Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 2, pp. 250–264
doi: https://doi.org/10.15407/mag20.02.250

A Reilly Type Integral Formula Associated with Diffusion-Type Operators and Its Applications

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In this paper, we derive a Reilly type formula for the diffusion-type operator $\mathcal{L}\cdot = \frac{1}{B}\operatorname{div}(A\nabla\cdot)$ on weighted manifolds with boundary, where A and B are two positive smooth functions on manifolds. As its applications, some inequalities of Poincaré type, Colesanti type, Minkowski type and Heintze–Karcher type are provided.

Key words: Reilly type formula, diffusion-type operator, m-modified Ricci curvature, A-mean curvature

Mathematical Subject Classification 2020: 53C21, 58J32

1. Introduction

Let $(M^n, g, d\mu)$ be an n-dimensional compact weighted manifold with boundary ∂M. A weighted Riemannian manifold is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given compact n-dimensional Riemannian manifold (M^n, g) with the metric g, the triple $(M^n, g, d\mu)$ is called a compact weighted Riemannian manifold, where $d\mu = B dv$ is a weighted volume form, B is a positive smooth function on M, and dv is the Riemannian volume element related to g. Let $\mathbf{n}$ be the unit outward normal of ∂M. Define the second fundamental form of ∂M by $\Pi(X, Y) = \langle \nabla_X \mathbf{n}, Y\rangle$ for any two tangent vector fields X and Y on M, and the mean curvature by $H = \operatorname{tr}(\Pi)$.

In this paper, we consider the diffusion-type operator on $(M^n, g, d\mu)$ as follows:

$$\mathcal{L}\cdot = \frac{1}{B}\operatorname{div}(A\nabla\cdot) = \frac{A}{B}\left(\Delta\cdot + \frac{1}{A}\langle\nabla\cdot, \nabla A\rangle\right), \tag{1.1}$$

where A and B are two positive smooth functions on M, ∇ denotes the Levi-Civita connection, $\operatorname{div} = \operatorname{tr}(\nabla\cdot)$ denotes the Riemannian divergence operator, and $\Delta = \operatorname{div}\nabla$ is the Laplace-Beltrami operator.

There are two trivial cases among all $\mathcal{L}$. For the case $A = B =$ constant, one sees from (1.1) that the diffusion-type operator $\mathcal{L}$ is the usual Laplacian. For the case $A = e^{(n-2)f}$, $B = e^{nf}$, $n \geq 2$, one sees from (1.1) that the diffusion-type operator $\mathcal{L}$ is in consistence with the usual ones for conformal metric $e^{2f}g$.

We note that in the case $A = B = e^{-f}$, one sees from (1.1) that

$$\mathcal{L}\cdot = e^f \operatorname{div}(e^{-f}\nabla\cdot) = \Delta\cdot - \langle\nabla\cdot, \nabla f\rangle. \tag{1.2}$$

We call (1.2) the Witten Laplacian (also called drifting, weighted or Bakry-Émery Laplacian) with respect to the weighted volume form $d\mu = e^{-f}dv$. In recent years, the Witten Laplacian received much attention from many mathematicians (see [1, 2, 4–8, 11–13, 18–20] and the references therein).

For the case $A = e^{-\alpha f}$, $B = e^{-f}$, $\alpha > 1$, one sees from (1.1) that

$$\mathcal{L}\cdot = e^f \operatorname{div}(e^{-\alpha f}\nabla\cdot) = e^{-f(\alpha-1)}(\Delta\cdot - \alpha\langle\nabla\cdot, \nabla f\rangle). \tag{1.3}$$

This is in fact the Laplacian with density in the literature which was introduced by Ndiaye [15].

We notice that the Green formula (the integration by parts formula) for the diffusion-type operator $\mathcal{L}$ holds under the weighted measure $d\mu = Bd\Omega$, that is,

$$\begin{aligned}\int_M h\mathcal{L}u\, d\mu &= \int_{\partial M} \frac{A}{B} h\partial_{\mathbf{n}}u\, d\mu_\partial - \int_M \frac{A}{B}\langle\nabla u, \nabla h\rangle\, d\mu \\ &= \int_{\partial M} \frac{A}{B}(h\partial_{\mathbf{n}}u - u\partial_{\mathbf{n}}h)\, d\mu_\partial + \int_M u\mathcal{L}h\, d\mu,\end{aligned}$$

holds provided u or h belongs to $C^\infty(M)$, where $\partial_{\mathbf{n}}u = \langle\mathbf{n}, \nabla u\rangle$, $d\mu_\partial = Bdv_\partial$ and dv_∂ is the volume form on ∂M.

Following [14], to relate $\mathcal{L}$ with geometry we consider the m-modified Ricci curvature $\widehat{\mathrm{Ric}}_{A,m}$ given by

$$\widehat{\mathrm{Ric}}_{A,m} = \mathrm{Ric} - \frac{1}{A}\nabla^2 A + \frac{m-n-1}{m-n}\frac{1}{A^2}dA \otimes dA, \tag{1.4}$$

where m is a real constant, and $m = n$ if and only if A is a constant. Here ∇^2 and Ric denote the Hessian operator and Ricci curvature. When $m = \infty$, (1.4) gives the tensor

$$\widehat{\mathrm{Ric}}_{A,\infty} = \mathrm{Ric} - \frac{1}{A}\nabla^2 A + \frac{1}{A^2}dA \otimes dA, \tag{1.5}$$

which is called ∞-modified Ricci curvature. The A-mean curvature of ∂M is defined by

$$\widehat{H}_A = H + \frac{1}{A}\partial_{\mathbf{n}}A. \tag{1.6}$$

It should be noticed that in the case $A = B = e^{-f}$, the m-modified Ricci curvature $\widehat{\mathrm{Ric}}_{A,m}$, the ∞-modified Ricci curvature $\widehat{\mathrm{Ric}}_{A,\infty}$ and the A-mean curvature $\widehat{H}_A$ become the m-Bakry–Émery Ricci curvature

$$\mathrm{Ric}_f^m = \mathrm{Ric}_f - \frac{1}{m-n}\nabla f \otimes \nabla f,$$

the ∞-Bakry–Émery Ricci curvature $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$ and the f-mean curvature $H_f = H - \langle \nabla f, \mathbf{n}\rangle$, respectively (see [17]).

