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Radial Positive Solutions for Problems Involving ϕ-Laplacian Operators with Weights

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Using the potential theory, we establish the existence and the asypmtotic behavior of radial solutions for the following boundary value problem:

$$\begin{cases} -\dfrac{1}{A}(A\phi(\mid u' \mid)u')' = a(t)u^{\sigma} \quad \text{on } (0,1), \\ A\phi(\mid u' \mid)u'(0) = 0, \\ u(1) = 0, \end{cases}$$

where $\sigma > 0$, A is a positive differentiable function on $(0,1)$ and the nonnegative function ϕ is continuously differentiable on $[0,\infty)$ such that for each $t > 0$,

$$k_1 \leqslant \frac{(t\phi(t))'}{\phi(t)} \leqslant k_2,$$

where $k_1 > 0$ and $k_2 > 0$. The nonnegative nonlinearity a is required to satisfy some appropriate assumptions related to the Karamata regular variation theory. We end this paper by giving applications.

Key words: positive solutions, asymptotic behavior, ϕ-Laplacian, Karamata class

Mathematical Subject Classification 2020: 26A12, 34A34, 34B15, 34B18, 34B27

1. Introduction

In order to explain many physical problems which arise from nonlinear elasticity, plasticity and both Newtonian and Non-Newtonian fluids, a particular attention was paid to problems driving the ϕ-Laplacian operator $u \mapsto -(\phi(u'))'$, where ϕ is an increasing homeomorphism [1, 3, 6, 8, 12–14, 16]. Particular cases of the ϕ-Laplacian are the p-Laplacian and the curvature operators in Euclidean and Minkowski space :

1. $\phi : (-\infty,\infty) \to (-\infty,\infty)$, $u \mapsto u \mid u \mid^{p-2}$, $p > 1$;

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2. $\phi : (-\infty, \infty) \to (-1, 1)$, $u \mapsto \dfrac{u}{\sqrt{1+u^2}}$;
3. $\phi : (-1, 1) \to (-\infty, \infty)$, $u \mapsto \dfrac{u}{\sqrt{1-u^2}}$.

On the other hand, many researchers investigated the existence of positive solutions for boundary value problems of second order ordinary differentiable equations involving the p-Laplacian operator with a positive function weight given by

$$L_p u = -\frac{1}{A}(A|u'|^{p-2}u')',$$

where $p > 1$ and the weight function A is positive satisfying some appropriate assumptions [2, 4, 5, 11, 17, 18]. For instance, in [18] Reichel and Walter studied the equation

$$-\frac{1}{A}\left(A|u'|^{p-2}u'\right)' = f(t, u),$$

where $p > 1$ and $A(t) = t^{\alpha}$, $\alpha \geqslant 0$. For the case where f is increasing in u, a sharp comparison theorem is proved. It leads to maximal solutions, uniqueness and nonuniqueness results and so on. Using these results, a strong comparison principle for the boundary value problem as well as a variety of properties of blow-up solutions are settled under weak assumptions on the nonlinearity f. In [17], Pucci et al. generalized this result and established some uniqueness results for the particular case $A(t) = t^{\alpha-1}r(t)$, $\alpha \geqslant 1$, $r \in C^1([0, \infty))$ and $f(t, u) = u^{\sigma}$, $\sigma > -1$.

Later, in [4], Ben Othman et al. studied the existence of radial solutions for the p-Laplacian problem given as follows:

$$\begin{cases} -\dfrac{1}{A}(A\Phi_p(u'))' = a(t)u^{\sigma} \quad \text{on } (0, 1), \\ A\Phi_p(u')(0) = 0, \\ u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\Phi_p(t) = t|t|^{p-2}$ for $t \in \mathbb{R}$, A is a positive differentiable function on $(0, 1)$ and $\sigma < p - 1$. Applying Karamata regular variation theory and using some potential theory tools, the authors proved in [4] that (1.1) has a unique positive continuous solution and gave sharp estimates on this solution. The p-Laplacian operator was also studied in the vicinity of infinity [5], where the authors established a result of the existence of a positive radial solution. They proved that such a solution verifies a certain asymptotic behavior similar to that of the source function. Dhifli et al. [10] generalized this result to the so-called ϕ-Laplacian problem.

Motivated by the above works, our main purpose is to improve the result given in [4] in the sense that we enlarge the class of the nonlinearity of a and extend the class of operators to the following problem :

$$\begin{cases} -\dfrac{1}{A}(A\phi(|u'|)u')' = a(t)u^{\sigma}, \quad \text{on } (0, 1), \\ A\phi(|u'|)u'(0) = 0, \\ u(1) = 0, \end{cases} \tag{1.2}$$

where the function a satisfies appropriate assumptions related to the Karamata class $\mathcal{K}$, the set of all Karamata functions L defined on $[0,\eta)$ by

$$L(t) := c\exp\left(\int_t^\eta \frac{z(x)}{x}\,dx\right)$$

for some $\eta > 0$, $c > 0$ and $z \in C([0,\eta])$ such that $\lim_{x\to 0} z(x) = 0$.

Throughout the paper, the function ϕ is in $C^1([0,\infty),[0,\infty))$ and it satisfies the condition:

(ϕ) There exist $k_1,\ k_2 > 0$ such that for $t > 0$,

$$k_1 \leqslant \frac{(t\phi(t))'}{\phi(t)} \leqslant k_2.$$

A large class of nonhomogenous differentiable ϕ-Laplacian operators with various types of nonlinearity of the function ϕ, satisfying the condition (ϕ), arises in several physical applications. For the case $\phi(t) = (1+|t|^2)^{p-1}$, $t \in \mathbb{R}$, $p > \frac{1}{2}$, then $k_1 = \min(1, 2p-1)$ and $k_2 = \max(1, 2p-1)$. This operator appears in nonlinear elasticity problems [12]. If $\phi(t) = |t|^{p-2} + |t|^{q-2}$, $t \in \mathbb{R}$ and $1 < p < q$, then $k_1 = p-1$ and $k_2 = q-1$. This operator is called the (p,q)-Laplacian operator and it models the phenomena of quantum physics [3].

By means of fixed point methods, potential theory tools and Karamata regular variation theory, we obtain the existence of positive continuous radial solutions of (1.2) for $0 < \sigma < k_1$ and give estimates on such solutions. To simplify our statements, we need to fix some notations. Let f and h be two nonnegative functions defined on a set S. We write $f(t) \approx h(t)$, $t \in S$, if there exists a constant $c > 1$ such that $c^{-1}h(t) \leqslant f(t) \leqslant ch(t)$ for all $t \in S$. It should be noticed that c denotes a generic positive constant which may vary from line to line.

