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An Analog of Multiplier Sequences for the Set of Totally Positive Sequences

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*To our dear teacher Iossif Vladimirivich Ostrovskii
with love and gratitude. We will always remember and admire him
not only as an outstanding mathematician and a brilliant teacher,
but also as a person of amazing kindness, generosity and nobility.*

A real sequence $(b_k)_{k=0}^{\infty}$ is called totally positive if all minors of the infinite matrix $\|b_{j-i}\|_{i,j=0}^{\infty}$ are nonnegative (here $b_k = 0$ for $k < 0$). In this paper, we investigate the problem of describing the set of sequences $(a_k)_{k=0}^{\infty}$ such that for every totally positive sequence $(b_k)_{k=0}^{\infty}$ the sequence $(a_k b_k)_{k=0}^{\infty}$ is also totally positive. We obtain the description of such sequences $(a_k)_{k=0}^{\infty}$ in two cases: when the generating function of the sequence $\sum_{k=0}^{\infty} a_k z^k$ has at least one pole, and when the sequence $(a_k)_{k=0}^{\infty}$ has not more than 4 nonzero terms.

Key words: totally positive sequences, multiply positive sequences, real-rooted polynomials, multiplier sequences, Laguerre–Pólya class

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1. Introduction

We start with the definition of multiply positive and totally positive sequences.

Definition 1.1. A real sequence $(a_k)_{k=0}^{\infty}$ is called m-times positive ($m \in \mathbb{N}$), if all minors of the infinite matrix

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \right\| \tag{1.1}$$

of orders less than or equal to m are nonnegative. The class of m-times positive sequences is denoted by TP_m, the class of the generating functions of m-times positive sequences ($f(x) = \sum_{k=0}^{\infty} a_k x^k$) is denoted by $\widetilde{TP}_m$.

Definition 1.2. A real sequence $(a_k)_{k=0}^\infty$ is called totally positive if all minors of the infinite matrix (1.1) are nonnegative. The class of totally positive sequences is denoted by TP_∞. The class of the generating functions of totally positive sequences is denoted by $\widetilde{TP}_\infty$.

Multiply positive sequences (also called Pólya frequency sequences) were introduced by Fekete in 1912 (see [7]) in connection with the problem of exact calculation of the number of positive zeros of a real polynomial. Multiply positive and totally positive sequences arise in many areas of mathematics and its applications, see, for example, [2, 10, 20].

The class TP_∞ was completely described by Aissen, Schoenberg, Whitney and Edrei in [1] (see also [10, p. 412]).

Theorem ASWE. *A function $f \in \widetilde{TP}_\infty$ if and only if*

$$f(z) = Cz^q e^{\gamma z} \prod_{k=1}^{\infty} \frac{(1+\alpha_k z)}{(1-\beta_k z)}, \tag{1.2}$$

where $C \geq 0, q \in \mathbb{Z}, \gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, \sum_{k=1}^\infty (\alpha_k + \beta_k) < \infty$.

Theorem ASWE gives the description of the class TP_∞ in terms of independent parameters $C, q, \gamma, \alpha_k, \beta_k$. It is easy to see that the class TP_2 consists of the sequences $(a_k)_{k=0}^\infty$ of the form $a_n = e^{-\psi(n)}$, where $\psi : \mathbb{N} \cup \{0\} \to (-\infty, +\infty]$ is a convex function. In [19] the description of the subclass of TP_3, which consists of the sequences all of whose sections belong to TP_3, in terms of independent parameters was obtained. The problem of the description of the classes TP_m, $3 \leq m < \infty$, in terms of independent parameters has not been solved until now.

By theorem ASWE a polynomial $p(z) = \sum_{k=0}^n a_k z^k,\ a_k \geq 0$, has only real non-positive zeros if and only if $(a_0, a_1, \ldots, a_n, 0, 0, \ldots) \in TP_\infty$.

In general, the problem of understanding whether a given polynomial has only real zeros is not trivial. Often such problems are very difficult. However, in 1926, J. I. Hutchinson found the following simple sufficient condition in terms of coefficients for an entire function with positive coefficients to have only real zeros.

Theorem A (J.I. Hutchinson, [9])**.** *Let $f(x) = \sum_{k=0}^\infty a_k x^k$, $a_k > 0$ for all k. Then $\frac{a_{n-1}^2}{a_{n-2}a_n} \geq 4$ for all $n \geq 2$, if and only if the following two conditions are fulfilled:*

(i) *the zeros of $f(x)$ are all real, simple and negative, and*
(ii) *the zeros of any polynomial $\sum_{k=m}^n a_k x^k$, $m < n$, formed by taking any number of consecutive terms of $f(x)$, are all real and non-positive.*

For some extensions of Hutchinson's results see, for example, [3, §4] and [16].

The question about whether or not a given polynomial has only real zeros is of great importance in many areas of mathematics. So, the problem to describe the set of operators that preserve this set of polynomials is of the great interest. In connection with this problem, we define multiplier sequences.

Definition 1.3. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, whenever a real polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ has only real zeros, the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros. The class of multiplier sequences is denoted by $\mathcal{MS}$.

A simple example of a multiplier sequence is the following sequence: $\gamma_k = k$, $k = 0, 1, 2, \ldots$. For an arbitrary polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ with real coefficients and only real zeros we have $\sum_{k=0}^{n} k a_k z^k = zP'(z)$, and this polynomial obviously also has only real zeros.

The full description of the set of multiplier sequences was given by G. Pólya and J. Schur in 1914. To formulate this famous result, we need the notion of the Laguerre-Pólya class of entire functions.

Definition 1.4. A real entire function f is said to be in the *Laguerre-Pólya class of type I*, written $f \in \mathcal{L}\text{-}\mathcal{P}I$, if it can be expressed in the following form

$$f(z) = cz^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{1.3}$$

where $c \in \mathbb{R}$, $\beta \geq 0$, $x_k > 0$, n is a nonnegative integer, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

Note that the product on the right-hand side can be finite or empty (in the latter case, the product equals 1).

This class is essential in the theory of entire functions since the polynomials with only real and nonpositive zeros converge locally uniformly to these and only these functions. The following prominent theorem provides an even stronger result.

Theorem B (E. Laguerre and G. Pólya, see, for example, [8, p. 42–46] and [14, Chap. VIII, §3])**.**

(i) *Let $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, be a sequence of real polynomials having only real negative zeros which converges uniformly on the disc $|z| \leq A, A > 0$. Then this sequence converges locally uniformly in $\mathbb{C}$ to an entire function from the class $\mathcal{L}\text{-}\mathcal{P}I$.*

(ii) *For any $f \in \mathcal{L}\text{-}\mathcal{P}I$ there is a sequence of real polynomials with only real nonpositive zeros, which converges locally uniformly to f.*

The following theorem fully describes multiplier sequences.

