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Principal $\mathrm{SO}(2n, \mathbb{C})$-Bundle Fixed Points over a Compact Riemann Surface

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Let X be a compact connected Riemann surface of genus $g \geq 2$ equipped with a holomorphic involution σ_X, and let G be a semisimple complex Lie group which admits an outer involution σ. A principal (G, σ_X, σ)-bundle over X is a pair (E, ρ), where E is a principal G-bundle over X and $\rho : E \to \sigma_X^*(\sigma(E))$ is an isomorphism such that $(\sigma_X^*\rho) \circ \rho : E \to E$ is an automorphism of E which acts as the product by an element of the center of G. In this paper, principal (G, σ_X, σ)-bundles over X are introduced and the study is particularized to the case of $G = \mathrm{SO}(2n, \mathbb{C})$. It is shown that the stability and polystability conditions for a principal $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-bundle coincide with those of the corresponding principal $\mathrm{SO}(2n, \mathbb{C})$-bundle. Finally, the explicit form that a principal $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-bundle takes is provided, and the stability of these principal $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-bundles is discussed.

Key words: principal bundle, orthogonal group, moduli space, Riemann surface, automorphism

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1. Introduction

Given a compact connected Riemann surface X of genus $g \geq 2$ and a semisimple complex Lie group G. A principal G-bundle over X is a holomorphic manifold E which admits a holomorphic projection over X and a fiber-preserving right action of G on it such that for each point $x \in X$. The restriction of E to U is G-equivariantly isomorphic to $U \times G$ for some open connected subset U of X with $x \in U$. The group G is called a structure group of E. From certain suitable notions of stability and polystability, it follows that the set of isomorphism classes of polystable principal G-bundles over X admits a structure of complex algebraic variety, called the moduli space of principal G-bundles over X, which is denoted by $M(G)$. These notions were first introduced by Ramanathan [17], who also constructed the moduli space $M(G)$ of principal G-bundles over a compact Riemann surface [18, 19]. In [8], Behrend extended these notions to group schemes in a way that his notions coincide with those of Ramanathan [17] for principal G-bundles over X, provided one is working over the complex numbers. Notice that if z is an element of the center of G and E is a principal G-bundle over X, then the map $e \mapsto ez$ defines an automorphism of E. A principal G-bundle is said to be simple if the only automorphisms that E admits are those induced by elements of the center of G. The concept of a simple principal bundle is relevant in the study of the geometry of the moduli spaces of principal G-bundles over X since the subvariety of stable and simple G-bundles over X is a dense open subset formed by smooth points of $M(G)$.

The notion of the principal G-bundle over X is essential for the definition of geometric objects. For example, a G-Higgs bundle is a pair (E, φ), where E is a principal G-bundle over X and φ is a holomorphic global section of $E(\mathfrak{g}) \otimes K$, where $E(\mathfrak{g})$ is the vector bundle defined by E through the adjoint representation of G, whose typical fiber is the Lie algebra $\mathfrak{g}$ of G and K is the canonical line bundle over X. Higgs bundles were first introduced by Hitchin [13] in the context of the study of Yang-Mills equations for $G = \mathrm{SL}(2, \mathbb{C})$. The concept was later extended by Simpson [20], who introduced G-Higgs bundles for a general G and constructed the corresponding moduli space for these objects. When the adjoint representation is replaced by another complex representation of G in the definition of G-Higgs bundles, then G-Higgs pairs arise. His study contributes to the study of the geometry of Higgs bundles because they define subvarieties of the moduli space of G-Higgs bundles over X [6].

The specialized literature includes several lines of research on the geometry and topology of moduli spaces of principal G-bundles over a compact Riemann surface. Among them, there is the definition and analysis of geometric objects linked to the principal G-bundles over X. This is the case, for example, of real and pseudo-real bundles over curves introduced by Biswas and Hurtubise in [9]. Given is a real form σ of G, that is, an anti-holomorphic involution of G. It defines a $\mathcal{C}^\infty$-bundle $\sigma(E)$ by changing the right action of G on E through σ. If an anti-holomorphic involution σ_X of the base curve X is also given, then a principal G-bundle E over X defines a pseudo-real principal bundle if it is isomorphic to $\sigma_X^*(\sigma(E))$, where σ_X^* denotes the pullback, through an isomorphism ρ which satisfies that $(\sigma_X^* \rho) \circ \rho : E \to E$ is the automorphism of E induced by an element of the center of G. The pseudo-real structure is called real if this central element is the identity element. Notice that, although $\sigma(E)$ is not holomorphic, $\sigma_X^*(\sigma(E))$ is, since σ_X and σ are both anti-holomorphic. These objects are interesting as they allow the study of principal bundles which reduce its structure group to a real form of G and which admit an automorphism that lifts the anti-holomorphic involution of X.

Another fruitful line of the study of principal bundles is through the automorphisms of the moduli space $M(G)$ of principal G-bundles over X. Fringuelli [10] proved that for $g \geq 4$, every automorphism of $M(G)$ is one of the following or a composition of some of them:

1. If σ is an outer automorphism of G (that is, an element of the quotient of the group of automorphisms of G by the normal subgroup of inner automorphisms of G), and E is a polystable principal G-bundle over X, then a polystable principal G-bundle $\sigma(E)$ over X is defined by taking the same total space as that of E but changing the action of G through a represen-

tative s of σ in the group of automorphisms of G [3,5]. Then σ induces an automorphism of $M(G)$.

