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Positive Matrix Representations of Rational Positive Real Functions of Several Variables

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A rational homogeneous (of degree one) positive real matrix-valued function of several variables can be represented as a Schur complement to the diagonal block of a linear homogeneous matrix-valued function with positive semidefinite real matrix coefficients (the long-resolvent representation). The numerators of the partial derivatives of a positive real function are sums of squares of polynomials.

Key words: positive real function, matrix-valued function, Schur complement, long-resolvent representation, sum of squares

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1. Introduction

The long-resolvent representation theorem (see [4–6]) asserts that each rational $m \times m$ matrix-valued function $f(z_1, \dots, z_d)$ is a Schur complement

$$f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z) \tag{1.1}$$

of the block $A_{22}(z)$ of a linear $(m + l) \times (m + l)$ matrix-valued function (linear pencil)

$$A(z) = \{A_{ij}(z)\}_{i,j=1}^{2} = A_0 + z_1 A_1 + \cdots + z_d A_d. \tag{1.2}$$

If, moreover, $f(z)$ satisfies additional conditions from the list:

(i) $\overline{f(\bar{z}_1, \dots, \bar{z}_d)} = f(z_1, \dots, z_d)$,
(ii) $f(z)^T = f(z)$,
(iii) $f(\lambda z_1, \dots, \lambda z_d) = \lambda f(z_1, \dots, z_d)$, $\lambda \in \mathbb{C} \setminus \{0\}$,

then one can choose matrices A_k, $k = 0, 1, \dots, d$, to be (i) real (respectively, (ii) symmetric, (iii) such that $A_0 = 0$). Another proof of this theorem has been recently obtained in [22].

A particular role is played by the Bessmertnyĭ class $\mathbb{R}\mathcal{B}_d^{m\times m}$ (see [1,16,17,22]) of functions (1.1) with a positive real homogeneous matrix pencil:

$$A(z) = z_1 A_1 + \cdots + z_d A_d, \quad A_k^T = \overline{A}_k = A_k \geq 0, \quad k = 1, \dots, d.$$

Positive definiteness is understood in the sense of quadratic forms. Functions of class $\mathbb{R}\mathcal{B}_d^{m\times m}$ are characteristic functions of electric circuits containing ideal transformers and elements of d type, where each element of the kth type has an impedance z_k [4,5,9]. If $f(z) \in \mathbb{R}\mathcal{B}_d^{m\times m}$, then, from (1.1), we get:

(iv) $f(z) + f(z)^* \geq 0$, $z \in \Pi^d = \{z \in \mathbb{C}^d \mid \operatorname{Re} z_1 > 0, \dots, \operatorname{Re} z_d > 0\}$,

(v) $f(z)$ is holomorphic on Π^d.

A function $f(z)$ satisfying conditions (i)–(v) is called *positive real* [5]. The class of rational positive real functions is denoted by $\mathbb{R}\mathcal{P}_d^{m\times m}$.

It is clear that $\mathbb{R}\mathcal{B}_d^{m\times m} \subseteq \mathbb{R}\mathcal{P}_d^{m\times m}$. For $d = 1, 2$, we have $\mathbb{R}\mathcal{B}_d^{m\times m} = \mathbb{R}\mathcal{P}_d^{m\times m}$. If $d \geq 3$, then the question of the coincidence of the classes $\mathbb{R}\mathcal{B}_d^{m\times m}$ and $\mathbb{R}\mathcal{P}_d^{m\times m}$ still remains open (see [2,16,22]), with the exception of functions of degree 2 and some others [5,7,8]. In this paper, we prove $\mathbb{R}\mathcal{B}_d^{m\times m} = \mathbb{R}\mathcal{P}_d^{m\times m}$ for all $d \geq 1$.

It was proved in [2, Theorem 4.1] that $f(z) \in \mathbb{R}\mathcal{B}_d^{m\times m}$ if and only if there exist rational matrix-valued functions $\Phi_k(z)$, $k = 1, \dots, d$, holomorphic on Π^d that satisfy the conditions:

$$\begin{aligned} \Phi_k(\lambda z_1, \dots, \lambda z_d) &= \Phi_k(z_1, \dots, z_d), \quad \lambda \in \mathbb{C} \setminus \{0\}, \\ \overline{\Phi_k(\bar{z}_1, \dots, \bar{z}_d)} &= \Phi_k(z_1, \dots, z_d), \\ f(z) &= \sum_{k=1}^{d} z_k \Phi_k(z) \Phi_k(w)^*, \quad w, z \in \mathbb{C}^d. \end{aligned} \tag{1.3}$$

Characterizations of the form (1.3) for various generalizations of the class $\mathbb{R}\mathcal{B}_d^{m\times m}$ were obtained in [2, 3, 16, 17]. In [16, 17], non-rational analogs of the classes $\mathbb{R}\mathcal{P}_d^{m\times m}$ and $\mathbb{R}\mathcal{B}_d^{m\times m}$ were studied, where the coefficients of long-resolvent representations are bounded linear operators on a Hilbert space. In [2], for rational Cayley inner Herglotz–Agler functions over the right poly-halfplain (here the term "Cayley inner" means that the Cayley transform over the values of function is an inner function), long-resolvent representation (1.1) was obtained, in which the matrix A_0 is skew-symmetric and the other matrices A_k are symmetric positive semidefinite. Thus, the class of Cayley inner rational Herglotz–Agler functions is an extension of the class $\mathbb{R}\mathcal{B}_d^{m\times m}$. In [3], non-rational analogs of Cayley inner Herglotz–Agler functions were studied, where the coefficients of the long-resolvent representation are linear operators on a Hilbert space, with A_0 possibly unbounded.

For rational functions, relation (1.3) requires the representation of nonnegative polynomials as a sum of squares of rational functions holomorphic in Π^d. The Artin solution of Hilbert's 17th problem on the representation of a non-negative polynomial as a sum of squares of rational functions says nothing on the location of the singularities of functions in the decomposition [19, Ch. XI, Corollary 3.3]. A similar class of positive real functions in d variables (without condition (iii) of homogeneity of degree 1) was considered by T. Koga [18]. Koga's method is based on the following statement.

Lemma 1.1 (Koga's Sum-of-Squares Lemma). *Let $p(x_1, \dots, x_d)$ be a polynomial with real coefficients, quadratic in each variable. If $p(x) \geq 0$ for real x_i,*

$i = 1, \dots, d$, then $p(x) = \sum h_j(x)^2$, where $h_j(x)$ are polynomials linear in each variable.

As noted by N.K. Bose, Koga's proof is wrong [11]. A counterexample is the non-negative polynomial not representable as a sum of polynomial squares constructed by M.-D. Choi [14]:

$$x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1y_1x_2y_2 + x_2y_2x_3y_3 + x_1y_1x_3y_3) \\ + 2(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2). \quad (1.4)$$

In Koga's method, a nonnegative polynomial is a partial Wronskian

$$F_k(x) = W_{x_k}[q,p] = q(x)\frac{\partial p(x)}{\partial x_k} - p(x)\frac{\partial q(x)}{\partial x_k} \quad (1.5)$$

of a pair of polynomials such that $q(z), p(z) \neq 0$, $z \in \Pi^d$. The polynomial (1.4) does not satisfy this condition. The representation (1.5) strongly restricts the class of nonnegative polynomials. In this paper, we will prove a theorem that "rehabilitates" T. Koga's method.

Theorem 1.2 (Sum-of-Squares Theorem)**.** *If $P(z)/q(z) \in \mathbb{R}\mathcal{P}_d^{m\times m}$, then the partial Wronskians*

$$W_{z_k}[q,P] = q(z)\frac{\partial P(z)}{\partial z_k} - P(z)\frac{\partial q(z)}{\partial z_k}$$

are sums of squares of polynomials.

This theorem made it possible to prove the main result: $\mathbb{R}\mathcal{P}_d^{m\times m} = \mathbb{R}\mathcal{B}_d^{m\times m}$ for every $d \geq 1$.

The paper is organized as follows. In Section 2, we explain terminology and provide preliminary information. In Section 3, we recall the simplest properties of functions of the class $\mathbb{R}\mathcal{P}_d^{m\times m}$ and properties of the degree reduction operator of a rational function. For a multi-affine function, a criterion for belonging to the class $\mathbb{R}\mathcal{P}_d^{m\times m}$ is obtained (Theorem 3.7). Section 4 studies the properties of the denominators of rational functions in the Artin decomposition into the sum of squares. Theorem 4.3 and Proposition 4.4 allow localizing the singularities of rational functions in the Artin decomposition. A convenient representation for the partial Wronskians $W_{z_k}[q;p]$ is given in Theorem 5.1 (Product Polarization Theorem) in Section 5. In fact, for a rational function $f = p/q$, this theorem implies Hefer's expansion $f(z) - f(\zeta) = \sum(z_k - \zeta_k)F_k(z,\zeta)$ with additional conditions of symmetry $F_k(z,\zeta) = F_k(\zeta,z)$.

In Section 6, we study the set of Gram matrices of a given $2n$-form and prove the Representation Defect Lemma (Lemma 6.7). This lemma allows one to obtain a new long-resolvent representation from a given representation if one of the matrices of the new representation is known.

In Section 7, a representation of a rational function with one nonnegative partial Wronskian is obtained in Theorem 7.1. This representation contains the Artin denominator of the nonnegative partial Wronskian in explicit form.

In Section 8 on the basis of Theorem 7.1, the Sum-of-Squares Theorem (Theorem 8.2) is proved.

In Section 9, a long-resolvent representation of a rational positive real matrix-valued function with a positive semidefinite matrix pencil is obtained in Theorem 9.1.

2. Terminology, notations and preliminaries

Let $\mathbb{R}[z]$ be a ring of polynomials in the variables $(z_1, \dots, z_d) \in \mathbb{C}^d$ with real coefficients. We say $p(z) \in \mathbb{R}[z]$ is *affine* in z_k if $\deg_{z_k} p(z) = 1$, and we say $p(z)$ is *multi-affine* if it is affine in z_k for all $k = 1, \dots, d$.

Recall that *a circular region* is a proper subset of the complex plane, which is bounded by circles (straight lines). In particular, the half-plane is a circular region. We need the following statement about symmetric multiaffine polynomials.