Theorem C (G. Pólya and J. Schur, cf. [22], [21, pp. 100–124], and [17, pp. 29–47])**.** *Let $(\gamma_k)_{k=0}^{\infty}$ be a given real sequence. The following three statements are equivalent.*

1. *$(\gamma_k)_{k=0}^{\infty}$ is a multiplier sequence.*
2. *For every $n \in \mathbb{N}$ the polynomial $P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ has only real zeros of the same sign.*

3. *The power series* $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ *converges absolutely in the whole complex plane and the entire function* $\Phi(z)$ *or the entire function* $\Phi(-z)$ *admits the representation*
$$cz^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{1.4}$$
where $c \in \mathbb{R}$, $\beta \geq 0$, $n \in \mathbb{N} \cup \{0\}$, $0 < x_k \leq \infty$, $\sum_{k=1}^{\infty} \frac{1}{x_k} < \infty$.

Strikingly, the following fact is an obvious consequence.

Corollary of Theorem C. *The sequence* $(\gamma_0, \gamma_1, \ldots, \gamma_l, 0, 0, \ldots)$ *is a multiplier sequence if and only if the polynomial* $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ *has only real zeros of the same sign.*

As we mentioned before, the set of polynomials with nonnegative coefficients having only real nonpositive roots is a subset of the set $\widetilde{TP}_\infty$. In this paper, we discover an analog of the multiplier sequences for the set of totally positive sequences. To formulate the problem, we need the next definition.

Definition 1.5. Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence. We define the following linear convolution operator on the set of real sequences:
$$\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) = (a_k b_k)_{k=0}^{\infty}.$$

The following problem was posed by Alan Sokal during the inspiring AIM workshop "Theory and applications of total positivity", July 24-July 28, 2023 (see [24] for more details).

Problem 1.6. Describe the set of nonnegative sequences $\mathbf{A} = (a_k)_{k=0}^{\infty}$, such that the corresponding convolution operator $\Lambda_{\mathbf{A}}$ preserves the set of TP_∞-sequences: for every $(b_k)_{k=0}^{\infty} \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in TP_\infty$.

For some questions connected with the problem above see [6] by A. Dyachenko and A. Sokal (see also previous works of A. Dyachenko [4, 5]).

We consider the multiplier sequence $\mathbf{\Gamma} = (k)_{k=0}^{\infty}$ and the corresponding convolution operator $\Lambda_{\mathbf{\Gamma}}((b_k)_{k=0}^{\infty}) = (kb_k)_{k=0}^{\infty}$. As we mentioned earlier, this operator preserves the set of finite totally positive sequences (in other words, the set of coefficients of polynomials with nonnegative coefficients and only real zeros). But this operator does not preserve the set of all totally positive sequences. Indeed, let us consider the function $f(z) = \frac{1}{(1-z)(2-z)} = \sum_{k=0}^{\infty} b_k z^k$ (we have $b_k = 1 - \frac{1}{2^{k+1}}$). By theorem ASWE, $(b_k)_{k=0}^{\infty} \in TP_\infty$. But $\sum_{k=0}^{\infty} kb_k z^k = zf'(z) = \frac{z(3-2z)}{(1-z)^2(2-z)^2}$. This function has a positive zero, so the sequence of its coefficients is not a TP_∞-sequence.

We will denote by A the generating function of a sequence $\mathbf{A} = (a_k)_{k=0}^{\infty}$: $A(z) = \sum_{k=0}^{\infty} a_k z^k$.

Suppose that the sequence $\mathbf{A}$ has the property that the corresponding convolution operator $\Lambda_{\mathbf{A}}$ preserves the set of TP_∞-sequences. Then, since the constant sequence of all ones is the TP_∞-sequence, by theorem ASWE, the generating function $A(x)$ is a meromorphic function having the representation (1.2). The following theorem gives the full description of the generating functions of TP_∞-preservers that have at least one pole.

Theorem 1.7. *Let $\mathbf{A} = (a_k)_{k=0}^\infty$ be a nonnegative sequence, and suppose its generating function is a meromorphic function with at least one pole. Then for every $(b_k)_{k=0}^\infty \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$ if and only if $A(z) = \frac{C}{1-\beta z}$, $C > 0$, $\beta > 0$.*

It remains to describe TP_∞-preservers whose generating functions are entire functions. We start with an obvious case of one or two term sequences. Let us consider a nonnegative sequence $\mathbf{A} = (a_k)_{k=0}^\infty$, such that $a_0 \geq 0, a_1 \geq 0$, and $a_k = 0$ for $k \geq 2$. Then, obviously, for every $(b_k)_{k=0}^\infty \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$. The case of three term sequences is also simple. The following statement is obvious.

Statement 1.8. *Let $\mathbf{A} = (a_k)_{k=0}^\infty$ be a nonnegative sequence, such that $a_k > 0$ for $k = 0, 1, 2$, and $a_k = 0$ for $k \geq 3$. Then for every $(b_k)_{k=0}^\infty \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$ if and only if $A(z) = a_0 + a_1 z + a_2 z^2$ has only real (and negative) zeros. Moreover, $\Lambda_{\mathbf{A}} : TP_\infty \to TP_\infty$ if and only if $\Lambda_{\mathbf{A}} : TP_2 \to TP_\infty$.*

The following theorem gives the description of TP_∞-preservers whose generating functions are polynomials of degree 3.

Theorem 1.9. *Let $\mathbf{A} = (a_k)_{k=0}^\infty$ be a nonnegative sequence, such that $a_k > 0$ for $0 \leq k \leq 3$, and $a_k = 0$ for $k \geq 4$. Then for every $(b_k)_{k=0}^\infty \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$ if and only if both polynomials $\sum_{k=0}^3 a_k x^k$ and $\sum_{k=1}^3 a_k x^k$ have only real (and nonpositive) zeros. Moreover, $\Lambda_{\mathbf{A}} : TP_\infty \to TP_\infty$ if and only if $\Lambda_{\mathbf{A}} : TP_3 \to TP_\infty$.*

Using the methods analogous to those that were used in the proof of Theorem 1.9, we can prove the following statement.

Theorem 1.10. *Let $\mathbf{A} = (a_k)_{k=0}^\infty$ be a nonnegative sequence, such that $a_k > 0$ for $0 \leq k \leq 4$, and $a_k = 0$ for $k \geq 5$. Then for every $(b_k)_{k=0}^\infty \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$ if and only if the three polynomials $\sum_{k=0}^4 a_k x^k$, $\sum_{k=1}^4 a_k x^k$ and $\sum_{k=2}^4 a_k x^k$ have only real (and nonpositive) zeros. Moreover, $\Lambda_{\mathbf{A}} : TP_\infty \to TP_\infty$ if and only if $\Lambda_{\mathbf{A}} : TP_4 \to TP_\infty$.*

We will not present the proof of the above result here, since it is very cumbersome and does not provide a complete solution to the problem of the description of all entire TP_∞-preservers.

The following example was given by Alan Sokal.