2. Given a holomorphic automorphism σ_X of the base curve X and a polystable principal G-bundle E, then $\sigma_X^* E$ is defined by taking the pullback which induces an action of σ_X on the moduli of polystable G-bundles.
3. If Z is the center of G, then $H^1(X, Z)$ (which is identified with the set of isomorphism classes of principal Z-bundles over X) also acts on $M(G)$ [5,7]. Given a polystable principal G-bundle E and an element $L \in H^1(X, Z)$, then the variety $E \times_X L$ of pairs, whose elements are over the same point of X, is naturally a $(G \times Z)$-bundle over X. Then the G-bundle $E \otimes L$ is defined to be the quotient of $E \times_X L$, where (e_1, z_1) is identified with (e_2, z_2) if there exists $\lambda \in Z$ such that $(e_2, z_2) = (e_1\lambda, z_1\lambda^{-1})$.

The results of Fringuelli [10] generalize analogous results for vector bundles [15] and for the groups E_6 and F_4 [5], and allow the study of the geometry of the moduli spaces of principal G-bundles over X through the study of the subvarieties of fixed points of their automorphisms in the spirit of [2,4,7].

Given a holomorphic involution σ_X of X and an outer automorphism σ of G, the notion of principal (G, σ_X, σ)-bundles over X is introduced in Definition 2.1.

Definition. *Let G be a complex semisimple Lie group, σ be a nontrivial outer automorphism of G of order 2, and σ_X be a holomorphic involution of X. A principal (G, σ, σ_X)-bundle over X is a pair (E, ρ), where E is a holomorphic principal G-bundle over X and $\rho : E \to \sigma_X^*(\sigma(E))$ is an isomorphism of holomorphic principal G-bundles over X such that the composition $(\sigma_X^*\rho) \circ \rho : E \to \sigma_X^2\sigma^2(E) = E$ is an automorphism of E induced by an element of the center of G fixed by an involution of G which represents σ.*

Notice that thus defined, the underlying principal bundle of a principal (G, σ, σ_X)-bundle over X is fixed by the automorphism of $M(G)$ defined by the composition of the automorphisms induced by the actions of σ and σ_X on it. Proper notions of stability and polystability for this kind of pairs are introduced in Definitions 3.1 and 3.2.

In this work, the moduli space of principal G-bundles over X are considered when $G = \mathrm{SO}(2n, \mathbb{C})$ for $n > 2$. A principal $\mathrm{SO}(2n, \mathbb{C})$-bundle can be understood as a holomorphic rank $2n$ vector bundle over X with a trivial determinant bundle and equipped with a globally-defined nondegenerate holomorphic quadratic form. Ramanan [16] proved that a principal $\mathrm{SO}(2n, \mathbb{C})$-bundle is stable when the vector bundle can be written as an orthogonal direct sum of mutually nonisomorphic orthogonal bundles which are stable as vector bundles. This is also true even in the odd rank case. The group of outer automorphisms of $\mathrm{SO}(2n, \mathbb{C})$ is isomorphic to $\mathbb{Z}_2$ since it is isomorphic to the group of graph automorphisms of the corresponding Dynkin diagram [11, Proposition D. 40]. Therefore, there is a unique nontrivial outer automorphism σ of $\mathrm{SO}(2n, \mathbb{C})$ of order 2. This is clear when $n \neq 4$ since, in this case, the group of symmetries of the corresponding Dynkin diagram is isomorphic to $\mathbb{Z}_2$. When $n = 4$, in the case when $G = \mathrm{SO}(8, \mathbb{C})$, the Dynkin diagram admits two symmetries of order 3 that are mutually inverse and induce an outer automorphism of order 3 of the universal cover $\mathrm{Spin}(8,\mathbb{C})$ of $\mathrm{SO}(8,\mathbb{C})$ called a triality automorphism. This triality automorphism comes from a (not unique) order 3 automorphism of $\mathrm{Spin}(8,\mathbb{C})$ which induces an order 3 permutation of the elements of order 2 of the center of $\mathrm{Spin}(8,\mathbb{C})$ (which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$). Therefore, this order 3 automorphism cannot induce an automorphism of $\mathrm{SO}(8,\mathbb{C})$, so no outer automorphism of order 3 of $\mathrm{SO}(8,\mathbb{C})$ can exist [2]. Given a holomorphic involution σ_X of X, in Proposition 3.9, it is proved that a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle (E,ρ) over X is stable or polystable if and only if the principal $\mathrm{SO}(2n,\mathbb{C})$-bundle is stable or polystable.

Proposition. *Let σ_X be a holomorphic involution of X and σ be the nontrivial outer automorphism of order 2 of $\mathrm{SO}(2n,\mathbb{C})$ for $n > 2$. Let (E,ρ) be a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle over X. Then (E,ρ) is stable (respectively, semistable, polystable) if and only if the principal $\mathrm{SO}(2n,\mathbb{C})$-bundle E over X is stable (respectively, semistable, polystable).*

Finally, an explicit form of the underlying vector bundle of E is provided for any principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle (E,ρ) over X (Theorem 4.2). Moreover, as a consequence of this explicit description, it is proved that the underlying vector bundle of a stable principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle admits a decomposition into a direct sum of even rank vector subbundles $E_1 \oplus E_2$ (Corollary 4.3).