Among the important formulae in differential geometry, the Reilly formula [16] is an important tool in the study of various geometric and analytical problems on a Riemannian manifold with smooth boundary. Ma and Du [13] extended the Reilly formula for the Witten Laplacian and applied it to study eigenvalue estimates for the Witten Laplacian on compact Riemannian manifolds with boundary. Kolesnikov and Milman [9, 10] obtained new Poincaré type inequalities for weighted manifolds by systematically using Ma-Du's Reilly-type formula combined with various conditions on the boundary of the manifold and boundary conditions for elliptic equations. Further more recent applications may be found in [3, 8, 21].

The purpose of this paper is to study some integral inequalities for the diffusion-type operator $\mathcal{L}$ and their applications on weighted manifolds with boundary. Firstly, we derive a Reilly type formula for the diffusion-type operator $\mathcal{L}$ on weighted manifolds with boundary, which is the important tool to prove our main theorems.

Theorem 1.1. *Let A and B be two positive smooth functions on a given compact weighted Riemannian manifold $(M^n, g, d\mu)$ of dimension $n \geq 2$ with the boundary ∂M. For any smooth function u, we have the following equality:*

$$\begin{aligned}\int_M \left[\frac{B}{A}|\mathcal{L}u|^2 - \frac{A}{B}|\nabla^2 u|^2\right] d\mu &= \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u)\, d\mu \\ &+ \int_{\partial M} \big(\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}}u)^2 + \mathcal{L}_\partial u \partial_{\mathbf{n}} u\big)\, d\mu_\partial \\ &+ \int_{\partial M} \frac{A}{B}(\Pi(\nabla_\partial u, \nabla_\partial u) - \langle \nabla_\partial u, \nabla_\partial(\partial_{\mathbf{n}} u)\rangle)\, d\mu_\partial. \qquad (1.7)\end{aligned}$$

Remark 1.2. Clearly, if $A = B =$ constant, our Reilly type formula (1.7) degenerates into the classical Reilly's formula in [16]; if $A = B = e^{-f}$, our formula (1.7) degenerates into the formula (3) of Ma and Du in [13]; if $A = e^{-\alpha f}$ and $B = e^{-f}$, our formula (1.7) degenerates into the formula (4) of Ndiaye in [15].

A simple computation shows that

$$\begin{aligned}&\frac{A}{B}|\nabla^2 u|^2 + \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u) \\ &= \frac{A}{B}|\nabla^2 u|^2 + \frac{A}{B}\left(\mathrm{Ric} - \frac{1}{A}\nabla^2 A + \frac{1}{A^2} dA \otimes dA\right)(\nabla u, \nabla u) \\ &= \frac{A}{B}|\nabla^2 u - \frac{\Delta u}{n} g|^2 + \frac{1}{m}\frac{A}{B}(\frac{B}{A}\mathcal{L}u)^2 + \frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u) \\ &+ \frac{A}{B}\left(\sqrt{\frac{m-n}{mn}}\Delta u - \sqrt{\frac{n}{m(m-n)}}\frac{1}{A}\langle \nabla u, \nabla A\rangle\right)^2 \\ &\geq \frac{1}{m}\frac{B}{A}(\mathcal{L}u)^2 + \frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u) \qquad (1.8)\end{aligned}$$

provided $m \in (-\infty, 0) \cup [n, +\infty)$. The equality holds if and only if

$$\nabla^2 u = \frac{\Delta u}{n} g \tag{1.9}$$

and

$$\Delta u - \frac{n}{\sqrt{(m-n)^2}} \frac{1}{A} \langle \nabla u, \nabla A \rangle = 0. \tag{1.10}$$

Inserting (1.8) into (1.1), we immediately obtain the following result:

Corollary 1.3. *Let A, B and u be as in Theorem 1.1 and $m \in (-\infty, 0) \cup [n, +\infty)$. Then we have the following inequalities:*

$$\begin{aligned} 0 \geq & \int_M \left(\frac{A}{B} \widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u) - \frac{m-1}{m} \frac{B}{A} |\mathcal{L} u|^2 \right) d\mu \\ & + \int_{\partial M} \left(\frac{A}{B} \widehat{H}_A (\partial_{\mathbf{n}} u)^2 + \mathcal{L}_\partial u \partial_{\mathbf{n}} u \right) d\mu_\partial \\ & + \int_{\partial M} \frac{A}{B} (\Pi(\nabla_\partial u, \nabla_\partial u) - \langle \nabla_\partial u, \nabla_\partial (\partial_{\mathbf{n}} u) \rangle) \, d\mu_\partial, \end{aligned} \tag{1.11}$$

where the equality occurs if and only if (1.9) and (1.10) hold.

Remark 1.4. In [12] (or see [9,10]), Li and Wei provide a Reilly type inequality for the Witten Laplacian and give some applications. In particular, if $A = B = e^{-f}$, then (1.11) becomes the formula (9) of Li and Wei in [12].