For $\phi \in C^1([0,\infty),[0,\infty))$ satisfying (ϕ), we put $\Phi(t) = t\phi(|t|)$ for $t \in \mathbb{R}$. It follows from (ϕ) that $\Phi : \mathbb{R} \to \mathbb{R}$ is an odd C^1-increasing homeomorphism.

We refer to $G_\Phi f$ for a nonnegative measurable function f on $(0,1)$ as the function defined on $(0,1)$ by

$$G_\Phi f(t) := \int_t^1 \Phi^{-1}\left(\frac{1}{A(s)}\int_0^s A(r)f(r)\,dr\right)ds, \tag{1.3}$$

where Φ^{-1} is the inverse of Φ. For the special case $\phi(t) = |t|^{p-2},\ t \in \mathbb{R}$ with $p > 1$, we have for $t \in \mathbb{R}$, $\Phi(t) = t|t|^{p-2}$ and $\Phi^{-1}(t) = t|t|^{\frac{2-p}{p-1}}$. Then we shall denote G_Φ by G_p. That is, for a nonnegative measurable function f on $(0,1)$, G_pf is given by

$$G_pf(t) := \int_t^1 \left(\frac{1}{A(s)}\int_0^s A(r)f(r)\,dr\right)^{\frac{1}{p-1}} ds, \quad t \in (0,1). \tag{1.4}$$

Remark 1.1. For a nonnegative continuous function f defined on $(0,1)$ such that the mapping $x \mapsto A(x)f(x)$ is integrable in a neighborhood of 0, $G_\Phi f$ is the solution of the problem

$$\begin{cases} L_\Phi u = \dfrac{1}{A}(A\Phi(u'))' = -f & \text{on } (0,1), \\ A\Phi(u')(0) = 0, \\ u(1) = 0. \end{cases}$$

In what follows, we will define a function that plays a crucial role in this work. Let $p > 1$, $\mu < 0$ and $\beta \leqslant p$. For $L \in \mathcal{K}$, defined on $(0, \eta]$, $\eta > 1$, such that

$$\int_0^\eta t^{\frac{1-\beta}{p-1}} (L(t))^{\frac{1}{p-1}}\, dt < \infty,$$

we denote by $\psi_{p,\beta,L}$ the function defined on $(0,1]$ by

$$\psi_{p,\beta,L}(t) = \begin{cases} 1 & \text{if } \beta < \mu + 1, \\ \left(\displaystyle\int_t^\eta \frac{L(s)}{s} ds\right)^{\frac{1}{p-1}} & \text{if } \beta = \mu + 1, \\ (L(t))^{\frac{1}{p-1}} & \text{if } \mu + 1 < \beta < p, \\ \displaystyle\int_0^t \frac{(L(s))^{\frac{1}{p-1}}}{s} ds & \text{if } \beta = p. \end{cases} \tag{1.5}$$

Our main objective is to study the existence of solutions for problem (1.2). To this end, let us introduce our hypotheses:

($\mathbf{H}_1$) A is a positive differentiable function on $(0,1)$ verifying the following:

$$A(t) \approx t^\lambda (1-t)^\mu, \quad \lambda \geqslant 0 \text{ and } \mu < 0.$$

($\mathbf{H}_2$) a is a positive continuous function on $[0,1)$ such that there exists a constant $c > 1$ satisfying for $t \in [0,1)$,

$$\frac{L_1(1-t)}{c(1-t)^{\beta_1}} \leqslant a(t) \leqslant c\, \frac{L_2(1-t)}{(1-t)^{\beta_2}},$$

where $L_i \in \mathcal{K}$ with $\beta_1 \leqslant \beta_2 < 1$ for $i \in \{1,2\}$ or

$$\beta_1 = k_2 + 1 = \beta_2 = k_1 + 1 \quad \text{and}$$
$$\int_0^1 \frac{(L_1(s))^{\frac{1}{k_2}}}{s}\, ds < \infty, \quad \int_0^1 \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds < \infty.$$

($\mathbf{H}_3$) $\beta_1 - k_1 \leqslant \beta_2 - k_2$.

Now we are ready to state our main results.

Theorem 1.2. *Assume* (ϕ), $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *hold.*
If $\beta_1 \leqslant \beta_2 < 1$, *then there exists* $c > 0$ *such that for* $t \in (0,1)$,

$$\frac{1}{c}(1-t)^{\nu_1} \psi_{k_1+1,\beta_1,L_1}(1-t) \leqslant G_\Phi a(t) \leqslant c(1-t)^{\nu_2} \psi_{k_2+1,\beta_2,L_2}(1-t),$$

where $\nu_i = \min(\frac{k_i+1-\beta_i}{k_i}, \frac{k_i-\mu}{k_i})$ *and* ψ_{k_i+1,β_i,L_i} *is the function given by* (1.5) *for* $i \in \{1,2\}$.

If $\beta_1 = \beta_2 = k_1+1 = k_2+1$, *then there exists* $c > 0$ *such that*

$$\frac{1}{c}\int_0^{1-t} \frac{(L_1(s))^{\frac{1}{k_2}}}{s}\, ds \leqslant G_\Phi a(t) \leqslant c\int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds.$$

Theorem 1.3. *Assume* (ϕ), ($\mathbf{H}_1$)–($\mathbf{H}_3$) *hold. Then for* $0 < \sigma < k_1$, *problem* (1.2) *has a positive continuous solution* u *such that there exists* $c > 1$ *satisfying*

$$\frac{1}{c}\theta_1(t) \leqslant u(t) \leqslant c\,\theta_2(t), \quad t \in (0,1), \tag{1.6}$$

where for $i \in \{1,2\}$, θ_i *is defined on* $(0,1)$ *by*

$$\theta_i(t) := (1-t)^{\delta_i}\tilde{\psi}_i(1-t), \tag{1.7}$$

where $\delta_i = \min(\frac{k_i+1-\beta_i}{k_i-\sigma}, \frac{k_i-\mu}{k_i})$ *and* $\tilde{\psi}_i$ *is defined on* $(0,1]$ *by*