Theorem 2.1 (Grace–Walsh–Szegö, [13, Theorem 2.12])**.** *Let p be a symmetric multi-affine polynomial in n complex variables, let $\mathcal{C}$ be an open or a closed circular region in $\mathbb{C}$, and let $z_1, \dots, z_n$ be any fixed points in the region $\mathcal{C}$. If $\deg p = n$ or $\mathcal{C}$ is convex, then there exists at least one point $\xi \in \mathcal{C}$ such that $p(z_1, \dots, z_n) = p(\xi, \dots, \xi)$.*

A polynomial $p(z)$ is called *a form* (*n-form*) if $p(\lambda z_1 \dots, \lambda z_d) = \lambda^n p(z_1 \dots, z_d)$, $\lambda \in \mathbb{C}$.

A rational matrix-valued function will be written in the form $f(z) = P(z)/q(z)$, where $P(z) = \{p_{ij}(z)\}_{i,j=1}^m$ is a matrix polynomial and $q(z)$ is a scalar polynomial. In fact, division $P(z)/q(z)$ is the standard operation of multiplying of the matrix $P(z)$ by the number $q(z)^{-1}$.

The matrix A is called *real* if $\overline{A} = A$ (where the bar denotes the replacement of each element of A by a complex conjugate number). The symbol A^T denotes the transpose of A. If A is a matrix with complex elements, then $A^* = \overline{A}^T$ is the Hermitian conjugate matrix.

A real symmetric $m \times m$ matrix A is called *positive semidefinite* ($A \geq 0$) if the inequality $\eta^T A \eta \geq 0$ holds for all $\eta \in \mathbb{R}^m$, and *positive definite* ($A > 0$) if $\eta^T A \eta > 0$ for all $\eta \neq 0$.

A matrix-valued form $F(z)$ is called *positive semidefinite* or PSD if $F(x) \geq 0$ for all $x \in \mathbb{R}^d$. A matrix-valued PSD form $F(z)$ is called *a sum of squares* or SOS if $F(z) = H(z)H(z)^T$, where $H(z)$ is some matrix-valued polynomial.

If $\alpha = (\delta_1, \dots, \delta_d) \in \mathbb{N}_0^d$, then $z^\alpha = z_1^{\delta_1} \cdots z_d^{\delta_d}$ is a monomial. Let $\{z^{\alpha_j}\}_{j=1}^M$ be a set of all monomials of degree n in variables $z_1, \dots, z_d$. Each $2n$-form $F(z)$ can be represented as

$$F(z) = \begin{pmatrix} z^{\alpha_1} & \cdots & z^{\alpha_M} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \dots & a_{MM} \end{pmatrix} \begin{pmatrix} z^{\alpha_1} \\ \vdots \\ z^{\alpha_M} \end{pmatrix}. \tag{2.1}$$

The symmetric matrix $A = \{a_{jk}\}_{j,k=1}^M$ is called *a Gram matrix* of a $2n$-form $F(z)$. The Gram matrix is not uniquely determined by the $2n$-form. It is known [20, Theorem 1] that PSD form $F(z)$ is a SOS form if and only if $F(z)$ has a positive semidefinite Gram matrix.

If K is a field, then $K(x_1, \dots, x_d)$ denotes the set of rational functions in variables $x_1, \dots, x_d$ with coefficients from the field K.

Theorem 2.2 (Artin, [19, Chap. XI, Corollary 3.3]). *Let K be a real field admitting only one ordering. Let $f(x) \in K(x_1, \dots, x_d)$ be a rational function that does not take negative values: $f(a) \geq 0$ for all $a = (a_1, \dots, a_d) \in K^d$, in which $f(a)$ is defined. Then $f(x)$ is a sum of squares in $K(x_1, \dots, x_d)$.*

If $F(z)$ is a SOS form, then $s(z)^2 F(z)$ is also a SOS form for each form $s(z)$. If $F(z)$ is not representable as a sum of squares of polynomials, then the question arises: *for which $s(z)$ is the form $s(z)^2 F(z)$ also not a SOS form?*

Proposition 2.3 ([15, Lemma 2.1]). *Let $F(x)$ be a PSD not SOS form and let $s(x)$ be an irreducible indefinite form in $\mathbb{R}[x_1, \dots, x_d]$. Then $s^2 F$ is also a PSD not SOS form.*

Proof. Clearly, $s^2 F$ is PSD. If $s^2 F = \sum_k h_k^2$, then for every real tuple a with $s(a) = 0$, it follows that $s^2 F(a) = 0$. This implies $h_k(a)^2 = 0 \ \forall k$. So, on the real variety $s = 0$, we have $h_k = 0$ as well. Thus, (see [10], Theorem 4.5.1) for each k, there exists g_k such that $h_k = s g_k$. This gives $F = \sum_k g_k^2$, which is a contradiction. □

Corollary 2.4. *Let $F(x)$ be a matrix-valued PSD not SOS form and let $s(x)$ be an irreducible indefinite form in $\mathbb{R}[x_1, \dots, x_d]$. Then $s^2 F$ is also a PSD not SOS form.*

In the univariate case, coprime polynomials have no common zeros. For several variables, the situation is different (for example, examine the polynomials z_1 and z_2). Let $\mathcal{Z}(h) = \{z \in \mathbb{C}^d \mid h(z) = 0\}$ be a zero set of the polynomial h.

Theorem 2.5 ([21, Theorem 1.3.2]). *Suppose that $d > 1$ and $s(z)$, $h(z) \in \mathbb{C}[z_1, \dots, z_d]$ are coprime polynomials such that $s(0) = h(0) = 0$. If Ω is a neighborhood of zero in $\mathbb{C}^d$, then:*

(a) *neither of the sets $\mathcal{Z}(s) \cap \Omega$ and $\mathcal{Z}(h) \cap \Omega$ is a subset of the other,*

(b) *for any $a \in \mathbb{C}$, there exists $z \in \Omega$ such that $h(z) \neq 0$, $s(z)/h(z) = a$.*

The assertion (a) of this theorem remains valid for polynomials coprime in the ring $\mathbb{R}[z_1, \dots, z_d]$.

A polynomial $p(z) \in \mathbb{R}[z_1, \dots, z_d]$ is called *a polynomial with Hurwitz property* or *a stable polynomial* if $p(z) \neq 0$ on Π^d [12, 13]. A homogeneous stable polynomial is called *a Hurwitz form*. If $F(z)$ is a Hurwitz form, then the polynomial

$$F(x_1, \dots, x_{k-1}, z_k, x_{k+1}, \dots, x_d)$$

has only real zeros in z_k for fixed $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d \in \mathbb{R}$.

3. Degree reduction operator and positivity

Proposition 3.1. *If $f(z) = P(z)/q(z) \in \mathbb{R}\mathcal{P}_d^{m\times m}$, then partial Wronskians*

$$W_{z_k}[q, P] = q(z)\frac{\partial P(z)}{\partial z_k} - P(z)\frac{\partial q(z)}{\partial z_k}, \quad k = 1, \dots, d, \tag{3.1}$$

are PSD forms.

Proof. Suppose $k = 1$. If $\widehat{x} = (x_2, \dots x_d) \in \mathbb{R}^{d-1}$, then $\operatorname{Im}\varphi(\zeta) = \operatorname{Im} f(\zeta, \widehat{x}) \geq 0$, $\operatorname{Im}\zeta > 0$, and $\operatorname{Im}\varphi(\zeta) = 0$, $\operatorname{Im}\zeta = 0$. Hence the inequality $d\varphi(\zeta)/d\zeta|_{\zeta\in\mathbb{R}} \geq 0$ holds. From this,

$$W_{z_k}[q, P](x) = q(x)^2 d\varphi(\zeta)/d\zeta|_{\zeta=x_1} \geq 0, \quad x \in \mathbb{R}^d. \qquad \square$$

Proposition 3.2. *Assume $f = P/q \in \mathbb{R}\mathcal{P}_d^{m\times m}$. If $\deg_{z_k} P > \deg_{z_k} q$, then there exists a real $m \times m$ matrix $A_k \geq 0$ and a matrix form $P_1(z)$ with the following properties:*

(a) $\deg_{z_k} P_1(z) = \deg_{z_k} q(z)$,
(b) $f_1(z) = P_1(z)/q(z) \in \mathbb{R}\mathcal{P}_d^{m\times m}$,
(c) $f(z) = z_k A_k + f_1(z)$.

Proof. Suppose that $k = 1$. If $\widehat{z} = (z_2, \dots, z_d) \in \Pi^{d-1}$, then, for the function $\varphi(\zeta) = f(\zeta, \widehat{z})$, the inequality $\varphi(\zeta) + \varphi(\zeta)^* \geq 0$ holds for $\operatorname{Re}\zeta > 0$. The degrees of the numerator and denominator of such function cannot differ by more than 1. It follows that $\lim_{z_1\to\infty} f(z)/z_1 = A_1(\widehat{z}) = \operatorname{res}_{\zeta=\infty}\varphi(\zeta) \geq 0$. Since $A_1(\widehat{z})$ is holomorphic on Π^{d-1}, we see that $A_1(\widehat{z}) \equiv A_1 \geq 0$ is a constant matrix and $f_1(z_1, \widehat{z}) = f(z_1, \widehat{z}) - z_1 A_1$ is positive real. $f_1(z)$ is homogeneous. Then we have $f_1(z) \in \mathbb{R}\mathcal{P}_d^{m\times m}$. $\square$

Further simplification is based on the use of the degree reduction operator [13, 18]. In some cases, we can restrict ourselves to considering multi-affine functions. An example of multi-affine k-forms are the elementary symmetric polynomials:

$$\sigma_k(\zeta_1, \dots, \zeta_n) = \sum_{i_1<i_2<\cdots<i_k} \zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_k}, \quad \sigma_0(\zeta_1, \dots, \zeta_n) \equiv 1. \tag{3.2}$$

Definition 3.3. Let $p(z_0, z) = \sum_{k=0}^{n_0} p_{l-k}(z) z_0^k$ be an l-form. A map

$$\mathbf{D}_{z_0}^{n_0} : \sum_{k=0}^{n_0} p_{l-k}(z) z_0^k \mapsto \sum_{k=0}^{n_0} p_{l-k}(z) \binom{n_0}{k}^{-1} \sigma_k(\zeta_1, \dots, \zeta_{n_0}) \tag{3.3}$$

is called a *degree reduction operator* in the variable z_0. If $f(z_0, z) = p(z_0, z)/q(z_0, z)$, $\deg_{z_0} f(z_0, z) = n_0$, then the degree reduction operator is defined as

$$\mathbf{D}_{z_0}^{n_0}[p(z_0, z)/q(z_0, z)] = \mathbf{D}_{z_0}^{n_0}[p(z_0, z)]/\mathbf{D}_{z_0}^{n_0}[q(z_0, z)]. \tag{3.4}$$

Under the condition $\zeta_1 = \cdots = \zeta_{n_0} = z_0$, we get the original function. Thus, the operator $\mathbf{D}_{z_0}^{n_0}$ is invertible. It turns out that the degree reduction operator (3.4) has the following property.