Throughout this work, we employ Einstein summation convention. By abuse of notation, $\widehat{\mathrm{Ric}}_{A,m}$ may denote the 2-covariant tensor $(\widehat{\mathrm{Ric}}_{A,m})_{pq}$, but also may denote its 1-contravariant version $(\widehat{\mathrm{Ric}}_{A,m})_p^q$, as in

$$\begin{aligned} \left\langle \widehat{\mathrm{Ric}}_{A,m} \nabla u, \nabla u \right\rangle &= g_{ij} \left(\widehat{\mathrm{Ric}}_{A,m} \right)_k^i \nabla^k u \nabla^j u \\ &= \left(\widehat{\mathrm{Ric}}_{A,m} \right)_{ij} \nabla^i u \nabla^j u = \widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u). \end{aligned}$$

Similarly, the 2-contravariant tensors $\left(\Pi^{-1}\right)^{\alpha\beta}$ and $\left(\left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1}\right)^{pq}$ are defined by

$$\left(\Pi^{-1}\right)^{ij} \Pi_{jk} = \delta_k^i, \quad \left(\left(\widehat{\mathrm{Ric}}_{A,m} \right)^{-1} \right)^{ij} (\widehat{\mathrm{Ric}}_{A,m})_{jk} = \delta_k^i.$$

Given an integrable function φ on $(M^n, g, d\mu)$, the dimensional mean-value and dimensional variance of φ on $(M^n, g, d\mu)$ are defined by

$$\overline{\varphi} = \frac{\displaystyle\int_M \frac{A}{B} \varphi \, d\mu}{\displaystyle\int_M \frac{A}{B} \, d\mu}, \qquad \mathrm{Var}_A(\varphi) = \int_M \frac{A}{B} (\varphi - \overline{\varphi})^2 \, d\mu.$$

Next, by applying the above Reilly type inequality (1.11), we obtain some new Poincaré type inequalities for the diffusion-type operator $\mathcal{L}$ on weighted Riemannian manifolds $(M^n, g, d\mu)$.

Theorem 1.5. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with $\widehat{\mathrm{Ric}}_{A,m} > 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Let A and B be two positive smooth functions on M. Then, for any $\varphi \in C^\infty(M)$, we have:*

(I) *Assume that $\partial M = \varnothing$ and $\int_M \frac{A}{B}\varphi \, d\mu = 0$. Then*

$$\frac{m}{m-1} \mathrm{Var}_A(\varphi) \leq \int_M \frac{A}{B} \left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1} (\nabla\varphi, \nabla\varphi) \, d\mu.$$

(II) *Assume that $\partial M \neq \varnothing$ and $\Pi \geq 0$ (M is locally convex). Then*

$$\frac{m}{m-1} \mathrm{Var}_A(\varphi) \leq \int_M \frac{A}{B} \left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1} (\nabla\varphi, \nabla\varphi) \, d\mu.$$

(III) *Assume that $\partial M \neq \varnothing$, $\widehat{H}_A \geq 0$ (M is generalized mean-convex), $\varphi \equiv 0$ on ∂M. Then*

$$\frac{m}{m-1} \int_M \frac{A}{B}\varphi^2 \, d\mu \leq \int_M \frac{A}{B} \left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1} (\nabla\varphi, \nabla\varphi) \, d\mu.$$

(IV) *Assume that $\partial M \neq \varnothing$, $\widehat{H}_A > 0$ (M is strictly generalized mean-convex). Then*

$$\frac{m}{m-1} \int_M \frac{A}{B}\varphi^2 \, d\mu \leq \int_M \frac{A}{B} \left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1} (\nabla\varphi, \nabla\varphi) \, d\mu + \int_{\partial M} \frac{A}{B} \frac{\varphi^2}{\widehat{H}_A} \, d\mu_\partial.$$

Remark 1.6. Particularly, when $A = B = e^{-f}$, then Theorem 1.5 reduces to Theorem 1.2 of Kolesnikov and Milman in [9].

By using the above Reilly type inequality (1.11), we also give the following Colesanti type inequalities on the boundary of weighted Riemannian manifolds $(M^n, g, d\mu)$.

Theorem 1.7. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $\widehat{\mathrm{Ric}}_{A,m} \geq \rho g$, where $\rho \in \mathbb{R}$ and $m \in (-\infty, 0) \cup [n, +\infty)$. Let A and B be two positive smooth functions on M. Assume that $\widehat{H}_A > 0$ on ∂M, then, for any $\psi \in C^\infty(\partial M)$, we have*

$$\int_{\partial M} \frac{A}{B}\Pi(\nabla_\partial\psi, \nabla_\partial\psi) \, d\mu_\partial \leq \int_{\partial M} \frac{A}{B\widehat{H}_A} \left(\frac{\rho}{2}\psi + \frac{B}{A}\mathcal{L}_\partial\psi\right)^2 d\mu_\partial. \tag{1.12}$$

We also obtain a dual-version of Theorem 1.7:

Theorem 1.8. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $\widehat{\mathrm{Ric}}_{A,m} \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M, then, for any $\psi \in C^\infty(\partial M)$, we have*

$$\int_{\partial M} \frac{A}{B}\Pi^{-1}(\nabla_\partial\psi, \nabla_\partial\psi) \, d\mu_\partial \geq \int_{\partial M} \frac{A}{B}\widehat{H}_A\psi^2 \, d\mu_\partial - \frac{m-1}{m} \frac{1}{V_A(M)} \left(\int_{\partial M} \frac{A}{B}\psi \, d\mu_\partial\right)^2, \tag{1.13}$$

where $V_A(M) = \int_M \frac{A}{B} \, d\mu$.

Remark 1.9. Particularly, when $A = B = e^{-f}$, then Theorem 1.7 and Theorem 1.8 reduce to Theorem 1.1 and Theorem 1.2 of Kolesnikov and Milman in [10].

In particular, taking $\psi = 1$ in (1.13), we obtain the following Minkowski type inequalities:

Theorem 1.10. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $\widehat{Ric}_{A,m} \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M, then*

$$\int_{\partial M} \frac{A}{B} \widehat{H}_A \, d\mu_\partial \leq \frac{m-1}{m} \frac{(V_A(\partial M))^2}{V_A(M)}, \tag{1.14}$$

where $V_A(\partial M) = \int_{\partial M} \frac{A}{B} \, d\mu_\partial$.

