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Generalized Fourier Quasicrystals, Almost Periodic Sets, and Zeros of Dirichlet Series

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Let S be an absolutely convergent Dirichlet series with bounded spectrum and a real zero set A, let μ be the sum of the unit masses at the points of the set A. The main result of the paper states that the Fourier transform of μ in the sense of distributions is a pure point measure. Conversely, given a sequence A of real points, a sufficient condition on the Fourier transform of μ is found for A to be the zero set of an absolutely convergent Dirichlet series with bounded spectrum, besides a criterion on the Fourier transform of μ is found for A to be the zero set of an almost periodic entire function of exponential growth. These results are based on a new representation of almost periodic sets.

Key words: Fourier quasicrystal, Fourier transform in the sense of distributions, pure point measure, almost periodic entire function, almost periodic set, zero set of an entire function

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1. Introduction

A crystalline measure on $\mathbb{R}^d$ is a complex measure μ, which is a tempered distribution, and both μ and its distributional Fourier transform $\hat{\mu}$ are measures with discrete locally finite support. When, in addition, both $|\mu|$ and $|\hat{\mu}|$ are tempered distributions, μ is called a Fourier quasicrystal.

Fourier quasicrystals can be considered as mathematical models for atomic arrangement having a discrete diffraction pattern. There are a lot of papers devoted to the study of properties of Fourier quasicrystals or, more generally, crystalline measures. See, for example, papers [2, 19] and, in particular, the fundamental paper [14].

Measures of the form

$$\mu = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \qquad c_\lambda \in \mathbb{N}, \tag{1.1}$$

constitute the most important case of a Fourier quasicrystal. In [10], P. Kurasov and P. Sarnak discovered the existence of nontrivial measures of the form 1.1 with a uniformly discrete support, whose Fourier transform is an atomic measure with locally finite support. A complete description of these measures was given by A. Olevsky and A. Ulanovsky in [17,18]. Namely, they showed that the supports Λ of these measures are precisely the zero sets of exponential polynomials with pure imaginary exponents and only real zeros with multiplicities c_λ at points $\lambda \in \Lambda$. Conversely, zero sets of such exponential polynomials are the supports of some Fourier quasicrystals of the form (1.1).

In this paper, using the same methods, we present similar results for measures (1.1) with the distribution Fourier transform

$$\hat{\mu} = \sum_{\gamma\in\Gamma} b_\gamma \delta_\gamma, \tag{1.2}$$

where Γ is an arbitrary countable set. In this case, the corresponding Poisson's formula

$$\sum_{\lambda\in\Lambda} c_\lambda \hat{f}(\lambda) = \sum_{\gamma\in\Gamma} b_\gamma f(\gamma)$$

holds for every function f from Schwartz' class. For the description of these measures, we apply the concept of almost periodic sets due to M. Krein and B. Levin [12, Appendix VI]. In modern notation (cf. [15,20]), a locally finite set Λ with multiplicities c_λ at points $\lambda \in \Lambda$ is almost periodic if the convolution of measure (1.1) with every continuous function with compact support is an almost periodic function. Therefore, almost periodic sets are in fact multisets. We write an almost periodic set as a sequence $A = \{a_n\}_{n\in\mathbb{Z}}$, where each point $a_n = \lambda$ occurs c_λ times.

In Section 2, we give the original definition of almost periodic sets due to Krein and Levin, which is equivalent to the above one. We also establish some properties of almost periodic sets. In particular, we show that such sets have the form $\{\alpha n + \phi(n)\}_{n\in\mathbb{Z}}$ with $\alpha > 0$ and an almost periodic mapping $\phi : \mathbb{Z} \to \mathbb{R}$.

In Section 3, we consider an absolutely convergent Dirichlet series with bounded spectrum and a real set of zeros $A = \{a_n\}_{n\in\mathbb{Z}}$. We prove that the Fourier transform $\hat{\mu}_A$ of the corresponding measure $\mu_A = \sum_n \delta_{a_n}$ is always a pure point measure. Notice that the zero set of any absolutely convergent Dirichlet series (or, more generally, of any holomorphic function with almost periodic modulus) is almost periodic (cf. [8]).

In Section 4, we study the inverse problem. Given a locally finite set $A = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$, let $\mu_A = \sum_n \delta_{a_n}$ of form (1.1) with the Fourier transform $\hat{\mu}_A$ of form (1.2). We show that the conditions

i) $|\hat{\mu}_A|$ is a tempered distribution, and
ii) $\int_0^1 s^{-2}|\hat{\mu}_A|(0,s)ds < \infty$

imply that A is the zero set of an absolutely convergent Dirichlet series with bounded spectrum.

By [5, Lemma 1], the first condition implies that the multiset $A = \{a_n\}_{n\in\mathbb{Z}}$, where each point $a_n = \lambda$ occurs c_λ times, is almost periodic. As it was proven in [8], every almost periodic set $A \subset \mathbb{R}$ is exactly the zero set of some entire almost periodic function. Every almost periodic function is bounded on the real line and, according to the Phragmen–Lindelöf principle, any almost periodic entire function grows at least exponentially fast. In Section 4, we find a criterion for A to be the zero set of an almost periodic entire function of the exponential growth through Γ and β_γ from equality (1.2).

2. Almost periodic sets

Definition 2.1 (see [1, 13])**.** A continuous function $g(x)$ on the real line is almost periodic if for any $\varepsilon > 0$ the set of ε-almost periods

$$E_\varepsilon = \{\tau \in \mathbb{R} : \sup_{x\in\mathbb{R}} |g(x+\tau) - g(x)| < \varepsilon\}$$

is relatively dense, i.e., $E_\varepsilon \cap (x, x+L) \neq \varnothing$ for all $x \in \mathbb{R}$ and some L depending on ε.

For example, every sum

$$Q(x) = \sum q_n e^{2\pi i x\omega_n}, \quad \omega_n \in \mathbb{R},\ q_n \in \mathbb{C},\ \sum_n |q_n| < \infty,$$

is an almost periodic function.

Spectrum of an almost periodic function g is the set

$$\operatorname{sp} g = \left\{ w \in \mathbb{R} : \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-2\pi i\omega x} g(x) dx \neq 0 \right\}.$$

It is easy to see that $\operatorname{sp} Q(x) = \{\omega_n : q_n \neq 0\}$. Notice that spectra of almost periodic functions are at most countable set.

Definition 2.2 (see [1, 13])**.** A continuous function $g(z)$ on the strip

$$S = \{z = x + iy : -\infty \le a < y < b \le +\infty\} \subset \mathbb{C}$$

is almost periodic if for any α, β such that $[\alpha, \beta] \subset (a, b)$ and $\varepsilon > 0$ the set of ε-almost periods

$$E_{\alpha,\beta,\varepsilon} = \{\tau \in \mathbb{R} : \sup_{x\in\mathbb{R}, \alpha\le y\le\beta} |g(x+\tau+iy) - g(x+iy)| < \varepsilon\}$$

is relatively dense, i.e., $E_{\alpha,\beta,\varepsilon} \cap (x, x+L) \neq \varnothing$ for all $x \in \mathbb{R}$ and some L depending on $\varepsilon, \alpha, \beta$.

