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Real Analytic Bergman Spaces

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The theory of CR wedge extension is combined with a study of moment conditions to construct a new class of Bergman-type spaces which are characterized by real analyticity, rather than holomorphicity. The spaces have dense subsets of real analytic functions which contain entire functions as a proper subset.

Key words: Bergman spaces, CR wedge extension, bounded point evaluation

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1. Introduction

In this paper, the idea of a Bergman space is given as a new class of examples with novel properties. The main theorems are extensions of the author's results obtained in [6]. The distinguishing property of all of the examples is that real analyticity is preserved under closure due to moment conditions, but without requiring holomorphicity.

This paper can be considered as a combination of two strands of mathematical research which unexpectedly converged. On the one hand, there is the theory of Bergman spaces, now 100 years old; a very rich theory with potent applications in sciences and an active area of research in its own right. The other strand is the polynomial approximation theory in several complex variables. With Weierstrass' theorem, mathematicians began studying the problem of finding the closure of a finitely generated algebra of functions on a compact set. A basic application of the n-dimensional version of the Weierstrass approximation theorem is the observation that on $\mathbf{C^n}$, the algebra generated by $z, \overline{z}$ is dense in continuous functions on any compact set. The idea that $\overline{z}$ makes an algebra consist of continuous functions is also seen in the Wermer theorem, which states that if $\overline{z}$ is added to the disc algebra $A(D)$, considered as an algebra of functions on the circle, then the continuous functions are the closure of the new algebra. For general information about Bergman spaces, see [5]. For more information on the polynomial approximation theory, see [2] and [8].

Polynomial approximation in several complex variables has new features, especially when considering approximation on submanifolds. In that case, as is well known, the existence of complex tangent directions often guarantees that

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functions in the polynomial algebra extend to be holomorphic on some defined set independent of the function. These theorems exist for smooth manifolds, but also for piecewise smooth unions of manifolds in $\mathbf{C^n}$. The theory of analytic continuation in this setting falls under the heading of "edge of the wedge" theory. See [3] for material about polynomial approximation in the CR setting as well as wedge extension.

Starting around the year 2000, Agranovsky and Globevnik [1] among others noted that analytic continuation techniques in several variables could be used for studying regularity problems involving moments. A notable success of this idea was Tumanov's solution of the "strip problem" [9], which showed that several complex variable techniques could be used to solve problems in one variable where no one-variable technique seemed to suffice. The author learned of these techniques while working on the proof of the strip problem for L^p functions (most of the theory in this area is for continuous or real analytic functions). In 2015, in his paper [6], he showed the first examples of using CR wedge extension to prove *real analyticity* of a function which satisfied some moment conditions, where there were the examples satisfying the moment conditions that were not holomorphic. Although the results are expressed in terms of moment conditions, another formulation is to say that for certain entire functions $g(z)$, the closure of the algebra generated by z and $\overline{z}g(z)$ consists of real analytic functions with an infinite radius of convergence. This comes back a full circle to the very early notion that $\overline{z}$ is "too much" to put into an algebra and get an interesting example. If there is a twist with the multiplication by $g(z)$, then real analyticity is maintained under closure. In this paper, further restrictions on the function g, as well as new techniques, allow us to prove a bounded point evaluation, from which the closure in the L^p space follows easily by normal families.

Another way of looking at the result on Fréchet algebras is to pose the following question. Consider algebras of functions $\mathcal{A}$ on $\mathbf{C}$ with the compact open topology. Are there examples of $\mathcal{A}$ with $\mathcal{O}(\mathbf{C}) \subsetneqq \mathcal{A} \subsetneqq C(\mathbf{C})$ closed in the compact open topology such that every function in $\mathcal{A}$ has some guaranteed smoothness more than continuity? As far as the author knows, this is a new question—at least for the case of real analyticity this question may be new. There is some analogy with the idea of Douglas algebras. These are the algebras in $L^\infty(S^1)$ which are between H^∞ and $C(S^1)$. A very rich structure was discovered in the setting of Douglas algebras. Our research shows that there are some tractable questions about Fréchet algebras between $\mathcal{O}(\mathbf{C})$ and $C(\mathbf{C})$ whose answers hint at a new theory. See [4] for background material on Douglas algebras.

After some background material, the basic result is proved in Sections 3 and 4. Various examples and generalizations are considered in the subsequent sections. The full potential of the new technique is not completely outlined here. In particular, Bergman spaces of real analytic functions on $\mathbf{C^n}$ and on certain Stein spaces can be constructed in a fairly straightforward fashion by following the methods of [6]. There are two methods of proving the estimates here. One of them uses the well-known integral formulas of Airapetyan and Henkin along with estimates which are specific to the 1-dimensional extension problem (and do not hold in general). With these techniques we can show that the bounded point evaluation holds in L^2 for suitable $g(z)$ and with the Gaussian weight. Then we demonstrate a different technique based on a standard type of the construction of analytic discs with boundary in the CR wedge. The advantage of the second method is that one gets control over the region of integration corresponding to the point evaluation for a given point. This allows us to prove some precise theorems about order of growth for a natural class of examples.