Theorem. *Let σ_X be a holomorphic involution of X and σ be the nontrivial outer automorphism of order 2 of $\mathrm{SO}(2n,\mathbb{C})$ for $n > 2$. Let (E,ρ) be a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle over X. Then the underlying holomorphic vector bundle of E admits a nontrivial decomposition into a direct sum of vector subbundles over X of one of the following forms:*

1. *$E_1 \oplus E_2$, where both subbundles have even rank, or*
2. *$E_1 \oplus E_2 \oplus E_3 \oplus E_4$, where the ranks of E_1 and E_2 are even, and E_3 and E_4 are isotropic subbundles with the same rank.*

The subbundles of the decomposition satisfy $\sigma_X^* E_i \cong E_i$ for every i.

The paper is structured as follows. In Section 2, the notion of principal (G,σ_X,σ)-bundle over X is introduced in the general case, where G is a semisimple complex Lie group. The notions of stability and polystability for these objects are included in Section 3, where it is also proved that polystability of $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundles coincides with that of principal $\mathrm{SO}(2n,\mathbb{C})$-bundles. Finally, in Section 4, the explicit form of principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundles over X is given.

2. Principal (G,σ,σ_X)-bundles over X

Let X be a compact connected Riemann surface equipped with a nontrivial holomorphic involution $\sigma_X : X \to X$. Let G be a semisimple complex Lie group equipped with a fixed holomorphic involution s of G, and E be a principal G-bundle over X. The principal G-bundle $s(E)$ over X is defined to be

$$s(E) = E \times^s G, \tag{2.1}$$

where $\times^s$ denotes the quotient of $E \times G$ in which the element (e_1, g_1) is identified with (e_2, g_2) if there exists $h \in G$ such that $e_2 = e_1 h$ and $g_2 = s(h)^{-1} g_1$. This is equivalent to defining that $s(E)$ has the same total space of E and the same projection map over X, but with the action of G defined by $e \cdot g = es(g)$ for $e \in s(E)$ and $g \in G$ [4, Definition 2.1]. Suppose also that G admits a nontrivial outer automorphism σ. Recall that the group $\mathrm{Inn}(G)$ of inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G such that the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is a group called the group of outer automorphisms of G and denoted by $\mathrm{Out}(G)$. The group $\mathrm{Out}(G)$ acts on the moduli space $M(G)$ in the following way: if $\tau \in \mathrm{Out}(G)$ and $t \in \mathrm{Aut}(G)$ is a representative of τ, then $\tau(E)$ is defined to be the principal G-bundle $t(E)$ in the sense defined in (2.1) [3, Section 3]. This is a good definition since clearly $t(E) \cong E$ if $t \in \mathrm{Inn}(G)$ and $t(E)$ naturally inherits a holomorphic structure from that of E and it is polystable as a principal G-bundle if so is E [3]. Each outer automorphism of G induces then a holomorphic automorphism of $M(G)$, which is defined through the choice of a representative of the outer automorphism, but it is independent of the choice of a representative. Notice also that for a given outer automorphism σ of G of order 2, it is always possible to take an order 2 automorphism s of G which represents σ [12, Proposition 2.6].

In addition, the holomorphic involution σ_X of X also induces an automorphism of order 2 of the moduli space $M(G)$ of principal G-bundles over X by $E \mapsto \sigma_X^* E$ since the holomorphic structure of $\sigma_X^* E$ comes as σ_X is holomorphic and its polystability comes from that of E.

Definition 2.1. Let G be a complex semisimple Lie group, σ be a nontrivial outer automorphism of G of order 2, and s be a holomorphic nontrivial involution of G representing σ. Suppose that X is equipped with a holomorphic involution σ_X. Then a *principal* (G, σ, σ_X)*-bundle* over X is a pair (E, ρ), where E is a holomorphic principal G-bundle over X and $\rho : E \to \sigma_X^* \sigma(E)$ is an isomorphism of holomorphic principal G-bundles over X, σ_X^* denoting the pullback and $\sigma(E)$ being defined in (2.1) such that the composition $(\sigma_X^* \rho) \circ \rho : E \to \sigma_X^2 \sigma^2(E) = E$ is an automorphism of E induced by an element of the center of G fixed by s.

Remark 2.2. Notice that every element z of the center Z of G induces an automorphism of E in this way: $e \mapsto ez$ for $e \in E$. The last condition of Definition 2.1 requires that $(\sigma_X^* \rho) \circ \rho$ act as the automorphism of E induced by an element $z \in Z \cap G^s$, where G^s denotes the subgroup of G of fixed elements of s, to which, at least, 1 belongs.

Remark 2.3. Thus defined, the isomorphism ρ, with which a principal (G, σ, σ_X)-bundle (E, ρ) over X is equipped, induces a lift to E of the holomorphic involution σ_X of X (that is, an isomorphism $\widetilde{\rho} : E \to E$ which moves the fiber over each $x \in X$ to the fiber over $\sigma_X(x)$) such that $\widetilde{\rho}(eg) = \widetilde{\rho}(e)s(g)$ for $e \in E$ and $g \in G$, and $\widetilde{\rho}^2$ is an automorphism of E induced by a central element of E.

