Q}$ denotes the closure of the linear set Q (with respect to the operator norm). Since $\widetilde{X}$ is a closed subspace of a Banach space $\mathcal{L}(X)$, it also is a Banach space. It is clear that $\widetilde{X}$ does not depend on the choice of μ. For the given semigroup $\{T(t)\}_{t\geq 0}$ on the space X, let us introduce a semigroup on the space $\widetilde{X}$ by:

$$\widetilde{T}(t)\widetilde{B} = \widetilde{B}T(t), \quad \widetilde{B} \in \widetilde{X}, \quad t \geq 0. \tag{2.5}$$

Important properties of this semigroup were shown in [14], namely that $\{\widetilde{T}(t)\}_{t\geq 0}$ forms a C_0-semigroup on $\widetilde{X}$, and that

(a) for A and $\widetilde{A}$ being the generators of $\{T(t)\}_{t\geq 0}$ and $\{\widetilde{T}(t)\}_{t\geq 0}$, respectively, it holds that
$$\sigma(\widetilde{A}) \subset \sigma(A); \tag{2.6}$$

(b) for any $\widetilde{B} \in \widetilde{X}$ and any $\mu \notin \sigma(A)$, it holds that
$$(\widetilde{A} - \mu I)^{-1}\widetilde{B} = \widetilde{B}(A - \mu I)^{-1}. \tag{2.7}$$

We note here that the proof of Theorem 2.1 is based on the idea of analyzing the behavior of the semigroup truncated to the images of the Riesz projections from [20]. Now, assume that (2.4) does not hold, which means that

$$0 \neq \limsup_{t\to+\infty} \frac{\|T(t)R_\mu\|}{f(t)} = \limsup_{t\to+\infty} \frac{\|R_\mu T(t)\|}{f(t)} = \limsup_{t\to+\infty} \frac{\|\widetilde{T}(t)R_\mu\|}{f(t)}. \tag{2.8}$$

Let us define a following seminorm on $\widetilde{X}$:

$$l(\widetilde{B}) = \limsup_{t\to+\infty} \frac{\|\widetilde{T}(t)\widetilde{B}\|}{f(t)}, \quad \widetilde{B} \in \widetilde{X},$$

which is well-defined due to (2.2). The technique of constructing a seminorm and using the quotient space defined by it appeared first in [3] and has been further developed in many papers, such as [4, 9, 15, 16, 20]. It follows from (2.8) that the quotient space $\widetilde{X}/\ker l = \{\widehat{B} = \widetilde{B} + \ker l : \widetilde{B} \in \widetilde{X}\}$ is non-zero. This space can be equipped with a norm different from the natural one ($\|\widehat{B}\|_N := \inf\{\|\widetilde{B}\| : \widetilde{B} \in \widehat{B}\}$) of the following form

$$\|\widehat{B}\|' := l(\widetilde{B}), \quad \widetilde{B} \in \widetilde{X}.$$

The space $(\widetilde{X}/\ker l, \|\cdot\|')$ may be incomplete. Its completion with respect to the norm $\|\cdot\|'$ is denoted by $\widehat{X}$. Let us define the family of operators $\widehat{T}(t)$, $t \geq 0$, by the formula

$$\widehat{T}(t)\widehat{B} = \widetilde{T}(t)\widetilde{B} + \ker l, \quad \widehat{B} \in \widetilde{X}/\ker l \subset \widehat{X}.$$

By applying the property (2.1) for $\omega_0 = 0$, we get

$$\|\widehat{T}(t)\widehat{B}\|' = \limsup_{s\to+\infty} \frac{\|\widetilde{T}(t+s)\widetilde{B}\|}{f(t+s)} \frac{f(t+s)}{f(s)} = \|\widehat{B}\|' \quad \text{for } \widehat{B} \in \widetilde{X}/\ker l.$$

Thus, $\{\widehat{T}(t)\}_{t\geq 0}$ is a family of isometries on $\widetilde{X}/\ker l$ with respect to the norm $\|\cdot\|'$. It is easy to check that for each $t \geq 0$, $\widehat{T}(t)$ extends to an isometry on $\widehat{X}$ and the family $\{\widehat{T}(t)\}_{t\geq 0}$ is a C_0-semigroup of isometries. Moreover, one can check that

$$\begin{aligned} \widehat{A}\widehat{B} &= \widetilde{A}\widetilde{B} + \ker l, \text{ and} \\ R(\widehat{A},\mu)\widehat{B} &= R(\widetilde{A},\mu)\widetilde{B} + \ker l \end{aligned} \tag{2.9}$$

for $\widehat{B} \in \widetilde{X}$, where $\widetilde{A}$ and $\widehat{A}$ are generators of $\{\widetilde{T}(t)\}_{t\geq 0}$ and $\{\widehat{T}(t)\}_{t\geq 0}$, respectively, and $R(\widetilde{A},\mu)$ and $R(\widehat{A},\mu)$ are the respective resolvent operators at the point μ. Now, due to condition (a) of the Theorem, there are points on the imaginary axis not contained in $\sigma(A)$. Thus we get