Theorem 3.4. *Let $P(z_0,z)$, $q(z_0,z)$ be coprime. If $f = P/q \in \mathbb{R}\mathcal{P}^{m\times m}_{d+1}$, $\deg_{z_0} f = n_0$, then $\widehat{f}(\zeta_1,\dots,\zeta_{n_0},z) = \mathbf{D}^{n_0}_{z_0}[f(z_0,z)]$ is a function of class $\mathbb{R}\mathcal{P}^{m\times m}_{d+n_0}$, affine and symmetric in variables $\zeta_1,\dots,\zeta_{n_0}$.*

We need some lemmas. Recall that the Hurwitz form is a homogeneous stable polynomial.

Lemma 3.5. *The coprime numerator and denominator of a scalar positive real function are Hurwitz forms.*

Proof. The homogeneity of the polynomials is obvious. Stability easily follows from a similar fact for functions of one variable having a nonnegative real part in the right half-plane. □

Lemma 3.6. *Let $p(z_0,z)$ be a Hurwitz form. If $\deg_{z_0} p(z_0,z) = n_0$, then the polynomial $\widehat{p}(\zeta_1,\dots,\zeta_{n_0},z) = \mathbf{D}^{n_0}_{z_0}[p(z_0,z)]$ is also a Hurwitz form in variable $\zeta_1,\dots,\zeta_{n_0},z$.*

Proof. If $z \in \Pi^d$, then the polynomial $p(z_0,z)$ has no zeros for $\operatorname{Re} z_0 > 0$. Suppose $\operatorname{Re}\zeta_j > 0$, $j = 1,\dots,n_0$ are fixed. By the Grace-Walsh-Szegö Theorem, there exists a point ξ, $\operatorname{Re}\xi > 0$ such that $\widehat{p}(\zeta_1,\dots,\zeta_{n_0},z) = p(\xi,z) \neq 0$. The homogeneity of $\widehat{p}(\zeta_1,\dots,\zeta_{n_0},z)$ is obvious. □

Proof of Theorem 3.4. A matrix-valued function $f(z_0,z)$ is positive real if and only if for any real row vector η scalar function $\eta f(z_0,z)\eta^T$ is positive real. Since addition does not move out of the class of positive real functions, we see that $z_{d+1} + p(z_0,z)/q(z_0,z)$ is a positive real function in variables z_0,z,z_{d+1}. By Lemma 3.6, the polynomial

$$\mathbf{D}^{n_0}_{z_0}[z_{d+1}q(z_0,z) + p(z_0,z)] = z_{d+1}\mathbf{D}^{n_0}_{z_0}[q(z_0,z)] + \mathbf{D}^{n_0}_{z_0}[p(z_0,z)]$$

is a Hurwitz form. This implies $\operatorname{Re}\left(\mathbf{D}^{n_0}_{z_0}[p(z_0,z)]/\mathbf{D}^{n_0}_{z_0}[q(z_0,z)]\right) \geq 0$ for $(\zeta_1,\dots,\zeta_{n_0},z_1,\dots z_d) \in \Pi^{d+n_0}$. Symmetry in the variables $\zeta_1,\dots,\zeta_{n_0}$ is obvious. □

Theorem 3.7. *A multi-affine real homogeneous ($f(\lambda z) = \lambda f(z)$, $\lambda \in \mathbb{C}\setminus\{0\}$) rational matrix-values function $f(z) = P(z)/q(z)$ belongs to the class $\mathbb{R}\mathcal{P}^{m\times m}_d$ iff all Wronskians $W_{z_k}[q,P]$, $k = 1,\dots,d$, are PSD forms.*

Proof. The necessity is proved in Proposition 3.1. Let us prove the sufficiency. Since $f(z)$ is multi-affine, we see that

$$f(z) = \frac{z_k P_1(\widehat{z}) + P_2(\widehat{z})}{z_k q_1(\widehat{z}) + q_2(\widehat{z})}, \quad \widehat{z} = (z_1,\dots,z_{k-1},z_{k+1},\dots,z_d).$$

It follows from here that

$$\operatorname{Im} f(z_k,\widehat{x}) = \frac{W_{z_k}[q,P](\widehat{x})}{|z_k q_1(\widehat{x}) + q_2(\widehat{x})|^2}\operatorname{Im} z_k, \quad \widehat{x} \in \mathbb{R}^{d-1}.$$

Hence $\operatorname{Im} f(z_k,\widehat{x}) \geq 0$, $\operatorname{Im} z_k > 0$ for each $k = 1,\dots,d$ (for any other real variables). So, as seen from Theorem 2.4 in [6], $\operatorname{Im} f(z) \geq 0$ in the upper polyhalfplane. It follows from the homogeneity of $f(z)$ that $f(z) \in \mathbb{R}\mathcal{P}^{m\times m}_d$. □

4. Artin's denominators of PSD not SOS form

Let $F(z) \in \mathbb{R}[z_1, \dots, z_d]$ be a PSD not SOS form. By Artin's theorem, there exists a form $s(z)$ such that $s(z)^2 F(z)$ is a SOS form. The form $s(z)$ is called *Artin's denominator of* $F(z)$.

Proposition 4.1. *Suppose $s^2 F$ is a SOS form. If each irreducible factor of s is an indefinite form, then F is also a SOS form.*

Proof. Suppose F is a PSD not SOS form. Let $s = s_1 \cdots s_m$ be the decomposition of s into irreducible factors. Successively applying Proposition 2.3 to the forms

$$F_1 = s_1^2 F, \quad F_2 = s_2^2 F_1, \quad \dots, \quad F_m = s_m^2 F_{m-1},$$

we obtain $F_m = s^2 F$ is a PSD not SOS form, which is a contradiction. □

Definition 4.2. Artin's denominator s of a PSD not SOS form F is called *an Artin minimal denominator* if a form $\widehat{s} = s/s_j$ is not Artin's denominator of F for each irreducible factor s_j of s.

Theorem 4.3. *Each PSD not SOS form $F(z)$ has a non-constant Artin minimal denominator $s(z)$. The irreducible factors of Artin's minimal denominator do not change sign on $\mathbb{R}^d$.*

Proof. By the Artin theorem, there exists a form $r(z)$ for which $r(z)^2 F(z)$ is a SOS form. Each irreducible factor of the form $r(z)$ is either indefinite or it does not change sign on $\mathbb{R}^d$. Then $r(z) = r_0(z)s(z)$, where all irreducible factors of the form $s(z)$ do not change sign on $\mathbb{R}^d$, and irreducible factors of the form $r_0(z)$ are indefinite. Consider the form $F_1(z) = s(z)^2 F(z)$. By the assumption, $r_0^2 F_1 = r^2 F$ is a SOS. Since every irreducible factor of $r_0(z)$ is indefinite, we see that F_1 is a SOS form (Proposition 4.1). Then $s(z)$ is also the Artin denominator for F. Let s_0 be some irreducible factor of the form s. If s/s_0 remains the Artin denominator of F, then the factor s_0 is removed from s. Removing all "excess" irreducible factors from the form s, we obtain the Artin denominator with the required properties. □

Proposition 4.4. *Let $s(z)$ be a non-constant real irreducible form that does not change sign on $\mathbb{R}^d$. Then there exists a point z' from the open upper polyhalfplane such that $s(z') = 0$.*

Proof. Suppose that for any k such that $\partial s(z)/\partial z_k \not\equiv 0$, the polynomial $s(z_k, \widehat{x}) = s(x_1 \dots, x_{k-1}, z_k, x_{k+1}, \dots, x_d)$ has only real zeros in z_k for each fixed $\widehat{x} \in \mathbb{R}^{d-1}$. Then the equation $s = 0$ defines a real manifold of dimension $d-1$. By Theorem 4.5.1 from [10], the ideal generated by the polynomial s is real, and the irreducible form $s(z)$ is indefinite. This contradicts the assumption.

The complex zeros of $s(z_k, \widehat{x})$ form complex conjugate pairs. Then there exists $z = (x_1, \dots, x_{k-1}, \eta_k, x_{k+1}, \dots, x_d)$, $\operatorname{Im} \eta_k > 0$ such that $s(z) = 0$. Let $z'_j = x_j + iy_j$, $j \neq k$. If $y_j > 0$ is sufficiently small, then $s(z_k, \widehat{z}')$ still vanishes at some z'_k from the open upper half-plane. □

5. Product Polarization Theorem

The following statement is an analogue of the theorem on the long-resolvent representation of a rational function.

Theorem 5.1 (Product Polarization Theorem)**.** *Let $q(z)$, $p(z)$ be real forms of degree n and $n+1$ satisfy the conditions*

$$\deg_{z_k} q(z) \le n_k, \quad \deg_{z_k} p(z) \le n_k, \ k = 1, \dots, d.$$

Let $\Psi(z) = (z^{\alpha_1}, \dots, z^{\alpha_N})$ be a row vector of all monomials of degree n for which $\deg_{z_k} z^{\alpha_j} \le n_k$. Then there exist real symmetric matrices A_k, $k = 1, \dots, d$ such that

$$q(\zeta)p(z) = \Psi(\zeta)(z_1 A_1 + \cdots + z_d A_d)\Psi(z)^T, \qquad \zeta, z \in \mathbb{C}^d, \tag{5.1}$$

$$W_{z_k}[q,p] = \Psi(z) A_k \Psi(z)^T, \qquad k = 1, \dots, d. \tag{5.2}$$

We need some lemmas.