$$\tilde{\psi}_i(t) = \begin{cases} 1 & \text{if } \beta_i < \dfrac{k_i(\mu+1)+\sigma(k_i-\mu)}{k_i}, \\ \left(\displaystyle\int_t^\eta \frac{L_i(s)ds}{s}\right)^{\frac{1}{k_i-\sigma}} & \text{if } \beta_i = \dfrac{k_i(\mu+1)+\sigma(k_i-\mu)}{k_i}, \\ L_i^{\frac{1}{k_i-\sigma}}(t) & \text{if } \dfrac{k_i(\mu+1)+\sigma(k_i-\mu)}{k_i} < \beta_i < k_i+1 \end{cases} \tag{1.8}$$

for $\beta_1 \leqslant \beta_2 < 1$ *or by*

$$\tilde{\psi}_1(t) = \left(\int_0^t \frac{(L_1(s))^{\frac{1}{k_2}}}{s}\right)^{\frac{k_2}{k_2-\sigma}} \quad \text{and} \quad \tilde{\psi}_2(t) = \left(\int_0^t \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\right)^{\frac{k_1}{k_1-\sigma}} \tag{1.9}$$

for $\beta_1 = k_2+1 = \beta_2 = k_1+1$.

The outline of this paper is as follows. Some preliminary results on the Karamata class are stated in Section 2. Section 3 is devoted to proving Theorems 1.2 and 1.3 involving some technical lemmas. The last section contains an example illustrating our results.

2. Karamata class

In this section, we quote some fundamental properties of functions belonging to the class $\mathcal{K}$ taken from [7, 15, 20].

Lemma 2.1. *Let* $L \in \mathcal{K}$ *and* $\varepsilon > 0$. *Then we have*

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0.$$

Proposition 2.2. *A function L is a Karamata function if and only if there exists $\eta > 0$ such that L is a positive function in $C^1((0,\eta])$ satisfying*

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

Lemma 2.3.

1. *Let $L_1, L_2 \in \mathcal{K}$, and let $p \in \mathbb{R}$. Then the functions L_1L_2, $L_1 + L_2$ and L_1^p are in $\mathcal{K}$.*
2. *Let $L \in \mathcal{K}$ be defined on $(0,\eta]$, $\eta > 0$. Then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\displaystyle\int_t^\eta \frac{L(x)}{x}\,dx} = 0.$$

In particular,

$$t \mapsto \int_t^\eta \frac{L(x)}{x}\,dx \in \mathcal{K}.$$

If further $\displaystyle\int_0^\eta \frac{L(x)}{x}dx$ converges, then we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(x)}{x}\,dx} = 0.$$

In particular,

$$t \mapsto \int_0^t \frac{L(x)}{x}\,dx \in \mathcal{K}.$$

Lemma 2.4. *Let $\gamma \in \mathbb{R}$, and let L be a function in $\mathcal{K}$ defined on $(0,\eta]$, $\eta > 0$. We have*

1. *If $\gamma > -1$, then $\displaystyle\int_0^\eta x^\gamma L(x)\,dx$ converges and $\displaystyle\int_0^t x^\gamma L(x)\,dx \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}$.*
2. *If $\gamma < -1$, then $\displaystyle\int_0^\eta x^\gamma L(x)\,dx$ diverges and $\displaystyle\int_t^\eta x^\gamma L(x)\,dx \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}$.*

Remark 2.5. We point out that, due to Lemmas 2.3 and 2.4, the functions $\psi_{p,\beta,L}$ and $\tilde{\psi}_i$, $i \in \{1,2\}$, given respectively by (1.5), (1.8) and (1.9), are in $\mathcal{K}$.

3. Proofs of theorems

3.1. Technical lemmas. The purpose of this paragraph is to provide some technical lemmas which will be useful in the proof of our main results. We notice that if $\phi \in C^1([0,\infty),[0,\infty))$ satisfies the condition (ϕ), then for each $t > 0$, $(t\phi(t))' > 0$. We recall that $\Phi(t) = t\phi(|t|)$, $t \in \mathbb{R}$.

Lemma 3.1 ([19]). *Assume* (ϕ) *holds. Then for* $s,t>0$,

$$\min(t^{k_1},t^{k_2})\Phi(s)\leqslant\Phi(st)\leqslant\max(t^{k_1},t^{k_2})\Phi(s)$$

and

$$\min(t^{\frac{1}{k_1}},t^{\frac{1}{k_2}})\Phi^{-1}(s)\leqslant\Phi^{-1}(st)\leqslant\max(t^{\frac{1}{k_1}},t^{\frac{1}{k_2}})\Phi^{-1}(s).$$

Remark 3.2.

1. There exists a positive constant c_0 such that for $0<t\leqslant 1$,

$$\frac{1}{c_0}\,t^{\frac{1}{k_1}}\leqslant\Phi^{-1}(t)\leqslant c_0\,t^{\frac{1}{k_2}}. \tag{3.1}$$

2. There exists a positive constant c_1 such that for $t\geqslant 1$,

$$\frac{1}{c_1}\,t^{\frac{1}{k_2}}\leqslant\Phi^{-1}(t)\leqslant c_1\,t^{\frac{1}{k_1}}. \tag{3.2}$$

Lemma 3.3 ([4, Proposition 2.4]). *Let $p>1$ and $\beta\leqslant p$. We suppose that A is continuous on $[0,1)$, differentiable and positive on $(0,1)$ such that*

$$A(t)\approx t^{\lambda}(1-t)^{\mu},$$

where $\lambda\geqslant 0$ and $\mu<p-1$. Let q be the function defined on $[0,1)$ by

$$q(t)=(1-t)^{-\beta}L(1-t)$$

such that $L\in\mathcal{K}$ is defined on $(0,\eta]$, $\eta>1$, and let it satisfy

$$\int_0^{\eta}t^{\frac{1-\beta}{p-1}}(L(t))^{\frac{1}{p-1}}\,dt<\infty.$$

Then we have

$$G_pq(t)\approx(1-t)^{\frac{\tilde\beta}{p-1}}\psi_{p,\beta,L}(1-t),\quad t\in[0,1), \tag{3.3}$$

where $\psi_{p,\beta,L}$ is the function given by (1.5) *and* $\tilde\beta=\min(p-\beta,p-1-\mu)$.