$$(i\mathbb{R}) \not\subset \sigma(A).$$

This, taking into account (2.6), gives

$$(i\mathbb{R}) \not\subset \sigma(\widetilde{A}). \tag{2.10}$$

Further, it can be shown analogously as in [21,22] that

$$\partial(\sigma(\widehat{A})) \cap (i\mathbb{R}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}),$$

where ∂ denotes the boundary of a set. This along with (2.10) and the fact that $\widehat{A}$ is a generator of a semigroup of isometries implies that

$$\partial\sigma(\widehat{A}) = \sigma(\widehat{A}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}) \neq (i\mathbb{R}) \tag{2.11}$$

and that $\{\widehat{T}(t)\}_{t\geq 0}$ extends to a C_0-group of isometries (see, e.g. IV, Lemma 2.19 from [2]). Now, since $\widehat{A}$ is a generator of a C_0-group of isometries, its spectrum has to be non-empty (see, e.g. V, Theorem 5.1.2 from [1])

$$\sigma(\widehat{A}) \neq \varnothing.$$

By combining the above with (2.11) and (2.6), we obtain:

$$\varnothing \neq \sigma(\widehat{A}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}) \subset \sigma(A) \cap (i\mathbb{R}). \tag{2.12}$$

Note that for the case $\sigma(A) \cap (i\mathbb{R}) = \varnothing$, we obtain here a contradiction. This means that

$$\lim_{t \to +\infty} \frac{\|T(t)R_\mu\|}{f(t)} = 0.$$

Now assume $\sigma(A) \cap (i\mathbb{R}) \neq \varnothing$. Let us fix λ such that

$$\lambda \in \sigma(\widehat{A}) \subset \sigma(A) \cap (i\mathbb{R}).$$

It follows from the condition (a) of the theorem that there exists a bounded curve Γ_λ enclosing λ such that

$$\Gamma_\lambda \cap \sigma(\widehat{A}) = \Gamma_\lambda \cap \sigma(\widetilde{A}) = \Gamma_\lambda \cap \sigma(A) = \varnothing.$$

Let $\widetilde{P}_{\Gamma_\lambda}$ and $\widehat{P}_{\Gamma_\lambda}$ be the Riesz projections in $\widetilde{X}$ and $\widehat{X}$, respectively, corresponding to the curve Γ_λ. One can see from (2.9) that for $\widehat{B} \in \widetilde{X}/\ker l$, we have

$$\widehat{P}_{\Gamma_\lambda}\widehat{B} = \widetilde{P}_{\Gamma_\lambda}\widetilde{B} + \ker l. \tag{2.13}$$

Furthermore, the projections $\widetilde{P}_{\Gamma_\lambda}$ and $\widehat{P}_{\Gamma_\lambda}$ split the spaces $\widetilde{X}$ and $\widehat{X}$ into direct sums $\widetilde{Z}_1 + \widetilde{Z}_2$ and $\widehat{Z}_1 + \widehat{Z}_2$, respectively, so that

$$\begin{aligned}
\widetilde{Z}_1 &:= \widetilde{P}_{\Gamma_\lambda}\widetilde{X}, \\
\widetilde{Z}_2 &:= (I - \widetilde{P}_{\Gamma_\lambda})\widetilde{X}, \\
\widehat{Z}_1 &:= \widehat{P}_{\Gamma_\lambda}\widehat{X}, \\
\widehat{Z}_2 &:= (I - \widehat{P}_{\Gamma_\lambda})\widehat{X}.
\end{aligned}$$

Clearly, the spectra of the restricted operators $\widetilde{A}|_{\widetilde{Z}_1}$ and $\widetilde{A}|_{\widetilde{Z}_2}$ are intersections of $\sigma(\widetilde{A})$ with regions inside and outside Γ_λ, respectively, with an analogous property for $\sigma(\widehat{A})$. Now, since the set $\sigma(\widehat{A})$ is a boundary set, it consists only of approximate eigenvalues (see, e.g. IV, Proposition 1.10, [2]). This means that for the chosen λ there exists a sequence $\{\widehat{B}_k : \|\widehat{B}_k\|' = 1\}$ such that

$$\|\widehat{A}\widehat{B}_k - \lambda\widehat{B}_k\|' \to 0 \text{ as } k \to \infty. \tag{2.14}$$

Now, $\{\widehat{B}_k\}$ can be split into a sequence

$$\widehat{B}_k = \widehat{B}_k^{(1)} + \widehat{B}_k^{(2)},$$

where

$$\widehat{B}_k^{(1)} \in \widehat{Z}_1, \quad \widehat{B}_k^{(2)} \in \widehat{Z}_2.$$