The first definition of almost periodic sets due to M. Krein and B. Levin appeared in [12]. Here we give this definition in a simplified form. The most general definition, where the behavior of A near the boundary is taken into account, was given by H. Tornehave in [21]. For the connection between almost periodic sets of general form and zeros of holomorphic almost periodic functions in terms of Chern cohomology, see [3].

Definition 2.3 ([12, Appendix VI]). Let S be a horizontal strip of finite width. A discrete locally finite multiset $A = \{a_n\}_{n\in\mathbb{Z}} \subset S$ is called almost periodic if for any $\varepsilon > 0$ there is $L_\varepsilon < \infty$ such that the set of its ε-almost periods

$$E_\varepsilon = \{\tau \in \mathbb{R} : \exists\, \sigma : \mathbb{Z} \to \mathbb{Z} \quad \sigma \text{ is a bijection and } \sup_n |a_n + \tau - a_{\sigma(n)}| < \varepsilon\} \quad (2.1)$$

has a nonempty intersection with every interval $(x, x + L_\varepsilon)$.

In our paper, we primarily consider the case of almost periodic sets on the real line.

Set $\mu_A = \sum_n \delta_{a_n}$. Clearly, the mass of μ_A in any point $x \in \mathbb{R}$ is equal to the multiplicity of this point in the sequence $\{a_n\}_{n\in\mathbb{Z}}$.

It was proven in [8] that almost periodicity of A is equivalent to the almost periodicity of the convolution $\mu_A \star \varphi$ for every C^∞-function $\varphi(x)$, $x \in \mathbb{R}$, with compact support. It is easy to see that C^∞-functions can be replaced by continuous functions with compact support. Indeed, let us take a C^∞-function $\varphi \geq 0$ such that $\varphi(x) \equiv 1$ for $0 < x < 1$. If $\mu_A \star \varphi$ is almost periodic, then it is uniformly bounded, hence $\mu_A[x, x+1] < k_1$ for all $x \in \mathbb{R}$ with some constant k_1. For any continuous function ψ with support in $(0, 1)$ we can take $\varphi \in C^\infty$ such that $\sup_{x\in\mathbb{R}} |\psi(x) - \varphi(x)| < \varepsilon/k_1$. We obtain that every ε-almost period of $\mu_A \star \varphi$ is the 2ε-almost period of $\mu_A \star \psi$.

In fact, we have proven the following proposition

Proposition 2.4 ([8]). *For any almost periodic set there exists $k_1 \in \mathbb{N}$ such that $\#A \cap [x, x+1] \leq k_1$. In addition, $\#A \cap [x, x+h) \leq k_1(h+1)$.*

Here and below, $\#H$ means the number of points in the multiset H, taking into account their multiplicities.

Proposition 2.5. *For any almost periodic set there exists $k_2 \in \mathbb{N}$ such that for every $h > 0$ and every half-intervals $[x_1, x_1 + h)$, $[x_2, x_2 + h)$ we have*

$$|\#A \cap [x_1, x_1 + h) - \#A \cap [x_2, x_2 + h)| \leq k_2.$$

Also, for every $x \in \mathbb{R}$, $h > 0$, $M \in \mathbb{N}$,

$$|\#A \cap [x, x+h) - (1/M)\#A \cap [x, x + Mh)| \leq k_2.$$

Proof. Let L_1, E_1 be defined in (2.1), and $\tau \in E_1 \cap [x_1 - x_2, L_1 + x_1 - x_2)$. Since

$$[x_2, x_2 + h) + \tau \subset [x_1, x_1 + L_1 + h),$$

we see that each $a_n \in [x_2, x_2 + h)$ can be associated with the point $a_{\sigma(n)} \in [x_1 - 1, x_1 + L_1 + h + 1)$. Therefore,

$$\#A\cap[x_2, x_2+h) \leq \#A\cap[x_1, x_1+h)+\#A\cap[x_1-1, x_1)+\#A\cap[x_1+h, x_1+h+L_1+1).$$

By Proposition 2.4, the last two terms are bounded by $k_1 + (L_1 + 2)k_1$. The opposite inequality can be proved in a similar way.

To prove the second statement we have to sum up all the inequalities

$$\#A \cap [x, x+h) - k_2 \le \#A \cap [x+(m-1)h, x+mh) \le k_2 + \#A \cap [x, x+h)$$

for $m = 1, 2, \dots, M$. □

Proposition 2.6. *Let A be an almost periodic set. There is a strictly positive density d such that for any $\eta > 0$ and any half-interval I with length $l(I) > N_\eta$ we have*

$$\left| \frac{\#A \cap I}{l(I)} - d \right| < \eta.$$

Proof. Let

$$I_1 = [x_1, x_1 + h_1), \quad I_2 = [x_2, x_2 + h_2)$$

be two half-intervals such that $h_1/h_2 = p/q$, $p, q \in \mathbb{N}$. We have

$$\frac{\#A \cap I_1}{h_1} - \frac{\#A \cap I_2}{h_2} = \frac{\#A \cap I_1}{h_1} - \frac{\#A \cap qI_1}{qh_1} + \frac{\#A \cap qI_1}{qh_1} - \frac{\#A \cap pI_2}{ph_2} + \frac{\#A \cap pI_2}{ph_2} - \frac{\#A \cap I_2}{h_2}.$$

Applying Proposition 2.5, we get

$$\left| \frac{\#A \cap I_1}{h_1} - \frac{\#A \cap I_2}{h_2} \right| \le \frac{k_2}{h_1} + \frac{k_2}{qh_1} + \frac{k_2}{h_2} \le k_2 \left(\frac{2}{h_1} + \frac{1}{h_2} \right). \tag{2.2}$$

For arbitrary h_1, h_2 take a half-interval $I' = [x_1, x_1 + h')$ such that $h_1 < h' < h_1 + 1$ and h'/h_2 rational. We have

$$\left| \frac{\#A \cap I_1}{h_1} - \frac{\#A \cap I'}{h'} \right| \le \frac{\#A \cap [x_1 + h_1, x_1 + h')}{h'} + \frac{\#A \cap I_1}{h_1 h'}.$$

By Proposition 2.4, we get

$$\left| \frac{\#A \cap I_1}{h_1} - \frac{\#A \cap I'}{h'} \right| \le \frac{k_1}{h'} + \frac{k_1(h_1 + 1)}{h_1 h'}.$$

Applying (2.2) with I' instead of I_1, for all I_1, I_2, we obtain

$$\left| \frac{\#A \cap I_1}{l(I_1)} - \frac{\#A \cap I_2}{l(I_2)} \right| \le k_2 \left(\frac{2}{l(I_1)} + \frac{1}{l(I_2)} \right) + k_1 \left(\frac{2}{l(I_1)} + \frac{1}{l(I_1)^2} \right).$$

Therefore the limit exists

$$d = \lim_{l(I) \to \infty} \frac{\#A \cap I}{l(I)}.$$

It is easy to check that the set A has nonempty intersection with every interval of length $2 + L_1$, hence this limit is strictly positive. □

This result was generalized to multidimensional Euclidean spaces in [7].