Lemma 3.4. *Let θ_1, θ_2 be the functions given by* (1.7). *Assume* (ϕ) *and* ($\mathbf{H}_1$)–($\mathbf{H}_3$) *hold. Then the function $\frac{\theta_1}{\theta_2}$ is bounded above. That is, there exists $c>0$ such that for $t\in[0,1)$,*

$$\frac{\theta_1(t)}{\theta_2(t)}\leqslant c.$$

Proof. We divide the proof into two cases.

Case 1: If $\beta_1\leqslant\beta_2<1$. For $i\in\{1,2\}$ and $t\in[0,1)$, we consider

$$\theta_i(t)=(1-t)^{\delta_i}\tilde\psi_i(1-t),$$

where $\delta_i=\min\left(\frac{k_i-\mu}{k_i},\frac{k_i+1-\beta_i}{k_i-\sigma}\right)$ and $\tilde\psi_i$ is the function given by (1.8). One can easily see that

$$\frac{k_2-\mu}{k_2}\leqslant\frac{k_1-\mu}{k_1}, \tag{3.4}$$

and we deduce from ($\mathbf{H}_3$) that

$$\frac{k_2+1-\beta_2}{k_2-\sigma} \leqslant \frac{k_2+1-\beta_2}{k_1-\sigma} \leqslant \frac{k_1+1-\beta_1}{k_1-\sigma}. \tag{3.5}$$

Equations (3.4) and (3.5) imply that $\delta_2 \leqslant \delta_1$.

For $\delta_2 < \delta_1$, by using Lemma 2.1, we get that the function $\frac{\tilde{\psi}_1}{\tilde{\psi}_2}$ is in $\mathcal{K}$ and that

$$\lim_{t\to 1^-}\left(\frac{\theta_1}{\theta_2}\right)(t) = \lim_{t\to 1^-}(1-t)^{\delta_1-\delta_2}\left(\frac{\tilde{\psi}_1}{\tilde{\psi}_2}\right)(1-t) = 0.$$

Since $\frac{\theta_1}{\theta_2} \in C([0,1])$, we obtain that $\frac{\theta_1}{\theta_2}$ is bounded above on $(0,1)$. It only remains to prove the result when $\delta_1 = \delta_2$. We split the proof into the following subcases:

Subcase 1. Assume $\frac{k_1-\mu}{k_1} = \frac{k_2-\mu}{k_2} < \frac{k_2+1-\beta_2}{k_2-\sigma}$. Therefore we obtain that $k_1 = k_2$, $\beta_1 < \frac{k_1(\mu+1)+\sigma(k_1-\mu)}{k_1}$ and $\beta_2 < \frac{k_2(\mu+1)+\sigma(k_2-\mu)}{k_2}$. It follows that for $t \in (0,1)$, we have

$$\frac{\theta_1(t)}{\theta_2(t)} = 1.$$

Subcase 2. Assume $\frac{k_1-\mu}{k_1} = \frac{k_2-\mu}{k_2} = \frac{k_2+1-\beta_2}{k_2-\sigma} < \frac{k_1+1-\beta_1}{k_1-\sigma}$. This implies that $k_1 = k_2$, $\beta_1 < \frac{k_1(\mu+1)+\sigma(k_1-\mu)}{k_1}$ and $\beta_2 = \frac{k_2(\mu+1)+\sigma(k_2-\mu)}{k_2}$. We have

$$\frac{\theta_1(t)}{\theta_2(t)} = \left(\int_{1-t}^{\eta}\frac{L_2(s)}{s}\,ds\right)^{\frac{-1}{k_2-\sigma}}.$$

Since $0 < \sigma < k_1 = k_2$, we obtain that

$$\lim_{t\to 1^-}\left(\frac{\theta_1}{\theta_2}\right)(t) < \infty.$$

This implies that $\dfrac{\theta_1}{\theta_2}$ is bounded on $(0,1)$.

Subcase 3. Assume $\frac{k_1-\mu}{k_1} = \frac{k_2-\mu}{k_2} = \frac{k_2+1-\beta_2}{k_2-\sigma} = \frac{k_1+1-\beta_1}{k_1-\sigma}$. In this case, we conclude that $k_1 = k_2$ and $\beta_1 = \beta_2 = \frac{k_1(\mu+1)+\sigma(k_1-\mu)}{k_1}$. Thus, from (H_2), we have $L_1 \leqslant L_2$ and for $t \in [0,1)$,

$$\frac{\theta_1(t)}{\theta_2(t)} = \left(\frac{\displaystyle\int_{1-t}^{\eta}\frac{L_1(s)}{s}\,ds}{\displaystyle\int_{1-t}^{\eta}\frac{L_2(s)}{s}\,ds}\right)^{\frac{1}{k_1-\sigma}} \leqslant 1.$$

Subcase 4. Assume $\frac{k_1+1-\beta_1}{k_1-\sigma} = \frac{k_2+1-\beta_2}{k_2-\sigma} < \frac{k_2-\mu}{k_2}$. Then for $i \in \{1,2\}$, $\beta_i > \frac{k_i(\mu+1)+\sigma(k_i-\mu)}{k_i}$. We deduce from (3.5) that $k_1 = k_2$ and $\beta_1 = \beta_2$. It follows from ($\mathbf{H}_2$) that $L_1 \leqslant L_2$. Therefore, for $t \in [0,1)$, we have

$$\frac{\theta_1(t)}{\theta_2(t)} = \left(\frac{L_1(1-t)}{L_2(1-t)}\right)^{\frac{1}{k_1-\sigma}} \leqslant 1.$$

Case 2: If $\beta_1 = k_2 + 1, \beta_2 = k_1 + 1$, then $L_1 \leqslant L_2$ and $k_1 = k_2$. This implies that for $t \in [0, 1]$,

$$\frac{\theta_1(t)}{\theta_2(t)} = \frac{\left(\int_0^{1-t} \frac{(L_1(s))^{\frac{1}{k_2}}}{s}\, ds\right)^{\frac{k_2}{k_2-\sigma}}}{\left(\int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds\right)^{\frac{k_1}{k_1-\sigma}}} = \left(\frac{\int_0^{1-t} \frac{(L_1(s))^{\frac{1}{k_1}}}{s}\, ds}{\int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds}\right)^{\frac{k_1}{k_1-\sigma}} \leqslant 1,$$

which ends the proof. □

3.2. Proof of Theorem 1.2. We distinguish two cases.

Case 1: Assume $\beta_1 \leqslant \beta_2 < 1$. For $t \in (0, 1)$, we have

$$G_\Phi a(t) = \int_t^1 \Phi^{-1}\left(\frac{1}{A(s)} \int_0^s A(r)a(r)\, dr\right) ds.$$