Then it follows from (2.14) that

$$\begin{aligned}
\left\|\widehat{A}\widehat{B}_k^{(1)} - \lambda\widehat{B}_k^{(1)}\right\|' &\to 0, \\
\left\|\widehat{A}\widehat{B}_k^{(2)} - \lambda\widehat{B}_k^{(2)}\right\|' &\to 0
\end{aligned}$$

as $k \to \infty$. Subsequently,

$$\left\| \widehat{B}_k^{(2)} \right\| \to 0$$

since otherwise λ would belong to $\sigma\left(\widehat{A}|_{\widehat{Z}_2}\right)$ giving a contradiction. In consequence,

$$\left\| \widehat{B}_k^{(1)} \right\|' \geq \frac{1}{2}$$

for k large enough. Furthermore, by the density of $\widetilde{X}/\ker l$ in $\widehat{X}$ and by the boundedness of $\widehat{A}|_{Z_1}$, $\widehat{B}_k^{(1)}$ can be chosen from $\widehat{P}_{\Gamma_\lambda}(\widetilde{X}/\ker l) \subset \widehat{Z}_1$. Subsequently, from (2.13), we get

$$\widehat{B}_k^{(1)} = \widehat{P}_{\Gamma_\lambda}\widehat{B}_k = \widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k + \ker l$$

for some sequence $\widetilde{B}_k \in \widetilde{X}$. Then the following estimate holds:

$$\begin{aligned} \frac{1}{2} \leq \left\| \widehat{B}_k^{(1)} \right\|' &= \| \widehat{P}_{\Gamma_\lambda}\widehat{B}_k \|' = \| \widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k + \ker l \|' = l(\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k) \\ &= \limsup_{t\to+\infty} \frac{\| \widetilde{T}(t)\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k \|}{f(t)} \end{aligned} \tag{2.15}$$

for k large enough. By integrating the equation (2.7), we obtain

$$\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k = \int_{\Gamma_\lambda} (\widetilde{A} - \mu I)^{-1}\widetilde{B}_k\, d\mu = \int_{\Gamma_\lambda} \widetilde{B}_k (A - \mu I)^{-1}\, d\mu = \widetilde{B}_k P_{\Gamma_\lambda}, \tag{2.16}$$

where we have used the analyticity of the resolvent operator function and the boundedness of $\widetilde{B}_k$ as an operator from $\mathcal{L}(X)$ to $\mathcal{L}(X)$ (treated as a multiplication operator). Using (2.16) and the definition of $\widetilde{T}(t)\widetilde{B} = \widetilde{B}T(t)$ in (2.15), we get

$$\begin{aligned} \frac{1}{2} \leq \limsup_{t\to+\infty} \frac{\| \widetilde{T}(t)\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k \|}{f(t)} &= \limsup_{t\to+\infty} \frac{\| \widetilde{B}_k P_{\Gamma_\lambda} T(t) \|}{f(t)} \\ &\leq \limsup_{t\to+\infty} \frac{\| \widetilde{B}_k \| \| P_{\Gamma_\lambda} T(t) \|}{f(t)} = 0, \end{aligned}$$

where to evaluate the limit we have used (2.3). This yields a contradiction, thus

$$\lim_{t\to+\infty} \frac{\| \widetilde{T}(t) R_\mu \|}{f(t)} = \lim_{t\to+\infty} \frac{\| T(t) R_\mu \|}{f(t)} = 0. \qquad \square$$

Corollary 2.2. *Let the function f satisfy the conditions of Theorem 2.1. If*

$$c f(t) \leq \|T(t)\| \leq C f(t), \quad t \geq 0, \tag{2.17}$$

for some $C, c > 0$ (from here on, relation (2.17) will be denoted by $\|T(t)\| \sim f(t)$), then the assertion of Theorem 2.1 takes the form of (1.5):

$$\frac{\|T(t)R_\mu\|}{\|T(t)\|} \to 0 \quad \text{as } t \to \infty.$$

Below we give a simple example of an unbounded C_0-semigroup with $\omega_0 = 0$ for which (1.5) holds, despite the fact that $\sigma(A) \cap (i\mathbb{R}) \neq \varnothing$.

Example 2.3. Consider a separable Hilbert space H with the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, and put

$$T(t)e_0 = e^{it}e_0, \quad T(t)e_{2k-1} = e^{(ik-\frac{1}{k})t}e_{2k-1}, \quad T(t)e_{2k} = e^{(ik-\frac{1}{k})t}(te_{2k-1} + e_{2k})$$

for $k = 1, 2, \ldots$. The above defines a C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on H. It is easy to see that on the invariant subspace

$$H_1 = \operatorname{span}\{e_0\},$$

the operators $T(t)$ and $T(t)R_\mu$ are uniformly bounded for $t \geq 0$. It is less obvious that on the complementary subspace

$$H_2 = \overline{\operatorname{span}\{e_1, e_2, \ldots\}},$$

the norm of the semigroup behaves as follows:

$$\|T(t)\| \sim t. \tag{2.18}$$

In particular, (2.18) implies $\omega_0 = 0$. Also, direct computations show that

$$\|T(t)R_\mu\| \leq M, \quad t \geq 0.$$

This means that (1.5) holds despite

$$\{i\} \subset \sigma(A) \cap (i\mathbb{R}) \neq \varnothing.$$

Now we will give a simple example of the application of Theorem 2.1, for which conditions (a) and (b) can be easily verified.