2. Basic notions

Let $X \subseteq \mathbf{C}$ be a discrete set, and let $\mathcal{A}_X$ be the set of all continuous $f(z), z \in \mathbf{C}$ such that for every $a \in X$ and every $r > 0$, $f|_{|z-a|=r}$ extends holomorphically to $|z-a| < r$. If the set X contains vertices of arbitrarily large triangles which contain 0 roughly in the center, none of whose angles degenerate near ∞, then every function in $\mathcal{A}(X)$ is real analytic with infinite radius of convergence. The precise statement is in Theorems 1 and 2 of [6]. The basic ideas will be explained anew, because we modify them for the construction of Bergman spaces. In case these conditions are met, then we further show that $\mathcal{A}_X$ is generated by z and $\overline{z}g(z)$, where $g(z)$ is an entire function vanishing exactly on X with only simple zeroes.

A *Bergman space* means a closed subspace of L^p of a domain (possibly with a weight) such that there is a bounded point evaluation at every point in the domain for functions in the space. This definition was given to the author by F. Haeslinger, who also posed the question of whether one could construct Bergman spaces of real analytic functions based on the example $\mathcal{A}_X$. We are very grateful to Professor Haeslinger for suggesting the problem.

3. Construction of the algebras; the CR extension with L^p estimates

There are always non-holomorphic functions in $\mathcal{A}_X$, for if $g(z)$ is an entire function whose zero set contains X, then $\overline{z}g(z) \in \mathcal{A}_X$. Given a discrete set $X \subseteq \mathbf{C}$ and a point $a \in X$, let $M(a)$ denote the associated half CR manifold of extension of functions in $\mathcal{A}_a$.

$$M_a = \left\{ (z,w) : w = \frac{t}{z-a} + \overline{a},\ t \geq |z-a|^2 \right\}. \tag{3.1}$$

M_a is a union of punctured analytic discs, and every function which is in $\mathcal{A}_a$ has a CR lift to M_a (see [6] for details). Given any two points, $a, b \in X$, there is CR wedge extension from $M_a \cup M_b$. For any z, the extension is into the region between the rays of M_a and M_b. In the exceptional case, where z is on the line joining a and b, the wedge extension does not (directly) show any analytic continuation. If compact subsets of $M_a \cup M_b$ are taken, then the local CR extension occurs. This is how the extensions will be used in this paper.

By a change of variables, we can consider the local model $M_1 = \{(z_1, z_2) : y_1 = 0,\ y_2 \geq 0\}$, $M_2 = \{(z_1, z_2) : y_1 \geq 0,\ y_2 = 0\}$. Given a continuous CR function on $M_1 \cup M_2$, we consider the analytic extension to the wedge $y_1 > 0$, $y_2 > 0$. The analytic extension can be realized by analytic discs with boundary in $M_1 \cup M_2$. Among other choices, here we use dilations and translations of a fixed disc. Let $\Delta = \{\zeta : |\zeta| \leq 1\}$. Let R be the upper half of a disc of radius 1, symmetric about the y-axis, with the bottom edge on the real axis. Let $\phi : \Delta \to R$ be the unique conformal map with the property that $|\phi(\zeta)| = 1$ exactly on the upper half circle, $\mathrm{Im}(\phi) = 0$ exactly on the lower half circle, and $Re(\phi(0)) = 0$. Denote $\phi(0) = it$. For any point $z = a + ib$ in the upper half-plane, the map $\phi_{a,b}(\zeta) = \frac{b}{t}(\phi(\zeta) + \frac{at}{b})$ also satisfies $\mathrm{Im}(\phi) = 0$ exactly on the lower half-circle, maps the upper half-circle to the upper half-plane, and $\phi_{a,b}(0) = a + ib$.

Now consider a point $(z, w) = (a + ib, c + id)$ with $b > 0$, $d > 0$. The map $\Phi_{a,b,c,d}(\zeta) = (\phi_{a,b}(\zeta), \phi_{c,d}(-\zeta))$ provides an analytic disc such that:

1. $\Phi(0) = (z, w)$;
2. $\Phi(e^{i\theta}) \in M_1$, $\pi \leq \theta \leq 2\pi$;
3. $\Phi(e^{i\theta}) \in M_2$, $0 \leq \theta \leq \pi$.

We observe that as $z \to \pm 1$ in $\overline{\Delta}$, $\phi'_{a,b}$ and $\phi'_{c,d}$ are comparable. Therefore, in Cauchy integrals over $\Phi(|\zeta| = 1)$, $|dz|$ and $|dw|$ are comparable.

For sharper estimates we need an improvement. Let $\epsilon > 0$ be given, and $R_1 >> \epsilon$. Given a real horizontal or vertical dilation of the upper half-circle $\widetilde{R}$, there is a corresponding t, and discs $\widetilde{\Phi}$ can be constructed in the same way. We state as a proposition the fact we will use later. The proof is straightforward.

Proposition 3.1. *Given ϵ, R_1, there exist R_2, $\epsilon < R_2 < R_1$, $C < 1$ such that if $Z = (a + ib, c + id)$ satisfies $b > \epsilon$, $d > \epsilon$, $|a| > \epsilon$, $|b| > \epsilon$, $|Z| < R_1$, then there exists a dilation of the upper half-disc $\widetilde{R}$ such that the associated disc $\widetilde{\Phi}$ satisfies $|Re(\Phi_1)| > C\epsilon$, $|Re(\Phi_2)| > C\epsilon$, and $|\Phi| < R_2$.*

The point of this proposition is that the real parts of the coordinates determine the leaf in the Levi foliation. For better estimates for the point 0, we will want to stay away from the leaves which contain 0. A disc given by Proposition 3.1 gives an L^p estimate, where the region of integration in each coordinate plane does not contain 0.