Put

$$h(s) = \frac{1}{A(s)} \int_0^s A(r)a(r)\, dr, \quad s \in (0, 1).$$

We claim that $\lim_{s \to 1^-} h(s) = 0$. Using $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we have for $s \in (\frac{1}{2}, 1)$,

$$\begin{aligned} h(s) &\leqslant c\, s^{-\lambda}(1-s)^{-\mu} \int_0^s r^\lambda (1-r)^{\mu-\beta_2} L_2(1-r)\, dr \\ &\leqslant c(1-s)^{-\mu}\left(\int_{\frac{1}{2}}^1 (1-r)^\lambda\, dr + \int_{1-s}^{\frac{1}{2}} r^{\mu-\beta_2} L_2(r)\, dr\right) \\ &\leqslant c(1-s)^{-\mu}\left(1 + \int_{1-s}^{\frac{1}{2}} r^{\mu-\beta_2} L_2(r)\, dr\right). \end{aligned}$$

We distinguish the following subcases:

Subcase 1: Let $\beta_2 < \mu + 1$. Then, using Lemma 2.4, we get that

$$\int_0^{\frac{1}{2}} r^{\mu-\beta_2} L_2(r)\, dr < \infty.$$

Thus we have

$$h(s) \leqslant c\, (1-s)^{-\mu}\left(1 + \int_{1-s}^{\frac{1}{2}} r^{\mu-\beta_2} L_2(r)\, dr\right) \leqslant c\, (1-s)^{-\mu}.$$

Since $\mu < 0$, we obtain that $\lim_{s \to 1^-} h(s) = 0$.

Subcase 2: Let $\beta_2 = \mu + 1$. We have

$$h(s) \leqslant c\, (1-s)^{-\mu}\left(1 + \int_{1-s}^{\frac{1}{2}} \frac{L_2(r)}{r}\, dr\right).$$

By Lemma 2.3, we obtain that

$$s \mapsto 1 + \int_{1-s}^{\frac{1}{2}} \frac{L_2(r)}{r}\, dr \in \mathcal{K}.$$

By Lemma 2.1, it implies that $\lim\limits_{s\to 1^-} h(s) = 0$.

Subcase 3: Let $\mu + 1 < \beta_2 < 1$. By Lemma 2.4, for $s \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned} h(s) &\leqslant c\,(1-s)^{-\mu}\left(1 + (1-s)^{1+\mu-\beta_2} L_2(1-s)\right) \\ &\leqslant c\,(1-s)^{1-\beta_2} L_2(1-s)\left(1 + \frac{(1-s)^{\beta_2-1-\mu}}{L_2(1-s)}\right). \end{aligned}$$

By Lemma 2.3, we obtain that $\frac{1}{L_2} \in \mathcal{K}$. Using the fact that $\beta_2 - 1 - \mu > 0$, Lemma 2.1 implies that

$$h(s) \leqslant c\,(1-s)^{1-\beta_2} L_2(1-s).$$

Finally, applying again Lemma 2.1, we deduce that $\lim\limits_{s\to 1^-} h(s) = 0$.

The claim is proved. So, there exists $0 < \delta_0 < 1$ such that $h(s) \leqslant 1$ for $s \in [\delta_0, 1)$. By (3.1), for $t \in [\delta_0, 1)$, we have

$$G_\Phi a(t) = \int_t^1 \Phi^{-1}(h(s))\, ds \leqslant c \int_t^1 (h(s))^{\frac{1}{k_2}}\, ds.$$

Hence, from (1.4), it follows that

$$G_\Phi a(t) \leqslant c\, G_{k_2+1} a(t), \quad t \in [\delta_0, 1).$$

By Lemma 3.3, using the fact that $\beta_2 < 1 < k_2 + 1$, we can deduce that for $t \in [\delta_0, 1)$,

$$G_{k_2+1} a(t) \leqslant c(1-t)^{\nu_2} \psi_{k_2+1,\beta_2,L_2}(1-t),$$

where $\nu_2 = \min\left(\frac{k_2-1-\beta_2}{k_2}, \frac{k_2-\mu}{k_2}\right)$ and ψ_{k_2+1,β_2,L_2} is the function given by (1.5). This implies that

$$G_\Phi a(t) \leqslant c\,(1-t)^{\nu_2} \psi_{k_2+1,\beta_2,L_2}(1-t), \quad t \in [\delta_0, 1). \tag{3.6}$$

Now, since the functions $G_\Phi a$ and $t \mapsto \psi_{k_2+1,\beta_2,L_2}(1-t)$ are positive and continuous in $[0, \delta_0]$, then inequality (3.6) remains true for $t \in [0, \delta_0]$. Hence, for $t \in (0, 1)$, we conclude that

$$G_\Phi a(t) \leqslant c\,(1-t)^{\nu_2} \psi_{k_2+1,\beta_2,L_2}(1-t).$$

The same arguments are used to prove the lower estimates:

$$\frac{1}{c}(1-t)^{\nu_1} \psi_{k_1+1,\beta_1,L_1}(1-t) \leqslant G_\Phi a(t), \quad t \in (0, 1).$$

Case 2: Assume $\beta_1 = k_2 + 1 = \beta_2 = k_1 + 1$. Using **(H$_1$)** and **(H$_2$)**, we have for $s \in (\frac{1}{2}, 1)$,

$$h(s) \geqslant \frac{1}{c}(1-s)^{-\mu} \int_{\frac{1}{2}}^{s} (1-t)^{\mu-\beta_1} L_1(1-t)\, dt \geqslant \frac{1}{c}(1-s)^{-\mu} \int_{1-s}^{\frac{1}{2}} r^{\mu-\beta_1} L_1(r)\, dr.$$

Since $\mu - \beta_1 < -1$, by Lemma 2.4, we obtain

$$h(s) \geqslant \frac{1}{c}(1-s)^{1-\beta_1} L_1(1-s).$$

This implies by Lemmas 2.1, 2.3 and the fact that $\beta_1 - 1 > 0$ and

$$\lim_{s\to 1^-} h(s) = \lim_{s\to 1^-} (1-s)^{1-\beta_1} L_1(1-s) = \lim_{s\to 1^-} \frac{1}{(1-s)^{\beta_1-1} \dfrac{1}{L_1(1-s)}} = \infty.$$