Example 1.11. Let f be an entire function of the form $f(z) = \sum_{k=0}^\infty a_k z^k$ with $a_0 = a_1 = 1$, $a_k = \frac{1}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ for $k \geq 2$, where $(q_k)_{k=2}^\infty$ is a sequence of arbitrary parameters under the following conditions: $q_k \geq 4$ for all k. Suppose that $(b_k)_{k=0}^\infty \in TP_\infty$ is an arbitrary sequence. For an entire function $(A *$

$B)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$ we have $\frac{(a_{n-1}b_{n-1})^2}{(a_{n-2}b_{n-2})(a_n b_n)} = \frac{a_{n-1}^2}{a_{n-2}a_n} \frac{b_{n-1}^2}{b_{n-2}b_n} \geq 4$ for all $n \geq 2$, since $\frac{a_{n-1}^2}{a_{n-2}a_n} = q_n \geq 4$ by our assumption, and $\frac{b_{n-1}^2}{b_{n-2}b_n} \geq 1$, because every TP_∞-sequence is, in particular, a 2-times positive sequence. Thus, using Theorem A by Hutchinson, we get $(A * B)(z) \in TP_\infty$.

We formulate the following conjecture, which is consistent with Theorems 1.9, 1.10 and Example 1.11.

Conjecture 1.12. *Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence. Then this sequence is a TP_∞-preserver, i.e. for every $(b_k)_{k=0}^{\infty} \in TP_\infty$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in TP_\infty$ if and only if for every $l \in \mathbb{N} \cup \{0\}$ the formal power series $\sum_{k=l}^{\infty} a_k z^k$ is an entire function from the $\mathcal{L}$-$\mathcal{P}I$ class (in particular, it has only real nonpositive zeros).*

We note that entire functions whose Taylor sections have only real zeros were studied in various works (see, for example, [12, 13]), but entire functions whose remainders have only real zeros have been studied less (some results can be found in the very interesting survey [18]). We mention here a way to construct such a function. The entire function $g_a(z) = \sum_{j=0}^{\infty} z^j a^{-j^2}$, $a > 1$, is called the *partial theta-function*. The survey [23] by S. O. Warnaar contains the history of investigation of the partial theta-function and some of its main properties. The paper [11] answers the question: for which $a > 1$ do the functions g_a belong to the class $\mathcal{L}$-$\mathcal{P}I$. In particular, in [11] it is proved that there exists a constant $q_\infty \approx 3.23363666\ldots$, such that $g_a \in \mathcal{L}$-$\mathcal{P}I$ if and only if $a^2 \geq q_\infty$. In [15] the following theorem is proved. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_0 = a_1 = 1$, $a_k = \frac{1}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ for $k \geq 2$, where $(q_k)_{k=2}^{\infty}$ is a sequence of arbitrary parameters under the following conditions: $q_2 \geq q_3 \geq q_4 \geq \cdots$ and $\lim_{n\to\infty} q_n \geq q_\infty$. Then $f \in \mathcal{L}$-$\mathcal{P}I$. Using this theorem we conclude that such an entire function f has all remainders with only real zeros.

2. Proof of Theorem 1.7

Suppose at first that $A(z) = \frac{C}{1-\beta z}, C > 0, \beta > 0$. Then we have $A(z) = \sum_{k=0}^{\infty} C\beta^k z^k$, whence for every $\mathbf{B} = (b_k)_{k=0}^{\infty} \in TP_\infty$ with the generation function B, the generation function of $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty})$ is equal to $\sum_{k=0}^{\infty} C\beta^k b_k z^k = CB(\beta z) \in \widetilde{TP}_\infty$. The sufficiency is proved.

Let us prove necessity. Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a sequence such that the corresponding convolution operator $\Lambda_{\mathbf{A}}$ preserves the set of TP_∞-sequences, and $\mathbf{A}$ is not identical zero.

Definition 2.1. For a nonnegative sequence $\mathbf{A} = (a_k)_{k=0}^{\infty}$ with the generating function $A(z) = \sum_{k=0}^{\infty} a_k z^k$ and a nonnegative sequence $\mathbf{B} = (b_k)_{k=0}^{\infty}$ with the generating function $B(z) = \sum_{k=0}^{\infty} b_k z^k$ we will denote by $A * B$ the generating function of the sequence $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty})$:

$$(A * B)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

We mention some simple properties of the generating functions of TP_∞-preservers.

Lemma 2.2. *Suppose that a sequence* $\mathbf{A} = (a_k)_{k=0}^\infty$ *is such that for every* $(b_k)_{k=0}^\infty \in TP_\infty$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^\infty) \in TP_\infty$. *Then the following are true*:

(1) $A(z) \in \widetilde{TP}_\infty$.

(2) $A'(z) \in \widetilde{TP}_\infty$.

(3) $(zA(z))' \in \widetilde{TP}_\infty$.

(4) $\dfrac{1}{1-c}(A(z) - cA(cz)) \in \widetilde{TP}_\infty$ *for all* $c \in (0,1) \cup (1,\infty)$.

(5) $\dfrac{1}{1-c}(A(z) - A(cz)) \in \widetilde{TP}_\infty$ *for all* $c \in (0,1) \cup (1,\infty)$.

(6) $(A(z) - \dfrac{d}{d+1}a_0) \in \widetilde{TP}_\infty$ *for all* $d \geq 0$.

(7) *For all* $n \in \mathbb{N} \cup \{0\}$ *we have* $(A(z) - \sum_{k=0}^{n} a_k z^k) \in \widetilde{TP}_\infty$.

Proof of Lemma 2.2. (1) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k \equiv 1, B(z) = \frac{1}{1-z}$. We have $(A * B)(z) = A(z) \in \widetilde{TP}_\infty$.

(2) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k = k, B(z) = \frac{z}{(1-z)^2}$. We have $(A * B)(z) = zA'(z) \in \widetilde{TP}_\infty$.

(3) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k = k+1, B(z) = \frac{1}{(1-z)^2}$. We have $(A * B)(z) = (zA(z))' \in \widetilde{TP}_\infty$.

(4) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k = \frac{c^{k+1}-1}{c-1}$, $c \in (0,1) \cup (1,\infty), B(z) = \frac{1}{(1-z)}\frac{1}{(1-cz)}$. We have $(A * B)(z) = \frac{1}{1-c}(A(z) - cA(cz)) \in \widetilde{TP}_\infty$.

(5) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k = \frac{c^k-1}{c-1}, c \in (0,1) \cup (1,\infty), B(z) = \frac{z}{(1-z)}\frac{1}{(1-cz)}$. We have $(A * B)(z) = \frac{1}{1-c}(A(z) - A(cz)) \in \widetilde{TP}_\infty$.

(6) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_0 = 1, b_k = d + 1$ for $k \geq 1, d > 0, B(z) = \frac{1+dz}{1-z}$. We have $(A * B)(z) = (1+d)(A(z) - \frac{d}{d+1}a_0) \in \widetilde{TP}_\infty$.