Example 2.4. Let $\{e_n\}_{n=1}^\infty$ be the orthonormal basis of a Hilbert space H. Define the operator $A : D(A) \subset H \to H$ as follows:

$$A|_{H_n} := A_n := \begin{bmatrix} ni + \frac{i}{n} & 1 \\ 0 & ni - \frac{i}{n} \end{bmatrix},$$

where $H_n = \operatorname{span}\{e_{2n-3}, e_{2n-2}\}$, $n = 2, 3, 4 \ldots$. For each $n \geq 2$, consider the curve Γ_n enclosing the pair of eigenvalues $i\big(n + \frac{1}{n}\big), i\big(n - \frac{1}{n}\big)$. Then the image of the Riesz projection corresponding to the curve is H_n. One can directly check that

$$e^{A_n t} := T_n(t) = e^{tni} \begin{bmatrix} e^{i\frac{t}{n}} & n \sin \frac{t}{n} \\ 0 & e^{-i\frac{t}{n}} \end{bmatrix}.$$

Since $\|T(t)\| = \sup\limits_{n\geq 2} \|T_n(t)\|$, we have

$$\|T(t)\| \sim t. \tag{2.19}$$

It is easy to see that $f(t) := t$ satisfies the desired conditions required by Theorem 2.1. Therefore (1.5) holds (cf. Corollary 2.2), i.e.

$$\lim_{t\to+\infty} \frac{\|T(t)A^{-1}\|}{\|T(t)\|} = \lim_{t\to+\infty} \frac{\|T(t)A^{-1}\|}{t} = 0. \tag{2.20}$$

Moreover, for this simple case, we can calculate the decay rate of (2.20), namely

$$T_n(t)A_n^{-1} = \frac{in}{1-n^4} e^{tni} \begin{bmatrix} (n^2-1)e^{i\frac{t}{n}} & (n^2-1)n\sin\frac{t}{n} + ine^{-i\frac{t}{n}} \\ 0 & (n^2+1)e^{-i\frac{t}{n}} \end{bmatrix},$$

hence

$$\|T(t)A^{-1}\| = \sup_{n\geq 2} \|T_n(t)A_n^{-1}\| \sim 1, \quad t \geq 0.$$

Due to (2.19), it follows that

$$\frac{\|T(t)A^{-1}\|}{\|T(t)\|} \sim \frac{1}{t} \to 0, \quad t \to \infty.$$

3. Application of the main results to a family of semigroups with countable pure imaginary simple spectrum

In this section, we give an example of a family of unbounded semigroups having a countable pure imaginary simple spectrum for which our result can be applied. For the elements of this family the eigenvectors are linearly dense but do not form a Riesz basis, which is due to the fact that the eigenvalues are not uniformly separated (cf. [18, 19]). These semigroups were described in [23, 24]. We recall here the main steps of the construction. Let $(H, \|\cdot\|)$ be a Hilbert space with the orthonormal basis $\{e_n\}_{n=2}^{\infty}$. For the sequence

$$\lambda_n = i\log n,\ n = 2, 3, \dots,$$

define the semigroup $T = \{T(t)\}_{t\geq 0}$ by

$$T(t)e_n = e^{t\lambda_n} e_n.$$

Note that the eigenvalues are not uniformly separated. For a given $N \in \mathbb{N}/\{0\}$, we are able to choose a new norm $\|\cdot\|_N$ on H dominated by $\|\cdot\|$ such that:

(a) The semigroup T naturally extends to a C_0-semigroup $\widetilde{T}$ on the completion of $(H, \|\cdot\|_N)$, say $\widetilde{H}_N$;

(b) there exist constants $m, M > 0$ such that

$$mt^N \leq \|\widetilde{T}(t)\| \leq Mt^N + 1, \quad t \geq 0. \tag{3.1}$$

The norm $\|\cdot\|_N$. is constructed as follows. Let $(H, \|\cdot\|)$ be a Hilbert space with the orthonormal basis $\{e_n\}_{n=2}^{\infty}$. Any element $x \in H$ has a unique decomposition

$\sum_{n=2}^{\infty} c_n e_n$ for some sequence $\{c_n\} \in l_2$. This correspondence is a bijection. Since the basis $\{e_n\}$ is orthonormal, we have

$$\|x\|^2 = \sum_{n=2}^{\infty} |c_n|^2 = \|\{c_n\}\|_{l_2}.$$

Consider the backward difference operator $\Delta : l_2 \to l_2$:

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For any $N \in \mathbb{N}$, we define the function $\|\cdot\|_N$ on H by

$$\|x\|_N^2 = \left\|\Delta^N(\{c_n\})\right\|_{l_2}. \tag{3.2}$$

It is easy to check that (3.2) defines a norm on H and that the space $(H, \|\cdot\|_N)$ is not complete. The space $\left(\widetilde{H}_N, \|\cdot\|_N\right)$ is defined as the completion of H with respect to the norm $\|\cdot\|_N$.