Denote this analytic disc by $D_{a,b,c,d}$, or D_p for short, where $p = (a+ib, c+id)$. Let D_{1p} and D_{2p} be the projections of D_p onto the z and w planes, the images of the components of Φ. Given a function f which is continuous and CR on $M_1 \cup M_2$, we can use the Cauchy formula

$$f(p) = \frac{1}{2\pi i} \int_{\partial D_{1p}} f(z, g(z))\, dz,$$

where $g(z) = \phi_{c,d} \circ (\phi_{a,b})^{-1}(z)$. In order to estimate $|f(p)|$, we will first consider the integral over the circular part of ∂D_{1p}. Then we use w and the same estimates over the circular part of D_{2p}. Since $|dz|$ and $|dw|$ are comparable, this gives one estimate. The details are in the next few paragraphs.

The L^p estimates are derived from growth estimates for Hardy space functions. If $f \in H^p(S^1)$, then

$$|f(z)| \leq C \frac{\|f\|_p}{(1-|z|)^{1/p}}. \tag{3.2}$$

Suppose $f \in \mathcal{A}_a$, and for convenience, set $a = 0$. Let $\gamma(t), t \in [0, b]$ be a smooth curve such that $|\gamma(0)| = r_0$, $|\gamma(t)| < r_0 - t$, and $r_0 - |\gamma(t)| \equiv kt$ for some $k > 1$. The radii can also be decreasing to r with the same growth rate. Let f_r denote the holomorphic extension of $f|_{|z|=r}$ to the disc $|z| < r$. Then

$$|f_r(\gamma(t))| \leq \frac{\|f_r\|_p}{(r-|\gamma(t)|)^{1/p}}$$

and

$$\int_0^b |f_{r-t}(\gamma(t))|\, dt \leq C \int_0^b \frac{\|f_{r-t}\|_p}{t^{1/p}}\, dt. \tag{3.3}$$

The following lemma gives what is needed to get the point evaluation estimates.

Lemma 3.2. *Fix $p > 2$. Let $\gamma(t) = (r(t), z(t))$, $0 \leq t \leq b$, $r(t) > 0$, $z(t) \in$ $\mathbf{C}$, $|z(t)| < r(t)$; $r(t)$ can be assumed to be monotonic in t. Suppose that $\gamma(0) = (r_0, z_0), |z_0| = r_0$, and for some $0 < K < 1$, $0 < Kt < r(t) - |z(t)| < \frac{1}{K}t$. If $f \in \mathcal{A}_0$ and $F(\gamma(t)) = f_{r(t)}(z(t))$, where f_r is the holomorphic extension of f on the disc of radius r, then*

$$\int_0^b |F(\gamma(t))|\, dt \leq C \left(\int_{r_1<|z|<r_2} |f(z)|^p \, dA \right)^{1/p}. \tag{3.4}$$

Here r_1 and r_2 are the lowest and the highest values of $r(t)$.

Proof. Let q be the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. First apply the growth estimate with $p = 2$ to get

$$\begin{aligned} \int_0^b |F(\gamma(t))| dt &\leq C \int_0^b \frac{\|f_{r(t)}\|_2}{t^{\frac{1}{2}}}\, dt \leq C \int_0^b \frac{||f_{r(t)}||_p}{t^{\frac{1}{2}}}\, dt \\ &\leq C \left(\int_0^b \frac{1}{t^{\frac{q}{2}}}\, dt \right)^{\frac{1}{q}} \left(\int_0^b \int_{|z|=r(t)} |f(z)|^p |dz|\, dt \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{r_1 \leq |z| \leq r_2} |f(z)|^p dA \right)^{\frac{1}{p}}. \end{aligned}$$

The lemma is proved. □

This lemma is applied to estimate the contribution of a Cauchy integral on the boundary of some $\Phi_{a,b,c,d}$ on a part of the curve near the edge. By the construction, the approach at the edge is transverse, which allows the estimate of the lemma to be applied. (The only difficulty with estimation is near the edge.)