3. Stability of principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundles

In this section, proper notions of stability, semistability and polystability for principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundles over a compact connected Riemann surface admitting a holomorphic involution σ_X for $n>2$, are discussed. In particular, it is proved that the polystability of a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle (E,ρ) is equivalent to the polystability of the principal G-bundle E over X.

First, let G be a semisimple complex Lie group and σ be an order 2 outer automorphism of G. Given a principal (G,σ,σ_X)-bundle (E,ρ) over X, the holomorphic principal G-bundle $\mathrm{Ad}(E)=E\times^G G$ is defined as the quotient of $E\times G$, where (e_1,g_1) and (e_2,g_2) are identified if there exists $h\in G$ such that $e_2=e_1h$ and $g_2=h^{-1}g_1h$. A choice of a representative $s\in\mathrm{Aut}(G)$ of order 2 of σ induces an isomorphism of principal G-bundles $s_{\mathrm{Ad}}:\mathrm{Ad}(E)\to\mathrm{Ad}(\sigma(E))$ defined by $s_{\mathrm{Ad}}(e,g)=(e,s(g))$. The isomorphism ρ induces then an isomorphism of principal G-bundles $\rho_{\mathrm{Ad}}:\mathrm{Ad}(E)\to\mathrm{Ad}(\sigma_X^*\sigma(E))=\sigma_X^*\,\mathrm{Ad}(\sigma(E))$ such that $(\sigma_X^*\rho_{\mathrm{Ad}})\circ\rho_{\mathrm{Ad}}:\mathrm{Ad}(E)\to\mathrm{Ad}(E)$ is the identity. Then $\left(\sigma_X^*s_{\mathrm{Ad}}^{-1}\right)\circ\rho_{\mathrm{Ad}}:\mathrm{Ad}(E)\to\sigma_X^*\,\mathrm{Ad}(E)$ is an isomorphism of principal G-bundles over X. So, it defines an order 2 isomorphism $\rho':\mathrm{Ad}(E)\to\mathrm{Ad}(E)$ which moves the fiber over a point $x\in X$ to the fiber over $\sigma_X(x)$. Now, let $\mathrm{ad}(E)=E\times^G\mathfrak{g}$ be the holomorphic vector bundle over X associated to E for the adjoint action of G on $\mathfrak{g}$ whose typical fiber is $\mathfrak{g}$. This bundle can be understood as the quotient of $E\times\mathfrak{g}$, where two elements (e_1,v_1) and (e_2,v_2) are identified exactly when there exists $h\in G$ such that $e_2=e_1h$ and $v_2=\mathrm{ad}(h^{-1})(v_1)$. Therefore, it is the Lie algebra bundle corresponding to $\mathrm{Ad}(E)$. The holomorphic isomorphism ρ' of $\mathrm{Ad}(E)$ gives rise to a holomorphic isomorphism $\rho'':\mathrm{ad}(E)\to\mathrm{ad}(E)$ defined by taking the quotient of $\rho\times ds:E\times\mathfrak{g}\to E\times\mathfrak{g}$ that moves the fiber over each $x\in X$ to the fiber over $\sigma_X(x)$. Of course, this is an involution that preserves the Lie algebra structure on the fibers of $\mathrm{ad}(E)$. Notice that ρ'' induces an isomorphism

$$\mathrm{ad}(E)\to\mathrm{ad}(\sigma_X^*\sigma(E))=\sigma_X^*\sigma(\mathrm{ad}(E)). \tag{3.1}$$

Given any $x\in X$, a complex linear subspace M of the fiber $\mathrm{ad}(E)_x$ of $\mathrm{ad}(E)$ over x is called parabolic subalgebra if it is the Lie subalgebra of a parabolic subalgebra $\mathfrak{p}$ of $\mathrm{ad}(E)_x$ (that is, if $M=\mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathrm{ad}(E)_x$ which corresponds with certain parabolic subgroup P of G). Similarly, a holomorphic vector subbundle $F\subseteq\mathrm{ad}(E)$ is called the parabolic subalgebra bundle if for each base point $x\in X$, the fiber F_x over x is a parabolic subalgebra of $\mathrm{ad}(E)_x$.

Definition 3.1. A principal (G,σ,σ_X)-bundle (E,ρ) over X is *semistable* (respectively, *stable*) if for every parabolic subbundle F of $\mathrm{ad}(E)$ with $\rho''(F)=F$ it is satisfied that $\deg F\le 0$ (respectively, $\deg F<0$), where ρ'' is defined in (3.1).

Let now $F\subseteq\mathrm{ad}(E)$ be a parabolic subalgebra bundle such that $\rho''(F)=F$. For each $x\in X$, let $R_x(F)\subseteq F$ be the nilpotent radical of the parabolic subalgebra F_x. The sheaf $R(F)$, given by all these nilpotent radicals, satisfies that $\rho''(R(F)) = R(F)$ and the quotient $F/R(F)$ is a bundle of reductive Lie algebras over X. A Levi subalgebra bundle of F is a holomorphic subbundle $L(F) \subseteq F$ such that each fiber $L(F)_x$ is a Lie subalgebra of F_x isomorphic to $F_x/R_x(F)$ through the composition

$$L(F) \hookrightarrow F \to F/R(F).$$

Notice that the fibers of a Levi subalgebra bundle are reductive subalgebras. This allows us to extend the notion of stability and semistability to this kind of Lie algebra bundles: a Levi subalgebra bundle $L(F)$ with $\rho''(F) = F$ is semistable (respectively, stable) if for every parabolic subalgebra bundle $S \subseteq L(F)$ such that $\rho''(S) = S$ it is satisfied that $\deg(S) \leq 0$ (respectively, $\deg(S) < 0$).