Now, denote the generator of $\widetilde{T}$ by $\widetilde{A}$. It was shown in [24] that

$$\sigma(\widetilde{A}) = \sigma_P(\widetilde{A}) = \bigcup_{n\geq 2}\{i \log n\}. \tag{3.3}$$

Clearly, the spectrum of $\widetilde{A}$ is purely imaginary and the semigroup $\widetilde{T}$ grows exactly like a polynomial. We are going to check that the semigroup $\widetilde{T}$ with $f(t) := Mt^N + 1$ meet the conditions of Theorem 2.1 for arbitrary $N \in \mathbb{N}/\{0\}$. Indeed, for each $\lambda_n = i \log n$, see (3.3), one can choose Γ_n surrounding only one point of $\sigma(\widetilde{A})$, namely, λ_n. Note also that, for $x \in H \subset \widetilde{H}_N$,

$$\begin{aligned} \widetilde{A}x &= Ax, \\ \widetilde{R}(\lambda)x &= R(\lambda)x, \\ \widetilde{P}_{\Gamma_n}x &= P_{\Gamma_n}x. \end{aligned}$$

Hence, due to density of H in $\widetilde{H}_N$,

$$\widetilde{T}(t)\widetilde{P}_{\Gamma_n}\widetilde{x} = e^{it\log n}\widetilde{P}_{\Gamma_n}\widetilde{x}, \quad \widetilde{x} \in \widetilde{H}_N.$$

It is easy to see that the function $f(t) := Mt^N + 1$ has the properties (2.1), (2.2), and that the following holds:

$$\frac{\|\widetilde{T}(t)\widetilde{P}_{\Gamma_n}\|}{f(t)} \leq \frac{\|\widetilde{P}_{\Gamma_n}\|}{f(t)} \leq \frac{\|\widetilde{P}_{\Gamma_n}\|}{Mt^N} \to 0 \quad \text{as } t \to \infty, \quad n > 0.$$

This means that the semigroup and $f(t) = Mt^N + 1$ meet the conditions of Theorem 2.1. Application of the presented result (see Corollary 2.2) yields

$$0 = \lim_{t\to+\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{Mt^N + 1} = \lim_{t\to+\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{t^N} = \lim_{t\to+\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|}$$

for any $\mu \notin \sigma(\widetilde{A})$. □

If we want a more precise estimation, it is difficult even in the simplest case of $N = 1$. A more accurate estimate for this case (4.1) can be found in the appendix. For larger values of N one can only expect the calculations to be more cumbersome. From the above considerations follows a corollary concerning the sufficient condition for (1.5) to hold.

Corollary 3.1. *Let $T = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup with the generator A and $\omega_0(T) = 0$. If*

$$\sigma(A) = \sigma_P(A) = \bigcup_{n\in\mathbb{N}} \{\lambda_n\}\ , \quad \lambda_n \in (i\mathbb{R}),$$

where all eigenvalues λ_n are simple, and

$$\|T(t)\| \sim f(t),$$

for an unbounded function $f(t)$ (see Corollary 2.2). Then the following holds

$$\frac{\|T(t)R_\mu\|}{\|T(t)\|} \to 0 \quad \text{as } t \to \infty,$$

for any $\mu \notin \sigma(A)$.

4. Appendix

Here we give detailed computations of the rate of decay of

$$\frac{\left\|\widetilde{T}(t)\widetilde{R}_\mu\right\|}{\|\widetilde{T}(t)\|} \quad \text{as } t \to +\infty,$$

where $\widetilde{T}(t)$ is acting on the space $\widetilde{H}_1$. This is the simplest case from previous section when $N = 1$. The action of the generator, resolvent at the point 0 and product of the semigroup with the resolvent are as follows:

$$\begin{aligned}
\widetilde{A}e_n &= i\log(n)e_n, \quad n \geq 2,\\
\widetilde{A}^{-1}e_n &= \frac{1}{i\log(n)}e_n, \quad n \geq 2,\\
\widetilde{T}(t)\widetilde{A}^{-1}e_n &= \frac{e^{it\log(n)}}{i\log(n)}e_n, \quad n \geq 2,\\
\widetilde{T}(t)\widetilde{A}^{-1}x &= \sum_{n=2}^{\infty} c_n \frac{e^{it\log(n)}}{i\log(n)}e_n.
\end{aligned}$$