Lemma 5.2. *Suppose $k \ge 0$ is an integer. If $\zeta^{\mu_1} = \zeta_2\zeta_4\cdots\zeta_{2k}$, $\zeta^\nu = \zeta_1\zeta_3\cdots\zeta_{2k+1}$, then there exist real symmetric $(2k+1)\times(2k+1)$ matrices C_j, $j = 1, 2, \dots, 2k+1$ and multi-affine monomials $\{\zeta^{\mu_j}\}_{j=2}^{2k+1}$ of degree k in variables $\zeta_1, \dots, \zeta_{2k+1}$ such that*

$$(\zeta_1 C_1 + \cdots + \zeta_{2k+1} C_{2k+1}) \begin{pmatrix} \zeta^{\mu_1} \\ \zeta^{\mu_2} \\ \vdots \\ \zeta^{\mu_{2k+1}} \end{pmatrix} = \begin{pmatrix} \zeta^\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.3}$$

Proof. If $k = 0$, then $\zeta^{\mu_1} = 1$ (empty product) and $\zeta^\nu = \zeta_1$. We have $\zeta^\nu = \zeta_1 \cdot 1$, and the matrix pencil $C(z) = \zeta_1 \cdot 1$ has a size of 1×1. For $k \ge 1$, the multi-affine monomials $\{\zeta^{\mu_j}\}_{j=2}^{2k+1}$ are defined by the relations

$$\zeta^{\mu_2} = \zeta_3\zeta_5\cdots\zeta_{2k+1}, \quad \zeta^{\mu_j} = \zeta_{j-2}\zeta^{\mu_{j-2}}/\zeta_{j-1}, \quad j = 3, \dots, 2k+1.$$

Notice that $\zeta^{\mu_{2k+1}} = \zeta^\nu/\zeta_{2k+1}$, $\zeta^{\mu_{2k}} = \zeta_{2k+1}\zeta^{\mu_1}/\zeta_{2k}$. Let us define the matrix pencil $C(\zeta) = \{c_{ij}(\zeta)\}_{i,j=1}^{2k+1}$:

$$c_{ij}(\zeta) = \begin{cases} (-1)^{\max\{i,j\}}\zeta_{\min\{i,j\}}/2, & \text{if } |i-j| = 1, \\ \zeta_{\max\{i,j\}}/2 & \text{if } |i-j| = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\overline{C(\bar\zeta)} = C(\zeta) = C(\zeta)^T$. Let us calculate the components $b_i = b_i(\zeta)$ of the right-hand side of (5.3):

$$b_1 = \sum_{j=1}^{2k+1} c_{1j}(\zeta)\zeta^{\mu_j} = c_{12}(\zeta)\zeta^{\mu_2} + c_{1,2k+1}(\zeta)\zeta^{\mu_{2k+1}} = \zeta^\nu.$$

For $2 \le i \le 2k$, we obtain

$$\begin{aligned} b_i &= \sum_{j=1}^{2k+1} c_{ij}(\zeta)\zeta^{\mu_j} = c_{i,i-1}(\zeta)\zeta^{\mu_{i-1}} + c_{i,i+1}(\zeta)\zeta^{\mu_{i+1}} \\ &= (-1)^i(\zeta_{i-1}\zeta^{\mu_{i-1}} - \zeta_i\zeta^{\mu_{i+1}})/2 = (-1)^i(\zeta_{i-1}\zeta^{\mu_{i-1}} - \zeta_{i-1}\zeta^{\mu_{i-1}})/2 = 0. \end{aligned}$$

For $i = 2k+1$, we get

$$b_{2k+1} = \sum_{j=1}^{2k+1} c_{2k+1,j}(\zeta)\zeta^{\mu_j} = c_{2k+1,1}(\zeta)\zeta^{\mu_1} + c_{2k+1,2k}(\zeta)\zeta^{\mu_{2k}} = 0. \qquad \square$$

Lemma 5.3. *Let z^{α_1}, z^β be monomials of degree n and $n+1$ satisfy the conditions*

$$\deg_{z_k} z^{\alpha_1} \le n_k, \quad \deg_{z_k} z^\beta \le n_k, \ k = 1, \dots, d.$$

Then there exist matrices $B_k = \overline{B_k} = B_k^T$, $k = 1, \dots, d$, such that

$$(z_1 B_1 + \cdots + z_d B_d) \begin{pmatrix} z^{\alpha_1} \\ z^{\alpha_2} \\ \vdots \\ z^{\alpha_N} \end{pmatrix} = \begin{pmatrix} z^\beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.4}$$

where $\{z^{\alpha_j}\}_{j=1}^N$ are all monomials of degree n for which $\deg_{z_k} z^{\alpha_j} \le n_k$.

Proof. Let z^γ be the greatest common divisor of the monomials z^{α_1}, z^β. Then $z^{\alpha_1} = m_1(z)z^\gamma$, $z^\beta = m_2(z)z^\gamma$, where the subsets of the variables of the monomials $m_1(z)$, $m_2(z)$ do not intersect. If $\deg m_1(z) = k$, then $\deg m_2(z) = k+1$. By Lemma 5.2, there exist matrices $C_k = \overline{C_k} = C_k^T$ such that

$$(\zeta_1 C_1 + \cdots + \zeta_{2k+1} C_{2k+1}) \begin{pmatrix} \zeta_2\zeta_4\cdots\zeta_{2k} z^\gamma \\ \zeta^{\mu_2} z^\gamma \\ \vdots \\ \zeta^{\mu_{2k+1}} z^\gamma \end{pmatrix} = \begin{pmatrix} \zeta_1\zeta_3\cdots\zeta_{2k+1} z^\gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.5}$$

In (5.5), we replace the variables $\zeta_2, \zeta_4, \dots, \zeta_{2k}$ by the variables of the monomial $m_1(z)$, and the variables $\zeta_1, \zeta_3, \dots, \zeta_{2k+1}$ by the variables of the monomial $m_2(z)$ such that

$$\zeta_2\zeta_4\cdots\zeta_{2k} z^\gamma \mapsto m_1(z)z^\gamma = z^{\alpha_1}, \quad \zeta_1\zeta_3\cdots\zeta_{2k+1} z^\gamma \mapsto m_2(z)z^\gamma = z^\beta.$$

From (5.5), we obtain

$$\left(z_{j_1} D_{j_1} + \cdots + z_{j_r} D_{j_r}\right) \begin{pmatrix} z^{\widehat{\alpha}_1} \\ z^{\widehat{\alpha}_2} \\ \vdots \\ z^{\widehat{\alpha}_{2k+1}} \end{pmatrix} = \begin{pmatrix} z^\beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad z^{\widehat{\alpha}_1} = z^{\alpha_1}, \tag{5.6}$$

where $z_{j_1}, \dots, z_{j_r}$ are the variables of the monomials $m_1(z)$, $m_2(z)$. The matrices D_{j_s} are the sums of the corresponding matrices C_i from (5.5).

Since each monomial ζ^{μ_j}, $j = 2, \dots, 2k+1$ contains only a part of the variables with even indices, we see that for the variables z_i that are present in the monomial $m_1(z)z^\gamma$ inequality $\deg_{z_i} z^{\widehat{\alpha}_j} \le \deg_{z_i} z^{\alpha_1} \le n_i$ holds. Similarly, for z_k that are present in $m_2(z)z^\gamma$ we get $\deg_{z_k} z^{\widehat{\alpha}_j} \le \deg_{z_k} z^\beta \le n_k$.

If $z^{\alpha_1}, z^{\alpha_2}, \dots, z^{\alpha_M}$ are pairwise distinct monomials from the set $z^{\widehat{\alpha}_1}, z^{\widehat{\alpha}_2}, \dots, z^{\widehat{\alpha}_{2k+1}}$, then there exists a matrix $B = \{b_{ij}\}$ such that

$$\begin{pmatrix} z^{\widehat{\alpha}_1} \\ z^{\widehat{\alpha}_2} \\ \vdots \\ z^{\widehat{\alpha}_{2k+1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{2k+1,1} & b_{2k+1,2} & \cdots & b_{2k+1,M} \end{pmatrix} \begin{pmatrix} z^{\alpha_1} \\ z^{\alpha_1} \\ \vdots \\ z^{\alpha_M} \end{pmatrix}.$$

From (5.6), we get

$$\big(z_{j_1}B_{j_1} + \cdots + z_{j_r}B_{j_r}\big) \begin{pmatrix} z^{\alpha_1} \\ z^{\alpha_1} \\ \vdots \\ z^{\alpha_M} \end{pmatrix} = \begin{pmatrix} z^\beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.7}$$

where $\big(z_{j_1}B_{j_1} + \cdots + z_{j_r}B_{j_r}\big) = B^T\big(z_{j_1}D_{j_1} + \cdots + z_{j_r}D_{j_r}\big)B$. We extend the set $\{z^{\alpha_j}\}_{j=1}^M$ to the set $\{z^{\alpha_i}\}_{i=1}^N$ of all pairwise distinct monomials of degree n for which $\deg_{z_k} z^{\alpha_i} \le n_k$, $k = 1, \dots, d$. Supplementing the matrices in (5.7) with zero entries, we obtain (5.4). □

Proof of Theorem 5.1. Let $q(z) = \sum_{j=1}^N a_j z^{\alpha_j}$, $p(z) = \sum_{\nu=1}^l b_\nu z^{\beta_\nu}$. By Lemma 5.3, for fixed monomials z^{α_j}, z^{β_ν}, there exists a symmetric real linear matrix pencil $B_{j\nu}(z)$ such that

$$B_{j\nu}(z) \begin{pmatrix} z^{\alpha_1} \\ \vdots \\ z^{\alpha_j} \\ \vdots \\ z^{\alpha_N} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ z^{\beta_\nu} \\ \vdots \\ 0 \end{pmatrix}, \quad j = 1, \dots, N, \ \nu = 1, \dots, l. \tag{5.8}$$

We define $A(z) = z_1A_1 + \cdots + z_dA_d = \sum_{j=1}^N a_j \sum_{\nu=1}^l b_\nu B_{j\nu}(z)$. Then

$$A(z) \begin{pmatrix} z^{\alpha_1} \\ \vdots \\ z^{\alpha_N} \end{pmatrix} = \begin{pmatrix} a_1p(z) \\ \vdots \\ a_Np(z) \end{pmatrix}. \tag{5.9}$$

Since $a_j, b_\nu \in \mathbb{R}$ and $\overline{B_{j\nu}(\overline{z})} = B_{j\nu}(z) = B_{j\nu}(z)^T$, we see that

$$\overline{A(\overline{z})} = A(z) = A(z)^T.$$

Multiplying (5.9) on the left by the row vector $(\zeta^{\alpha_1}, \dots, \zeta^{\alpha_N})$, $\zeta \in \mathbb{C}^d$, we obtain (5.1). In addition, (5.2) follows from (5.1). □

6. Representation Defect Lemma

Let $\{z^{\alpha_j}\}_{j=1}^N$ be a set of all monomials of degree n satisfying the conditions $\deg_{z_k} z^{\alpha_j} \le n_k$, $k = 1, \dots, d$, and $F(z)$ a real $2n$-form such that $\deg_{z_k} F(z) \le 2n_k$. Suppose $F(z)$ has two Gram matrices A_1, A_2: $F(z) = \Psi(z)A_1\Psi(z)^T = \Psi(z)A_2\Psi(z)^T$, where $\Psi(z) = (z^{\alpha_1}, \dots, z^{\alpha_N})$. The symmetric matrix $S = \{s_{ij}\}_{i,j=1}^N = A_1 - A_2$ satisfies the relation

$$\Psi(z)S\Psi(z)^T = \sum_{i,j=1}^{N} s_{ij} z^{\alpha_i} z^{\alpha_j} \equiv 0. \tag{6.1}$$

The set of matrices S satisfying (6.1) is a linear space L_0. Now we construct a special basis of this linear space.