So, there exists $0 < \delta_0 < 1$ such that $h(s) \geqslant 1$ for $s \in [\delta_0, 1)$. By (3.2), we have for $t \in [\delta_0, 1)$,

$$G_\Phi a(t) \leqslant c \int_t^1 (h(s))^{\frac{1}{k_1}}\, ds.$$

From (1.4), we have

$$G_\Phi a(t) \leqslant c\, G_{k_1+1} a(t), \quad t \in [\delta_0, 1).$$

Using (H_2) and the fact that $\beta_2 = k_1 + 1$, we can deduce by Lemma 3.3 that for $t \in [\delta_0, 1)$,

$$G_{k_1+1} a(t) \leqslant c \int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds.$$

This implies that

$$G_\Phi a(t) \leqslant c \int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds, \quad t \in [\delta_0, 1). \tag{3.7}$$

Now, since the functions $G_\Phi a$ and

$$t \mapsto \int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds$$

are positive and continuous on $[0, \delta_0]$, then inequality (3.7) remains true for $t \in [0, \delta_0]$. Hence, for $t \in (0, 1)$, we conclude that

$$G_\Phi a(t) \leqslant c \int_0^{1-t} \frac{(L_2(s))^{\frac{1}{k_1}}}{s}\, ds.$$

The same arguments are used to prove the lower estimates:

$$\frac{1}{c} \int_0^{1-t} \frac{(L_1(s))^{\frac{1}{k_2}}}{s}\, ds \leqslant G_\Phi a(t), \quad t \in (0, 1).$$

This leads to the following.

Corollary 3.5. *Assume* (ϕ), $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Let* θ_1 *and* θ_2 *be the functions given by* (1.7). *Then there exists* $c > 0$ *such that for* $t \in (0,1)$,

$$\frac{1}{c}\theta_1(t) \leqslant G_\Phi(a\theta_1^\sigma)(t) \tag{3.8}$$

and

$$G_\Phi(a\theta_2^\sigma)(t) \leqslant c\,\theta_2(t). \tag{3.9}$$

Proof. Using (H_2), we obtain that for $t \in (0,1)$,

$$(1-t)^{-\tilde{\beta}_1}\tilde{L}_1(1-t) \leqslant c\,(a\theta_1^\sigma)(t)$$

and

$$\frac{1}{c}\,(a\theta_2^\sigma)(t) \leqslant (1-t)^{-\tilde{\beta}_2}\tilde{L}_2(1-t),$$

where for $i \in \{1,2\}$, $\tilde{\beta}_i = \beta_i - \sigma\delta_i$ and $\tilde{L}_i = L_i\tilde{\psi}_i^\sigma$ such that $\tilde{\psi}_i$ are the functions given by (1.8) and (1.9).

In what follows, we distinguish two cases :

Case 1: If $\beta_1 \leqslant \beta_2 < 1$, then it follows from Theorem 1.2 that there exists $c > 1$ such that for $t \in (0,1)$,

$$G_\Phi(a\theta_1^\sigma)(t) \geqslant \frac{1}{c}(1-t)^{\tilde{\nu}_1}\psi_{k_1+1,\tilde{\beta}_1,\tilde{L}_1}(1-t) \tag{3.10}$$

and

$$G_\Phi(a\theta_2^\sigma)(t) \leqslant c\,(1-t)^{\tilde{\nu}_2}\psi_{k_2+1,\tilde{\beta}_2,\tilde{L}_2}(1-t) \tag{3.11}$$

such that for $i \in \{1,2\}$, $\psi_{k_i+1,\tilde{\beta}_i,\tilde{L}_i}$ is the function given by (1.5), and $\tilde{\nu}_i = \min\left(\frac{k_i+1-\tilde{\beta}_i}{k_i}, \frac{k_i-\mu}{k_i}\right)$. By simple calculus, we get that $\psi_{k_i+1,\tilde{\beta}_i,\tilde{L}_i} = \tilde{\psi}_i$ and $\tilde{\nu}_i = \delta_i$. Finally, (1.7) leads to (3.8) and (3.9).

Case 2: Let $\beta_1 = k_2 + 1 = \beta_2 = k_1 + 1$. From Theorem 1.2, there exists $c > 1$ such that for $t \in (0,1)$,

$$G_\Phi(a\theta_1^\sigma)(t) \geqslant \frac{1}{c}\left(\int_0^t \frac{(\tilde{L}_1(s))^{\frac{1}{k_2}}}{s}\,ds\right) \tag{3.12}$$

and

$$G_\Phi(a\theta_2^\sigma)(t) \leqslant c\,\left(\int_0^t \frac{(\tilde{L}_2(s))^{\frac{1}{k_1}}}{s}\,ds\right). \tag{3.13}$$

By simple calculus, we get (3.8) and (3.9). □

3.3. Proof of Theorem 1.3. Let θ_1 and θ_2 be the functions defined by (1.7) and let a be a function satisfying $(\mathbf{H}_2)$. By Corollary 3.5, there exists $m > 1$ such that for $t \in (0,1)$,

$$\frac{1}{m}\theta_1(t) \leqslant G_\Phi(a\theta_1^\sigma)(t), \tag{3.14}$$

$$G_\Phi(a\theta_2^\sigma)(t) \leqslant m\theta_2(t). \tag{3.15}$$

Besides, Lemma 3.4 implies that there exists $M > 1$ satisfying on $(0,1)$,

$$\theta_1 \leqslant M\theta_2.$$

Now we look at the existence of positive solution of problem (1.2) satisfying (1.6). We consider the following closed convex:

$$Y = \left\{ u \in C([0,1]) \;\middle|\; \frac{1}{c}\theta_1 \leqslant u \leqslant c\,\theta_2 \right\},$$

where $c = \max\left(M, m^{\frac{k_1}{k_1-\sigma}}\right)$. Using Proposition 2.3 and Lemma 2.1, we get that $\theta_1 \in C([0,1])$. Moreover, the fact that

$$\frac{1}{c}\theta_1 \leqslant \theta_1 \leqslant M\theta_2 \leqslant c\,\theta_2$$

implies that $\theta_1 \in Y$. Then Y is non empty. On Y, we define the integral operator T by

$$Tu := G_\Phi(au^\sigma).$$

In order to prove that T has a fixed point in Y, we should first show that T leaves invariant the convex Y. Let u be a function in Y. Since G_Φ is nondecreasing, we obtain that