Definition 3.2. Let (E, ρ) be a semistable principal (G, σ, σ_X)-bundle over X. Then it is *polystable* if either (E, ρ) is stable as a principal (G, σ, σ_X)-bundle or there is a proper parabolic subalgebra bundle $F \subseteq \mathrm{ad}(E)$ and a Levi subalgebra bundle $L(F) \subseteq F$ such that $\rho''(F) = F$, $\rho''(L(F)) = L(F)$, and $L(F)$ is stable, where ρ'' is defined in (3.1).

Remark 3.3. The definitions given for principal (G, σ, σ_X)-bundles over X are compatible with the notions introduced by Ramanathan [17] and Behrend [8].

The following results explore the notions of stability, semistability and polystability of principal (G, σ, σ_X)-bundles over X and relate them to those of the corresponding principal G-bundles. The main objective is to prove that thus defined principal (G, σ, σ_X)-bundle (E, ρ) is stable (respectively, semistable, polystable) if and only if the principal G-bundle E over X is stable (respectively, semistable, polystable) when $G = \mathrm{SO}(2n, \mathbb{C})$ for $n > 2$.

Lemma 3.4. *Let (E, ρ) be a principal (G, σ, σ_X)-bundle over X. Then it is semistable if and only if the vector bundle* $\mathrm{ad}(E)$ *is semistable.*

Proof. In the case, where $\mathrm{ad}(E)$ is semistable as a vector bundle, (E, ρ) is semistable by definition. For the converse, suppose that (E, ρ) is semistable and $\mathrm{ad}(E)$ is not semistable. Let

$$0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathrm{ad}(E)$$

be the Harder-Narasimhan filtration of the vector bundle $\mathrm{ad}(E)$. Then, as a consequence of [1, Lemma 2.11], n must be odd and $F_{(n+1)/2}$ be a parabolic subalgebra bundle of $\mathrm{ad}(E)$ over a dense open subset U of X such that the complement $X \setminus U$ is a complex analytic subset of codimension at least 2. Moreover, $\{\rho''(F_i)\}_{i=1}^n$ is a Harder-Narasimhan filtration of the same vector bundle because ρ'' is an isomorphism of $\mathrm{ad}(E)$. Therefore, from the uniqueness of the Harder-Narasimhan filtration, it is deduced that $\rho''(F_{(n+1)/2}) = F_{(n+1)/2}$. Also, from the properties of the filtration, $\deg F_{(n+1)/2} > 0$. Then $F_{(n+1)/2}$ is a parabolic subalgebra bundle, which contradicts the semistability condition for (E, ρ). $\square$

Lemma 3.5. *Let (E,ρ) be a stable principal (G,σ,σ_X)-bundle over X. Then* $\mathrm{ad}(E)$ *is polystable as a vector bundle.*

Proof. Assume that (E,ρ) is stable. Then, by Lemma 3.4, $\mathrm{ad}(E)$ is semistable. Suppose that $\mathrm{ad}(E)$ is not polystable. Then there exists a unique filtration

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathrm{ad}(E)$$

such that, for each i with $1 \le i \le n$, the quotient sheaf F_i/F_{i-1} is the socle of the sheaf $\mathrm{ad}(E)/F_{i-1}$. Recall that every semistable sheaf admits a unique maximal polystable subsheaf of the same slope as the sheaf called the socle of the sheaf as proved in [14, Lemma 1.5.5]. As in Lemma 3.4, the subbundle $F_{(n+1)/2}$ is a parabolic subalgebra bundle such that $\rho''(F_{(n+1)/2}) = F_{(n+1)/2}$, by uniqueness of the filtration, and $\deg F_{(n+1)/2} > 0$. Therefore, (E,ρ) is not stable. Then the polystability of $\mathrm{ad}(E)$ is concluded. □

Lemma 3.6. *Let (E,ρ) be a polystable principal (G,σ,σ_X)-bundle over X. Then the vector bundle* $\mathrm{ad}(E)$ *is polystable.*

Proof. If (E,ρ) is strictly polystable (if it is stable, the statement holds by Lemma 3.5), then it is semistable. So, $\mathrm{ad}(E)$ is semistable by Lemma 3.4. There exists a proper parabolic subalgebra bundle F and a Levi subalgebra bundle $L(F)$ as in Definition 3.2. Suppose that $\mathrm{ad}(E)$ is not polystable. Let

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathrm{ad}(E)$$

be the unique filtration as in the proof of Lemma 3.5. Then F_1 is the socle of $\mathrm{ad}(E)$. The Levi subalgebra bundle $L(F)$ is stable, and thus $L(F) \subseteq F_1$.