Next, we apply this estimate to get L^p bounds inside the wedge extension of F associated to $f \in \mathcal{A}_{01}$. For z with $\mathrm{Im}(z) \neq 0$, we have an extension on a CR wedge consisting locally of half-spaces in $\mathrm{Im}(zw) = 0$, $\mathrm{Im}(zw - z - w) = 0$. Apply the change of coordinates $\Psi(z, w) = (\zeta, \tau) = (zw, zw - z - w)$. The Jacobian of this map is $z - w$, which is non-zero in a neighborhood of any point $(z, \overline{z})$ with z not real. Let B_1 be a small ball centered at P and pick such a $P = (z, \overline{z})$ that Ψ is 1-1 on B_1 with $|\det(\Psi')|$ bounded away from zero. By a translation, multiplying the components of the map by -1 if necessary, we may assume that $\Phi(z, \overline{z}) = (0, 0)$ and the image of our CR wedge is locally in $\{(z', w') : \mathrm{Im}(z') \geq 0,\ \mathrm{Im}(w') = 0\} \cup \{(z', w') : \mathrm{Im}(z') = 0,\ \mathrm{Im}(w') \geq 0\}$. Let $B_2 = B(0, \epsilon)$ be a ball in the (z', w') coordinates such that $U = B_2 \cap \{(z', w') : \mathrm{Im}(z') > 0,\ \mathrm{Im}(w') > 0\}$ is contained in the region of the wedge extension of $\Psi(B_1) \cap (\{z', w') : \mathrm{Im}(z') = 0,\ \mathrm{Im}(w') \geq 0\} \cup \{(z', w') : \mathrm{Im}(z') \geq 0,\ \mathrm{Im}(w') = 0\})$. Fix a $\delta << 1$. By shrinking ϵ if necessary, for any point $(z, w) \in U$, there is a disc $\Phi_{a,b,c,d}$ such that $\Phi(|\zeta| = 1) \subseteq \frac{1}{2}K$. Use the discs $\Phi_{a,b,c,d}$ whose images lie in $\Psi(B)$ for all points in U. The comparability of $\mathrm{Im}(\phi_1)(e^{i\theta})$ and $Re(\phi_2)(e^{i\theta})$ for θ close to 0 or π in the range $[0, \pi]$ and the corresponding statement for $Re(\phi_1)$ and $\mathrm{Im}(\phi_2)$ on the lower half of the circle allow us to apply the lemma giving the following proposition. We let M_{ab} denote the region of wedge extension from $M_a \cup M_b$.

Proposition 3.3. *Given a point $(z, \overline{z})$ with $Rez \neq 0$, there exists a ball $B_1 \subseteq B_2$ containing $(z, \overline{z})$ such that:*

1. *In the local CR extension from $B_2 \cap M_0 \cap M_1$, every point in $M_{01} \cap B_1$ is contained in a disc Φ. whose boundary is contained in B_1.*
2. *For any point $(u, v) \in B_2 \cap M_{01}$, $v \neq \overline{u}$, there exists a constant $C_{u,v}$ independent of f and $R > 0$ independent of f such that*

$$|f(u, v)| \leq C_{u,v} \left(\int_{|z| \leq R} |f(z)|^p \, dA \right)^{\frac{1}{p}}. \tag{3.5}$$

The constants $C_{u,v}$ are bounded on any compact set of B which does not intersect $\{(z, w) : (\mathrm{Im}(zw) = 0\} \cup \{(z, w) : \mathrm{Im}(zw - z - w) = 0\}$.

The holomorphic extension to B is simply what is given by the CR wedge extension. The bounds on the constants $C_{u,v}$ follow from the bounds on the maps Φ.

The estimates blow up near the CR wedge. In the original paper, if the set X associated to $\mathcal{A}_X$ contained vertices of arbitrary large triangles, one could prove real analyticity. To get a bounded point evaluation, we need some overlap: the analytic continuation from a wedge $M_1 \cup M_2$ will cover a third CR half-space M_3 which is in the region of wedge extension. We need at least 5 local CR wedges to get the estimates. Here is a theorem which gives L^p estimates for point evaluation in a neighborhood of a suitable point for $p > 2$.

Theorem 3.4. *Let $X \subseteq \mathbf{C}$ be a discrete set, and let $\mathcal{A}_X$ be the associated algebra. Let $z_0 \in \mathbf{C}$ be a point with the properties listed below. Then for some*

$\epsilon > 0$, there exist continuously varying constants C_z for $|z - z_0| < \epsilon$ and $R > 0$ such that for all $f \in \mathcal{A}_X$,

$$|f(z)| < C_z \left(\int_{|z-z_0|<R} |f(z)|^p dA \right)^{\frac{1}{p}}.$$

There are 5 points $a_1, \ldots, a_5, a_i \in X$, such that in the fiber over z of $\cup M_i$, every ray of an M_i is contained between two other rays on the side with the angle less than π.

Proof. Take z to be 0. Consider a circle $\gamma = \{(z, w) : w = e^{i\alpha z},\ |z| = \eta\}$, where α is chosen so that the intersections with the M_{a_i}'s are isolated points. Then for small enough η (which may require shrinking ϵ and δ), we can find an L^p estimate on every point of γ. On points on or close enough to a particular M_{a_i}, we can use the wedge extension estimates from two other CR manifolds as in the proposition. By Cauchy's formula, this gives the estimate for $z = 0$; for nearby points it follows from continuity of all parameters involved. □

It is a straightforward matter to use normal families to show that functions in $B_{p,X}$ are in $\mathcal{A}_X$. Since this is an important fact, we state it explicitly