Theorem 2.7. *Let $A = \{a_n\} \subset \mathbb{R}$ be an almost periodic set of density d such that $a_n \le a_{n+1}$ for all $n \in \mathbb{Z}$. Then*

$$a_n = n/d + \phi(n) \quad \text{with an almost periodic mapping } \phi : \mathbb{Z} \to \mathbb{R}. \tag{2.3}$$

Proof. We can assume that $a_0 < a_1$. It follows from Proposition 2.4 that every interval of length 1 contains at least one subinterval of length $1/(2k_1)$ that does not intersect A. Take

$$\varepsilon < \min\{1/(6k_1), (a_1 - a_0)/3\}.$$

Divide $\mathbb{R}$ into an infinite number of disjoint half-intervals $I_j = (t_j, t_{j+1}]$, $j \in \mathbb{Z}$ such that $t_{j+1} - t_j < 2$ and $A \cap (t_j - 2\varepsilon, t_j + 2\varepsilon) = \varnothing$ for all j.

Let τ be any number from E_ε in (2.1), and σ be the corresponding bijection. Then $\rho(j) \in \mathbb{Z}$ corresponds to any j such that σ is the bijection of $A \cap I_j$ to $A \cap I_{\rho(j)}$. Hence, $\#(A \cap I_j) = \#(A \cap I_{\rho(j)})$. Let σ_j be the monotone increasing bijection of $A \cap I_j$ on $A \cap I_{\rho(j)}$. Check that

$$|a_n + \tau - a_{\sigma_j(n)}| < \varepsilon \quad a_n \in I_j. \tag{2.4}$$

Assume the opposite. Let n_0 be the minimal number such that (2.4) does not satisfy. If $a_{n_0} + \tau + \varepsilon \le a_{\sigma_j(n_0)}$, then $a_n + \tau + \varepsilon \le a_k$ for all $n \le n_0$ and $k \ge \sigma_j(n_0)$, $a_n \in I_j$, $a_k \in I_{\rho(j)}$. Therefore, $k \ne \sigma(n)$ for these numbers, and σ gives a correspondence between points of the set $\{n \le n_0 : a_n \in I_j\}$ and points of the set $\{k < \sigma_j(n_0) : a_k \in I_{\rho(j)}\}$. But, by the definition of σ_j, we have

$$\begin{aligned}\#\{n \le n_0 : a_n \in I_j\} &= \#\{k \le \sigma_j(n_0) : a_k \in I_{\rho(j)}\} \\ &= 1 + \#\{k < \sigma_j(n_0) : a_k \in I_{\rho(j)}\}.\end{aligned}$$

We come to a contradiction.

If $a_{n_0} + \tau \ge a_{\sigma_j(n_0)} + \varepsilon$, then $a_n + \tau \ge a_k + \varepsilon$ for all $n \ge n_0$ and $k \le \sigma_j(n_0)$, $a_n \in I_j$, $a_k \in I_{\rho(j)}$. Therefore, $k \ne \sigma(n)$ for these numbers, and σ gives a correspondence between points from the set $\{n \ge n_0 : a_n \in I_j\}$ and points of the set $\{k > \sigma_j(n_0) : a_k \in I_{\rho(j)}\}$. But, by the definition of σ_j, we have

$$\begin{aligned}\#\{n \ge n_0 : a_n \in I_j\} &= \#\{k \ge \sigma_j(n_0) : a_k \in I_{\rho(j)}\} \\ &= 1 + \#\{k > \sigma_j(n_0) : a_k \in I_{\rho(j)}\}.\end{aligned}$$

We also get a contradiction.

Since the numbers $\#(A \cap I_j)$ and $\#(A \cap I_{\rho(j)})$ coincide, we see that the differences between indices of the first elements in these sets coincide for all j. Hence, there is a number $h \in \mathbb{Z}$ such that inequality (2.1) satisfies for all $n \in \mathbb{N}$ with $\sigma(n) = n + h$.

From the definition of τ for all $k \in \mathbb{N}$ and $N \in \mathbb{N}$ it follows that

$$-\varepsilon < a_{kh} - \tau - a_{(k-1)h} < \varepsilon \quad \text{and} \quad -N\varepsilon < a_{Nh} - N\tau - a_0 < N\varepsilon.$$

Let I be the smallest segment containing a_0 and a_{Nh}. The last inequality implies that its length satisfies the inequality

$$N\tau - N\varepsilon < l(I) < N\tau + N\varepsilon.$$

On the other hand, taking into account that ends of I can be points A with multiplicity no more than k_1, we have

$$Nh - 2(k_1 - 1) \le \#A \cap I \le Nh + 2(k_1 - 1).$$

Therefore,

$$\frac{Nh - 2(k_1 - 1)}{N\tau + N\varepsilon} \le \frac{\#A \cap I}{l(I)} \le \frac{Nh + 2(k_1 - 1)}{N\tau - N\varepsilon}.$$

Passing to the limit as $N \to \infty$ and using Proposition 2.6, we obtain the inequality

$$\tau - \varepsilon \le h/d \le \tau + \varepsilon.$$

Set $\phi(n) := a_n - n/d$. Then, for all $n \in \mathbb{Z}$, we get

$$\phi(n+h) - \phi(n) = a_{n+h} - (n+h)/d - a_n + n/d = a_{\sigma(n)} - (a_n + \tau) + (\tau - h/d).$$

Using (2.1), we obtain $|\phi(n+h) - \phi(n)| < 2\varepsilon$. Therefore, h is a 2ε-almost period of the function ϕ. The set of ε-almost periods τ of A is relatively dense, therefore the set of such integers h is also relatively dense. □

Remark 2.8. The proof of this theorem in [6] contains gaps.

Remark 2.9. The converse assertion is simple since for every ε-almost period $\tau \in \mathbb{Z}$ of the mapping $\phi(n)$ the number τ/d is an ε-almost period of the almost periodic set $A = \{n/d + \phi(n)\}$ with any $d > 0$ and bijections in (2.1) of the form $\sigma(n) = n + \tau$.

Corollary 2.10. *For any almost periodic set $A = \{a_n\}$ such that $0 \notin A$ there is a finite limit*

$$L := \lim_{N\to\infty} \sum_{|a_n|<N} 1/a_n.$$

Moreover, the sum

$$\frac{1}{z - a_0} + \sum_{n\in\mathbb{N}} \left[\frac{1}{z - a_n} + \frac{1}{z - a_{-n}}\right]$$

converges absolutely and uniformly on every disjoint with A compact set K.