Example 4.1. Consider $\widetilde{T} : \widetilde{H}_1 \to \widetilde{H}_1$, then (see (3.2))

$$\|\widetilde{x}\|_1 = \left(\sum_{n=2}^{\infty} |c_{n+1} - c_n|^2 + |c_2|^2 \right)^{\frac{1}{2}}, \quad \widetilde{x} \in \widetilde{H}_1.$$

We will prove that, for this case,

$$\frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|} \sim \frac{1}{\log(t)}. \tag{4.1}$$

In further considerations, we will use the following inequality

$$\sum_{n=1}^{\infty} \frac{|c_n|^2}{n^2} \leq 4 \sum_{n=1}^{\infty} |c_{n+1} - c_n|^2, \quad \{c_n\}_{n=1}^{\infty} \subset \mathbb{C}, \tag{4.2}$$

which is a special case of Hardy's inequality:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,$$

for p = 2. To prove (4.1) we will estimate $\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1^2$. It is given by

$$\begin{aligned}
\left\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\right\|_1^2 &= \sum_{n=2}^{\infty} \left| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_n \frac{e^{it\log(n)}}{i\log(n)} \right|^2 + |c_2|^2 \\
&\leq 2 \sum_{n=2}^{\infty} \left| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_{n+1} \frac{e^{it\log(n)}}{i\log(n)} \right|^2 \\
&\quad + 2 \sum_{n=2}^{\infty} \left| (c_{n+1} - c_n) \frac{e^{it\log(n)}}{i\log(n)} \right|^2 + |c_2|^2.
\end{aligned}$$

The second and third elements of the right-hand side of the inequality are clearly bounded by $B\left(\frac{t}{\log(t)}\right)^2 \|\widetilde{x}\|_1^2$ and $C\left(\frac{t}{\log(t)}\right)^2 \|\widetilde{x}\|_1^2$, $B, C > 0$ for $t > e$. We only need to look at the first sum then.

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_{n+1} \frac{e^{it\log(n)}}{i\log(n)} \right|^2 \\
&\quad = \sum_{n=2}^{\infty} \left| \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n+1) - e^{it\log(n)}\log(n))}{\log(n+1)\log(n)} \right. \\
&\qquad + \left. \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n) - e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)} \right|^2 \\
&\quad \leq 2 \sum_{n=2}^{\infty} \left| \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n+1) - e^{it\log(n)}\log(n))}{\log(n+1)\log(n)} \right|^2
\end{aligned}$$

$$+\left|\frac{c_{n+1}}{n}\frac{n(e^{it\log(n)}\log(n)-e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)}\right|^2$$

$$=2\sum_{n=2}^{\infty}\left|\frac{c_{n+1}}{n}\frac{n\log(1+\frac{1}{n})}{\log(n+1)\log(n)}\right|^2+2\sum_{n=2}^{\infty}\left|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\right|^2.$$

The first of the above sums, due to Hardy's inequality (see (4.2)), is bounded by $D\|\widetilde{x}\|^2$, and thus by $D\Big(\frac{t}{\log(t)}\Big)^2\|\widetilde{x}\|^2$ for $t>e$. We estimate the remaining sum by splitting it into two t-dependent sums.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\right|^2\\
&\quad=\sum_{2\le n<t}\left|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\right|^2+\sum_{n\ge t}\left|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\right|^2\\
&\quad\le E\sum_{2\le n<t}\left|\frac{c_{n+1}}{n}\right|^2\left(\frac{t}{\log(t)}\right)^2+\sum_{n\ge t}\left|\frac{c_{n+1}}{n}\frac{tn\log(1+\frac{1}{n})(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)t\log(1+\frac{1}{n})}\right|^2\\
&\quad\le E\sum_{2\le n<t}\left|\frac{c_{n+1}}{n}\right|^2\left(\frac{t}{\log(t)}\right)^2+F\sum_{n\ge t}\left|\frac{c_{n+1}}{n}\frac{(1-e^{it\log(1+\frac{1}{n})})}{t\log(1+\frac{1}{n})}\right|^2\left(\frac{t}{\log(t)}\right)^2\\
&\quad\le (E+G)\sum_{n=2}^{\infty}\left|\frac{c_{n+1}}{n}\right|^2\left(\frac{t}{\log(t)}\right)^2.
\end{aligned}$$

Where we have used the boundedness of $s\log(1+\frac{1}{s})$ and $\frac{1-e^{is}}{s}$ for $s\in\mathbb{R}^+$. Thus, again due to (4.2),

$$\big\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\big\|_1\le(B+C+D+4E+4G)^{\frac{1}{2}}\frac{t}{\log(t)}\|\widetilde{x}\|_1.$$