Theorem 3.5. *We have $B_{p,X} \subseteq \mathcal{A}_X$.*

4. Estimates derived from Airapetyan–Henkin formulas

There is a different way of getting estimates which includes the case $p = 2$. The proof uses integral formulas due to Airapetyan and Henkin for wedge extension from a pair of Levi-flats. The exponent is improved, but the weak mean-value property using integration over an annulus cannot be proved with this method. Given M_{a_1}, M_{a_2} and some point p not on the line determined by a_1 and a_2, there is a wedge extension on one side. Locally, this is biholomorphically equivalent to the case $M_1 = \{(z, w) : \mathrm{Re}(w) = 0,\ \mathrm{Re}(z) > 0\}$, $M_2 = \{\mathrm{Re}(z) = 0,\ \mathrm{Re}(w) > 0\}$. Intersecting M_1 and M_2 with a small ball centered at p, then for a point (z, w) close enough to 0, with $\mathrm{Re}(z) > 0$, $\mathrm{Re}(w) > 0$, then for any function f which is CR on $M_{a_1} \cup M_{a_2}$ (thus automatically extending to a holomorphic function on a region including (z, w) for (z, w) close enough to 0), we have

$$\begin{aligned} f(z, w) = &\int_{(M_{a_1} \cap M_{a_2} \times [0,1]} K_1(\zeta, \tau, t, z, w) f(\zeta, \tau) dm(z, w)\, dt \\ &+ \int_{M_{a_1}} K_2(\zeta, \tau, z, w) f(\zeta, \tau)\, dm_1(\zeta, \tau) \\ &+ \int_{M_{a_2}} K_3(\zeta, \tau, z, w) f(\zeta, \tau)\, dm_2(\zeta, \tau). \end{aligned}$$

Implicit in these formulas is a cut-off, so the integrations are taken over compact sets. The kernel K_1 has $(z-\zeta)(w-\tau)$ in the denominator, which is non-vanishing.

The kernel K_2 has $(w-\tau)$ in the denominator, and K_3 has $(z-\zeta)$ in the denominator. In both cases, the fraction is non-bounded and vanishing. We mean here the boundedness with the point (z, w) fixed.

Now we derive a bounded point evaluation for $1 \leq p < \infty$. Estimates of the following type are contained in [7] and depend on the 1-dimensional extension property. Most likely they do not hold in general.

Lemma 4.1. *Given $\zeta \in M_a \cap (w = \mathrm{Im}(z))$, then for sufficiently small balls $B_r(\zeta)$,*

$$\int_{B_\zeta(r)\cap M_a} |f^2| \leq K \int_{B_\zeta(r)\cap(w=\bar{z})} |f|^2. \tag{4.1}$$

Proof. Integrals on slices parallel to the totally real plane are bounded by the integral on the totally real plane. This is an application of the standard H^p theory on the disc. For details, we refer readers to [7]. □

We conclude that there is a bounded point evaluation in $L^p(\mathbf{C}, e^{-|z|^2})$, $1 \leq p < \infty$.

5. An explicit example with growth estimates

Let $X = \mathbf{Z} \cup (\omega\mathbf{Z}) \cup (\omega^2\mathbf{Z})$, where $\omega^3 = 1, \omega \neq 1$. Take

$$g(z) = \frac{\sin z \sin(\omega z) \sin(\omega^2 z)}{z^2}.$$

From the order of g, we see that

$$B_X = \left\{ f \in \mathcal{A}_X : \int_{\mathbf{C}} |f(z)|^2 e^{-|z|^2} dA \right\}$$

is a Bergman space of real analytic functions which contains non-holomorphic functions.

The first step is not necessary for the coefficient estimates but seems independently interesting. In Proposition 5.1, we prove a kind of weak maximum principle.

Proposition 5.1. *Let $L \subseteq \mathbf{C}$ be a regular n-gon centered at 0, $n \geq 5$, with vertices $p_1, \ldots, p_n$. Denote by Y the set of vertices of L. There exists $\delta > 0$ and $0 < r_1 < r_2$, $C > 0$ such that if $f \in \mathcal{A}_Y$ and $|z| < \delta$, then*

$$|f(z)| \leq C \left(\int_{r_1<|z|<r_1} |f(z)|^p \, dA(z) \right)^{\frac{1}{p}}. \tag{5.1}$$

Proof. The technique of the main lemma gives the estimate. The lower bound r_1 results from choosing an analytic disc (affine) whose boundary avoids the leaves of foliations of the M_i's containing 0 over the part of the arc, where an estimate depends on a particular M_i. Let $M_1, \ldots, M_n$ be the associated CR half-manifolds for the one-dimensional extensions. In a small neighborhood of $z = 0$, let $\widetilde{M}_i$ be the CR manifolds whose fiber over z is the same as the fiber of M_i over 0. Each $\widetilde{M}_i$ is also a Levi-flat. If construct an analytic disc with the desired properties with respect to the $\widetilde{M}_i$'s, then the same relation will hold between the analytic disc and the M_i's in a small enough neighborhood of 0.

The fiber over z of $\widetilde{M}_i$ is a ray from 0 in the direction $[0, \overline{p_i}]$. The angle between two adjacent rays is $\frac{2\pi}{n}$.

Consider the circle S, $z = 0, |w| = \delta$ for a small δ to be fixed later. For each point q on S, there is an L^p estimate for the value of F expressed in terms of an integral in the z-plane. This estimate depends on the two closest $\widetilde{M}_i$'s, unless q lies in a small neighborhood of $\widetilde{M}_i$, (the size of the neighborhood determines the coefficient in the L^p estimate). In that case, the adjacent $\widetilde{M}_i$'s are used. Because $n \geq 5$, the region of analytic continuation of those $\widetilde{M}_i$'s will cover q. Each $\widetilde{M}_i$ is determined by $\operatorname{Im}\phi_i = 0$, where ϕ is a linear form. In general, the forms are quadratic; the linear approximation will suffice to understand the geometry. The holomorphic leaf on M_i is determined by $\operatorname{Re}\phi_i$. Therefore, if we show that $\operatorname{Re}\phi_i(q) \neq 0, \operatorname{Re}\phi_j(q) \neq 0$, where $\widetilde{M}_i$ and $\widetilde{M}_j$ are the Levi-flats used for the estimate at q, then using the disc from Proposition 3.1, we get an integral over a region which does not contain $z = 0$. By continuity, we can find a non-zero lower bound.