(7) We choose the sequence $\mathbf{B} = (b_k)_{k=0}^\infty \in TP_\infty$ such that $b_k = 0$ for $k = 0, 1, \ldots, n$, and $b_k = 1$ for $k \geq n+1$, $B(z) = \frac{z^{n+1}}{1-z}$. We have $(A * B)(z) = A(z) - \sum_{k=0}^{n} a_k z^k \in \widetilde{TP}_\infty$. The lemma is proved. □

By Lemma 2.2(1), $A(z) \in \widetilde{TP}_\infty$, whence by theorem ASWE we have

$$A(z) = Cz^q e^{\gamma z} \prod_{k=1}^{\infty} \frac{1+\alpha_k z}{1-\beta_k z}, \tag{2.1}$$

were $C > 0$, $q \in \mathbb{N} \cup \{0\}$, $\gamma \geq 0$, $\alpha_k \geq 0$, $\beta_k \geq 0$, $\sum_{k=1}^{\infty}(\alpha_k + \beta_k) < \infty$.

By Lemma 2.2(2), $A'(z) \in \widetilde{TP}_\infty$. By the assumption of Theorem 1.7, the function A has at least one pole. Suppose A has at least 2 different positive poles. Since A does not have positive zeros, then A' has a positive zero (between poles), which is impossible. Thus, A has one (maybe, multiple) pole, and we have

$$A(z) = Cz^q e^{\gamma z} \frac{\prod_{k=1}^{\infty}(1+\alpha_k z)}{(1-\beta z)^m} =: \frac{F(z)}{(1-\beta z)^m},$$

were $C > 0$, $q \in \mathbb{N} \cup \{0\}$, $\gamma \geq 0$, $\alpha_k \geq 0$, $\beta > 0$, $\sum_{k=1}^{\infty} \alpha_k < \infty$.

Since F is an entire function with nonnegative Taylor coefficients, we have $M(r, F) = \max_{|z|\leq r} |F(z)| = F(r)$. If F is not a nonnegative constant, then $\lim_{x\to+\infty} F(x) = +\infty$. If $\lim_{x\to+\infty} \frac{F(x)}{(1-\beta x)^m} = +\infty$, then A' has a positive zero on $(\beta, +\infty)$, which is impossible. We conclude that

$$A(z) = \frac{Cz^q \prod_{k=1}^{n}(1+\alpha_k z)}{(1-\beta z)^m} =: \frac{P(z)}{(1-\beta z)^m},$$

were $C > 0$, $q \in \mathbb{N} \cup \{0\}$, $\alpha_k \geq 0, \beta > 0$, $n \in \mathbb{N} \cup \{0\}$, $q + n \leq m$.

Suppose that $m = 2s, s \in \mathbb{N}$. By Lemma 2.2(5), $(A * B)(z) = \frac{1}{1-c}(A(z) - A(cz)) \in \widetilde{TP}_\infty, c \neq 1$. For $c > 1$ we have $(A*B)(z) = \frac{1}{c-1}\left(\frac{P(cz)}{(1-\beta cz)^{2s}} - \frac{P(z)}{(1-\beta z)^{2s}}\right)$. The function $(A * B)$ has two different poles: $\frac{1}{\beta c} < \frac{1}{\beta}$. Since $P(x) > 0$ for $x > 0$, and $c > 1$, we have $\lim_{x\to\frac{1}{\beta c}+0}(A * B)(x) = +\infty$, and $\lim_{x\to\frac{1}{\beta}-0}(A * B)(x) = -\infty$, so $(A * B)$ has a root in the interval $\left(\frac{1}{\beta c}, \frac{1}{\beta}\right)$. This contradicts the fact that $A * B \in \widetilde{TP}_\infty$. Thus, $m = 2s + 1, s \in \mathbb{N} \cup \{0\}$.

Suppose that $A(z) = \frac{P(z)}{(1-\beta z)^{2s+1}}$ with $\deg P = 2s + 1$. Then $\lim_{x\to+\infty} A(x) = -L$, $L > 0$. By Lemma 2.2(4), $(A * B)(z) = \frac{1}{c-1}(cA(cz) - A(z)) \in \widetilde{TP}_\infty, c \neq 1$. For $c > 1$ we have $(A * B)(z) = \frac{1}{c-1}\left(\frac{cP(cz)}{(1-\beta cz)^{2s+1}} - \frac{P(z)}{(1-\beta z)^{2s+1}}\right)$. We observe that $\lim_{x\to\frac{1}{\beta}+0}(A * B)(x) = +\infty$, and $\lim_{x\to+\infty}(A * B)(x) = \frac{1}{c-1}(-cL + L) < 0$. So, $(A * B)$ has a root in the interval $\left(\frac{1}{\beta}, +\infty\right)$. This contradicts the fact that $A * B \in \widetilde{TP}_\infty$. Thus, $\deg P < 2s + 1$.

We have proved that $A(z) = \frac{P(z)}{(1-\beta z)^n}$, where $\deg P < n$, $n = 2s + 1$, $s \in \mathbb{N} \cup \{0\}$, and P is a polynomial with nonnegative coefficients and all nonpositive roots. It remains to prove that $s = 0$.

We observe that

$$A(z) = \frac{B_0}{(1-\beta z)^n} - \frac{B_1}{(1-\beta z)^{n-1}} + \cdots + (-1)^{n-1}\frac{B_{n-1}}{(1-\beta z)}, \tag{2.2}$$

where $P(z) = B_0 - B_1(1-\beta z) + \cdots + (-1)^{n-1}(1-\beta z)^{n-1} = B_0 + \beta B_1(z - \frac{1}{\beta}) + \beta^2 B_2(z - \frac{1}{\beta})^2 + \cdots + \beta^{n-1}B_{n-1}(z - \frac{1}{\beta})^{n-1}$, whence $\beta^j B_j = \frac{P^{(j)}(\frac{1}{\beta})}{j!} > 0$ for all $j = 0, 1, \cdots, n-1$.