Using the Cauchy–Schwarz inequality

$$(V(\partial M))^2 \leq \int_{\partial M} \frac{A}{B} \widehat{H}_A \, d\mu_\partial \int_{\partial M} \frac{A}{B} \frac{1}{\widehat{H}_A} \, d\mu_\partial$$

in (1.14) gives the following Heintze–Karcher type inequalities:

Theorem 1.11. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $\widehat{\mathrm{Ric}}_{A,m} \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M, then*

$$\int_{\partial M} \frac{A}{B} \frac{1}{\widehat{H}_A} \, d\mu_\partial \geq \frac{m}{m-1} V_A(M). \tag{1.15}$$

On the other hand, we can replace the assumption $\Pi > 0$ in Theorem 1.11 by a weaker condition $\widehat{H}_A > 0$ to obtain the following theorem on ∂M.

Theorem 1.12. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $\widehat{\mathrm{Ric}}_{A,m} \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\widehat{H}_A > 0$ on ∂M, then*

$$\int_{\partial M} \frac{A}{B} \frac{1}{\widehat{H}_A} \, d\mu_\partial \geq \frac{m}{m-1} V_A(M). \tag{1.16}$$

Remark 1.13. Particularly, when $A = B = e^{-f}$, then Theorems 1.10-1.12 reduce to some previous results in [10, Theorem 4.4] and [4, Theorem 1.1].

This paper is organized as follows. In Section 2, we prove Theorem 1.1. Theorem 1.5 is proved in Section 3. Theorem 1.7 and Theorem 1.8 are proved in Section 4. In Section 5, we prove Theorem 1.12.

2. Proof of Theorem 1.1

In this section, we give the proof of our main tool, Theorem 1.1 from the Introduction. For the proof of Theorem 1.1, the following lemma will be used.

Lemma 2.1. *Let A, B and u be as in Theorem 1.1. We have*

$$\frac{1}{2}\mathcal{L}|\nabla u|^2 = \frac{A}{B}|\nabla^2 u|^2 + \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathcal{L}u\rangle + \frac{A}{B}\left\langle \nabla \frac{B}{A}, \nabla u\right\rangle \mathcal{L}u. \tag{2.1}$$

Proof. From the definition of $\mathcal{L}$ and the Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle,$$

we have

$$\begin{aligned}
\mathcal{L}|\nabla u|^2 &= \frac{A}{B}\Delta|\nabla u|^2 + \frac{1}{B}\langle \nabla A, \nabla|\nabla u|^2\rangle \\
&= 2\frac{A}{B}|\nabla^2 u|^2 + 2\frac{A}{B}\mathrm{Ric}(\nabla u, \nabla u) + 2\frac{A}{B}\langle \nabla u, \nabla \Delta u\rangle + \frac{1}{B}\langle \nabla A, \nabla|\nabla u|^2\rangle \\
&= 2\frac{A}{B}|\nabla^2 u|^2 + 2\frac{A}{B}\mathrm{Ric}(\nabla u, \nabla u) + 2\langle \nabla u, \nabla \mathcal{L}u\rangle + \frac{1}{B}\langle \nabla A, \nabla|\nabla u|^2\rangle \\
&\quad + 2\frac{A}{B}\mathcal{L}u\langle \nabla u, \nabla\frac{B}{A}\rangle - 2\frac{A}{B}\langle \nabla u, \nabla(\frac{1}{A}\langle \nabla u, \nabla A\rangle)\rangle.
\end{aligned} \tag{2.2}$$

By direct computations, we have

$$\frac{1}{B}\langle \nabla A, \nabla|\nabla u|^2\rangle = 2\frac{1}{B}\nabla^2 u(\nabla u, \nabla A) \tag{2.3}$$

and

$$\begin{aligned}
&-2\frac{A}{B}\left\langle \nabla u, \nabla\left(\frac{1}{A}\langle \nabla u, \nabla A\rangle\right)\right\rangle \\
&\qquad = 2\frac{1}{AB}\langle \nabla u, \nabla A\rangle^2 - 2\frac{1}{B}(\nabla^2 A(\nabla u, \nabla u) + \nabla^2 u(\nabla u, \nabla A)).
\end{aligned} \tag{2.4}$$

Inserting (2.3) and (2.4) into (2.2), we obtain (2.1). □

Notice that the Bochner-type formula (2.1) looks very similar to the Bochner formula for the Ricci tensor of an n-dimensional manifold. This is our motivation for the ∞-modified Ricci curvature $\widehat{\mathrm{Ric}}_{A,\infty}$.

Proof of Theorem 1.1. We integrate equality (2.1). On the left-hand side, we have

$$\begin{aligned}
\frac{1}{2}\int_M \mathcal{L}|\nabla u|^2\, d\mu &= \frac{1}{2}\int_M \frac{A}{B}\left(\Delta|\nabla u|^2 + \frac{1}{A}\langle \nabla A, \nabla|\nabla u|^2\rangle\right) d\mu \\
&= \frac{1}{2}\int_{\partial M} \partial_{\mathbf{n}}(|\nabla u|^2)A\, dv_{\partial} - \frac{1}{2}\int_M \langle \nabla|\nabla u|^2, \nabla A\rangle\, dv
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_M \frac{1}{B}\langle \nabla A, \nabla|\nabla u|^2\rangle \, d\mu \\
&= \int_{\partial M} \frac{A}{B}\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle \, d\mu_\partial. \qquad (2.5)
\end{aligned}
$$