$$Tu \geqslant G_\Phi(c^{-\sigma}a\theta_1^\sigma) \tag{3.16}$$

and

$$Tu \leqslant G_\Phi(c^{\sigma}a\theta_2^\sigma). \tag{3.17}$$

By Lemma 3.1, using the fact that $c^{-\sigma} \leqslant 1$, we conclude that on $(0,1)$,

$$G_\Phi(c^{-\sigma}a\theta_1^\sigma)(t) = \int_t^1 \Phi^{-1}\left(c^{-\sigma}\frac{1}{A(s)}\int_0^s A(r)(a\theta_1^\sigma)(r)\,dr\right)ds \tag{3.18}$$

$$\geqslant c^{-\frac{\sigma}{k_1}} G_\Phi(a\theta_1^\sigma). \tag{3.19}$$

Using the same arguments as above, we obtain

$$G_\Phi(c^{\sigma}a\theta_2^\sigma) = \int_t^1 \Phi^{-1}\left(c^{\sigma}\frac{1}{A(s)}\int_0^s A(r)(a\theta_2^\sigma)(r)dr\right)ds \leqslant c^{\frac{\sigma}{k_1}} G_\Phi(a\theta_2^\sigma). \tag{3.20}$$

Combining (3.16), (3.18) with (3.14) and (3.17), (3.20) with (3.15), we conclude the following:

$$Tu \geqslant c^{-\frac{\sigma}{k_1}} G_\Phi(a\theta_1^\sigma) \geqslant c^{-\frac{\sigma}{k_1}} \frac{1}{m}\theta_1 \geqslant \frac{1}{c}\theta_1 \tag{3.21}$$

and

$$Tu \leqslant c^{\frac{\sigma}{k_1}} G_\Phi(a\theta_2^\sigma) \leqslant c^{\frac{\sigma}{k_1}}\; m\;\theta_2 \leqslant c\;\theta_2. \tag{3.22}$$

This yields

$$\frac{1}{c}\theta_1 \leqslant Tu \leqslant c\,\theta_2.$$

With the fact that $TY \subset C([0,1])$, we conclude that $TY \subset Y$. Now, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([0,1])$ defined by

$$\begin{cases} u_0 = \dfrac{1}{c}\theta_1, \\ u_{n+1} = Tu_n \quad \text{for } n \in \mathbb{N}. \end{cases}$$

Moreover, since $\sigma > 0$, the operator T is nondecreasing on Y. This implies that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant u_{n+1} \leqslant c\theta_2.$$

It follows from the monotone convergence theorem that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a function $u \in Y$ which satisfies

$$u(t) = Tu(t) = G_\Phi(au^\sigma)(t), \quad t \in [0,1].$$

Thus, we deduce that problem (1.2) has a positive continuous solution u verifying (1.6).

4. Example

Let $1 < p < q$, $0 < \sigma < p-1$ and $\phi(t) = t^{p-2} + t^{q-2}$, $t > 0$. Consider the $\phi-$Laplacian problem

$$\begin{cases} -\dfrac{1}{A}\left(A\,(|u'|^{p-2} + |u'|^{q-2})u'\right)' = a(t)u^\sigma \quad \text{on } (0,1), \\ \left(A\,(|u'|^{p-2} + |u'|^{q-2})u'\right)(0) = 0, \\ u(1) = 0, \end{cases} \tag{4.1}$$

where

$$A(t) = t^{\frac{1}{2}}(1-t)^{\frac{-1}{2}}, \quad t \in [0,1),$$

and a is a continuous function on $[0,1)$ such that for $t \in [0,1)$,

$$\frac{\log^{-1}(\frac{3}{1-t})}{c(1-t)^{\beta_1}} \leqslant a(t) \leqslant c\frac{\log(\frac{3}{1-t})}{(1-t)^{\beta_2}},$$

where $\beta_1 < \beta_2 < 1$ such that $\beta_1 - p < \beta_2 - q$.

Let $a_1(t) = (1-t)^{-\beta_1}\log^{-1}(\frac{3}{1-t})$ and $a_2(t) = (1-t)^{-\beta_2}\log(\frac{3}{1-t})$ be the functions defined on $[0,1)$. By Theorem 1.3, problem (4.1) has a positive solution u satisfying for $t \in [0,1)$,

$$\frac{1}{c}\,\theta_1(t) \leqslant u(t) \leqslant c\,\theta_2(t),$$

where

$$\theta_1(t) = (1-t)^{\min\left(\frac{p-\beta_1}{p-1-\sigma}, \frac{p-\frac{1}{2}}{p-1}\right)} \begin{cases} 1 & \text{if } \beta_1 < \frac{1}{2} + \sigma\frac{2p-1}{2(p-1)}, \\ \left(\log\left(\log\left(\frac{3}{1-t}\right)\right)\right)^{\frac{1}{p-1-\sigma}} & \text{if } \beta_1 = \frac{1}{2} + \sigma\frac{2p-1}{2(p-1)}, \\ \left(\log\left(\frac{3}{1-t}\right)\right)^{\frac{-1}{p-1-\sigma}} & \text{if } \frac{1}{2} + \sigma\frac{2p-1}{2(p-1)} < \beta_1 < p \end{cases}$$

and

$$\theta_2(t) = (1-t)^{\min\left(\frac{q-\beta_2}{q-1-\sigma}, \frac{q-\frac{1}{2}}{q-1}\right)} \begin{cases} 1 & \text{if } \beta_2 < \frac{1}{2} + \sigma\frac{2q-1}{2(q-1)}, \\ \left(\log\left(\frac{3}{1-t}\right)\right)^{\frac{2}{q-1-\sigma}} & \text{if } \beta_2 = \frac{1}{2} + \sigma\frac{2q-1}{2(q-1)}, \\ \left(\log\left(\frac{3}{1-t}\right)\right)^{\frac{1}{q-1-\sigma}} & \text{if } \frac{1}{2} + \sigma\frac{2q-1}{2(q-1)} < \beta_2 < q. \end{cases}$$

In what follows, we give numerical illustrations of our example.

(I) We consider $\sigma = \frac{1}{12}$, $p = 2$, $q = \frac{9}{4}$, $\beta_1 = \frac{19}{36}$, and $\beta_2 = \frac{15}{16}$. The functions a_1 and a_2 are drawn in Fig. 4.1.