Write $q = (0, \delta e^{i\alpha})$. Fix an angle θ_0; we can see that $\theta_0 = \frac{\pi}{18}$ will suffice. Thus, we have two cases:

1. $\left|\alpha - \dfrac{2l\pi}{n}\right| > \theta_0$, $l = 1, 2, \ldots, n$;
2. $\left|\alpha - \dfrac{2l\pi}{n}\right| < \theta_0$ for some l.

In the first case, we use the two closest M_i's. In the second, we are close to one M_i and use the adjacent ones on either side for estimation. It is evident that if $n \geq 5$ and $\theta < \frac{\pi}{10}$ for the pentagon (the worst case), then you have $\operatorname{Re}(w_i q) > c\delta$ for some small $c > 0$.

An estimate for the original M_i's and for all z close enough to 0 follows by continuity. □

By bounded point evaluation for B_X and by scaling, for any p, $|p| > 1$, we have that

$$|f(p)) \leq C \int_{|z| \leq K|p|} |f|^2 \, dA(z),$$

where C and K are independent of p. This integral does not contain the Gaussian weight since it depends only on the geometry of the M_a's. We can derive a restriction on the growth of a function in B_X. As with analytic functions, set $M_r = \sup_{|z|=r} |f(z)|$.

Theorem 5.2. *Let $f \in B_X$. Then for any $t > \frac{K^2}{2}$,*

$$\limsup_{r \to \infty} \frac{M_r}{e^{tr^2}} = 0.$$

Proof. Assume the contrary. Then for some $\delta > 0$, there is a sequence z_n, $|z_n| \to \infty$ and $|f(z_n)| \geq \delta e^{t|z_n|^2}$. We gave

$$\begin{aligned}\delta e^{t|z_n|^2} &\leq C \int_{|z|\leq K|z_n|} |f(z)|^2\, dA(z)\\ &\leq Ce^{\frac{K^2}{2}|z_n|^2} \int_{|z|\leq K|z_n|} |f(z)|^2 e^{-|z|^2}\, dA(z).\end{aligned}$$

From this we derive that $\frac{\delta}{C} e^{(t-\frac{K^2|z_n|^2}{2})} \leq \|f\|_2$, which forces the right-hand side to be ∞. □

Following the method of Levin, we can make progress in coefficient estimation. Let $f \in B_X$. Pick $M < \frac{K^2}{2}$. Then $|f(z)| < Ce^{M|z|^2}$. For some K_1,

$$\sup_{|z|<r,|w|<r} < C \sup_{|z|<K_1 r} |f(z)|.$$

Here, $f(z,w)$ is the complexified power series for f. The constant K_1 is not strictly related to the bounded point evaluation—we only need the Frèchet space theorem here.

Putting these estimates together, we can say that for $t > \frac{K^2}{2}$ we have estimates of the form

$$\sup_{|z|<r,|w|<r} |f(z,w)| \leq Ce^{tM^2r^2}. \tag{5.2}$$

Writing $f(z,w) = \sum_{n,m\geq 0} a_{mn} z^n (wg(z))^m)$, we can obtain some Cauchy estimates for coefficients. One would like to apply the Levin method, but the zeroes of $g(z)$ make it difficult to get a clean answer. In our future work, we hope to give conditions on the coefficient sequence of a function in B_X, and in reverse, to show how to estimate the norm based on coefficient estimates.

We also offer the following observation. Suppose that the weight $\omega = e^{-k|z|}$ is used. Then, depending on k, the growth of $\sin(z)$ suggests that $L^p(\mathbf{C}, \omega)$ consists of series which are polynomial in $\overline{z}$. In order to prove this, one needs coefficient estimation. This example appears to relate to the topic of polyanalytic functions. What we would show is that with a weight $e^{-t|z|}$ we can construct Bergman spaces of polyanalytic functions, all of them real analytic and with bounded point evaluation.

6. Hybrid Bergman–Hardy spaces on CR manifolds

Let $M \subseteq \mathbf{C^2}$ be a smooth CR manifold such that the fiber over each point z is a simple closed curve. Denote by Ω the domain which is the union of the interiors of the M_z's. Assume that $\{(z,w) : w = 0\} \subseteq \Omega$. Given a discrete X which satisfies the conditions of Theorem 3.4. Let $A_{M,X} = \overline{\pi^*(\mathcal{A}_X) \otimes \mathcal{O}(\mathbf{C})}$, where the closure is taken in the topology of uniform convergence on compacta. From [6], $f \in A_{M,X}$ is a continuous function on M with the following properties:

1. $\displaystyle\int_{M_z} f(z,w)\, dw \in \mathcal{A}_X$, where M_z is the fiber of M over z;

2. every $f \in A_{M,X}$ extends continuously to a function $f(z,w)$ on Ω which is holomorphic in w and real analytic in both variables.