Thus

$$\big\|\widetilde{T}(t)\widetilde{A}^{-1}\big\|\le M_0\frac{t}{\log(t)},\tag{4.3}$$

for some $M_0>0$ and $t>e$. We will now prove the opposite inequality

$$m_0\frac{t}{\log(t)}\le\big\|\widetilde{T}(t)\widetilde{A}^{-1}\big\|,\tag{4.4}$$

for some $m_0>0$. First, we observe that due to the reverse triangle inequality, it holds that

$$\begin{aligned}
\big\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\big\|_1&=\left(\sum_{n=2}^{\infty}\left|c_{n+1}\frac{e^{it\log(n+1)}}{i\log(n+1)}-c_n\frac{e^{it\log(n)}}{i\log(n)}\right|^2+|c_2|^2\right)^{\frac{1}{2}}\\
&\ge\left(\sum_{n=2}^{\infty}\left|c_{n+1}\frac{(e^{it\log(n)}\log(n)-e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)}\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$- \left(\sum_{n=2}^{\infty} \left| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_{n+1} \frac{e^{it\log(n)}}{i\log(n)} \right|^2 \right)^{\frac{1}{2}}$$

$$- \left(\sum_{n=2}^{\infty} \left| (c_{n+1} - c_n) \frac{e^{it\log(n)}}{i\log(n)} \right|^2 \right)^{\frac{1}{2}} - |c_2|.$$

It follows from previous considerations that

$$\left\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\right\|_1 \geq \left(\sum_{n=2}^{\infty} \left| c_{n+1} \frac{(e^{it\log(n)}\log(n) - e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)} \right|^2 \right)^{\frac{1}{2}} - C\|\widetilde{x}\|_1$$

for some $C > 0$. Thus, in order to prove (4.4), it suffices to show that

$$\sum_{n=2}^{\infty} \left| c_{n+1} \frac{(e^{it\log(n)} - e^{it\log(n+1)})}{\log(n+1)} \right|^2 \geq m_1^2 \Big(\frac{t}{\log(t)} \Big)^2 \|\widetilde{x}\|_1^2 \tag{4.5}$$

for some $m_1 > 0$ and $t > e$. To this end, we construct for each $t > e$ an element in $\widetilde{H}_1$ in the following way

$$\widetilde{x}^{(t)} = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n^{(t)} e_n, \quad j \in \mathbb{N},$$

where

$$c_n^{(t)} = \begin{cases} n & \text{if } n \leq 2t, \\ 4t - n & \text{if } 2t < n \leq 4t, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\left\|\widetilde{x}^{(t)}\right\|_1^2 \leq 4t. \tag{4.6}$$

Now, the following estimate holds (see (4.5)):

$$\sum_{n=2}^{\infty} \left| c_{n+1}^{(t)} \frac{(e^{it\log(n)} - e^{it\log(n+1)})}{\log(n+1)} \right|^2 \geq \sum_{t \leq n \leq 2t} \left| t \frac{1 - e^{it\log(1+\frac{1}{n})}}{\log(n+1)} \right|^2$$

$$\geq \left(\frac{t}{\log(4t)} \right)^2 \sum_{t \leq n \leq 2t} \left| \frac{1 - e^{it\log(1+\frac{1}{n})}}{it\log(1+\frac{1}{n})} it\log(1+\frac{1}{n}) \right|^2$$

$$\geq \left(\frac{t}{\log(4t)} \right)^2 \sum_{t \leq n \leq 2t} \left| \frac{1 - e^{it\log(1+\frac{1}{n})}}{t\log(1+\frac{1}{n})} \log(1+\frac{1}{2t})^t \right|^2$$

$$\geq \left(\frac{Ct}{\log(4t)} \right)^2 \sum_{0 \leq n \leq t} \left| \frac{1 - e^{it\log(1+\frac{1}{n+t})}}{t\log(1+\frac{1}{n+t})} \right|^2$$

$$\geq \left(\frac{Ct^2}{\log(4t)}\right)^2 \sum_{0\leq n\leq t} D \geq \left(\frac{Ct}{\log(4t)}\right)^2 \frac{t}{2} D$$

for $t > e$ and some $C, D > 0$ independent of $t > e$. Combining the above with (4.5) and (4.6) gives

$$m_0 \frac{t}{\log(t)} \leq \frac{\left\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}^{(t)}\right\|_1}{\|\widetilde{x}^{(t)}\|_1}$$

for $t > e$. Together with (4.3) this shows that

$$m_0 \frac{t}{\log(t)} \leq \left\|\widetilde{T}(t)\widetilde{A}^{-1}\right\| \leq M_0 \frac{t}{\log(t)}$$

for $t > e$. This implies, due to (3.1), that

$$m_0' \frac{1}{\log(t)} \leq \frac{\left\|\widetilde{T}(t)\widetilde{A}^{-1}\right\|}{\left\|\widetilde{T}(t)\right\|} \leq M_0' \frac{1}{\log(t)}, \tag{4.7}$$

for some $m_0', M_0' > 0$ and $t > e$ or, equivalently,

$$\frac{\left\|\widetilde{T}(t)\widetilde{R}_\mu\right\|}{\left\|\widetilde{T}(t)\right\|} \sim \frac{1}{\log(t)},$$

for $t > e$ and arbitrary $\mu \notin \sigma(\widetilde{A})$. □

References

[1] J. van Neerven, *The asymptotic behaviour of semigroups of linear operators*, Birkhäuser, Basel, 1996.