The vector bundle $F_{(n-1)/2}$ of the filtration is the nilpotent radical bundle of the parabolic subalgebra bundle $F_{(n+1)/2}$, so $F_{(n-1)/2}$ is composed of nilpotent elements. Since F_1 is contained in $F_{(n-1)/2}$, all the elements of F_1 are also nilpotent. This is also true for the Levi subalgebra bundle $L(F)$. But this is a contradiction because the fibers of $L(F)$ are reductive subalgebras of $\mathfrak{g}$. This concludes that $\mathrm{ad}(E)$ must be polystable as a vector bundle. □

Lemma 3.7. *Let (E,ρ) be a polystable principal* $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$*-bundle over X, for some $n > 2$, such that the principal* $\mathrm{SO}(2n,\mathbb{C})$*-bundle E is strictly polystable. Then (E,ρ) is strictly polystable.*

Proof. From the strict polystability of E, the existence of a filtration $0 \subsetneq I \subsetneq I^\perp \subsetneq E$ of E such that

$$E = I \oplus I^\perp/I \oplus E/I^\perp,$$

where I is an isotropic subbundle of E, is deduced. It is satisfied that $E/I^\perp \cong I^*$, so $I \oplus E/I^\perp \cong H(I) = I \oplus I^*$. Let $E' = I^\perp/I$. Then $E = H(I) \oplus E'$, where I, I^* and E' are stable vector bundles, and $H(I)$ and E' are mutually orthogonal subbundles of E. With the natural identification of total spaces, ρ leaves invariant the preceding filtration. Moreover, the preceding filtration induces a parabolic subbundle F of $\mathrm{ad}(E)$ such that $\widetilde{\alpha}''(F) = F$ in terms of the map defined in (3.1). This proves that (E,ρ) is strictly polystable. □

Lemma 3.8. *Let (E,ρ) be a principal (G,σ,σ_X)-bundle over X. Suppose that the principal G-bundle E is polystable. Then (E,ρ) is polystable.*

Proof. Let $F \subseteq \mathrm{ad}(E)$ be a proper parabolic subalgebra bundle of $\mathrm{ad}(E)$ such that the following conditions hold:

1. The bundle F reduces to a Levi subalgebra bundle $L(F)$;
2. F is minimal satisfying the preceding condition.

It is satisfied that $\rho''(F) = F$ (if it is not the case, where $F \cap \rho''(F) \subsetneq F$ is a parabolic subalgebra bundle which contradicts that F is minimal). Therefore, $\widetilde{\alpha}''(L(F)) = L(F)$. In addition, $L(F)$ is stable. If it is not so, let $\mathfrak{q}$ be the subalgebra bundle of $L(F)$ which violates the stability condition, and let $R(L(F))$ be the nilpotent radical. Then $\mathfrak{q} \oplus R(L(F)) \subsetneq L(F)$ and it contradicts the minimality assumption on F. □

Proposition 3.9. *Let σ_X be a holomorphic involution of X and σ be the nontrivial outer automorphism of order 2 of $\mathrm{SO}(2n,\mathbb{C})$ for $n > 2$. Let (E,ρ) be a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle over X. Then (E,ρ) is stable (respectively, semistable, polystable) if and only if the principal $\mathrm{SO}(2n,\mathbb{C})$-bundle E over X is stable (respectively, semistable, polystable).*

Proof. This is a consequence of Lemmas 3.4, 3.5, 3.6, 3.7, and 3.8 under the observation that the adjoint bundle $\mathrm{ad}(E)$ is stable, semistable, or polystable if and only if so is the principal bundle [1]. □

4. Form of principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundles and fixed points

For any semisimple complex Lie group G and any outer automorphism σ of G of order 2, the combination of the automorphism of $M(G)$ induced by the action of σ, defined in (2.1), and the automorphism induced by the action of the fixed holomorphic involution σ_X by pullback, allows the definition of a new automorphism of $M(G)$ given by

$$E \mapsto \sigma_X^*(\sigma(E)). \tag{4.1}$$

Notice that if s is a holomorphic involution of $\mathrm{SO}(2n,\mathbb{C})$ representing σ, then s leaves the elements of the center of $\mathrm{SO}(2n,\mathbb{C})$ fixed. In this situation, by Definition 2.1, a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle over X is a pair (E,ρ), where E is a fixed point of the automorphism defined in (4.1). This follows from Proposition 3.9 since the polystability of a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle (E,ρ) coincides with the polystability of the corresponding principal $\mathrm{SO}(2n,\mathbb{C})$-bundle E. Moreover, the condition that requires a fixed point E of the automorphism defined in (4.1) define a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle is that the isomorphism $\rho : E \to \sigma_X^*(\sigma(E))$, whose existence is guaranteed as E is fixed by (4.1), satisfies that $(\sigma_X^*\rho) \circ \rho$ is the automorphism of E induced by an element of the center of $\mathrm{SO}(2n,\mathbb{C})$. Consequently, the description of the principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundles over X gives a complete description of the simple fixed points of the automorphism of $M(\mathrm{SO}(2n,\mathbb{C}))$ defined in (4.1), where simple refers to a principal $\mathrm{SO}(2n,\mathbb{C})$ that does not admit any automorphism except those defined by the elements of the center of the structure group (that is, a change of sign).

In Theorem 4.2, in addition to providing an explicit expression of a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle over X, it is shown that the fixed points of the automorphism of $M(\mathrm{SO}(2n,\mathbb{C}))$ defined in (4.1) must be fixed points of both the automorphism defined in (2.1) and the automorphism induced by σ_X and defined by taking the pullback. The converse is obviously true.