For the next construction, we specialize to the case where the fibers of M are circles centered at 0. In this case, $\Omega = \{(z,w) : |w| \leq e^{-\phi(z)}\}$, where ϕ is plurisubharmonic on $\mathbf{C}$. We assume that ϕ is not constant, which means that the fibers M_z shrink to 0 as $|z| \to \infty$. Define a Bergman-Hardy space on M as follows. We work in $L^p(M)$ with

$$\int_M g(z,w)d\mu = \int_{\mathbf{C}} \int_{M_z} g(z,w) |\, dw| dA(z).$$

Definition 6.1. $B_{M,X,p}$ is the closure in $L^p(M, e^{-|z|^2}d\mu))$ of

$$\left\{ f \in A_{M,X} : \int_M |f(z,w)|^p e^{-|z|^2} d\mu < \infty \right\}.$$

Combining information about $A_{M,X}$ with basic Hardy space theory, we have the following proposition.

Proposition 6.2. *If $f \in B_{M,X,p}$, then the following hold:*

1. *f extends to a real analytic function on Ω which is holomorphic in w, whose non-tangential limits in the vertical (w) direction equal f almost everywhere*;
2. *for almost every z, $f|_{M_z} \in H^p(M_z)$;*
3. *as $t \to 1$,*
$$\int_{tM} |f(z,w)|^p e^{-|z|^2} t\, d\mu \to \int_M |f(z,w)|^p e^{-|z|^2} t\, d\mu,$$
where tM is the dilation of M in the vertical fiber, and $td\mu$ is the measure which is scaled by t on each fiber in the obvious way.

The third point follows from the monotone convergence theorem; the second follows from the standard theory, and the first was proved in [6]. Any $f \in B_{M,X,p}$ has a series representation $f(z,w) = \sum_{l,m,n} a_{lmn} z^m (\overline{z} g(z))^n w^l$. To find estimates for the coefficients, take

$$b_m(z) = \frac{1}{2\pi i} \int_{M_z} \frac{f(z,w)}{w^n} dw.$$

The b_n's may not be integrable in the given weight, but they are in $L^p(\mathbf{C}, e^{-|z|^2 - |z|})$. This means that coefficient estimates, similar to those for $B_{X,\omega}$, can be obtained.

7. A Bergman space on the unit disc

It is clear from the basic construction of algebras that a sufficiently dense discrete X set of zeros in the unit disc results in a closed Fréchet algebra of real analytic functions; with more restrictions on the zero set of g, local L^p estimates for bounded point evaluation will exist. The interesting case looks like $p = 2$ and the function $g(z)$ is bounded analytic. This of course implies a restriction on the zero set of g. It is not clear whether such an example could exist. However, let us assume such a situation and look at associated shift or shift-type operators.

So, we suppose that $g(z)$ is a bounded analytic function on the unit disc with zeroes at $X \subseteq \Delta$ such that there is L^2 bounded point evaluation for $\mathcal{A}_{X,2}$, which is what we will call this space of functions. An important distinction here is that a function $f \in \mathcal{A}_{X,2}$ need not be represented by a power series which converges on the entire disc. What we can do, is to say that $f(z) = \sum_{m,n \geq 0} a_{mn} z^m (\overline{z} g(z))^n$ in the neighborhood of 0. Furthermore, this representation is unique. Looking at the power series, we can construct shift operators:

1. The classical shift operator $f \to zf$.
2. Multiplication by $(\overline{z})(g(z))$. It is in this step that we would use the fact that g is bounded.
3. One can multiply the shifts from 1 and 2 together. You could also combine with projection onto the orthogonal complement in $\mathcal{A}_{X,2}$ of the usual Bergman space.

This suggestion is for mathematicians who know more about shift operators than the author. We cannot say whether any of these shift operators are interesting.

Definition 7.1. Let X be a discrete set a domain Ω. Denote by $\mathcal{A}_{\Omega,X}$ the set of continuous functions $f(z)$ on Ω such that if $p \in X$ and $\{|z-p| \leq r\} \subseteq \Omega$, then $f|_{|z-a|=r}$ extends holomorphically to $|z-a| < r$.

It is clear from the construction of real analytic Fréchet spaces that for X, which is thick enough at $\partial\Omega$, the functions in $\mathcal{A}_{\Omega,X}$ will be real analytic. Let $g(z)$ be an analytic function on Ω with simple zeroes exactly at the points of X. Then z and $\overline{z}g(z)$ generate an algebra, but it is not clear whether this algebra is $\mathcal{A}_{\Omega,X}$. With a suitable weight on Ω, one can construct a Bergman space. We describe a construction for the unit disc.

The proof of the following lemma is elementary.

Lemma 7.2. *Let $p1, p2, \ldots, p5$ be the vertices of a convex pentagon Q in $\mathbf{C}$. Let $L1, L2, \ldots, L5$ be the secants of Q, and let $s1, \ldots, s5$ be the intersection points of the secants which are in the interior of Q. They are the vertices of another convex pentagon $\widetilde{Q}$. Then the geometric conditions for applying Theorem 3.4 to $\mathcal{A}_{p_1 \cdots p_5}$ hold for any point in the interior of $\widetilde{Q}$. In the case of the algebra $\mathcal{A}_{\Omega,X}$, the condition holds if for each point $w \in Q$ and each p_i, the disc centered at p_i of radius $|p_i - w|$ is contained in Ω.*

Using this lemma, we can estimate how dense X has to be at the circle to obtain a real analytic Bergman space.