Proposition 6.1. *Suppose L_0 is a linear space of real symmetric matrices satisfying condition (6.1). Then there exists a basis in L_0 such that the nonzero submatrices of the basis matrices are located at the intersection of rows and columns corresponding to monomials $z_r^2 z^\gamma$, $z_r z_l z^\gamma$, $z_l^2 z^\gamma$ that are present as entries in the row vector $\Psi(z)$:*

$$\left(z_r^2 z^\gamma, z_r z_l z^\gamma, z_l^2 z^\gamma\right) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_r^2 z^\gamma \\ z_r z_l z^\gamma \\ z_l^2 z^\gamma \end{pmatrix} = 0, \tag{6.2}$$

and monomials $z_r z^{\gamma_1}$, $z_l z^{\gamma_1}$, $z_l z^{\gamma_2}$, $z_r z^{\gamma_2}$ $(z^{\gamma_1} \neq z^{\gamma_2})$ that are present as entries in $\Psi(z)$:

$$\left(z_r z^{\gamma_1}, z_l z^{\gamma_1}, z_l z^{\gamma_2}, z_r z^{\gamma_2}\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_r z^{\gamma_1} \\ z_l z^{\gamma_1} \\ z_l z^{\gamma_2} \\ z_r z^{\gamma_2} \end{pmatrix} = 0. \tag{6.3}$$

Remark 6.2. Since $\sum_{i,j=1}^N s_{ij} z^{\alpha_i} z^{\alpha_j} = \sum_k c_k z^{\beta_k} = 0$, and $z^{\beta_i} \neq z^{\beta_j}$, $i \neq j$, we see that

$$c_k = \sum_{\alpha_i + \alpha_j = \beta_k} s_{ij} = 0. \tag{6.4}$$

If the sum (6.4) contains $m \ge 2$ different elements s_{ij}, then $m-1$ elements can be chosen as arbitrary. Then multi-index β_k defines an $(m-1)$-dimensional subspace in L_0.

Let $\beta = (r_1, \dots, r_d)$ be a multi-index with non-negative components and $|\beta| = r_1 + \dots + r_d = 2n > 0$. Let Θ_β denote the set of all *unordered* pairs $\pi = (z^{\alpha_i}, z^{\alpha_j})$ of all monomials z^{α_i}, z^{α_j} ($|\alpha_i| = |\alpha_j| = n$) such that $z^{\alpha_i} z^{\alpha_j} = z^\beta$ and $\deg_{z_k} z^{\alpha_s} \le n_k$ for all s, k. If $2n_k < r_k$ for some k, then the set Θ_β is empty. Therefore, $\Theta_\beta \neq \varnothing$ if and only if

$$2n_k \ge r_k, \quad k = 1, \dots, d.$$

It is easy to see that the monomial $z^\alpha = z_1^{\delta_1} \cdots z_d^{\delta_d}$ ($|\alpha| = n$) is an element of the pair $\pi \in \Theta_\beta$ if and only if

$$0 \le \delta_k \le \min\{n_k, r_k\}, \quad 0 \le r_k - \delta_k \le \min\{n_k, r_k\}, \quad k = 1, \dots, d. \tag{6.5}$$

For each monomial $z^\alpha = z_1^{\delta_1} \cdots z_d^{\delta_d}$ ($|\alpha| = n$) satisfying condition (6.5), there is a unique monomial z^μ such that $(z^\alpha, z^\mu) \in \Theta_\beta$.

Definition 6.3. Let $\pi_i = (z^{\alpha_i}, z^{\mu_i}) \in \Theta_\beta$, $z^{\alpha_i} = z_1^{\delta_1} \cdots z_s^{\delta_s} \cdots z_l^{\delta_l} \cdots z_d^{\delta_d}$ and $\delta_s < \min\{n_s, r_s\}$, $\delta_l > 0$. The monomial

$$z^{\alpha_j} = z_1^{\delta_1} \cdots z_s^{\delta_s+1} \cdots z_l^{\delta_l-1} \cdots z_d^{\delta_d}$$

satisfies condition (6.5). Then there is a unique monomial z^{μ_j} such that $\pi_j = (z^{\alpha_j}, z^{\mu_j}) \in \Theta_\beta$. The map

$$(z^{\alpha_i}, z^{\mu_i}) = \pi_i \mapsto \pi_j = (z^{\alpha_j}, z^{\mu_j}) \in \Theta_\beta \tag{6.6}$$

is called *an elementary transformation* in Θ_β. Multi-indices α_i, α_j of monomials z^{α_i}, z^{α_j} are related by

$$\alpha_j = \alpha_i + (0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0).$$

Remark 6.4. Since we do not distinguish between the pairs (z^α, z^μ) and (z^μ, z^α), there may exist a pair $\pi \in \Theta_\beta$ that dots not change under an elementary transformation.

To prove Proposition 6.1 we need several lemmas.

Lemma 6.5. *For any $\pi, \widehat{\pi} \in \Theta_\beta$, $\pi \ne \widehat{\pi}$, there exists a "connecting" π and $\widehat{\pi}$ chain of elements $\pi_k \in \Theta_\beta$ such that each next element is an elementary transformation of the previous one.*

Proof. Let $\beta = (r_1, \dots, r_d)$ and $\pi = \big(z^\alpha, z^\mu\big)$, $\widehat{\pi} = \big(z^{\widehat{\alpha}}, z^{\widehat{\mu}}\big)$, where $z^\alpha = z_1^{\delta_1} \cdots z_d^{\delta_d}$ and $z^{\widehat{\alpha}} = z_1^{\widehat{\delta}_1} \cdots z_d^{\widehat{\delta}_d}$. Since $\pi \ne \widehat{\pi}$, we may assume that $z^\alpha \ne z^{\widehat{\alpha}}$. From the relations $|\beta| = 2n > 0$ and $|\alpha| = |\widehat{\alpha}| = n$, it follows that $\sum_{k=1}^d m_k = 0$, where $m_k = \widehat{\delta}_k - \delta_k$ ($k = 1, \dots, d$).

Let m be the sum of all positive components of $\gamma = (m_1, \dots, m_d)$. Then there exist m elementary tuples γ_l, $l = 1, 2, \dots, m$, not necessarily all different, such that:

(i) each elementary tuple $\gamma_l = \Big(e_1^{(l)}, \dots, e_d^{(l)}\Big)$ has only two non-zero components $+1$ and -1;

(ii) $|m_k - e_k^{(l)}| \le |m_k|$, $k = 1, \dots, d$, $l = 1, \dots, m)$;

(iii) $\gamma_1 + \cdots + \gamma_m = \gamma$.

If $m = 1$, then the tuple γ is elementary and $\gamma_1 = \gamma$.

Let $m > 1$ and let γ_1 be an elementary tuple containing $+1$ in place of some positive component and -1 in place of some negative component of the tuple γ.

The sum of all positive components of $\gamma - \gamma_1$ is equal to $m-1$ and the sum of all components is still zero.

Repeating the previous argument for $\gamma - \gamma_1$, we obtain the elementary tuple γ_2. At the $(m-1)$-th step, the elementary tuple $\gamma - \gamma_1 - \cdots - \gamma_{m-1} = \gamma_m$ is formed.

Consider a sequence of multi-indices:

$$\alpha,\ \alpha_1 = \alpha + \gamma_1,\ \ldots, \alpha_l = \alpha_{l-1} + \gamma_l,\ \ldots, \alpha_m = \alpha_{m-1} + \gamma_m = \widehat{\alpha}.$$

From (i)–(iii), it follows that the components $\nu_k^{(l)}$ of each multi-index $\alpha_l = (\nu_1^{(l)}, \ldots, \nu_d^{(l)})$, $l = 1, \ldots, m-1$, satisfy the condition

$$\min\{\delta_k, \widehat{\delta}_k\} \le \nu_k^{(l)} \le \max\{\delta_k, \widehat{\delta}_k\}, \quad k = 1, \ldots, d.$$

For $\alpha = (\delta_1, \ldots, \delta_d)$ and $\widehat{\alpha} = (\widehat{\delta}_1, \ldots, \widehat{\delta}_d)$, inequalities (6.5) are valid. Then

$$\begin{cases} 0 \le \min\{\delta_k, \widehat{\delta}_k\} \le \nu_k^{(l)} \le \max\{\delta_k, \widehat{\delta}_k\} \le \min\{n_k, r_k\}, \\ 0 \le r_k - \max\{\delta_k, \widehat{\delta}_k\} \le r_k - \nu_k^{(l)} \le r_k - \min\{\delta_k, \widehat{\delta}_k\} \le \min\{n_k, r_k\}. \end{cases}$$

Therefore, all "intermediate" multi-indices α_l $(l = 1, \ldots, m-1)$ also satisfy (6.5) and determine the sequence of elements

$$\pi, \quad \pi_1 = (z^{\alpha_1}, z^{\mu_1}), \quad \ldots, \quad \pi_{m-1} = (z^{\alpha_{m-1}}, z^{\mu_{m-1}}), \quad \widehat{\pi}$$

of the set Θ_β, in which neighboring elements are related by an elementary transformation of the form (6.6). □

Lemma 6.6. *If Θ_β contains $m \ge 2$ elements, then in the set Θ_β there exist $(m-1)$ different pairs $\{\pi_k, \pi_\nu\}$ such that π_ν is an elementary transformation of π_k.*

Proof. Let us associate the finite graph with the set Θ_β. The vertices are elements of the set Θ_β. The edges form pairs $\{\pi_k, \pi_\nu\}$ $(\pi_k \ne \pi_\nu)$ of elements are connected by an elementary transformation. By Lemma 6.1, the graph is connected. Then the graph tree contains $m-1$ edges. In the graph tree different edges are incident to different pairs of vertices. □

Proof of Proposition 6.1. The linear space L_0 is the direct sum of subspaces L_{β_k}, each of which corresponds to its own multi-index β_k $(|\beta_k| = 2n)$. Let us construct a basis in each of these subspaces. Suppose Θ_{β_k} contains $m \ge 2$ elements. By Lemma 6.6, in Θ_{β_k} there exist $m-1$ different pairs $\{\pi_k, \pi_\nu\}$ such that π_k and π_ν are connected by an elementary transformation. Let us show that each pair $\{\pi_k, \pi_\nu\}$ defines a basis matrix of the form (6.2) or (6.3). The following cases are possible:

(a) One of the elements of a pair $\{\pi_k, \pi_\nu\}$ has the form $\pi_k = (z^{\alpha_i}, z^{\alpha_i})$, and $\pi_\nu = (z^{\alpha_j}, z^{\alpha_l})$. Let z^γ be the greatest common divisor of the monomials z^{α_j}, z^{α_i},

z^{α_l}. Since π_ν is an elementary transformation of π_k, then there exist variables z_r, z_l such that

$$z^{\alpha_j} = z_r^2 z^\gamma, \quad z^{\alpha_i} = z_r z_l z^\gamma, \quad z^{\alpha_l} = z_l^2 z^\gamma.$$

This triplet of monomials defines a basis matrix of the form (6.2).