Lemma 4.1. *Let $n>2$, E be a principal $\mathrm{SO}(2n,\mathbb{C})$-bundle over X, and let $g\in\mathrm{SO}(2n,\mathbb{C})$ be a semisimple noncentral element such that E admits a reduction of structure group to the centralizer $Z(g)$ of g in $\mathrm{SO}(2n,\mathbb{C})$. Then the underlying vector bundle of E admits a nontrivial decomposition into a direct sum of vector subbundles of one of the following forms:*

1. *$E_1\oplus E_2$, where both subbundles have even rank;*
2. *$E_1\oplus E_2\oplus E_3\oplus E_4$, where the ranks of E_1 and E_2 are even, and E_3 and E_4 are isotropic subbundles with the same rank.*

In any case, in each fiber, the indicated subbundles are the eigenspaces of the action of g.

Proof. Since g is semisimple, it is diagonalizable with eigenvalues $\lambda_1,\dots,\lambda_{2n}$. These eigenvalues do not depend on the fiber of E chosen since they define locally constant functions and X is connected. Let $e_1,\dots,e_{2n}$ be a local basis on which g diagonalizes and such that $g(e_i)=\lambda_i e_i$ for every i. If this basis is orthogonal, then $\lambda_i^2=1$ for every i. So, the only eigenvalues are $+1$ and -1, and the induced decomposition $E=E_1\oplus E_2$ is orthogonal. Of course, since g is special orthogonal and the rank of E is even, both subbundles have even rank. If the basis above is not orthogonal, then there exists k with $2\le k\le 2n$ such that

$$\begin{aligned}&\lambda_1\lambda_2=1,\\&\lambda_2\lambda_3=1,\\&\cdots\\&\lambda_{k-1}\lambda_k=1,\\&\lambda_k\lambda_1=1,\end{aligned}$$

and $\lambda_j^2=1$ for $j>k$. The eigenvectors corresponding to $\lambda_1,\dots,\lambda_k$ are of course isotropic. If k is odd, then it can easily be checked that it must be $\lambda_1=\cdots=\lambda_k$ and that this eigenvalue is a square root of unity (so $+1$ or -1). In this case, the desired decomposition of the underlying vector bundle of E is of the form $E_1\oplus E_2$, where E_1 is the $(+1)$-eigenspace of E and E_2 is the (-1)-eigenspace of E. The ranks of E_1 and E_2 are even because g is special orthogonal. However, if k is even, then it must be $\lambda_1=\lambda_3=\cdots=\lambda_{k-1}$ and $\lambda_2=\lambda_4=\cdots=\lambda_k$. Therefore, there exist two eigenvalues, λ and μ, such that

$$\lambda=\lambda_1=\lambda_3=\cdots=\lambda_{k-1},$$

$$\mu = \lambda_2 = \lambda_4 = \cdots = \lambda_k,$$

and $\lambda\mu = 1$. If $\lambda = \mu$, the situation is analogous to the previous one, since $\lambda^2 = 1$ in this case. On the other hand, if $\lambda \neq \mu$, then the announced decomposition is of the form $E_1 \oplus E_2 \oplus E_3 \oplus E_4$, where E_1 is the $(+1)$-eigenspace, E_2 is the (-1)-eigenspace, E_3 is the λ-eigenspace, and E_4 is the μ-eigenspace, the last two being isotropic subbundles with the same rank. Notice that in any case, the decomposition of E is nontrivial since g is not a central element. □

Theorem 4.2. *Let σ_X be a holomorphic involution of X and σ be the non-trivial outer automorphism of order 2 of* SO$(2n,\mathbb{C})$ *for $n > 2$. Let (E,ρ) be a principal* $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$*-bundle over X. Then the underlying holomorphic vector bundle of E admits a nontrivial decomposition into a direct sum of vector subbundles over X of one of the following forms:*

1. *$E_1 \oplus E_2$, where both subbundles have even rank, or*
2. *$E_1 \oplus E_2 \oplus E_3 \oplus E_4$, where the ranks of E_1 and E_2 are even, and E_3 and E_4 are isotropic subbundles with the same rank.*

The subbundles of the decomposition satisfy that $\sigma_X^* E_i \cong E_i$ for every i.

Proof. Let $\rho : E \to \sigma_X^*\sigma(E)$ be the isomorphism announced in Definition 2.1. Given a choice of an element $e \in E$, there exists $g_e \in \mathrm{SO}(2n,\mathbb{C})$ such that $\rho(e) = eg_e$. This g_e is well defined up to conjugacy, that is, if other element $eh \in E$ (with $h \in \mathrm{SO}(2n,\mathbb{C})$) is chosen, then the induced element of $\mathrm{SO}(2n,\mathbb{C})$ is $h^{-1}g_eh$ since $\rho(eh) = eh(h^{-1}g_eh)$. Moreover, the conjugacy class of g_e does not depend on the fiber chosen of E since its trace is a constant function on X. Also, g_e is a semisimple element since the group $\mathrm{Out}(\mathrm{SO}(2n,\mathbb{C}))$ is finite, and it is not central because the actions of σ_X^* and σ are not trivial. Then the subvariety $\{\epsilon \in E : \rho(\epsilon) = \epsilon g_e\}$ of E is a reduction of structure group of E to the centralizer $Z(g_e)$. By Lemma 4.1, the underlying vector bundle of E decomposes as a direct sum of proper vector subbundles as announced in the statement, where the different summands are the eigenspaces of the action of g_e. Notice also that if s is an order-2 automorphism of $\mathrm{SO}(2n,\mathbb{C})$ which represents σ, then s admits every element of $Z(g_e)$ as fixed points. Indeed, if $e \in E$ is an element of the described reduction of structure group and $g \in Z(g_e)$, then $\rho(eg) = egg_e = eg_eg$ by the definition of the reduction and since g commutes with g_e. But, on the other hand, it must be $\rho(eg) = eg_es(g)$, so $s(g) = g$. Since $E \cong \sigma_X^*\sigma(E)$ and the action of σ on E leaves invariant its reduction to $Z(g_e)$, this finally proves that for each i, there exists j such that $\sigma_X^*E_i \cong E_j$. Of course, given an eigenvalue α of g_e since $\rho(e)$ is an α-eigenvector of g_e if e is an α-eigenvector of g_e, given that ρ commutes with the action of g_e, it must be $\sigma_X^*E_i \cong E_i$ for every i. □