The third and the fourth terms on the right-hand side give

$$
\begin{aligned}
\int_M &\left(\langle \nabla u, \nabla \mathcal{L}u\rangle + \frac{A}{B}\mathcal{L}u\left\langle \nabla\frac{B}{A}, \nabla u\right\rangle\right) d\mu \\
&= \int_M B\langle \nabla u, \nabla \mathcal{L}u\rangle \, dv + \int_M A\mathcal{L}u\left\langle \nabla\frac{B}{A}, \nabla u\right\rangle \, dv \\
&= \int_{\partial M} B\partial_{\mathbf{n}} u\mathcal{L}u \, dv_\partial - \int_M \Delta u\mathcal{L}uB \, dv - \int_M \langle \nabla u, \nabla B\rangle \mathcal{L}u \, dv \\
&\quad + \int_M \langle \nabla u, \nabla B\rangle \mathcal{L}u \, dv - \int_M \frac{B}{A}\langle \nabla u, \nabla A\rangle \mathcal{L}u \, dv \\
&= \int_{\partial M} B\partial_{\mathbf{n}} u\mathcal{L}u \, dv_\partial - \int_M B\Delta u\mathcal{L}u \, dv - \int_M \frac{B}{A}\langle \nabla u, \nabla A\rangle \mathcal{L}u \, dv \\
&= \int_{\partial M} \partial_{\mathbf{n}} u\mathcal{L}u \, d\mu_\partial - \int_M \frac{B}{A}|\mathcal{L}u|^2 \, d\mu. \qquad (2.6)
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\int_{\partial M} \frac{A}{B}\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle \, d\mu_\partial &= \int_M \frac{A}{B}|\nabla^2 u|^2 \, d\mu + \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u) \, d\mu \\
&\quad + \int_{\partial M} \partial_{\mathbf{n}} u\mathcal{L}u \, d\mu_\partial - \int_M \frac{B}{A}|\mathcal{L}u|^2 \, d\mu,
\end{aligned}
$$

that is,

$$
\begin{aligned}
\int_M \frac{B}{A}|\mathcal{L}u|^2 \, d\mu - \int_M \frac{A}{B}|\nabla^2 u|^2 \, d\mu &= \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u) \, d\mu \\
&\quad + \int_{\partial M} \partial_{\mathbf{n}} u\mathcal{L}u \, d\mu_\partial - \int_{\partial M} \frac{A}{B}\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle \, d\mu_\partial. \qquad (2.7)
\end{aligned}
$$

Now, it remains to estimate $\partial_{\mathbf{n}} u\mathcal{L}u - \frac{A}{B}\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle$ which is equal to

$$
\frac{A}{B}\left[\Delta u\partial_{\mathbf{n}} u + \frac{1}{A}\langle \nabla u, \nabla A\rangle \partial_{\mathbf{n}} u - \langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle\right].
$$

We notice that

$$
\Delta u = H\partial_{\mathbf{n}} u + \Delta_\partial u + \partial_{\mathbf{n}}^2 u \qquad (2.8)
$$

and

$$
\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle = \partial_{\mathbf{n}} u\partial_{\mathbf{n}}^2 u - \Pi(\nabla_\partial u, \nabla_\partial u) + \langle \nabla_\partial u, \nabla_\partial(\partial_{\mathbf{n}} u)\rangle. \qquad (2.9)
$$

We then combine equalities (2.8) and (2.9) to derive an expression for the last term in the right-hand side of (2.7)

$$
\partial_{\mathbf{n}} u\mathcal{L}u - \frac{A}{B}\langle \nabla u, \nabla(\partial_{\mathbf{n}} u)\rangle = \frac{A}{B}\left[(H\partial_{\mathbf{n}} u + \Delta_\partial u + \partial_{\mathbf{n}}^2 u)\partial_{\mathbf{n}} u + \frac{1}{A}\langle \nabla u, \nabla A\rangle \partial_{\mathbf{n}} u\right]
$$

$$
\begin{aligned}
&- \frac{A}{B}[\partial_{\mathbf{n}} u \partial_{\mathbf{n}}^2 u - \Pi(\nabla_\partial u, \nabla_\partial u) + \langle \nabla_\partial u, \nabla_\partial(\partial_{\mathbf{n}} u)\rangle] \\
= &\frac{A}{B}\left[(H\partial_{\mathbf{n}} u + \Delta_\partial u)\partial_{\mathbf{n}} u + \frac{1}{A}\langle \nabla u, \nabla A\rangle \partial_{\mathbf{n}} u\right] \\
&- \frac{A}{B}[-\Pi(\nabla_\partial u, \nabla_\partial u) + \langle \nabla_\partial u, \nabla_\partial(\partial_{\mathbf{n}} u)\rangle].
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\frac{A}{B}\left[(H\partial_{\mathbf{n}} u + \Delta_\partial u)\partial_{\mathbf{n}} u + \frac{1}{A}\langle \nabla u, \nabla A\rangle \partial_{\mathbf{n}} u\right] \\
&= \frac{A}{B}\left[(H\partial_{\mathbf{n}} u + \Delta_\partial u)\partial_{\mathbf{n}} u + \frac{1}{A}(\langle \nabla_\partial u, \nabla_\partial A\rangle + \partial_{\mathbf{n}} u \partial_{\mathbf{n}} A)\partial_{\mathbf{n}} u\right] \\
&= \frac{A}{B}\left[H(\partial_{\mathbf{n}} u)^2 + \frac{1}{A}(\partial_{\mathbf{n}} u)^2 \partial_{\mathbf{n}} A + \Delta_\partial u \partial_{\mathbf{n}} u + \frac{1}{A}\langle \nabla_\partial u, \nabla_\partial A\rangle \partial_{\mathbf{n}} u\right] \\
&= \frac{A}{B}\left(H + \frac{1}{A}\partial_{\mathbf{n}} A\right)(\partial_{\mathbf{n}} u)^2 + \frac{A}{B}\left(\Delta_\partial u + \frac{1}{A}\langle \nabla_\partial u, \nabla_\partial A\rangle\right)\partial_{\mathbf{n}} u \\
&= \frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2 + \mathcal{L}_\partial u \partial_{\mathbf{n}} u.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_M \frac{B}{A}|\mathcal{L}u|^2\,d\mu - \int_M \frac{A}{B}|\nabla^2 u|^2\,d\mu = &\int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,\infty}(\nabla u, \nabla u)\,d\mu \\
&+ \int_{\partial M}(\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2 + \mathcal{L}_\partial u \partial_{\mathbf{n}} u)\,d\mu_\partial \\
&+ \int_{\partial M}\frac{A}{B}(\Pi(\nabla_\partial u, \nabla_\partial u) - \langle \nabla_\partial u, \nabla_\partial(\partial_{\mathbf{n}} u)\rangle)\,d\mu_\partial.
\end{aligned}
$$

This completes the proof. □

3. Proof of Theorem 1.5

The idea in the proof of Theorem 1.5 is similar to that used by Kolesnikov and Milman in [9]. We use the Reilly type inequality (1.11) to prove Theorem 1.5 below.