(b) $\pi_k = (z^{\alpha_i}, z^{\alpha_j})$, $z^{\alpha_i} \neq z^{\alpha_j}$; $\pi_\nu = (z^{\alpha_l}, z^{\alpha_s})$, $z^{\alpha_l} \neq z^{\alpha_s}$, and π_ν is an elementary transformation of π_k. Then there exist variables z_r, z_l such that $z^{\alpha_i} = z_r z^{\gamma_1}$, $z^{\alpha_l} = z_l z^{\gamma_1}$, $z^{\alpha_j} = z_l z^{\gamma_2}$, $z^{\alpha_s} = z_r z^{\gamma_2}$. Notice that $z^{\gamma_1} \neq z^{\gamma_2}$. Indeed, if $z^{\gamma_1} = z^{\gamma_2}$, then $\pi_k = \pi_\nu$, which is impossible. This quadruple of monomials defines a basis matrix of the form (6.3).

All pairs $\{\pi_k, \pi_\nu\}$ are different. Then the constructed set of $(m-1)$ matrices is linearly independent. □

Lemma 6.7 (Representation Defect Lemma). *Suppose that a real symmetric $(N \times N)$-matrix S_1 satisfies the following conditions:*

(a) $\Psi(z) S_1 \Psi(z)^T \equiv 0$,

(b) $S_1 \dfrac{\partial^{n_1} \Psi(z)^T}{\partial z_1^{n_1}} \equiv 0$,

where $\Psi(z) = (z^{\alpha_1}, \dots, z^{\alpha_N})$ is a row vector of all monomials of degree n such that $\deg_{z_k} z^{\alpha_j} \le n_k$, $k = 1, \dots, d$. *Then there exist real symmetric $(N \times N)$-matrices S_k, $k = 2, \dots, d$ for which*

$$(z_1 S_1 + z_2 S_2 + \dots + z_d S_d) \Psi(z)^T \equiv 0. \tag{6.7}$$

Proof. Without loss of generality, we can assume

$$\Psi(z) = \left(z_1^{n_1} \varphi(\widehat{z}),\ \psi(z_1, \widehat{z}) \right),$$

where $\deg_{z_1} \psi(z_1, \widehat{z}) = (n_1 - 1)$, $\widehat{z} = (z_2, \dots, z_d)$. Then

$$S_1 \frac{\partial^{n_1} \Psi(z)^T}{\partial z_1^{n_1}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & \widehat{S}_1 \end{pmatrix} \begin{pmatrix} n_1! \varphi(\widehat{z})^T \\ 0 \end{pmatrix} \equiv 0.$$

Hence $S_{11} = 0$, $S_{12}^T = 0$ and $\psi(z_1, \widehat{z}) \widehat{S}_1 \psi(z_1, \widehat{z})^T = 0$. We rewrite (6.7) in block form

$$\begin{pmatrix} S_{11}(\widehat{z}) & S_{12}(\widehat{z}) \\ S_{12}(\widehat{z})^T & z_1 \widehat{S}_1 + S_{22}(\widehat{z}) \end{pmatrix} \begin{pmatrix} z_1^{n_1} \varphi(\widehat{z})^T \\ \psi(\widehat{z})^T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.8}$$

Let us find a solution $\{S_{ij}(\widehat{z})\}_{i,j=1}^2$ of (6.8) when in place of the matrix $\widehat{S}_1$ there are basis matrices $\widehat{S}_{1,j}$. For the basis matrix (6.3), there exist $z_1 z^{\gamma_1}, z_1 z^{\gamma_2} \in \Psi(z)$ such that the corresponding nonzero submatrix of the solution (together with the basis matrix) has the form

$$\left(\begin{array}{cc|cccc} 0 & 0 & -z_\nu & z_k & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_k & z_\nu \\ \hline -z_\nu & 0 & 0 & 0 & z_1 & 0 \\ z_k & 0 & 0 & 0 & 0 & -z_1 \\ 0 & -z_k & z_1 & 0 & 0 & 0 \\ 0 & z_\nu & 0 & -z_1 & 0 & 0 \end{array} \right) \begin{pmatrix} z_1 z^{\gamma_2} \\ z_1 z^{\gamma_1} \\ z_k z^{\gamma_1} \\ z_\nu z^{\gamma_1} \\ z_\nu z^{\gamma_2} \\ z_k z^{\gamma_2} \end{pmatrix} \equiv 0.$$

Similarly, for basis matrix (6.2) and monomials $z_1 z_r z^\gamma, z_1 z_l z^\gamma \in \Psi(z)$, we have

$$\left(\begin{array}{cc|ccc} 0 & 0 & 0 & -z_l & z_r \\ 0 & 0 & z_l & -z_r & 0 \\ \hline 0 & z_l & 0 & 0 & -z_1 \\ -z_l & -z_r & 0 & 2z_1 & 0 \\ z_r & 0 & -z_1 & 0 & 0 \end{array}\right) \begin{pmatrix} z_1 z_r z^\gamma \\ z_1 z_l z^\gamma \\ z_r^2 z^\gamma \\ z_r z_l z^\gamma \\ z_l^2 z^\gamma \end{pmatrix} \equiv 0.$$

The matrix $\widehat{S}_1$ is a linear combination of basis matrices $\widehat{S}_{1,j}$ of the forms (6.2), (6.3). Then any solution of equation (6.8) is a linear combination of such solutions. □

7. Functions with PSD not SOS Wronskian

Let $f(z) = p(z)/q(z)$ be a scalar function. If the partial Wronskian $W_{z_k}[q,p]$ is a PSD form, then the following statement holds.

Theorem 7.1. *Let $f(z) = p(z)/q(z)$, $z \in \mathbb{C}^d$, be a real homogeneous (of degree one) rational function such that $\deg_{z_k} p \le \deg_{z_k} q$, $k = 1, \ldots, d$.*

If $s(z)^2 W_{z_1}[q,p] = H(z)H(z)^T$ is a SOS form for some form $s(z)$, then there exist real symmetric matrices $A_1, A_2, \ldots, A_d$ (where A_1 is positive semidefinite) for which

$$f(z) = \frac{\Psi(\zeta)}{q(\zeta)s(\zeta)} \left(z_1 A_1 + z_2 A_2 + \cdots + z_d A_d\right) \frac{\Psi(z)^T}{q(z)s(z)}, \quad \zeta, z \in \mathbb{C}^d, \tag{7.1}$$

$$W_{z_k}[q,p] = \frac{\Psi(z)}{s(z)} A_k \frac{\Psi(z)^T}{s(z)}, \quad k = 1, \ldots, d, \tag{7.2}$$

$$\Psi(z) A_1 \Psi(z)^T = H(z)H(z)^T. \tag{7.3}$$

Here $\Psi(z) = (z^{\alpha_1}, \ldots, z^{\alpha_N})$ satisfies the conditions $\deg z^{\alpha_j} = \deg(qs)$ and $\deg_{z_k} z^{\alpha_j} \le \deg_{z_k}(qs)$, $k = 1, \ldots, d$.

Proof. By Theorem 5.1, there exists a matrix pencil $B(z)$ for which

$$q(\zeta)s(\zeta)p(z)s(z) = \Psi(\zeta)\left(z_1 B_1 + z_2 B_2 + \cdots + z_d B_d\right)\Psi(z)^T, \tag{7.4}$$

$$W_{z_k}[qs, ps] = \Psi(z) B_k \Psi(z)^T, \quad k = 1, \ldots, d. \tag{7.5}$$

Let $\deg_{z_1} \Psi(z) = n_1$. Differentiating (7.4) $(n_1 + 1)$ times in z_1, we obtain

$$B_1 \frac{\partial^{n_1} \Psi(z)^T}{\partial z_1^{n_1}} \equiv 0. \tag{7.6}$$

By the assumption, $s^2 W_{z_1}[q,p] = W_{z_1}[qs, ps]$ is a SOS form. Then there exists a matrix $A_1 \ge 0$ for which

$$\Psi(z) B_1 \Psi(z)^T = s^2 W_{z_1}[q,p] = H(z)H(z)^T = \Psi(z) A_1 \Psi(z)^T.$$

Since $W_{z_1}[qs, ps](x) \geq 0$, then, $\deg_{z_1} W_{z_1}[qs, ps] \leq 2(n_1 - 1)$. Therefore, if $\deg_{z_1}(z^{\alpha_i} z^{\alpha_i}) = 2n_1$, then the matrix $A_1 = \{a_{ij}\}$ has a corresponding diagonal elements $a_{ii} = 0$. Then, from $A_1 \geq 0$, we obtain

$$A_1 \frac{\partial^{n_1} \Psi(z)^T}{\partial z_1^{n_1}} \equiv 0. \tag{7.7}$$

By (7.7) and (7.6), it follows that the matrix $S_1 = A_1 - B_1$ satisfies the assumptions of Representation Defect Lemma. Then there exist real symmetric matrices $S_2, \dots, S_d$ such that

$$\Psi(\zeta)(z_1 S_1 + z_2 S_2 + \cdots + z_d S_d)\Psi(z)^T \equiv 0. \tag{7.8}$$

Adding (7.8) to (7.4) and dividing both sides of the resulting identity by the product $q(\zeta)s(\zeta)q(z)s(z)$, we obtain (7.1). Relations (7.2), (7.3) follow from the identities $s^2 W_{z_k}[q, p] = W_{z_k}[qs, ps] = \Psi(z) A_k \Psi(z)^T$. □

8. Sum-of-Squares Theorem

Let $(i\Pi)^d = \{z \in \mathbb{C}^d \mid \operatorname{Im} z_1 > 0, \dots, \operatorname{Im} z_d > 0\}$ be an open upper polyhalfplane.