Lemma 2.3. *Let $k \in \mathbb{N}$, $k \geq 2$, $A_{k,\beta}(z) = \frac{1}{(1-\beta z)^k}$, and $F_{\gamma,\delta}(z) = \frac{e^{\gamma z}}{(1-\delta z)}$, $\gamma > 0$, $\delta > 0$. Then*

$$(A_{k,\beta} * F_{\gamma,\delta})(z) = \frac{(k-1)! e^{\gamma\beta z}}{(1-\delta\beta z)^k} Q_{2k-2}(z),$$

where $Q_{2k-2}(z)$ is a polynomial of degree at most $2k-2$ of the form

$$Q_{2k-2}(z) = \sum_{s=0}^{k-1} \sum_{t=k-s-1}^{k-1} \frac{(\gamma\beta)^{k-1-s}(\delta\beta)^{t-k+s+1}}{(k-1-s)!(k-1-t)!t!} z^t (1-\delta\beta z)^{2k-t-s-2}.$$

Proof. We have

$$\begin{aligned}
A_{k,\beta}(z) &= \frac{1}{(k-1)!\beta^{k-1}} \left(\frac{1}{1-\beta z}\right)^{(k-1)} = \frac{1}{(k-1)!\beta^{k-1}} \sum_{j=0}^{\infty} \beta^j (z^j)^{(k-1)} \\
&= \frac{1}{(k-1)!\beta^{k-1}} \sum_{j=k-1}^{\infty} \beta^j j(j-1)\cdots(j-k+2) z^{j-k+1} \\
&= \frac{1}{(k-1)!\beta^{k-1}} \sum_{s=0}^{\infty} \beta^{s+k-1}(s+k-1)(s+k-2)\cdots(s+1) z^s \\
&= \frac{1}{(k-1)!} \sum_{s=0}^{\infty} \beta^s (s+k-1)(s+k-2)\cdots(s+1) z^s.
\end{aligned}$$

For a function $G(z) = \sum_{s=0}^{\infty} d_s z^s$ we obtain

$$\begin{aligned}
(A_{k,\beta} * G)(z) &= \frac{1}{(k-1)!} \sum_{s=0}^{\infty} \beta^s (s+k-1)(s+k-2)\cdots(s+1) d_s z^s \\
&= \frac{1}{(k-1)!} \left(z^{k-1} G(\beta z)\right)^{(k-1)}.
\end{aligned}$$

Thus, for $G(z) = F_{\gamma,\delta}(z)$ we have

$$(A_{k,\beta} * F_{\gamma,\delta})(z) = \frac{1}{(k-1)!} \left(e^{\gamma\beta z} z^{k-1} (1-\delta\beta z)^{-1}\right)^{(k-1)}.$$

Whence we get

$$\begin{aligned}
(A_{k,\beta} * F_{\gamma,\delta})(z) &= \frac{1}{(k-1)!} \\
&\quad \times \sum_{s=0}^{k-1} \sum_{l=0}^{s} \frac{(k-1)!}{(k-1-s)!(s-l)!l!} (e^{\gamma\beta z})^{(k-1-s)} (z^{k-1})^{(s-l)} ((1-\delta\beta z)^{-1})^{(l)} \\
&= \sum_{s=0}^{k-1} \sum_{l=0}^{s} \Bigg(\frac{1}{(k-1-s)!(s-l)!l!} (\gamma\beta)^{k-1-s} e^{\gamma\beta z} \\
&\quad \times \frac{(k-1)!}{(k-s+l-1)!} z^{k-s+l-1} l! (\delta\beta)^l \frac{1}{(1-\delta\beta z)^{l+1}} \Bigg) \\
&= \frac{(k-1)! e^{\gamma\beta z}}{(1-\delta\beta z)^k} \sum_{s=0}^{k-1} \sum_{l=0}^{s} \Bigg(\frac{1}{(k-1-s)!(s-l)!(k-s+l-1)!} \\
&\quad \times z^{k-s+l-1} (\gamma\beta)^{k-1-s} (\delta\beta)^l (1-\delta\beta z)^{k-l-1} \Bigg).
\end{aligned}$$

Changing the summation index l in the second sum by $k-s+l-1=t$, we obtain

$$\begin{aligned}(A_{k,\beta} * F_{\gamma,\delta})(z) &= \frac{(k-1)!e^{\gamma\beta z}}{(1-\delta\beta z)^k} \\ &\quad\times\left(\sum_{s=0}^{k-1}\sum_{t=k-s-1}^{k-1}\frac{(\gamma\beta)^{k-1-s}(\delta\beta)^{t-k+s+1}}{(k-1-s)!(k-1-t)!t!}z^t(1-\delta\beta z)^{2k-t-s-2}\right) \\ &=: \frac{(k-1)!e^{\gamma\beta z}}{(1-\delta\beta z)^k}Q_{2k-2}(z),\end{aligned}$$

where Q_{2k-2} is a polynomial of degree at most $2k-2$, since $\deg(z^t(1-\delta\beta z)^{2k-t-s-2}) = 2k-s-2$. The lemma is proved. □

By (2.2) we have

$$A(z) = B_0A_{n,\beta}(z) - B_1A_{n-1,\beta}(z) + \ldots + (-1)^{n-1}B_{n-1}A_{1,\beta}(z),$$

where $B_0 > 0, B_1 > 0, \ldots, B_{n-1} > 0$.

So, using Lemma 2.3, we get

$$\begin{aligned}&(A * F_{\gamma,\delta})(z) = B_0(A_{n,\beta} * F_{\gamma,\delta})(z) - B_1(A_{n-1,\beta} * F_{\gamma,\delta})(z) + \cdots \\ &\quad + (-1)^{n-1}B_{n-1}(A_{1,\beta} * F_{\gamma,\delta})(z) \\ &= B_0\frac{(n-1)!e^{\gamma\beta z}}{(1-\delta\beta z)^n}Q_{2n-2}(z) - B_1\frac{(n-2)!e^{\gamma\beta z}}{(1-\delta\beta z)^{n-1}}Q_{2n-4}(z) \\ &\quad + B_2\frac{(n-3)!e^{\gamma\beta z}}{(1-\delta\beta z)^{n-2}}Q_{2n-6}(z) - \cdots + (-1)^{n-1}B_{n-1}\frac{0!e^{\gamma\beta z}}{(1-\delta\beta z)}Q_0(z) \\ &= \frac{e^{\gamma\beta z}}{(1-\delta\beta z)^n}\big(B_0(n-1)!Q_{2n-2}(z) - B_1(n-2)!Q_{2n-4}(z)(1-\delta\beta z) \\ &\quad + B_2(n-3)!Q_{2n-6}(z)(1-\delta\beta z)^2 - \cdots + (-1)^{n-1}B_{n-1}0!Q_0(z)(1-\delta\beta z)^{n-1}\big) \\ &=: \frac{e^{\gamma\beta z}}{(1-\delta\beta z)^n}H_{2n-2}(z), \qquad (2.3)\end{aligned}$$

where the degree of a polynomial H_{2n-2} is at most $2n-2$.