Corollary 4.3. *Let σ_X be a holomorphic involution of X and σ be the non-trivial outer automorphism of order 2 of* SO$(2n,\mathbb{C})$ *for $n > 2$. Let (E,ρ) be a stable principal* $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$*-bundle over X. Then the underlying holomorphic vector bundle of E admits a nontrivial decomposition into a direct sum of vector subbundles of even rank $E_1 \oplus E_2$ such that $\sigma_X^*E_1 \cong E_1$ and $\sigma_X^*E_2 \cong E_2$.*

Proof. Under the conditions of the statement, the underlying vector bundle of E admits a nontrivial decomposition into a direct sum of vector subbundles of one of the two forms described in Theorem 4.2. Since the principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma,\sigma_X)$-bundle is stable, then the principal $\mathrm{SO}(2n,\mathbb{C})$-bundle E is stable by Proposition 3.9. If the decomposition of the underlying vector bundle of E obtained were of the form $E_1 \oplus E_2 \oplus E_3 \oplus E_4$ with E_3 and E_4 isotropic, then E would not be stable as a special orthogonal bundle. Hence, the decomposition into a direct sum of even rank vector subbundles must be of the form $E_1 \oplus E_2$. The conditions $\sigma_X^* E_1 \cong E_1$ and $\sigma_X^* E_2 \cong E_2$ follow from Theorem 4.2. □

5. Conclusion

Let X be a compact connected Riemann surface of genus $g \geq 2$ which admits a nontrivial holomorphic involution σ_X, and let G be a semisimple complex Lie group. Suppose, in addition, that G admits a nontrivial outer automorphism σ of order 2. There is introduced the notion of principal (G,σ_X,σ)-bundle over X as pairs (E,ρ), where E is a principal G-bundle over X and $\rho : E \to \sigma_X^*(\sigma(E))$ is a holomorphic isomorphism of G-bundles over X such that $(\sigma_X^*\rho)\circ\rho : E \to E$ acts as the product by an element of the center of G. Principal (G,σ_X,σ)-bundles over X are fixed points of the automorphism of the moduli space of polystable G-bundles over X defined by combining the actions of σ and σ_X on it. Moreover, every simple fixed point of such automorphism induces a principal (G,σ_X,σ)-bundle over X. Proper notions of stability and polystability are given for these geometric objects. When these definitions are applied to the case of $G = \mathrm{SO}(2n,\mathbb{C})$ for $n > 2$ and the unique outer automorphism of order 2 that this group admits, it is proved that a principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle (E,ρ) over X is stable or polystable if and only if so is the principal $\mathrm{SO}(2n,\mathbb{C})$-bundle. Finally, it is proved that if (E,ρ) is such principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundle over X, then the underlying vector bundle of E admits a decomposition into two or four vector subbundles over X. In the case of four vector subbundles, two of them are isotropic subbundles with the same rank. These principal $(\mathrm{SO}(2n,\mathbb{C}),\sigma_X,\sigma)$-bundles described as a direct sum of four vector subbundles are strictly polystable.

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Нерухомі точки головного $SO(2n;\mathbb{C})$-розшарування над компактною рімановою поверхнею

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Нехай X є компактною зв'язною рімановою поверхнею роду $g \geq 2$, оснащеною голоморфною інволюцією σ_X, та нехай G є напівпростою комплексною групою Лі, яка дозволяє зовнішню інволюцію σ. Головне (G, σ_X, σ)-розшарування над X є парою (E, ρ), де E є головним G-розшаруванням над X, а $\rho : E \to \sigma_X^*(\sigma(E))$ є таким ізоморфізмом, що $(\sigma_X^* \rho) \circ \rho : E \to E$ є автоморфізмом E, який діє як добуток з елементом центру групи G. У цій роботі головне (G, σ_X, σ)-розшарування над X введено в розгляд і досліджено у частковому випадку, коли $G = \mathrm{SO}(2n, \mathbb{C})$. Показано, що умови стійкості і мультистійкості для головного $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-розшарування збігаються з такими ж умовами для відповідного головного $\mathrm{SO}(2n, \mathbb{C})$-розшарування. Наприкінці, наведено явний вигляд, якого набирає головне $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-розшарування, і досліджено стійкість таких $(\mathrm{SO}(2n, \mathbb{C}), \sigma_X, \sigma)$-розшарувань.

Ключові слова: головне розшарування, ортогональна група, простір модулів, ріманова поверхня, автоморфізм