Lemma 8.1. *Let $h(z), s_0(z) \in \mathbb{R}[z_1, \dots, z_d]$ be coprime forms and let $s_0(z)$ be an irreducible non-constant form that does not change sign on $\mathbb{R}^d$. Then there exists a point $z' \in (i\Pi)^d$ for which $s_0(z') = 0$, $h(z') \neq 0$.*

Proof. By Proposition 4.4, there exists a point $z \in (i\Pi)^d$ such that $s_0(z) = 0$. Suppose that $h(z) = 0$. Let $\Omega \subset (i\Pi)^d$ be a neighborhood of z. By Theorem 2.5, $s_0(z') = 0$, $h(z') \neq 0$ for some point $z' \in \Omega \subset (i\Pi)^d$. □

Theorem 8.2 (Sum-of-Squares Theorem)**.** *If $P/q \in \mathbb{R}\mathcal{P}_d^{m\times m}$, then the partial Wronskians*

$$W_{z_k}[q, P] = q(z)\frac{\partial P(z)}{\partial z_k} - P(z)\frac{\partial q(z)}{\partial z_k}, \quad k = 1, \dots, d \tag{8.1}$$

are matrix-valued SOS forms.

Proof. By Proposition 3.1, the Wronskians $W_{z_k}[q, P]$ are PSD forms. If $d = 2$, then each PSD form is a SOS form. We will assume $d \geq 3$. By Proposition 3.2, $f(z) = z_1 A_1 + \cdots + z_d A_d + f_1(z)$, where $f_1 = P_1/q \in \mathbb{R}\mathcal{P}_d^{m\times m}$, $\deg_{z_k} P_1 = \deg_{z_k} q$ and $A_k \geq 0$, $k = 1, \dots, d$. If $W_{z_k}[q, P_1]$ is a SOS form, then $W_{z_k}[q, P] = q(z)^2 A_k + W_{z_k}[q, P_1]$ is also a SOS form.

We will assume that $\deg_{z_1} P(z) = \deg_{z_1} q(z) = n_1$. Let us act on the function $f(z)$ by the degree reduction operator in the variable z_1. We obtain a function

$$\widehat{f}(\zeta_1, \dots, \zeta_{n_1}, z_2, \dots, z_d) = \widehat{P}/\widehat{q} \in \mathbb{R}\mathcal{P}_{n_1+(d-1)}^{m\times m}.$$

$\widehat{f}(\zeta_1, \dots, \zeta_{n_1}, \widehat{z})$ is multi-affine and symmetric in variables $\zeta_1, \dots, \zeta_{n_1}$. If $W_{\zeta_1}[\widehat{q}, \widehat{P}]$ is a SOS, then $W_{z_1}[q, P] = n_1 W_{\zeta_1}[\widehat{q}, \widehat{P}]\Big|_{\zeta_1 = \cdots = \zeta_{n_1} = z_1}$ is also a SOS. Therefore, without loss of generality, we can assume $n_1 = 1$ and $\deg_{z_k} P = \deg_{z_k} q$, $k = 1,\dots,d$.

Suppose that $W_{z_1}[q,P]$ is a PSD not SOS form. Since $n_1 = 1$, we see that the PSD form $W_{z_1}[q,P]$ does not depend on the variable z_1. Let $s(\widehat{z}) = s(z_2,\dots,z_d)$ be its Artin's minimal denominator: $s(\widehat{z})^2 W_{z_1}[q,P] = G(\widehat{z})G(\widehat{z})^T$. Each irreducible factor $s_j(\widehat{z})$ of the form $s(z)$ cannot be a divisor of all elements of the polynomial matrix $G(\widehat{z})$, otherwise s/s_j is also Artin's denominator of $W_{z_1}[q,P]$, which contradicts the minimality of s. Then there exists a diagonal element $f_{ii}(z) = p(z)/q(z)$ of the matrix $f(z)$ such that

$$s(\widehat{z})^2 W_{z_1}[q,p] = H(\widehat{z})H(\widehat{z})^T, \tag{8.2}$$

where the polynomial row vector $H(\widehat{z})$ has at least one component (we denote it by $h(\widehat{z})$) such that $s_0(\widehat{z})$, $h(\widehat{z})$ are coprime (here $s_0(\widehat{z})$ is some irreducible factor of Artin's minimal denominator $s(\widehat{z})$).

By Theorem 7.1, there exists a symmetric matrix pencil

$$A(z) = z_1A_1 + z_2A_2 + \cdots + z_dA_d$$

with a positive semidefinite matrix $A_1 \geq 0$ and the monomial row vector $\Psi(z)$ such that

$$f_{ii}(z) = \frac{p(z)}{q(z)} = \frac{\Psi(\zeta)}{q(\zeta)s(\zeta)}(z_1A_1 + \cdots + z_dA_d)\frac{\Psi(z)^T}{q(z)s(z)}, \tag{8.3}$$

$$W_{z_1}[q,p] = \frac{\Psi(z)}{s(\widehat{z})}A_1\frac{\Psi(z)^T}{s(\widehat{z})} = \frac{H(\widehat{z})}{s(\widehat{z})}\frac{H(\widehat{z})^T}{s(\widehat{z})}. \tag{8.4}$$

From (8.3), we get

$$\operatorname{Im} f_{ii} = \operatorname{Im} z_1 \frac{H(\widehat{z})}{q(z)s(\widehat{z})}\frac{H(\widehat{z})^*}{\overline{q(z)s(\widehat{z})}} + \sum_{k=2}^{d} \operatorname{Im} z_k \frac{\Psi(z)}{q(z)s(\widehat{z})}A_k\frac{\Psi(z)^*}{\overline{q(z)s(\widehat{z})}}. \tag{8.5}$$

By Lemma 8.1, there exists $\widehat{z'} \in (i\Pi)^{d-1}$ for which $s_0(\widehat{z'}) = 0$, $h(\widehat{z'}) \neq 0$. Let $\Omega_{d-1} \subset (i\Pi)^{d-1}$ be a neighborhood of the point $\widehat{z'}$. Since $f_{ii}(z)$ is a positive real function, we see that for any fixed $x_1' \in \mathbb{R}$, the inequality $\operatorname{Im} f_{ii}(x_1',\widehat{z}) > 0$ holds for $\widehat{z} \in \Omega_{d-1} \subset (i\Pi)^{d-1}$. From (8.5), we obtain

$$\operatorname{Im} f_{ii}(x_1',\widehat{z}) = \sum_{k=2}^{d} \frac{\Psi(x_1',\widehat{z})}{q(x_1',\widehat{z})s(\widehat{z})}A_k\frac{\Psi(x_1',\widehat{z})^*}{\overline{q(x_1',\widehat{z})s(\widehat{z})}} \operatorname{Im} z_k > 0. \tag{8.6}$$

Then, for a sufficiently small positive fixed $y_1' > 0$, there exists a neighborhood $\Omega_{d-1}' \subseteq \Omega_{d-1}$ of the point $\widehat{z'}$ for which

$$\sum_{k=2}^{d} \operatorname{Im} z_k \frac{\Psi(x_1'+iy_1',\widehat{z})}{q(x_1'+iy_1',\widehat{z})s(\widehat{z})}A_k\frac{\Psi(x_1'+iy_1',\widehat{z})^*}{\overline{q(x_1'+iy_1',\widehat{z})s(\widehat{z})}} > 0, \quad \widehat{z} \in \Omega_{d-1}'. \tag{8.7}$$

Let $z_1' = x_1' + iy_1'$ ($\operatorname{Im} z_1' > 0$). From (8.5), we get

$$\operatorname{Im} f_{ii}(z_1', \widehat{z}) = \operatorname{Im} z_1' \sum_j \frac{|h_j(\widehat{z})|^2}{|q(z_1', \widehat{z})s(\widehat{z})|^2} + \sum_{k=2}^{d} \operatorname{Im} z_k \frac{\Psi A_k \Psi^*}{|q(z_1', \widehat{z})s(\widehat{z})|^2}. \tag{8.8}$$

According to (8.7), for $\widehat{z} \in \Omega_{d-1}'$, the second term in (8.8) is positive. Since $s(\widehat{z'}) = 0$ and $h(\widehat{z'}) \neq 0$, then the first term increases indefinitely at $\widehat{z} \to \widehat{z}'$. This implies $\operatorname{Im} f_{ii}(z) \to +\infty$. Then the diagonal element $f_{ii}(z)$ is not holomorphic in the open upper poly-halfplane. This is a contradiction. $\square$

9. Representations of positive real functions

Theorem 9.1 (Main theorem)**.** *Each rational function $f(z)$ of class $\mathbb{R}\mathcal{P}_d^{m\times m}$ is the Schur complement*

$$f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z) \tag{9.1}$$

of the block $A_{22}(z)$ of a linear $(m+l)\times(m+l)$ matrix pencil

$$A(z) = \{A_{ij}(z)\}_{i,j=1}^2 = z_1 A_1 + \cdots + z_d A_d$$

with real symmetric positive semidefinite matrix coefficients A_k, $k = 1, \ldots, d$.

Corollary 9.2. *We have $\mathbb{R}\mathcal{P}_d^{m\times m} = \mathbb{R}\mathcal{B}_d^{m\times m}$ for every $d \geq 1$.*

We need the following generalization of Darlington's theorem for functions of several variables (see also [18]).