Proof. Let $A = \{n/d + \phi(n)\}_{n\in\mathbb{Z}}$. Since the numbers $\phi(n)$ are uniformly bounded, we see that the sums

$$\sum_{n\in\mathbb{Z}, |a_n|<N} \frac{1}{a_n} \quad \text{and} \quad \sum_{n\in\mathbb{Z}, |n|<dN} \frac{1}{n/d + \phi(n)}$$

differ for a uniformly bounded with respect to N number of terms, and each of these terms tends to 0 as $N \to \infty$. Then

$$\sum_{n\in\mathbb{Z},0<|n|<N} \frac{1}{n/d+\phi(n)} = \sum_{n\in\mathbb{N},0<n<N} \frac{\phi(n)+\phi(-n)}{\phi(n)\phi(-n)+n\phi(-n)/d-n\phi(n)/d-(n/d)^2}.$$

The first assertion follows from the Cauchy criterion. The second one follows from the absolutely convergence of the series

$$\sum_{n\in\mathbb{N}} \left[\frac{1}{z-a_n}+\frac{1}{z-a_{-n}}\right] = \sum_{n\in\mathbb{N}} \left[\frac{2z-\phi(-n)-\phi(n)}{(n/d+\phi(n)-z)(-n/d+\phi(-n)-z)}\right].$$

□

In [12, Appendix VI], M. Krein and B. Levin considered zero sets Z_f of entire almost periodic functions f of exponential growth. They proved that if $Z_f \subset \mathbb{R}$, then zeros a_n form an almost periodic set satisfying (2.3) and

$$\sup_{\tau\in\mathbb{Z}} \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-1}[\phi(n+\tau)-\phi(n)] < \infty. \tag{2.5}$$

On the other hand, they proved that any almost periodic set $A \subset \mathbb{R}$ satisfying conditions (2.3) and (2.5) is the set of zeros of an entire almost periodic function of exponential growth.

It follows from Theorem 2.7 that condition (2.3) can be omitted in the last result.

Theorem 2.7 was generalized by W. Lawton [11] to almost periodic sets in $\mathbb{R}^m$, $m > 1$, whose spectrum is contained in a finitely generated additive group.

3. Zeros of infinite exponential sums

By $S(\mathbb{R})$, denote the Schwartz space of test functions $\varphi \in C^\infty(\mathbb{R})$ with the finite norms

$$N_{n,m}(\varphi) = \sup_{\mathbb{R}} \max_{k\le m} \left|(1+|x|^n)\varphi^{(k)}(x)\right|, \quad n,m = 0,1,2,\dots$$

These norms generate the topology on $S(\mathbb{R})$. Elements of the space $S^*(\mathbb{R})$ of continuous linear functionals on $S(\mathbb{R})$ are called tempered distributions.

The Fourier transform of a tempered distribution f is given by the equality

$$\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all} \quad \varphi \in S(\mathbb{R}),$$

where

$$\hat{\varphi}(t) = \int_{\mathbb{R}^d} \varphi(x)e^{-2\pi ixt}\,dx$$

is the Fourier transform of the function φ. The inverse Fourier transform of φ we denote by $\check{\varphi}$. The Fourier transform is a bijection of $S(\mathbb{R})$ onto itself and a bijection of $S^*(\mathbb{R})$ onto itself.

Let $\mathfrak{T}$ be the class of exponential sums

$$f(x) = \sum_n q_n e^{2\pi i\omega_n x}, \quad q_j \in \mathbb{C}\setminus\{0\}$$

with finite Wiener's norm $\|f\|_W := \sum_n |q_n|$ and a bounded spectrum $\Omega := \{\omega_n\} \subset \mathbb{R}$.

Any function $f(x) \in \mathfrak{T}$ can be extended to the whole complex plane as an entire almost periodic function $f(z)$ of exponential type $\sigma = \sup_n |\omega_n|$; the zero set $A = \{a_n\}$ of $f(z)$ lies in some horizontal strip of finite width if and only if $\inf\Omega \in \Omega$ and $\sup\Omega \in \Omega$ (cf. [12, Chap. VI, Corollary 2]). Moreover, A is an almost periodic set (cf. [12, Appendix VI, Lemma 1]).

If $0 \notin A$ and $a_n = \alpha n + \phi(n)$ with $\phi : \mathbb{Z} \to \mathbb{R}$ of the form

$$\phi(n) = \sum_j p_j e^{2\pi i\rho_j n}, \quad \rho_j \in [0,1), \ \sum_j |p_j| < \infty, \tag{3.1}$$

then the function

$$(1 - z/a_0)\prod_{n\in\mathbb{N}}(1 - z/a_n)(1 - z/a_{-n})$$

expands into an absolutely convergent exponential series (cf. [12, Appendix VI, Th.9]) and hence belongs to $\mathfrak{T}$.

Theorem 3.1. *Suppose that $f \in \mathfrak{T}$ has a zero set $A = \{a_n\} \subset \mathbb{R}$, and $\mu_A = \sum_n \delta_{a_n}$. Then the Fourier transform $\hat{\mu}_A$ is a pure point measure.*

Proof. It follows from Proposition 2.4 that the measure μ_A satisfies the condition $\mu_A([-r,r]) = O(r)$ as $r \to \infty$. Hence the measure μ_A and the distribution $\hat{\mu}_A$ are tempered distributions. To prove that $\hat{\mu}_A$ is a measure, we check the estimate

$$|(\hat{\mu}_A, \varphi)| \le C \max_{|t|<T} |\varphi(t)| \tag{3.2}$$

for any $T < \infty$ and any C^∞-function φ with support on the interval $(-T, T)$. If this is the case, the distribution $\hat{\mu}$ has a unique expansion to a linear functional on the space of continuous functions g on $[-T, T]$ such that $g(-T) = g(T) = 0$ with bound (3.2). Since we can extend this functional to the space of all continuous functions on $[-T, T]$ with bound (3.2), we see that $\hat{\mu}_A$ is a complex measure.