[2] K. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[3] G.M. Sklyar and V. Shirman, *On asymptotic stability of linear differential equation in Banach spaces*, (Russian) Teoria Funk., Funkt. Anal. Prilozh. **37** (1982), 127–132.

[4] Y. Lyubich and V. Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. **88** (1988), 37–42.

[5] W. Arendt and C.J.K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Am. Math. Soc. **306** (1988), 837–852.

[6] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347** (2010). 455–478.

[7] C.J.K. Batty, A. Borichev, and Y. Tomilov, *Lp-tauberian theorems and Lp-rates for energy decay*, J. Funct. Anal **270** (2016) 1153–1201.

[8] F. Huang, *Strong Asymptotic Stability of Linear Dynamical Systems in Banach Spaces*, J. Differ. Equ. **104**, (1993), 307–324.

[9] C.J.K Batty and S. Yeates, *Weighted and local stability of semigroups of operators*, Math. Proc. Cambridge Philos. Soc.**129** (2000), 85–98.

[10] J. Woźniak and M. Firkowski, *Optimal decay ratio of damped slowly rotating Timoshenko beams*, ZAMM Z. Angew. Math. Mech. **99** (2019),

[11] C.J.K. Batty, *Asymptotic behaviour of semigroups of operators*, Banach Center Publ. **30** (1994), 35–52.

[12] C.J.K. Batty and T. Duyckaerts, *Non-uniform stability for bounded semi-groups on Banach spaces*, J. Evol. Equ. **8**, 2008, 765–780.

[13] G.M. Sklyar and P. Polak, *On asymptotic estimation of a discrete type C_0-semigroups on dense sets: Application to neutral type systems*, Appl. Math. Optim. **75** (2017), 175–192.

[14] G.M. Sklyar, *On the decay of bounded semigroup on the domain of its generator*, Vietnam J. Math. **43** (2015), 207–213.

[15] V. Phóng, *Theorems of Katznelson–Tzafriri type for semigroups of operators*, J. Func. Anal. **103** (1992), 74–84.

[16] V. Phóng, *Semigroups with nonquasianalytic growth, Studia Mathematica* **104** (1993), 229–241.

[17] Yu.I. Lyubich, V.I Matsaev, and G.M. Fel'dman *On representations with a separable spectrum,* Funct. Anal. Its Appl. **7** (1973) 129–136.

[18] G.Q. Xu and S.P. Yung, *The expansion of a semigroup and a Riesz basis criterion*, J. Differ. Equ. **210** (2005), 1–24.

[19] H. Zwart, *Riesz basis for strongly continuous groups*, J. Differ. Equ. **249** (2010), 2397–2408.

[20] G.M. Sklyar and P. Polak, *Notes on the asymptotic properties of some class of unbounded strongly continuous semigroups*, J. Math. Phys. Anal. Geom. **15** (2019), 412–424.

[21] G.M. Sklyar, *On the maximal asymptotics for linear differential equations in Banach spaces*, Taiwan. J. Math. **14** (2010), 2203–2217.

[22] G.M. Sklyar. *Lack of maximal asymptotics for linear differential equations in Banach spaces.* Dokl. Math. **81** (2010), 265–267.

[23] G.M. Sklyar, V. Marchenko, and P. Polak, *Sharp polynomial bounds for certain C_0-groups generated by operators with non-basis family of eigenvectors*, J. Funct. Anal. **280** (2021)

[24] G.M. Sklyar and V. Marchenko, *Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors*, J. Func. Anal. **272** (2017), 1017–1043.

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Про відносне зниження норми необмеженої півгрупи в області визначення генератора

Grigory M. Sklyar, Piotr Polak, and Bartosz Wasilewski

Досліджується асимптотика C_0-півгрупи в області визначення генератора. Зокрема, ми аналізуємо поведінку $\|T(t)(A-\lambda I)^{-1}\|$, коли час прямує до нескінченності. Наші результати розширюють деякі наявні результати на випадок, коли перетин спектра генератора з уявною віссю є непорожнім. Наведено також конструктивний приклад класу необмежених C_0-напівгруп з чисто уявним точковим спектром, для яких наша теорема може бути застосована.

Ключові слова: C_0-півгрупи, асимптотична поведінка