Since $F_{\gamma,\delta}(z) = \frac{e^{\gamma z}}{(1-\delta z)} \in \widetilde{TP}_\infty$ for all $\gamma > 0, \delta > 0$, by our assumption we conclude that $(A * F_{\gamma,\delta}) \in \widetilde{TP}_\infty$, whence the polynomial H_{2n-2} has all nonnegative coefficients and all nonpositive roots. We denote by

$$H_{2n-2}(z) =: \sum_{s=0}^{2n-2} h_s z^s, h_s \geq 0.$$

For $n > 1$ we have $2n-3 > 0$, and we want to evaluate h_{2n-3}. Note that z^{2n-3} can be found only in the terms $B_0(n-1)!Q_{2n-2}(z)$ and $B_1(n-2)!Q_{2n-4}(z)(1-\delta\beta z)$ of formula (2.3). By (3.5),

$$Q_{2n-2}(z) = \sum_{s=0}^{n-1}\sum_{t=n-s-1}^{n-1}\frac{(\gamma\beta)^{n-1-s}(\delta\beta)^{t-n+s+1}}{(n-1-s)!(n-1-t)!t!}z^t(1-\delta\beta z)^{2n-t-s-2}.$$

We observe that $\deg(z^t(1-\delta\beta z)^{2n-t-s-2}) = 2n-s-2 < 2n-3$ for $s > 1$, so we will search for the term z^{2n-3} in the summands with $s = 0$ and $s = 1$. We have

$$\begin{aligned} Q_{2n-2}(z) = &\frac{(\gamma\beta)^{n-1}(\delta\beta)^0}{((n-1)!)^2} z^{n-1}(1-\delta\beta z)^{n-1} \\ &+ \frac{(\gamma\beta)^{n-2}(\delta\beta)^0}{((n-2)!)^2} z^{n-2}(1-\delta\beta z)^{n-1} + \frac{(\gamma\beta)^{n-2}(\delta\beta)^1}{(n-2)!(n-1)!} z^{n-1}(1-\delta\beta z)^{n-2} \\ &+ \sum_{s=2}^{n-1} \sum_{t=n-s-1}^{n-1} \frac{(\gamma\beta)^{n-1-s}(\delta\beta)^{t-n+s+1}}{(n-1-s)!(n-1-t)!t!} z^t(1-\delta\beta z)^{2n-t-s-2}. \end{aligned}$$

Thus, gathering the terms with z^{2n-3} in the first 3 summands of the above formula, we obtain the term with z^{2n-3} in $B_0(n-1)!Q_{2n-2}(z)$:

$$\begin{aligned} B_0(n-1)! \Bigg(&\frac{(\gamma\beta)^{n-1}(\delta\beta)^0}{((n-1)!)^2} (-1)^{n-2}(n-1)(\delta\beta)^{n-2} \\ &+ \frac{(\gamma\beta)^{n-2}(\delta\beta)^0}{((n-2)!)^2} (-1)^{n-1}(\delta\beta)^{n-1} + \frac{(\gamma\beta)^{n-2}(\delta\beta)^1}{(n-2)!(n-1)!} (-1)^{n-2}(\delta\beta)^{n-2} \Bigg) \\ &= B_0 \frac{(-1)^{n-2}(\gamma\beta)^{n-2}(\delta\beta)^{n-2}}{(n-2)!} (\gamma\beta - \delta\beta(n-1) + \delta\beta) \\ &= B_0 \frac{(-1)^{n-2}(\gamma\beta)^{n-2}(\delta\beta)^{n-2}}{(n-2)!} (\gamma\beta - \delta\beta(n-2)). \end{aligned}$$

By (3.5),

$$\begin{aligned} &Q_{2n-4}(z)(1-\delta\beta z) \\ &\qquad = (1-\delta\beta z) \sum_{s=0}^{n-2} \sum_{t=n-s-2}^{n-2} \frac{(\gamma\beta)^{n-2-s}(\delta\beta)^{t-n+s+2}}{(n-2-s)!(n-2-t)!t!} z^t(1-\delta\beta z)^{2n-t-s-4}. \end{aligned}$$

We observe that $\deg(z^t(1-\delta\beta z)^{2n-t-s-4}) = 2n-s-4 < 2n-4$ for $s \geq 1$, so we will search for the term z^{2n-4} in the summands with $s = 0$ and multiply it by $(-\delta\beta z)$. Thus, the term with z^{2n-3} in $(-B_1(n-2)!Q_{2n-4}(z)(1-\delta\beta z))$ equals

$$-B_1(n-2)! \frac{(-1)^{n-1}(\gamma\beta)^{n-2}(\delta\beta)^{n-1}}{((n-2)!)^2} = B_1 \frac{(-1)^n(\gamma\beta)^{n-2}(\delta\beta)^{n-1}}{(n-2)!}.$$

Finally, we get

$$\begin{aligned} h_{2n-3} &= B_0 \frac{(-1)^{n-2}(\gamma\beta)^{n-2}(\delta\beta)^{n-2}}{(n-2)!} (\gamma\beta - \delta\beta(n-2)) + B_1 \frac{(-1)^n(\gamma\beta)^{n-2}(\delta\beta)^{n-1}}{(n-2)!} \\ &= \frac{(-1)^{n-2}(\gamma\beta)^{n-2}(\delta\beta)^{n-2}}{(n-2)!} (B_0\gamma\beta - B_0(n-2)\delta\beta + B_1\delta\beta). \end{aligned}$$

Since $n = 2s+1, s \in \mathbb{N} \cup \{0\}$, and $\beta > 0, \gamma > 0, \delta > 0$, we obtain

$$\text{sign} h_{2n-3} = -\text{sign}(B_0\gamma\beta - B_0(n-2)\delta\beta - B_1\delta\beta) = -1$$

for $\gamma > 0$ being large enough and $\delta > 0$ being small enough. We get a contradiction. Thus, $n = 1$ and $A(z) = \frac{C}{1-\beta z}, C > 0, \beta > 0$.

Theorem 1.7 is proved. □

3. Proof of Theorem 1.9

Let us prove the necessity. Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence, such that $a_k > 0$ for $0 \le k \le 3$, and $a_k = 0$ for $k \ge 4$. Suppose that the operator $\Lambda_{\mathbf{A}}$ preserves the set of the TP_∞-sequences. Since the sequence $\mathbf{B}_1 = (1, 1, 1, 1, \ldots) \in TP_\infty$, we have $\Lambda_{\mathbf{A}}(\mathbf{B}_1) = (a_0, a_1, a_2, a_3, 0, 0, 0, \ldots) \in TP_\infty$, whence the polynomial $\sum_{k=0}^{3} a_k x^k$ has only real (and nonpositive) zeros. Further, since the sequence $\mathbf{B}_2 = (0, 1, 1, 1, 1, \ldots) \in TP_\infty$, we have $\Lambda_{\mathbf{A}}(\mathbf{B}_2) = (0, a_1, a_2, a_3, 0, 0, 0, \ldots) \in TP_\infty$, whence the polynomial $\sum_{k=1}^{3} a_k x^k$ has only real (and nonpositive) zeros. The necessity is proved.

Let us prove the sufficiency. Obviously, $(c_k)_{k=0}^{\infty} \in TP_\infty$ if and only if $(C\lambda^k c_k)_{k=0}^{\infty} \in TP_\infty$ for $C > 0$, $\lambda > 0$. Thus, without loss of generality we can assume that $a_0 = a_1 = 1$. Then we can rewrite our sequence $\mathbf{A}$ in the form $\left(1, 1, \frac{1}{a}, \frac{1}{a^2 b}, 0, 0, 0, \ldots\right)$, where $a = \frac{a_1^2}{a_0 a_2} = \frac{1}{a_2}$, $b = \frac{a_2^2}{a_1 a_3} = \frac{a_2^2}{a_3}$.