Proof. (I) We solve PDE

$$
\frac{B}{A}\mathcal{L}u = \varphi \quad \text{on } M. \tag{3.1}
$$

Thus, it follows from (1.11) that

$$
\int_M \frac{m-1}{m}\frac{B}{A}|\mathcal{L}u|^2\,d\mu \geq \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u)\,d\mu \tag{3.2}
$$

and

$$
\int_M \frac{m-1}{m}\frac{B}{A}|\mathcal{L}u|^2\,d\mu = \int_M \frac{m-1}{m}\frac{B}{A}\left(\frac{A}{B}\varphi\right)^2 d\mu = \frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\,d\mu. \tag{3.3}
$$

Then combining (3.2) and (3.3), we have

$$\frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\,d\mu \geq \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u,\nabla u)\,d\mu. \tag{3.4}$$

Using the divergence theorem, we have

$$\begin{aligned}\int_M \frac{A}{B}\varphi^2 d\mu &= \int_M B\varphi\mathcal{L}u\,dv = -\int_M \frac{A}{B}\langle\nabla\varphi,\nabla u\rangle\,d\mu\\ &\leq \left(\int_M \frac{A}{B}\langle\widehat{\mathrm{Ric}}_{A,m}\nabla u,\nabla u\rangle\,d\mu\right)^{\frac12}\left(\int_M \frac{A}{B}\langle(\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu\right)^{\frac12}\\ &\leq \left(\frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\,d\mu\right)^{\frac12}\left(\int_M \frac{A}{B}\langle(\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu\right)^{\frac12}. \end{aligned}\tag{3.5}$$

Thus,

$$\frac{m}{m-1}\int_M \frac{A}{B}\varphi^2\,d\mu \leq \int_M \frac{A}{B}\left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1}(\nabla\varphi,\nabla\varphi)\,d\mu.$$

By the assumption that $\int_M \frac{A}{B}\varphi\,d\mu = 0$, we obtain the assertion of Case (I).

(II) Since M^n is compact, by integration by parts, we have

$$\int_{\partial M}\partial_{\mathbf{n}}u\mathcal{L}_\partial u\,d\mu_\partial = -\int_{\partial M}\frac{A}{B}g(\nabla_\partial u,\nabla_\partial\partial_{\mathbf{n}}u)\,d\mu_\partial. \tag{3.6}$$

By (1.11) and (3.6), we can get

$$\begin{aligned}0 \geq &\int_M\left(\frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u,\nabla u) - \frac{m-1}{m}\frac{B}{A}|\mathcal{L}u|^2\right)d\mu\\ &+\int_{\partial M}\frac{A}{B}[\Pi(\nabla_\partial u,\nabla_\partial u) - 2g(\nabla_\partial u,\nabla_\partial\partial_{\mathbf{n}}u)]\,d\mu_\partial\\ &+\int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}}u)^2\,d\mu_\partial.\end{aligned}\tag{3.7}$$

Let u be a smooth solution to the Neumann problem

$$\begin{cases}\dfrac{B}{A}\mathcal{L}u = \varphi & \text{on } M,\\ \partial_{\mathbf{n}}u \equiv 0 & \text{on } \partial M.\end{cases}\tag{3.8}$$

Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_M \frac{A}{B}\varphi^2\,d\mu &= \int_M \frac{B}{A}|\mathcal{L}u|^2\,d\mu\\ &= -\int_M \frac{A}{B}\langle\nabla\varphi,\nabla u\rangle\,d\mu + \int_{\partial M}\frac{A}{B}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial\\ &\leq \left(\int_M \frac{A}{B}\langle\widehat{\mathrm{Ric}}_{A,m}\nabla u,\nabla u\rangle\,d\mu\right)^{\frac12}\left(\int_M \frac{A}{B}\langle(\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu\right)^{\frac12}\\ &\quad+\int_{\partial M}\frac{A}{B}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial.\end{aligned}\tag{3.9}$$

Since $\partial_{\mathbf{n}} u\big|_{\partial M} \equiv 0$ and $\Pi \geq 0$, we obtain from (3.7) that

$$\int_M \frac{B}{A}|\mathcal{L}u|^2\, d\mu \geq \frac{m-1}{m}\int_M \frac{A}{B}\left\langle \widehat{\mathrm{Ric}}_{A,m}\nabla u, \nabla u\right\rangle d\mu. \tag{3.10}$$

Consequently, we obtain

$$\frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\, d\mu \geq \int_M \frac{A}{B}\left\langle \widehat{\mathrm{Ric}}_{A,m}\nabla u, \nabla u\right\rangle d\mu. \tag{3.11}$$

Plugging this back into (3.9) and using that $\partial_{\mathbf{n}} u\big|_{\partial M} \equiv 0$ yields

$$\frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\, d\mu \leq \int_M \frac{A}{B}\left\langle \left(\widehat{\mathrm{Ric}}_{A,m}\right)^{-1}\nabla \varphi, \nabla \varphi\right\rangle d\mu.$$

By the fact that

$$\int_M \frac{A}{B}\varphi\, d\mu = \int_M \mathcal{L}u\, d\mu = \int_{\partial M} \frac{A}{B}\partial_{\mathbf{n}} u\, d\mu_\partial = 0,$$

we obtain the assertion of Case (II).