Let φ be a C^∞-function with support in $(-T, T)$. Set

$$\Phi(z) = \int_{-\infty}^{\infty} \varphi(t)e^{-2\pi itz}dt.$$

Clearly, $\Phi(z)$ is an entire function that equals the Fourier transform of the function $\varphi(t)e^{2\pi ty}$. Therefore, $\Phi(x + iy)$ belongs to $S(\mathbb{R})$ for each fixed $y \in \mathbb{R}$, and for its inverse Fourier transform we have

$$\check{\Phi}(\omega + iy) = \varphi(\omega)e^{2\pi\omega y}, \qquad \omega \in \mathbb{R}. \tag{3.3}$$

Let $\omega_1 = \inf\Omega$, $\omega_2 = \sup\Omega$. Then the corresponding coefficients q_1, q_2 do not vanish. Taking into account that $\sum_n |q_n| < \infty$, we can take a number M such that

$$\sum_{n>M} \frac{|q_n|}{\min|q_1|,|q_2|} < 1/3, \tag{3.4}$$

and then $s, s' > 0$ such that

$$\sum_{n\le M, n\ne 1} e^{2\pi(\omega_1-\omega_n)s}|q_n/q_1| < 1/3, \qquad \sum_{n\le M, n\ne 2} e^{2\pi(\omega_n-\omega_2)s'}|q_n/q_2| < 1/3. \tag{3.5}$$

Since $f(z)$ is almost periodic, it follows from [12, Ch.6,L.1] that for every $\varepsilon > 0$ there exists $m = m(\varepsilon) > 0$ such that

$$|f(z)| \ge m \quad \text{for } -s' \le \operatorname{Im} z \le s \text{ and } z \notin A(\varepsilon) := \{z : \operatorname{dist}(z, A) < \varepsilon\}.$$

By Proposition 2.4, for ε small enough, each connected component of $A(\varepsilon)$ contains no segment of length 1, hence its diameter is less than 1. Therefore, there are two sequences $R_k \to +\infty$, $R'_k \to -\infty$ such that

$$|f(x+iy)| > m \quad \text{for } x = R_k \text{ or } x = R'_k,\ -s' \le y \le s.$$

Consider the integrals of the function $\Phi(z)f'(z)f^{-1}(z)$ over the boundaries of rectangles

$$\Pi_k = \{z = x+iy : R'_k < x < R_k,\ -s' < y < s\}.$$

Since $\Phi(x \pm iy)$ tends to zero as $x = R_k \to +\infty$, $x = R'_k \to -\infty$ uniformly with respect to $-s' \le y \le s$, we get that these integrals tend to

$$\int_{+\infty}^{-\infty} \Phi(x+is)f'(x+is)f^{-1}(x+is)\,dx + \int_{-\infty}^{+\infty} \Phi(x-is')f'(x-is')f^{-1}(x-is')\,dx =: I_1 + I_2. \tag{3.6}$$

The Residue Theorem implies

$$\begin{aligned} I_1 + I_2 &= 2\pi i \sum_{\lambda: f(\lambda)=0} \operatorname{Res}_\lambda \Phi(z)f'(z)f^{-1}(z) \\ &= 2\pi i \sum_{\lambda: f(\lambda)=0} a(\lambda)\Phi(\lambda) = 2\pi i(\mu_A, \Phi), \end{aligned} \tag{3.7}$$

where $a(\lambda)$ is the multiplicity of the zero of $f(z)$ at the point λ.

We have for $z = x + is$, $s > 0$,

$$f(z) = q_1 e^{2\pi i(x+is)\omega_1}\left(1 + \sum_{n=2}^{M}(q_n/q_1)e^{2\pi i(\omega_n-\omega_1)(x+is)} + \sum_{n>M}(q_n/q_1)e^{2\pi i(\omega_n-\omega_1)(x+is)}\right).$$

Set

$$H(x) := \sum_{n=2}^{\infty}(q_n/q_1)e^{2\pi i(\omega_n-\omega_1)x}e^{2\pi(\omega_1-\omega_n)s} = \sum_{n=2}^{\infty} h_n(s)e^{2\pi i(\omega_n-\omega_1)x}.$$

By (3.4) and (3.5), we have $\|H\|_W < 2/3$. Since $\|\cdot\|_W$ is the norm in the algebra of all absolutely convergent exponential sums, we get

$$(1+H(x))^{-1} = \sum_{j=0}^{\infty}(-1)^j H^j(x), \qquad \|(1+H)^{-1}\|_W \le \sum_{j=0}^{\infty}\|H^j\|_W < 3. \quad (3.8)$$

We have

$$f^{-1}(x+is) = q_1^{-1}e^{2\pi\omega_1 s}e^{-2\pi i\omega_1 x}(1+H(x))^{-1},$$
$$f'(x+is) = \sum_{n=1}^{\infty} 2\pi i\omega_n q_n e^{-2\pi\omega_n s}e^{2\pi i\omega_n x}$$

and

$$f'(x+is)f^{-1}(x+is) = \sum_{n=1}^{\infty} 2\pi i\omega_n (q_n/q_1)e^{-2\pi(\omega_n-\omega_1)s}e^{2\pi i(\omega_n-\omega_1)x}(1+H(x))^{-1}. \quad (3.9)$$

Rewrite $f'f^{-1}$ in the form

$$f'(x+is)f^{-1}(x+is) = \sum_{\gamma\in\Gamma_1} p_\gamma e^{2\pi i\gamma x}, \quad p_\gamma = p_\gamma(s) \in \mathbb{C},$$

with some countable $\Gamma_1 \subset \mathbb{R}_+ \cup \{0\}$. Since Ω is bounded, we obtain from (3.8) and (3.9),

$$\sum_{\gamma\in\Gamma_1}|p_\gamma| = \|f^{-1}(x+is)f'(x+is)\|_W \le 6\pi \max_n\{|\omega_n|e^{2\pi(\omega_n-\omega_1)s}\}\sum_n |q_n/q_1| =: C_f.$$

The function $\Phi(x+is)$ belongs to $S(\mathbb{R})$ for s fixed, hence $|x|^2\Phi(x+is) \to 0$ as $|x| \to \infty$. Changing the order of integration and summation and taking into account (3.3), for the first integral in (3.6), we obtain

$$I_1 = -\sum_{\gamma\in\Gamma_1} p_\gamma \int_{-\infty}^{+\infty}\Phi(x+is)e^{2\pi i\gamma x}dx = -\sum_{\gamma\in\Gamma_1} p_\gamma e^{2\pi\gamma s}\varphi(\gamma).$$

Since $\operatorname{supp}\varphi \subset (-T,T)$, we get the bound

$$|I_1| \le C_f e^{2\pi T s}\max_{|t|\le T}|\varphi(t)|.$$

Similar reasoning shows that the second integral in (3.6) with the appropriate s' has the same estimate.