So, by assumption both the polynomial

$$P(x, a, b) = 1 + x + \frac{x^2}{a} + \frac{x^3}{a^2 b}, \quad a > 0,\ b > 0. \tag{3.1}$$

and the polynomial

$$T(x, a, b) = x + \frac{x^2}{a} + \frac{x^3}{a^2 b}. \tag{3.2}$$

have only real non-positive zeros.

Note that

$$P(x, a, b) = 1 + T(x, a, b). \tag{3.3}$$

Denote by

$$F(y, a, b) = 1 + ay\left(1 + y + \frac{y^2}{b}\right) \tag{3.4}$$

and by

$$t(y, b) = y + y^2 + \frac{y^3}{b} = y(1 + y + \frac{y^2}{b}). \tag{3.5}$$

We have

$$F(y, a, b) = 1 + at(y, b). \tag{3.6}$$

Statement 3.1. *Both P and T have only real zeros if and only if both F and t have only real zeros.*

Proof. Statement 3.1 follows from the two identities below.

$$F(y, a, b) = P(ay, a, b)$$

and

$$t(y, b) = \frac{1}{a} T(ay, a, b). \qquad \square$$

The following fact is obvious.

Statement 3.2. *The polynomials $T(x, a, b)$ and $t(y, b)$ have only real zeros if and only if $b \ge 4$.*

From now on we will assume that

$$b \geq 4. \tag{3.7}$$

Consider the derivative of the polynomial $F(y,a,b)$.

$$F'(y,a,b) = a\left(1 + 2y + \frac{3}{b}y^2\right) \tag{3.8}$$

Denote by

$$\alpha_1(b) := \frac{b}{3}\left(-1 + \sqrt{1 - \frac{3}{b}}\right) \tag{3.9}$$

and

$$\alpha_2(b) := \frac{b}{3}\left(-1 - \sqrt{1 - \frac{3}{b}}\right) \tag{3.10}$$

the roots of $F'(y,a,b)$. It follows from (3.7) that

$$\frac{\alpha_1(b)}{b} = \frac{1}{3}\left(-1 + \sqrt{1 - \frac{3}{b}}\right) \geq -\frac{1}{6}. \tag{3.11}$$

Since all roots of $F(y,a,b)$ are real, $\alpha_2(b) < \alpha_1(b)$, and $\lim_{y\to+\infty} F(y,a,b) = +\infty$, it is clear that

$$F\left(\alpha_1(b), a, b\right) \leq 0, \tag{3.12}$$

and that

$$F\left(\alpha_2(b), a, b\right) \geq 0. \tag{3.13}$$

The inequality (3.13) can be improved in the following way.

Statement 3.3. *We have*

$$F\left(\alpha_2(b), a, b\right) \geq 1. \tag{3.14}$$

Proof. It follows from (3.5) that

$$t'(y,b) = \frac{1}{a}F'(y,a,b). \tag{3.15}$$

Therefore, $\alpha_1(b)$ from (3.9) and $\alpha_2(b)$ from (3.10) are roots of $t'(y,b)$ too. By our assumption, $t(y,b)$ has only real roots. Since $\lim_{y\to-\infty} t(y,b) = -\infty$, we have

$$t\left(\alpha_2(b), b\right) \geq 0. \tag{3.16}$$

By virtue of (3.6), the following is true

$$F\left(\alpha_2(b), a, b\right) = 1 + at\left(\alpha_2(b), b\right) \geq 1.$$

Statement 3.3 is proved. □

In notations introduced in (3.1) and (3.2) the sufficiency in Theorem 1.9 could be equivalently reformulated in the following form.

Statement 3.4. *Assume that both $P(x,a,b)$ and $T(x,a,b)$ have only real non-positive zeros, and $G(x)=\sum_{k=0}^{\infty} c_k x^k \in \widetilde{TP}_\infty$. Then $P * G \in \widetilde{TP}_\infty$.*

First, consider the case when $c_0 = 0$. If additionally $c_1 = 0$, the Statement 3.4 is obvious.

Let $c_1 \neq 0$. Since $G(x) \in \widetilde{TP}_\infty$, and therefore, $G(x) \in \widetilde{TP}_2$, we conclude that

$$c_2^2 - c_1 c_3 \geq 0. \tag{3.17}$$

Since $T(x,a,b)$ has only real zeros we have $b \geq 4$ (see (3.7)). Thus,

$$bc_2^2 - 4c_1 c_3 \geq 0. \tag{3.18}$$

The last inequality means that the polynomial

$$P * G(x,a,b) = c_1 x + \frac{c_2}{a} x^2 + \frac{c_3}{a^2 b} x^3$$

has only real roots, that is by theorem ASWE $P * G \in \widetilde{TP}_\infty$. So, in this case Statement 3.4 is true.

From now on we will assume that $c_0 \neq 0$. For $G(x) = \sum_{k=0}^{\infty} c_k x^k$ we denote by

$$p_k = \frac{c_{k-1}}{c_k}, \ k \in \mathbb{N}, \tag{3.19}$$

and by

$$q_{k+1} = \frac{p_{k+1}}{p_k}, \ k \in \mathbb{N}. \tag{3.20}$$

Then we have

$$\begin{aligned} G(x) &= c_0 \left(1 + \frac{x}{p_1} + \frac{x^2}{p_1 p_2} + \cdots + \frac{x^k}{p_1 p_2 \cdots p_k} + \cdots \right) \\ &= c_0 \left(1 + \frac{x}{p_1} + \left(\frac{x}{p_1}\right)^2 \frac{1}{q_2} + \left(\frac{x}{p_1}\right)^3 \frac{1}{q_2^2 q_3} + \cdots \right. \\ &\qquad \left. + \left(\frac{x}{p_1}\right)^k \frac{1}{q_2^k q_3^{k-1} \cdots q_{k-1}^2 q_k} + \cdots \right). \end{aligned} \tag{3.21}$$

Denote by

$$g(y) = 1 + y + \frac{y^2}{q_2} + \frac{y^3}{q_2^2 q_3} + \ldots + \frac{y^k}{q_2^k q_3^{k-1} \cdots q_{k-1}^2 q_k} + \cdots, \tag{3.22}$$

so that by (3.21)

$$g(y) = \frac{1}{c_0} G(p_1 y) \in \widetilde{TP}_\infty. \tag{3.23}$$

Statement 3.5. *If* $g(y) = 1 + y + \sum_{k=2}^{\infty} \frac{y^k}{q_2^{k-1} q_3^{k-2} \cdots q_k} \in \widetilde{TP}_\infty$*, then*

$$q_2 \geq 1, \quad q_3 \geq 1, \quad q_2^2 q_3 - 2q_2 q_3 + 1 \geq 0. \tag{3.24}$$