Proposition 9.3. *Let a rational multi-affine matrix-valued function $f(z) \in \mathbb{R}\mathcal{P}_d^{m\times m}$ be represented as*

$$f(z) = \frac{P(z)}{q(z)} = \frac{z_1 P_1(\widehat{z}) + P_2(\widehat{z})}{z_1 q_1(\widehat{z}) + q_2(\widehat{z})}, \quad q_1(\widehat{z}) \neq 0, \quad \widehat{z} = (z_2, \ldots, z_d).$$

If $W_{z_1}[q, P] = \Phi_1(\widehat{z})\Phi_1(\widehat{z})^T$ is a SOS, and Φ_1 is of size $m \times r$, then

$$g(\widehat{z}) = \begin{pmatrix} g_{11}(\widehat{z}) & g_{12}(\widehat{z}) \\ g_{21}(\widehat{z}) & g_{22}(\widehat{z}) \end{pmatrix} = \frac{1}{q_1(\widehat{z})} \begin{pmatrix} P_1(\widehat{z}) & \Phi_1(\widehat{z}) \\ \Phi_1(\widehat{z})^T & q_2(\widehat{z}) I_r \end{pmatrix} \tag{9.2}$$

is a multi-affine function of class $\mathbb{R}\mathcal{P}_{d-1}^{(m+r)\times(m+r)}$, and

$$f(z) = g_{11}(\widehat{z}) - g_{12}(\widehat{z})\left(g_{22}(\widehat{z}) + z_1 I_r\right)^{-1} g_{21}(\widehat{z}). \tag{9.3}$$

Proof. Representation (9.3) follows from the obvious identity

$$f(z) = \frac{z_1 P_1(\widehat{z}) + P_2(\widehat{z})}{z_1 q_1(\widehat{z}) + q_2(\widehat{z})} = \frac{P_1}{q_1} - \frac{\Phi_1 \Phi_1^T}{q_1^2\left(z_1 + q_2/q_1\right)}. \tag{9.4}$$

The multi-affinity of $g(\widehat{z})$ is obvious. Let us prove that

$$g(\widehat{z}) \in \mathbb{R}\mathcal{P}_{d-1}^{(m+r)\times(m+r)}.$$

By Theorem 3.7, it suffices to prove that $W_{z_k} = q_1^2 \partial g(\widehat{z})/\partial z_k$, $k = 2, \dots, d$, are PSD forms. The function $f(z)$ is multi-affine. Then

$$f(z) = \frac{P(z)}{q(z)} = \frac{z_1 z_k \widehat{P}_1 + z_1 \widehat{P}_2 + z_k \widehat{P}_3 + \widehat{P}_4}{z_1 z_k \widehat{q}_1 + z_1 \widehat{q}_2 + z_k \widehat{q}_3 + \widehat{q}_4}. \tag{9.5}$$

From (9.5) and (9.2), we get

$$\begin{aligned} \Phi_1 \Phi_1^T &= z_k^2 \left(\widehat{q}_3 \widehat{P}_1 - \widehat{q}_1 \widehat{P}_3 \right) \\ &\quad + z_k \left(\widehat{q}_4 \widehat{P_1} - \widehat{q}_1 \widehat{P}_4 + \widehat{q}_3 \widehat{P_2} - \widehat{q}_2 \widehat{P}_3 \right) + \left(\widehat{q}_4 \widehat{P_2} - \widehat{q}_2 \widehat{P}_4 \right), \end{aligned} \tag{9.6}$$

$$W_{z_k} = q_1^2 \frac{\partial g(\widehat{z})}{\partial z_k} = \begin{pmatrix} \widehat{P}_1 \widehat{q}_2 - \widehat{P}_2 \widehat{q}_1 & \Phi_k(\widehat{z}) \\ \Phi_k(\widehat{z})^T & (\widehat{q}_2 \widehat{q}_3 - \widehat{q}_1 \widehat{q}_4)\, I_r \end{pmatrix}, \tag{9.7}$$

where $\Phi_k(\widehat{z}) = (z_k \widehat{q}_1 + \widehat{q}_2)\, \partial \Phi_1 / \partial z_k - \widehat{q}_1 \Phi_1$, $k = 2, \dots, d$.

Note the identity

$$\left(\widehat{P}_1 \widehat{q}_2 - \widehat{P}_2 \widehat{q}_1 \right) = \Phi_k(\widehat{z}) \left(\widehat{q}_2 \widehat{q}_3 - \widehat{q}_1 \widehat{q}_4 \right)^{-1} \Phi_k(\widehat{z})^T. \tag{9.8}$$

Indeed,

$$\begin{aligned} \Phi_k(\widehat{z}) \Phi_k(\widehat{z})^T &= (z_k \widehat{q}_1 + \widehat{q}_2)^2 \frac{\partial \Phi_1}{\partial z_k} \frac{\partial \Phi_1^T}{\partial z_k} \\ &\quad - \left(z_k \widehat{q}_1^2 + \widehat{q}_1 \widehat{q}_2 \right) \left(\frac{\partial \Phi_1}{\partial z_k} \Phi_1^T + \Phi_1 \frac{\partial \Phi_1^T}{\partial z_k} \right) + \widehat{q}_1^2 \Phi_1 \Phi_1^T. \end{aligned} \tag{9.9}$$

$\Phi_1(\widehat{z})$ is a multi-affine form. Differentiating (9.6) in z_k and substituting the obtained expressions into (9.9), we obtain (9.8). Since $q(z)$ is a Hurwitz form, then, (see [13], Proposition 2.8),

$$h = \left. \frac{q}{\partial q / \partial z_1} \right|_{z_1 = 0} = \frac{z_k \widehat{q}_3 + \widehat{q}_4}{z_k \widehat{q}_1 + \widehat{q}_2} \in \mathbb{R}\mathcal{P}_{d-1}.$$

Then $Q = (\widehat{q}_2 \widehat{q}_3 - \widehat{q}_1 \widehat{q}_4)$ is PSD. For $\widehat{x} \in \mathbb{R}^{d-2}$, from (9.7), (9.8), we get

$$W_{z_k}(\widehat{x}) = \begin{pmatrix} I_m & \Phi_k Q^{-1} \\ 0 & I_r \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I_m & 0 \\ Q^{-1} \Phi_k^T & I_r \end{pmatrix} \geq 0. \qquad \square$$

Lemma 9.4. *If* $f(z) = g_{11} - g_{12} \left(g_{22} + z_1 I_{r_1} \right)^{-1} g_{21}$ *and*

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \left(a_{33} + z_2 I_{r_2} \right)^{-1} \begin{pmatrix} a_{31} & a_{32} \end{pmatrix},$$

then

$$f(z) = A_{11} - \begin{pmatrix} A_{12} & A_{13} \end{pmatrix} \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}^{-1} \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix},$$

where

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_1 I_{r_1} & 0 \\ 0 & 0 & z_2 I_{r_2} \end{pmatrix}.$$

The assertion of the lemma can be verified by direct calculation.

Proof of Theorem 9.1. By Proposition 3.2, we may assume that

$$\deg_{z_k} P = \deg_{z_k} q = n_k, \quad k = 1, \dots, d.$$

Applying to $f(z)$ the degree reduction operator $\mathbf{D}_{z_k}^{n_k}$, in each variable z_k we obtain the multi-affine positive real function

$$\widehat{f}(\zeta_1, \dots, \zeta_n) = \mathbf{D}_{z_1}^{n_1} \cdots \mathbf{D}_{z_d}^{n_d}[f(z)] = \frac{\widehat{P}}{\widehat{q}} = \frac{\zeta_1 P_1(\widehat{\zeta}) + P_2(\widehat{\zeta})}{\zeta_1 q_1(\widehat{\zeta}) + q_2(\widehat{\zeta})}, \tag{9.10}$$

where $q_1(\widehat{\zeta}) \neq 0$. The matrix pencil representing the multi-affine function $\widehat{f}(\zeta)$ will be constructed step by step. By Theorem 8.2, there exists $m \times r_1$ matrix-valued polynomial $\Phi_1(\widehat{\zeta})$ such that $W_{\zeta_1}[\widehat{q}, \widehat{P}] = \Phi_1(\widehat{\zeta})\Phi_1(\widehat{\zeta})^T$. By Proposition 9.3, the function

$$g^{(1)}(\widehat{\zeta}) = \begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} \\ g_{21}^{(1)} & g_{22}^{(1)} \end{pmatrix} = \frac{1}{q_1(\widehat{\zeta})} \begin{pmatrix} P_1(\widehat{\zeta}) & \Phi_1(\widehat{\zeta}) \\ \Phi_1(\widehat{\zeta})^T & q_2(\widehat{\zeta}) I_{r_1} \end{pmatrix}$$

belongs to the class $\mathbb{R}\mathcal{P}_{n-1}^{(m+r_1)\times(m+r_1)}$, and

$$\widehat{f}(\zeta_1, \dots, \zeta_n) = g_{11}^{(1)} - g_{12}^{(1)} \left(g_{22}^{(1)} + \zeta_1 I_{r_1} \right)^{-1} g_{21}^{(1)}.$$

The matrix-valued function $g^{(1)}(\widehat{\zeta})$ depends only on $n-1$ variables and satisfies the conditions of Proposition 9.3. Then

$$g^{(1)}(\widehat{\zeta}) = \begin{pmatrix} g_{11}^{(2)} & g_{12}^{(2)} \\ g_{21}^{(2)} & g_{22}^{(2)} \end{pmatrix} - \begin{pmatrix} g_{13}^{(2)} \\ g_{23}^{(2)} \end{pmatrix} \left(g_{33}^{(2)} + \zeta_2 I_{r_2} \right)^{-1} \begin{pmatrix} g_{31}^{(2)} & g_{32}^{(2)} \end{pmatrix}.$$

By Lemma 9.4, we obtain

$$\widehat{f}(\zeta) = g_{11}^{(2)} - \begin{pmatrix} g_{12}^{(2)} & g_{13}^{(2)} \end{pmatrix} \begin{pmatrix} g_{22}^{(2)} + \zeta_1 I_{r_1} & g_{23}^{(2)} \\ g_{32}^{(2)} & g_{33}^{(2)} + \zeta_2 I_{r_2} \end{pmatrix}^{-1} \begin{pmatrix} g_{21}^{(2)} \\ g_{31}^{(2)} \end{pmatrix},$$

where the matrix-valued function $g^{(2)}$ depends only on $n-2$ variables and satisfies the conditions of Proposition 9.3.

Continuing the process, at the $n-1$ step we get a positive real matrix-valued function of one variable ζ_n: $g^{(n-1)}(\zeta_n) = \zeta_n A_n$, where $A_n = \{a_{ij}\}_{i,j=1}^N \geq 0$ (here a_{ij} are blocks of the appropriate size). Then

$$\widehat{f}(\zeta_1, \dots, \zeta_n) = A_{11}(\zeta) - A_{12}(\zeta) A_{22}(\zeta)^{-1} A_{21}(\zeta),$$

where the positive real matrix pencil $A(\zeta) = \{A_{ij}(\zeta)\}_{i,j=1}^{2}$ has the form

$$A(\zeta) = \zeta_n \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1N} \\ \hline a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{array}\right) + \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & \zeta_1 I_{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_{n-1} I_{r_{n-1}} \end{array}\right).$$

The degree reduction operator is invertible. Returning to the variables $z_1, \dots, z_d$, we obtain a positive long-resolvent representation of the original function $f(z)$. □

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Додатні матричні зображення раціональних позитивних дійсних функцій кількох змінних

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Раціональну однорідну (першого степеня) позитивну дійсну матричну функцію кількох змінних можна зобразити як доповнення Шура до діагонального блоку лінійної однорідної матричної функції з невід'ємно визначеними дійсними матричними коефіцієнтами (довго-резольвентне зображення). Чисельники частинних похідних позитивної дійсної функції є сумами квадратів многочленів.

Ключові слова: позитивна дійсна функція, матричнозначна функція, доповнення Шура, довго-резольвентне зображення, сума квадратів