Theorem 7.3. *There is a discrete set $X \subseteq D$, $X = \{a_1, a_2, \ldots\}$ such that there is a local bounded point evaluation in L_p for every point in the disc, and*

1. *for any $p > 2$, there is a local L_p bounded point evaluation estimate*;

2. *for any* $t > 1$, $\sum_{n=0}^{\infty}(1 - |a_n|)^t < \infty$.

Proof. Using Lemma 7.2, we can cover the annulus $1 - 2^{-n} \le z < 1 - 2^{-n-1}$, the union of $k2^n$ pentagons (k being independent of n) whose rectangular part has the side length $O(2^{-n-2})$ guaranteeing that the estimate of Lemma 7.2 will hold at interior points. Then the estimate on approach of zeros follows immediately. □

If one got a zero set with the Blaschke condition, then using a bounded $g(z)$, one could construct a Bergman space of real analytic functions in $L^p(D)$. The preceding theorem indicates that for some weight ω, there is a Bergman space of real analytic functions contained in $L_p(D, \omega)$. We denote this space by $B_{D,X,p,\omega}$.

It would be interested to find out whether $B_{D,X,p,\omega}$ is generated by z and $\overline{z}g(z)$.

8. Application to partial differential equations and further questions

On a formal level, one can construct many PDE's of evolution type with constant coefficients, both linear and non-linear, such that solutions with initial data in $\mathcal{A}_X$ remain in $\mathcal{A}_X$ for positive t. Consider the equation $u_t + u_{\overline{z}} = 0$, or $u_t + u_{\overline{z}\overline{z}}$ for an unknown function $u(t, z)$ with initial conditions $u(0, \cdot)$. If $u(t, z)$ has the form $\sum_{n,m\ge 0} u_{nm} t z^n (\overline{z}g(z)^n$, then any number of $\frac{\partial}{\partial \overline{z}}$ derivatives remains formally in $\mathcal{A}_X$, which means that if $u(0, \cdot) \in \mathcal{A}_X$, then $u(t, \cdot) \in \mathcal{A}_z$ for $t > 0$ formally. We take the modified transport equation $u_t + u_{\overline{z}} = 0$ to demonstrate how this can work for initial data which are polynomial in $\overline{z}$. Suppose $u(0, z) = \sum_{m=0}^{k} u_m(t, z)(\overline{z}g(z))^k$, and we want to find a solution of the transport equation with u_m holomorphic in z. If the initial conditions are constants $u_k(0, z) = U_k$, $0 \le k \le m$, $U_k = 0$, $k > m$, then it is easy to show that $u_l(t, z) = 0$ for $l > m$ and $u_k(t, z)$ is a polynomial in t of degree $m - k$ for $0 \le k \le m$. This is a simple example. Certainly, stronger results could be proved if there was some potential application.

9. Approximation of Fock spaces by real analytic Bergman spaces

The results of this section could be stated in greater generality, but we focus on the well-known example of Fock spaces to illustrate the principle. Let $X \subseteq \mathbf{C}$ be a discrete set having the property that $nX \subseteq X$. Suppose also that for every positive integer n, $\frac{1}{n}X \subseteq X$. With this assumption, the point evaluation estimates for $\mathcal{A}_X$ also hold for all $\mathcal{A}_n = \mathcal{A}_{\frac{1}{n}X}$ The following theorem is clear.

Theorem 9.1. *The space* $\cap_n \mathcal{A}_n = B$ *is the Fock space.*

Let T_n be the orthogonal projection onto $\mathcal{A}_n$. Because of bounded point evaluation, we can represent T_n by a kernel function

$$K_n(z, w) : T_n(f)(z) = \int_{\mathbf{C}} K_n(z, w) f(w) e^{-|z|^2|} \, dA(z). \tag{9.1}$$

Let $K(z,w)$ denote the kernel for the usual Bergman projection onto the Fock space. Because the point evaluation estimates are continuous, we have that $K_n(z,w)$ is a locally bounded sequence of functions. Clearly, $\lim_{n\to\infty} K_n(z,w) = K(z,w)$. From this and because normal families of real analytic spaces $\mathcal{A}_*$ behave like usual normal families, we deduce that $K_n(z,w) \to K(z,w)$ locally uniformly on compacta, and real analytically as well.

The Fock space is used in quantum mechanics in the Bargmann–Segal formalism. A possible application of our results is to extend the Bargmann–Segal formalism to these real analytic Bergman spaces. Perhaps there is an asymptotic formula as $n \to \infty$ in the limit which could be useful.

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Дійсно-аналітичні простори Бергмана

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Теорію продовження CR клину поєднано із дослідженням моментних умов для побудови нового класу просторів типу Бергмана, які характеризуються швидше дійсною аналітичністю ніж голоморфністю. Простори мають щільні підмножини дійсно-аналітичних функцій, які містять цілі функції як власну підмножину.

Ключові слова: простори Бергмана, продовження CR клина, обчислення обмеженої точки