Proof. By the definition, the statement $g(y) \in \widetilde{TP}_\infty$ means that all minors of the matrix

$$\begin{bmatrix} 1 & 1 & \frac{1}{q_2} & \frac{1}{q_2^2 q_3} & \frac{1}{q_2^3 q_3^2 q_4} & \dots \\ 0 & 1 & 1 & \frac{1}{q_2} & \frac{1}{q_2^2 q_3} & \dots \\ 0 & 0 & 1 & 1 & \frac{1}{q_2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{3.25}$$

are nonnegative. The first statement of (3.24) is equivalent to the fact that

$$\begin{vmatrix} 1 & \frac{1}{q_2} \\ 1 & 1 \end{vmatrix} \geq 0.$$

The second statement of (3.24) follows from the fact that

$$\begin{vmatrix} \frac{1}{q_2} & \frac{1}{q_2^2 q_3} \\ 1 & \frac{1}{q_2} \end{vmatrix} \geq 0.$$

The third statement of (3.24) can be easily obtained from the inequality below

$$\begin{vmatrix} 1 & \frac{1}{q_2} & \frac{1}{q_2^2 q_3} \\ 1 & 1 & \frac{1}{q_2} \\ 0 & 1 & \frac{1}{q_2} \end{vmatrix} \geq 0.$$

Statement 3.5 is proved. □

By (3.1) we have

$$(P * G)(x, a, b) = c_0 \left(1 + \frac{x}{p_1} + \left(\frac{x}{p_1}\right)^2 \frac{1}{q_2 a} + \left(\frac{x}{p_1}\right)^3 \frac{1}{(q_2 a)^2 q_3 b}\right). \tag{3.26}$$

Denote by

$$F_q(y, a, b) = 1 + q_2 a y + q_2 a y^2 + \frac{q_2 a}{q_3 b} y^3 = 1 + q_2 a y \left(1 + y + \frac{y^2}{q_3 b}\right). \tag{3.27}$$

By (3.26) we have

$$F_q(y, a, b) = \frac{1}{c_0} (P * G)(y p_1 q_2 a, a, b). \tag{3.28}$$

Now we can equivalently reformulate Statement 3.4 in the following way.

Statement 3.6. *Assume that both* $F(y, a, b)$ *and* $t(y, b)$ *from* (3.4) *and* (3.5) *have only real non-positive zeros, and* $g(y)$ *from* (3.22) *belongs to* $\widetilde{TP}_\infty$*. Then* $F_q(y, a, b) \in \widetilde{TP}_\infty$.

Proof. We will show that

$$F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) \leq 0, \tag{3.29}$$

and

$$F_q(\alpha_2(b), a, b) \geq 0. \tag{3.30}$$

Since $\lim_{x\to+\infty} F_q(x,a,b) = +\infty$ and $\lim_{x\to-\infty} F_q(x,a,b) = -\infty$, Statement 3.6 follows from (3.29) and (3.30).

Let us prove (3.29). It follows from (3.27) that

$$F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) = 1 + a\alpha_1(b) + \frac{a}{q_2}\left(\alpha_1(b)\right)^2 + \frac{a}{bq_2^2q_3}\left(\alpha_1(b)\right)^3, \tag{3.31}$$

and from (3.4) that

$$\begin{aligned} &F_q(\alpha_1(b), a, b) - F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) \\ &\qquad = a\left(\alpha_1(b)\right)^2\left(1 - \frac{1}{q_2}\right) + \frac{a}{b}\left(\alpha_1(b)\right)^3\left(1 - \frac{1}{q_2^2q_3}\right). \end{aligned} \tag{3.32}$$

By (3.11) we have

$$\begin{aligned} F_q(\alpha_1(b), a, b) - F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) &\geq a\left(\alpha_1(b)\right)^2\left(\left(1 - \frac{1}{q_2}\right) - \frac{1}{6}\left(1 - \frac{1}{q_2^2q_3}\right)\right) \\ &= \frac{a\left(\alpha_1(b)\right)^2}{6q_2^2q_3}\left(5q_2^2q_3 - 6q_2q_3 + 1\right). \end{aligned} \tag{3.33}$$

Note that by the first statement of (3.24) we obtain

$$\left(5q_2^2q_3 - 6q_2q_3 + 1\right) - \left(q_2^2q_3 - 2q_2q_3 + 1\right) = 4q_2q_3(q_2 - 1) \geq 0. \tag{3.34}$$

It follows from (3.33) and the third statement of (3.24) that

$$F_q(\alpha_1(b), a, b) - F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) \geq \frac{a\left(\alpha_1(b)\right)^2}{6q_2^2q_3}\left(q_2^2q_3 - 2q_2q_3 + 1\right) \geq 0. \tag{3.35}$$

Thus, by virtue of (3.12) we conclude that

$$F_q\left(\frac{\alpha_1(b)}{q_2}, a, b\right) \leq F_q(\alpha_1(b), a, b) \leq 0.$$

So, (3.29) is proved.

Let us prove (3.30). It follows from (3.27) and the second statement of (3.24) that

$$\begin{aligned} F_q(\alpha_2(b), a, b) &= 1 + q_2a\alpha_2(b)\left(1 + \alpha_2(b) + \frac{\left(\alpha_2(b)\right)^2}{q_3b}\right) \\ &> 1 + q_2a\alpha_2(b)\left(1 + \alpha_2(b) + \frac{\left(\alpha_2(b)\right)^2}{b}\right). \end{aligned}$$

By (3.4) and (3.14) we have

$$F_q(\alpha_2(b), a, b) > 1 + q_2(F(\alpha_2(b), a, b) - 1) \geq 1. \tag{3.36}$$

So, (3.30) and thereby Statement 3.6 is proved. □

Theorem 1.9 is proved. □

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Аналог послідовностей множників для множини тотально додатних послідовностей

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Дійсна послідовність $(b_k)_{k=0}^{\infty}$ називається тотально додатною, якщо всі мінори нескінченної матриці $\|b_{j-i}\|_{i,j=0}^{\infty}$ є невід'ємними (тут $b_k = 0$ для $k < 0$). У цій статті ми досліджуємо проблему опису множини послідовностей $(a_k)_{k=0}^{\infty}$, таких, що для кожної тотально додатної послідовності $(b_k)_{k=0}^{\infty}$ послідовність $(a_k b_k)_{k=0}^{\infty}$ також є тотально додатною. Ми отримуємо опис таких послідовностей $(a_k)_{k=0}^{\infty}$ у двох випадках: коли твірна функція послідовності $\sum_{k=0}^{\infty} a_k z^k$ має принаймні один полюс, а також коли послідовність $(a_k)_{k=0}^{\infty}$ має не більше чотирьох ненульових членів.

Ключові слова: тотально додатна послідовність, кратно додатна послідовність, многочлени з усіма дійсними коренями, послідовність множників, клас Лагерра–Поліа