Since $\hat\varphi(x)=\Phi(x)$, we obtain from (3.7) that

$$(\mu_A,\hat\varphi)=(2\pi i)^{-1}(I_1+I_2).$$

Therefore,

$$|(\hat\mu_A,\varphi)|=|(\mu_A,\hat\varphi)|\le C(f,T)\sup_{|y|\le T}|\varphi(y)|,$$

and $\hat\mu_A$ is a measure. Since μ_A is almost periodic, Theorem 5.5 from [16] implies that $\hat\mu_A$ is a pure point measure. □

4. Entire functions with given almost periodic zero sets

In this section, we assume that a measure μ of form (1.1) is a tempered distribution, its Fourier transform $\hat\mu$ is a pure point measure of form (1.2), the measure $|\hat\mu|$ is a tempered distribution, and $A=\{a_n\}_{n\in\mathbb{Z}}$ is a multiset in which each point $a_n=\lambda\in\operatorname{supp}\mu$ occurs c_λ times. In what follows, we will assume that $0\notin A\subset\mathbb{R}$. By [5, Lemma 1], μ is an almost periodic measure and A is an almost periodic set. Since $\hat\mu$ is also a measure, Theorem 5.5 from [16] implies that every number b_γ from (1.2) equals the corresponding Fourier coefficient of the measure μ, i.e.,

$$b_\gamma=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-2\pi i\gamma x}\mu(dx).$$

In particular, $b_{-\gamma}=\bar b_\gamma$ and $b_0\ge 0$. Moreover, b_0 coincides with the density d of the set A. Also, it can be checked (cf. [4]) that the condition $|\hat\mu|\in S^*(\mathbb{R})$ implies

$$|\hat\mu|(-r,r)=\sum_{|\gamma|<r}|b_\gamma|=O(r^\kappa)\quad\text{as}\quad r\to\infty \tag{4.1}$$

with some $\kappa<\infty$.

From Proposition 2.4 and Corollary 2.10 it follows that the set A satisfies the conditions

$$n_A(r):=\#(A\cap(-r,r))=O(r)\quad(r\to\infty),$$

and

$$\left|\sum_{n:|a_n|<r}1/a_n\right|\quad\text{is bounded in}\quad r>1.$$

By Lindelöf's Theorem (see [9]), the product

$$F(z):=\prod_{n=-\infty}^{\infty}(1-z/a_n)e^{z/a_n}$$

is an entire function of exponential growth. Taking Corollary 2.10 into account again, we see that the function

$$f(z)=(1-z/a_0)\prod_{n\in\mathbb{N}}(1-z/a_n)(1-z/a_{-n})=e^{-Lz}F(z) \tag{4.2}$$

is also a well-defined entire function of exponential growth with the zero set A. It should be noticed that $f(\bar{z}) = \overline{f(z)}$. Introduce the notation

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\},\ \mathbb{R}_- := -\mathbb{R}_+,\ \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\},\ \mathbb{C}_- := -\mathbb{C}_+.$$

Proposition 4.1. *For all $z = x + iy \in \mathbb{C}_+$,*

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_0} + \sum_{n\in\mathbb{N}} \left[\frac{1}{z - a_n} + \frac{1}{z - a_{-n}}\right] = -2\pi i \sum_{\gamma\in\Gamma\cap\mathbb{R}_+} b_\gamma e^{2\pi i\gamma z} - \pi i d, \quad (4.3)$$

and for all $z = x + iy \in \mathbb{C}_-$,

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_0} + \sum_{n\in\mathbb{N}} \left[\frac{1}{z - a_n} + \frac{1}{z - a_{-n}}\right] = 2\pi i \sum_{\gamma\in\Gamma\cap\mathbb{R}_-} b_\gamma e^{2\pi i\gamma z} + \pi i d, \quad (4.4)$$

where d is the density of the almost periodic set A.

The function $f'(z)/f(z)$ is almost periodic on each line $y = y_0 \neq 0$.

Proof. Set

$$\xi_z(t) = \begin{cases} -2\pi i e^{2\pi i t z} & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases},\ z \in \mathbb{C}_+, \quad \xi_z(t) = \begin{cases} 2\pi i e^{2\pi i t z} & \text{if } t < 0 \\ 0 & \text{if } t \ge 0 \end{cases},\ z \in \mathbb{C}_-.$$

It is not hard to check that in the sense of distributions $\hat{\xi}_z(\lambda) = 1/(z - \lambda)$ for $z \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Let $\varphi(t)$ be an even nonnegative C^∞-function such that $\operatorname{supp}\varphi \subset (-1, 1)$ and $\int \varphi(t)\,dt = 1$. Set $\varphi_\varepsilon(t) = \varepsilon^{-1}\varphi(t/\varepsilon)$ for $\varepsilon > 0$. Fix $z = x + iy \in \mathbb{C}_+$. The functions $\xi_z(t) \star \varphi_\varepsilon(t)$ and $\hat{\xi}_z(\lambda)\hat{\varphi}_\varepsilon(\lambda)$ belong to $S(\mathbb{R})$. Therefore,

$$(\hat{\mu}, \xi_z \star \varphi_\varepsilon(t)) = (\mu, \hat{\xi}_z(\lambda)\hat{\varphi}_\varepsilon(\lambda)).$$

Then, for any $T_0 < \infty$, we have

$$\begin{aligned} \frac{i}{2\pi}(\hat{\mu}(t), \xi_z \star \varphi_\varepsilon(t)) = d\int_{-\varepsilon}^{0} e^{-2\pi i s z}\varphi_\varepsilon(s)\,ds + \sum_{0<|\gamma|\le\varepsilon} b_\gamma e^{2\pi i\gamma z} \int_{-\varepsilon}^{\gamma} e^{-2\pi i s z}\varphi_\varepsilon(s)\,ds \\ + \sum_{\varepsilon<\gamma<T_0} b_\gamma e^{2\pi i\gamma z} \int_{-\varepsilon}^{\varepsilon} e^{-2\pi i s z}\varphi_\varepsilon(s)\,ds \\ + \sum_{\gamma\ge T_0} b_\gamma e^{2\pi i\gamma z} \int_{-\varepsilon}^{\varepsilon} e^{-2\pi i s z}\varphi_\varepsilon(s)\,ds = I_0 + I_1 + I_2 + I_3. \end{aligned}$$

Thus, we have

$$\frac{i}{2\pi}(\hat{\mu}, \xi_z) = \sum_{0<\gamma\le\varepsilon} b_\gamma e^{2\pi i\gamma z} + \sum_{\varepsilon<\gamma<T_0} b_\gamma e^{2\pi i\gamma z} + \sum_{\gamma\ge T_0} b_\gamma e^{2\pi i\gamma z} = S_1 + S_2 + S_3.$$

Set $m(s)=\sum_{\gamma\in\Gamma:\,0<\gamma\le s}|b_\gamma|$. Then

$$\sum_{\gamma\ge T_0}|b_\gamma|e^{-2\pi\gamma y}=\int_{T_0}^\infty e^{-2\pi sy}m(ds)\le\lim_{T\to\infty}m(T)e^{-2\pi Ty}+2\pi y\int_{T_0}^\infty e^{-2\pi sy}m(s)\,ds. \tag{4.5}$$