(III) Let u be a smooth solution to the Dirichlet problem

$$\begin{cases} \dfrac{B}{A}\mathcal{L}u = \varphi & \text{on } M, \\ u \equiv 0 & \text{on } \partial M. \end{cases} \tag{3.12}$$

Observe that (3.11) still holds since $u \equiv 0$ and $H_f^\alpha \geq 0$. Plugging (3.11) back into (3.9) and using that $\varphi\big|_{\partial M} \equiv 0$ yields the assertion of Case (III).

(IV) Let u be a smooth solution to the Dirichlet problem (3.12). If $\widehat{H}_A > 0$, by (3.7), we have

$$\frac{m-1}{m}\int_M \frac{A}{B}\varphi^2\, d\mu \geq \int_M \frac{A}{B}\widehat{\mathrm{Ric}}_{A,m}(\nabla u, \nabla u)\, d\mu + \int_{\partial M} \frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial. \tag{3.13}$$

On the other hand, we obtain for any $\varepsilon > 0$:

$$\begin{aligned}\int_M \frac{A}{B}\varphi^2\, d\mu &= -\int_M \frac{A}{B}\langle \nabla\varphi, \nabla u\rangle\, d\mu + \int_{\partial M}\frac{A}{B}\varphi\partial_{\mathbf{n}} u\, d\mu_\partial \\ &\leq \frac{\varepsilon}{2}\int_M \frac{A}{B}\langle \widehat{\mathrm{Ric}}_{A,m}\nabla u, \nabla u\rangle\, d\mu \\ &\quad + \frac{1}{2\varepsilon}\int_M \frac{A}{B}\langle (\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi, \nabla\varphi\rangle\, d\mu + \int_{\partial M}\frac{A}{B}\varphi\partial_{\mathbf{n}} u\, d\mu_\partial. \end{aligned} \tag{3.14}$$

By (3.13) and (3.14), using the Cauchy–Schwarz inequality, we can get

$$\begin{aligned}\left(1 - \frac{\varepsilon}{2}\frac{m-1}{m}\right)\int_M \frac{A}{B}\varphi^2\, d\mu &\leq \frac{1}{2\varepsilon}\int_M \frac{A}{B}\langle (\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi, \nabla\varphi\rangle\, d\mu \\ &\quad - \frac{\varepsilon}{2}\int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial + \int_{\partial M}\frac{A}{B}\varphi\partial_{\mathbf{n}} u\, d\mu_\partial \\ &\leq \frac{1}{2\varepsilon}\int_M \frac{A}{B}\langle (\widehat{\mathrm{Ric}}_{A,m})^{-1}\nabla\varphi, \nabla\varphi\rangle\, d\mu + \frac{1}{2\varepsilon}\int_{\partial M}\frac{A}{B}\frac{\varphi^2}{\widehat{H}_A}\, d\mu_\partial. \end{aligned}$$

Multiplying by 2ε and using the optimal $\varepsilon = \frac{m-1}{m}$, we obtain the assertion of Case (IV). This completes the proof. □

4. Proof of Theorems 1.7 and 1.8

We use the Reilly type inequality to prove Theorem 1.7 below.

Proof of Theorem 1.7. Let u be a smooth solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{on } M, \\ u \equiv \psi & \text{on } \partial M. \end{cases} \tag{4.1}$$

By (1.11), we have

$$\begin{aligned} 0 \geq \rho \int_M \frac{A}{B}|\nabla u|^2 \, d\mu &+ \int_{\partial M} \frac{A}{B}\left(\frac{B}{A}\mathcal{L}_\partial \psi + \widehat{H}_A \partial_{\mathbf{n}} u\right) \partial_{\mathbf{n}} u \, d\mu_\partial \\ &+ \int_{\partial M} \frac{A}{B}[\Pi(\nabla_\partial \psi, \nabla_\partial \psi) - g(\nabla_\partial \psi, \nabla_\partial \partial_{\mathbf{n}} u)] \, d\mu_\partial. \end{aligned}$$

By (3.6), we have

$$\begin{aligned} 0 \geq \rho \int_M \frac{A}{B}|\nabla u|^2 \, d\mu &+ \int_{\partial M} \frac{A}{B}\widehat{H}_A (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial \\ &+ \int_{\partial M} \frac{A}{B}\Pi(\nabla_\partial \psi, \nabla_\partial \psi) \, d\mu_\partial + 2\int_{\partial M} \partial_{\mathbf{n}} u \mathcal{L}_\partial \psi \, d\mu_\partial. \end{aligned} \tag{4.2}$$

On the other hand, note that

$$\int_M \frac{A}{B}|\nabla u|^2 \, d\mu = \int_{\partial M} \frac{A}{B} u \partial_{\mathbf{n}} u \, d\mu_\partial - \int_M u \mathcal{L} u \, d\mu.$$

It follows from (4.1) that

$$\int_M \frac{A}{B}|\nabla u|^2 \, d\mu = \int_{\partial M} \frac{A}{B}\psi \partial_{\mathbf{n}} u \, d\mu_\partial. \tag{4.3}$$

By (4.2) and (4.3), using the inequality $Ax^2 + Bx > -\frac{B^2}{4A}$ with $A > 0$, we can get

$$\begin{aligned} &\int_{\partial M} \frac{A}{B}\Pi(\nabla_\partial \psi, \nabla_\partial \psi) \, d\mu_\partial \\ &\qquad \leq -\rho \int_{\partial M} \frac{A}{B}\psi \partial_{\mathbf{n}} u \, d\mu_\partial - \int_{\partial M} \frac{A}{B}\widehat{H}_A (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial - 2\int_{\partial M} \partial_{\mathbf{n}} u \mathcal{L}_\partial \psi \, d\mu_\partial \\ &\qquad = -\int_{\partial M} \left(\frac{A}{B}\widehat{H}_A (\partial_{\mathbf{n}} u)^2 + \rho \frac{A}{B}\psi \partial_{\mathbf{n}} u + 2\partial_{\mathbf{n}} u \mathcal{L}_\partial \psi\right) d\mu_\partial \\ &\qquad = -\int_{\partial M} \left(\frac{A}{B}\widehat{H}_A (\partial_{\mathbf{n}} u)^2 + \left(\rho \frac{A}{B}\psi + 2\mathcal{L}_\partial \psi\right) \partial_{\mathbf{n}} u\right) d\mu_\partial \\ &\qquad \leq \int_{\partial M} \frac{\frac{A}{B}}{\widehat{H}_A}\left(\frac{\rho}{2}\psi + \frac{B}{A}\mathcal{L}_\partial \psi\right)^2 d\mu_\partial. \end{aligned}$$

This completes the proof. □

Next we use the Reilly type inequality to prove Theorem 1.8 below.