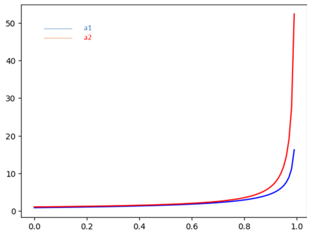

Fig. 4.1: The functions a_1 and a_2.

We notice that β_1 and β_2 satisfy the conditions $\beta_1 = \frac{1}{2} + \sigma\frac{2p-1}{2(p-1)}$ and $\frac{2(q-p)+1}{2} + \sigma\frac{2p-1}{2(p-1)} < \beta_2 < 1$. Therefore θ_1 and θ_2 are respectively given for $t \in [0, 1)$,

$$\theta_1(t) = (1-t)^{\frac{3}{2}}\left(\log\left(\log\left(\frac{3}{1-t}\right)\right)\right)^{\frac{12}{11}}$$

and

$$\theta_2(t) = (1-t)^{\frac{9}{8}}\log\left(\frac{3}{1-t}\right)^{\frac{6}{7}}.$$

The representation below shows the functions θ_1 and θ_2 (see Fig. 4.2).

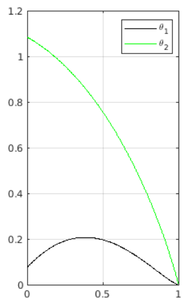

Fig. 4.2: The functions θ_1 and θ_2.

(II) We take $\sigma = \frac{1}{6}$, $p = \frac{3}{2}$, $q = \frac{4}{3}$, $\beta_1 = \frac{9}{10}$, and $\beta_2 = \frac{11}{12}$. Fig. 4.3 represents the functions a_1 and a_2.

We should notice that β_1 and β_2 satisfy the conditions $\frac{1}{2} + \sigma\frac{2p-1}{2(p-1)} < \beta_1 < p$ and $\beta_2 = \frac{1}{2} + \sigma\frac{2q-1}{2(q-1)}$. We conclude that the functions θ_1 and θ_2 are

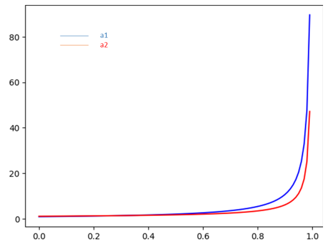

Fig. 4.3: The functions a_1 and a_2.

respectively defined on $[0,1)$ by

$$\theta_1(t) = (1-t)^{\frac{9}{5}} \left((\log(\frac{3}{1-t}) \right)^{-3}$$

and

$$\theta_2(t) = (1-t)^{\frac{5}{2}} \log(\frac{3}{1-t})^{12}.$$

The functions θ_1 and θ_2 are represented in Fig. 4.4.

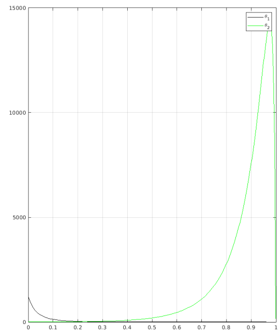

Fig. 4.4: The functions θ_1 and θ_2.

(III) For $\sigma = \frac{1}{7}$, $p = \frac{5}{4}$, $q = \frac{3}{2}$, $\beta_1 = \frac{5}{7}$, and $\beta_2 = \frac{181}{210}$. The functions a_1 and a_2 are represented in Fig. 4.5.

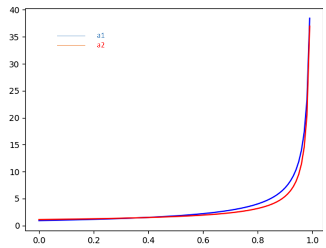

Fig. 4.5: The functions a_1 and a_2.

Here, β_1 and β_2 satisfy the conditions $\beta_1 < \frac{1}{2} + \sigma\frac{2p-1}{2(p-1)}$ and $\frac{1}{2} + \sigma\frac{2q-1}{2(q-1)} < \beta_2 < q$. Therefore we have for $t \in [0,1)$,

$$\theta_1(t) = (1-t)^3$$

and

$$\theta_2(t) = (1-t)^{\frac{134}{75}} \log(\frac{3}{1-t})^{\frac{14}{5}}.$$

The behavior of the functions θ_1 and θ_2 is shown in Fig 4.6.

Conclusion. We studied the existence and the asymptotic behavior of radial positive solution to a class of problems involving the ϕ-Laplacian operator. We recall that our work extends [4], where the authors considered the $p-$Laplacian with $\beta_1 = \beta_2$ and $L_1 = L_2$. However, we mention that due to technical difficulties, our results involve the extremal cases $\beta_i = k_i + 1$ for $i \in \{1, 2\}$, only for the particular case $\beta_1 = \beta_2$ and $L_1 \leqslant L_2$. Finally, some numerical simulations were given to illustrate our results.

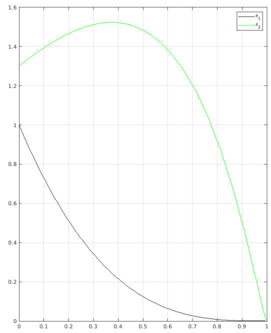

Fig. 4.6: The functions θ_1 and θ_2.

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Радіальні додатні розв'язки для задач, що включають вагові оператори ϕ-лапласіана

Sywar Belkahla, Bilel Khamessi, and Zagharide Zine El Abidine

Використовуючи теорію потенціалу, встановлюємо існування та асимптотичну поведінку радіальних розв'язків наступної крайової задачі:

$$\begin{cases} -\dfrac{1}{A}(A\phi(\mid u' \mid)u')' = a(t)u^{\sigma} \quad \text{on } (0,1), \\ A\phi(\mid u' \mid)u'(0) = 0, \\ u(1) = 0, \end{cases}$$

де $\sigma > 0$, A є додатною диференційовною функцією на $(0,1)$, а невід'ємна функція ϕ є неперервно диференційовною на $[0,\infty)$ так, що для кожного $t > 0$,

$$k_1 \leqslant \frac{(t\phi(t))'}{\phi(t)} \leqslant k_2,$$

де $k_1 > 0$ і $k_2 > 0$. Невід'ємна нелінійність a повинна задовольняти деякі відповідні припущення, пов'язані з теорією регулярних варіацій Карамати. Ми закінчуємо цю роботу розглядом застосувань.

Ключові слова: додатні розв'язки, асимптотична поведінка, ϕ-лапласіан, клас Карамати