Property (4.1) implies that I_3 and S_3 are less than a given $\eta>0$ for T_0 large enough. Taking into account that $\sum_{-\varepsilon<\gamma<T_0}|b_\gamma|<\infty$, we get

$$I_0\to d/2,\qquad I_1\to 0,\qquad S_1\to 0,$$
$$I_2-S_2=\sum_{\varepsilon<\gamma<T_0}b_\gamma e^{2\pi i\gamma z}\int_{-\varepsilon}^{\varepsilon}(e^{-2\pi isz}-1)\varphi_\varepsilon(s)ds\to 0\quad\text{as }\varepsilon\to 0,$$

and

$$\lim_{\varepsilon\to 0}(\mu,\hat\xi_z(\lambda)\hat\varphi_\varepsilon(\lambda))=(\hat\mu,\xi_z(t))-2\pi id/2=-2\pi i\sum_{\gamma\in\Gamma\cap\mathbb{R}_+}b_\gamma e^{2\pi i\gamma z}-\pi id.$$

On the other hand, we have

$$(\mu,\hat\xi_z(\lambda)\hat\varphi_\varepsilon(\lambda))=\frac{\hat\varphi(\varepsilon a_0)}{z-a_0}+\sum_{n\in\mathbb{N}}\left[\frac{\hat\varphi(\varepsilon a_n)}{z-a_n}+\frac{\hat\varphi(\varepsilon a_{-n})}{z-a_{-n}}\right]. \tag{4.6}$$

The function $\hat\varphi(t)$ tends to 1 as $t\to 0$ and $|\hat\varphi(t)|\le 1$. We have

$$\left[\frac{\hat\varphi(\varepsilon a_n)}{z-a_n}+\frac{\hat\varphi(\varepsilon a_{-n})}{z-a_{-n}}\right]=\hat\varphi(\varepsilon a_{-n})\left[\frac{1}{z-a_n}+\frac{1}{z-a_{-n}}\right]+\frac{1}{z-a_n}[\hat\varphi(\varepsilon a_n)-\hat\varphi(\varepsilon a_{-n})].$$

Since $\hat\varphi$ is even, we get with bounded $\theta(n)$ and $\phi(n)$,

$$\begin{aligned}\hat\varphi(\varepsilon a_n)-\hat\varphi(\varepsilon a_{-n})&=\hat\varphi(\varepsilon n/d+\varepsilon\phi(n))-\hat\varphi(-\varepsilon n/d+\varepsilon\phi(-n))\\&=\hat\varphi'(\varepsilon n/d+\varepsilon\theta(n))\varepsilon[\phi(n)+\phi(-n)].\end{aligned}$$

Since $\hat\varphi(t)$ belongs to the Schwartz space, we see that $\hat\varphi'(t)=O(1/|t|)$ as $t\to\infty$. Hence, for $\varepsilon|n/d+\theta(n)|>1$,

$$|\varepsilon[\hat\varphi'(\varepsilon n/d+\varepsilon\theta(n))]|\le C|n|^{-1}$$

with a constant $C<\infty$. The same estimate (with another constant C) is valid for $\varepsilon|n/d+\theta(n)|\le 1$. Then for all $n\in\mathbb{N}$ and $\varepsilon>0$,

$$|\hat\varphi(\varepsilon a_n)-\hat\varphi(\varepsilon a_{-n})|\le(C/n)2\sup_n|\phi(n)|.$$

Hence the right-hand side of (4.6) for all $\varepsilon>0$ is majorized by the sum

$$\frac{1}{|z-a_0|}+\sum_{n\in\mathbb{N}}\left|\frac{1}{z-a_n}+\frac{1}{z-a_{-n}}\right|+\sum_{n\in\mathbb{N}}\frac{C'}{n|z-a_n|}. \tag{4.7}$$

By Theorem 2.7, we have $1/(z-a_n)=O(1/n)$. Taking into account also Corollary 2.10, we get the convergence of both sums in (4.7). Therefore, we can pass to the limit in (4.6) as $\varepsilon\to 0$ and obtain (4.3).

By (4.5), for $y_0>0$,

$$\sum_{\gamma\ge 1}|b_\gamma|e^{-2\pi\gamma y_0}<\infty, \qquad \sum_{0<\gamma<1}|b_\gamma|<\infty.$$

Therefore the series in the right-hand side of (4.3) converges absolutely and uniformly for $\operatorname{Im} z\ge\alpha>0$, and $f'(z)/f(z)$ is almost periodic on the line $y=y_0$.

In the case $y_0<0$, we apply (4.3) to the function $\overline{f(\bar z)}$ and obtain (4.4). □

Theorem 4.2. *If*

$$\int_0^1 s^{-2}|\hat\mu_A|(0,s)\,ds<\infty, \tag{4.8}$$

then the function f from (4.2) with the zero set A can be rewritten in the form

$$f(z)=\sum_{\omega\in\Omega}q_\omega e^{2\pi i\omega z} \qquad q_\omega\in\mathbb{C}\setminus\{0\}, \quad \omega\in\mathbb{R}, \quad \sum_{\omega\in\Omega}|q_\omega|<\infty,$$

where the countable bounded set Ω satisfies the conditions $\sup\Omega\in\Omega$, $\inf\Omega\in\Omega$.

Proof. The sum in the right-hand side of (4.3) converges absolutely and uniformly in $x\in\mathbb{R}$ and $y\ge\alpha>0$. Changing the order of summation and integration, we get for $z=x+iy\in\mathbb{C}_+$,

$$\log f(z)-\log f(i)=\int_i^z\frac{f'(z)}{f(z)}dz=-\sum_{\gamma\in\Gamma\cap\mathbb{R}_+}b_\gamma\frac{e^{2\pi i\gamma z}-e^{-2\pi\gamma}}{\gamma}-id\pi z-\pi d. \tag{4.9}$$

It is easy to check that the convergence of the integral in (4.8) implies (in fact, is equivalent to) the convergence of the series

$$\sum_{0<\gamma<1}|b_\gamma|\gamma^{-1}=\int_0^1 s^{-1}d|\hat\mu|(0,s),$$

and (4.5) implies

$$\sum_{\gamma\ge 1}|b_\gamma|\gamma^{-1}e^{-2\pi\gamma}<\infty.$$

Therefore,

$$\log f(x+i)+id\pi x=-\sum_{0<\gamma<1}(b_\gamma/\gamma)e^{-2\pi\gamma}e^{2\pi i\gamma x}-\sum_{\gamma\ge 1}(b_\gamma/\gamma)e^{-2\pi\gamma}e^{2\pi i\gamma x}+C_0 \tag{4.10}$$

with some constant $C_0\in\mathbb{C}$, and $\|\log f(x+i)+id\pi x\|_W<\infty$. Since $\|FG\|_W\le\|F\|_W\|G\|_W$, we obtain

$$f(x+i)e^{id\pi x}=\sum_{\omega\in\Omega}p_\omega e^{2\pi i\omega x},$$

$$\sum_{\omega\in\Omega}|p_\omega| = \|f(x+i)e^{id\pi x}\|_W \le e^{\|\log f(x+i)+id\pi x\|_W} < \infty,$$

with $p_\omega \in \mathbb{C}$ and a countable spectrum $\Omega \subset \mathbb{R}_+ \cup \{0\}$. The entire function $f(z+i)e^{id\pi z}$ has exponential growth, therefore, by [12, § 1, Chap. VI], Ω is bounded. Hence the function

$$f(z) = \sum_{\omega\in\Omega} p_\omega e^{\pi(2\omega-d)} e^{\pi i(2\omega-d)z}$$

is also a Dirichlet series and $\|f\|_W < \infty$. Moreover, all zeros of f are real, therefore, by [12, Chap. VI, Corollary 2], we have $\sup\Omega \in \Omega$, $\inf\Omega \in \Omega$. □