Proof of Theorem 1.8. By the Cauchy–Schwarz inequality, we have

$$-2\frac{A}{B}g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u) \geq -\frac{A}{B}\Pi(\nabla_\partial u, \nabla_\partial u) - \frac{A}{B}\Pi^{-1}(\nabla_\partial \partial_{\mathbf{n}} u, \nabla_\partial \partial_{\mathbf{n}} u). \tag{4.4}$$

From (3.7) and (4.4), we have

$$\begin{aligned}\int_M \frac{m-1}{m}\frac{B}{A}|\mathcal{L}u|^2\, d\mu \geq & \int_{\partial M}\frac{A}{B}[\Pi(\nabla_\partial u, \nabla_\partial u) + \widehat{H}_A(\partial_{\mathbf{n}} u)^2]\, d\mu_\partial \\ & - 2\int_{\partial M}\frac{A}{B}g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)\, d\mu_\partial \\ \geq & \int_{\partial M}\frac{A}{B}[\Pi(\nabla_\partial u, \nabla_\partial u) + \widehat{H}_A(\partial_{\mathbf{n}} u)^2]\, d\mu_\partial \\ & - \int_{\partial M}\frac{A}{B}\Pi^{-1}(\nabla_\partial \partial_{\mathbf{n}} u, \nabla_\partial \partial_{\mathbf{n}} u)\, d\mu_\partial - \int_{\partial M}\frac{A}{B}\Pi(\nabla_\partial u, \nabla_\partial u)\, d\mu_\partial \\ = & \int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial - \int_{\partial M}\frac{A}{B}\Pi^{-1}(\nabla_\partial \partial_{\mathbf{n}} u, \nabla_\partial \partial_{\mathbf{n}} u)\, d\mu_\partial. \end{aligned} \tag{4.5}$$

Let u be a smooth solution to the Neumann problem

$$\begin{cases} \frac{B}{A}\mathcal{L}u = \frac{1}{V_A(M)}\int_{\partial M}\frac{A}{B}\psi\, d\mu_\partial & \text{on } M, \\ \partial_{\mathbf{n}} u \equiv \psi & \text{on } \partial M. \end{cases} \tag{4.6}$$

By (4.5) and (4.6), we have

$$\begin{aligned}\int_M \frac{m-1}{m}\frac{A}{B}\left(\frac{1}{V_A(\partial M)}\int_{\partial M}\frac{A}{B}\psi\, d\mu_\partial\right)^2 d\mu \\ \geq \int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial - \int_{\partial M}\frac{A}{B}\Pi^{-1}(\nabla_\partial \partial_{\mathbf{n}} u, \nabla_\partial \partial_{\mathbf{n}} u)\, d\mu_\partial.\end{aligned}$$

This completes the proof. □

5. Proof of Theorem 1.12

Proof. Let u be a smooth solution to the Dirichlet problem

$$\begin{cases} \frac{B}{A}\mathcal{L}u = 1 & \text{on } M, \\ u \equiv 0 & \text{on } \partial M. \end{cases} \tag{5.1}$$

By (3.7), we can get

$$\frac{m-1}{m}V_A(M) = \int_M \frac{m-1}{m}\frac{B}{A}|\mathcal{L}u|^2\, d\mu \geq \int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial. \tag{5.2}$$

On the other hand, note that

$$\begin{aligned}(V_A(M))^2 = \left(\int_M \frac{B}{A}|\mathcal{L}u|^2\, d\mu\right)^2 = \left(\int_{\partial M}\frac{A}{B}\partial_{\mathbf{n}} u\, d\mu_\partial\right)^2 \\ \leq \int_{\partial M}\frac{A}{B}\widehat{H}_A(\partial_{\mathbf{n}} u)^2\, d\mu_\partial \int_{\partial M}\frac{A}{B}\frac{1}{\widehat{H}_A}\, d\mu_\partial.\end{aligned} \tag{5.3}$$

By (5.2) and (5.3), the assertion follows. □

Acknowledgments. This work is supported by NSFC (No. 11971415), Natural Science Foundation of Henan (No. 212300410235), the Key Scientific Research Program in Universities of Henan Province (Nos. 21A110021, 22A110021), Nanhu Scholars Program for Young Scholars of XYNU (No. 2019), and Xinyang Normal University Graduate Student Innovation Fund Project (2024KYJJ061).

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Received November 25, 2022.

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Інтегральна формула типу Рейлі, пов'язана з операторами дифузійного типу, та її застосування

Fanqi Zeng, Huiting Chang, and Yujun Sun

У цій статті ми виводимо формулу типу Рейлі для оператора дифузійного типу $\mathcal{L}\cdot = \frac{1}{B}\operatorname{div}(A\nabla\cdot)$ на зважених многовидах із межею, де A і B — дві додатні гладкі функції на многовидах. В якості її застосування наведено деякі нерівності типу Пуанкаре, Колесанті, Мінковського та Хайнце–Карчера.

Ключові слова: формула типу Рейлі, оператор дифузійного типу, m-модифікована кривина Річчі, A-середня кривина