Set

$$g(z) := \sum_{\gamma\in\Gamma, 0<\gamma<1} b_\gamma \frac{e^{2\pi i\gamma z}-1}{\gamma}.$$

Since the sum $\sum_{\gamma\in\Gamma,0<\gamma<1}|b_\gamma|$ is bounded and

$$g(z) = \sum_{k=1}^{\infty}\frac{(2\pi i z)^k}{k!}\sum_{\gamma\in\Gamma,0<\gamma<1}\gamma^{k-1}b_\gamma,$$

we see that $g(z)$ is a well-defined entire function. Then, for $z \in \mathbb{C}_+ \cup \mathbb{R}$,

$$\begin{aligned} |g(z)| &\le \left[\sum_{\gamma\in\Gamma,0<\gamma<\varepsilon} + \sum_{\gamma\in\Gamma,\varepsilon\le\gamma<1}\right]\left|\frac{1-e^{2\pi i\gamma z}}{\gamma z}\right||z||b_\gamma| \\ &\le 4\pi|z|\sum_{0<\gamma<\varepsilon}|b_\gamma| + \frac{2}{\varepsilon}\sum_{\varepsilon\le\gamma<1}|b_\gamma|. \end{aligned} \tag{4.11}$$

The sum $\sum_{0<|\gamma|<\varepsilon}|b_\gamma|$ is arbitrary small for small ε, therefore, $|g(z)| = o(|z|)$ as $|z| \to \infty$, $z \in \mathbb{C}_+ \cup \mathbb{R}$. Moreover, in every strip $\{z : |\operatorname{Im} z| < M\}$,

$$|g(z) - g(x)| \le \sum_{\gamma\in\Gamma,0<\gamma<1}|b_\gamma|\left|e^{2\pi i\gamma x}\frac{e^{-2\pi\gamma y}-1}{\gamma}\right| < C(M). \tag{4.12}$$

Theorem 4.3. *If $g(z)$ is uniformly bounded for $z = x \in \mathbb{R}$, then the function (4.2) with the zero set A is almost periodic of exponential type πd.*

Conversely, if A is the zero set of an entire almost periodic function of exponential growth, then the function $g(z)$ is uniformly bounded on $\mathbb{R}$.

Proof. By (4.5), the sums

$$\sum_{\gamma\ge1} b_\gamma\gamma^{-1}e^{-2\pi\gamma}, \qquad \sum_{0<\gamma<1} b_\gamma\frac{1-e^{-2\pi\gamma}}{\gamma}$$

are finite. Hence we can rewrite (4.9) as

$$\log f(z) = -\sum_{0<\gamma<1} b_\gamma\frac{e^{2\pi i\gamma z}-1}{\gamma} - \sum_{\gamma\ge1} b_\gamma\frac{e^{2\pi i\gamma z}}{\gamma} - id\pi z + \text{const.} \tag{4.13}$$

Here the second sum is uniformly bounded in every half-plane $\operatorname{Im} z \geq \alpha > 0$. If the function $g(x)$ is uniformly bounded on $\mathbb{R}$, then, by (4.12), the first sum in (4.13) is bounded on every line $\operatorname{Im} z = M > 0$. Therefore the functions $\log|f(z)|$ and $f(z)$ are bounded on this line, and $f(z)$ is also bounded on every line $\operatorname{Im} z = -M < 0$ as well. Since the function $f(z)$ has exponential growth, the Phragmen–Lindelöf Theorem implies that f is bounded in every horizontal strip of bounded width.

Furthermore, combining (4.13) and (4.11), we get

$$\lim_{y\to+\infty} y^{-1}\log|f(iy)| = \pi d.$$

The function $f(z)$ is bounded on the real line, therefore the Phragmen–Lindelöf Theorem implies that $|f(z)| \leq Me^{y\pi d}$ for all $z \in \mathbb{C}_+$. Also, $|f(z)| \leq Me^{-y\pi d}$ for all $z \in \mathbb{C}_-$, therefore $f(z)$ has the exponential type πd.

Further, the function $\log f(z) + id\pi z$ is bounded on the line $\operatorname{Im} z = 1$. Its derivative $(\log f(x+i))' + id\pi$ is almost periodic, hence, by Bohr's Theorem (cf. [13, Theorem 1.2.1]), the functions $\log f(x+i) + id\pi x$ and $f(x+i)$ are almost periodic in the variable x. By [13, Theorem 1.2.3], the function $f(z)$ is almost periodic in every strip where it is bounded, and hence it is almost periodic in $\mathbb{C}$.

Now, let $G(z)$ be an entire almost periodic function of exponential growth with a zero set A. Clearly, $G(z) = K_1 e^{K_2 z} f(z)$ with $K_1, K_2 \in \mathbb{C}$. Taking into account that the second sum in (4.13) is bounded on the line $\operatorname{Im} z = 1$, we obtain

$$\log G(x+i) = K_2 x - g(x+i) - id\pi x + O(1) \quad \text{as} \quad x \to \infty.$$

Since $G(x+i)$ is almost periodic, we get $\inf_{\mathbb{R}} |G(x+i)| > 0$ and $\log G(x+i) = i\omega x + O(1)$, where ω is the mean motion ([13], Chap. 2). Then (4.12) implies

$$K_2 x - g(x) - id\pi x - i\omega x = O(1).$$

By (4.11), $g(x) = o(|x|)$, therefore, $K_2 = i\pi d + i\omega$ and $g(x)$ is bounded. □

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Узагальнені квазікристали Фур'є, майже періодичні множини та нулі рядів Діріхле

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Нехай S є абсолютно збіжним рядом Діріхле з обмеженим спектром і дійсною нульовою множиною A, а μ є сумою одиничних мас у точках множини A. Основний результат статті стверджує, що перетворення Фур'є μ у сенсі розподілів є чисто точковою мірою. І навпаки, для заданої послідовності A дійсних точок знайдено достатню умову на перетворення Фур'є μ для того, щоб A була нульовою множиною абсолютно збіжного ряду Діріхле з обмеженим спектром; окрім того, для перетворення Фур'є μ знайдено критерій того, що A є нульовою множиною майже періодичної цілої функції експоненціального зростання. Ці результати базуються на новому поданні майже періодичних множин.

Ключові слова: квазікристал Фур'є, перетворення Фур'є в сенсі розподілів, чисто точкова міра, майже періодична ціла функція, майже періодична множина, нульова множина цілої функції