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Arithmetic of a certain convolution semigroup of probability distributions on the group $\mathbb{R} \times \mathbb{Z}(2)$

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We consider a certain convolution semigroup Θ of probability distributions on the group $\mathbb{R} \times \mathbb{Z}(2)$, where $\mathbb{R}$ is the group of real numbers and $\mathbb{Z}(2)$ is the additive group of the integers modulo 2. This semigroup appeared in connection with the study of a characterization problem of mathematical statistics on $\boldsymbol{a}$-adic solenoids containing an element of order 2. We answer the questions that arise in the study of arithmetic of the semigroup Θ. Namely, we describe the class of infinitely divisible distributions, the class of indecomposable distributions, and the class of distributions which have no indecomposable factors.

Key words: indecomposable distribution, infinitely divisible distribution, semigroup of probability distributions

Mathematical Subject Classification 2020: 60B15

1. Introduction

A number of works are devoted to arithmetic of various semigroups of probability distributions (see, e.g., [6, 8, 9]). The purpose of this note is to study the arithmetic of a certain semigroup of probability distributions on the direct product of the group of real numbers and the additive group of the integers modulo 2. This semigroup appears in connection with the study of a characterization problem of mathematical statistics on $\boldsymbol{a}$-adic solenoids containing an element of order 2 ([1], see also [3, §11]).

Let X be a locally compact Abelian group. Denote by Y the character group of the group X and by (x, y) the value of a character $y \in Y$ at an element $x \in X$. Denote by $\mathrm{M}^1(X)$ the convolution semigroup of all distributions (probability measures) on the group X. Let $\mu \in \mathrm{M}^1(X)$. Denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y,$$

the characteristic function of the distribution μ. The characteristic function of a signed measure on the group X is defined in the same way. Denote by m_K the Haar distribution on a compact subgroup K of the group X.

Recall the following definitions. Let $\mu \in \mathrm{M}^1(X)$. A distribution $\mu_1 \in \mathrm{M}^1(X)$ is called a *factor* of μ if there is a distribution $\mu_2 \in \mathrm{M}^1(X)$ such that the equality

$$\mu = \mu_1 * \mu_2 \tag{1.1}$$

holds. A distribution with support only at a single point $x \in X$ is called *degenerate* and is denoted by E_x. A nondegenerate distribution $\mu \in \mathrm{M}^1(X)$ is called *indecomposable* if it has only degenerate distributions or shifts μ as factors. A distribution $\mu \in \mathrm{M}^1(X)$ is called *decomposable* if there are nondegenerate distributions μ_1 and μ_2 such that (1.1) holds. A distribution $\mu \in \mathrm{M}^1(X)$ is said to be *infinitely divisible* if, for each natural n, there are a distribution $\mu_n \in \mathrm{M}^1(X)$ and an element $x_n \in X$ such that $\mu = \mu_n^{*n} * E_{x_n}$. We note that this definition is slightly different from the classical one in the case of the group of real numbers. The shift by the element x_n is necessary, in particular, for all degenerate distributions to be infinitely divisible.

Denote by $\mathbb{R}$ the group of real numbers and by $\mathbb{Z}(2) = \{0, 1\}$ the additive group of the integers modulo 2. Consider the group $\mathbb{R} \times \mathbb{Z}(2)$. Denote by (t, k), where $t \in \mathbb{R}$, $k \in \mathbb{Z}(2)$, its elements. The character group of the group $\mathbb{R} \times \mathbb{Z}(2)$ is topologically isomorphic to the group $\mathbb{R} \times \mathbb{Z}(2)$. Denote by (s, l), $s \in \mathbb{R}$, $l \in \mathbb{Z}(2)$, elements of the character group of the group $\mathbb{R} \times \mathbb{Z}(2)$. The value of a character (s, l) at an element $(t, k) \in \mathbb{R} \times \mathbb{Z}(2)$ is defined by the formula

$$((t, k), (s, l)) = e^{its}(-1)^{kl}.$$

Let $\mu \in \mathrm{M}^1(\mathbb{R} \times \mathbb{Z}(2))$ and assume that the support of μ is contained in the subgroup $\mathbb{Z}(2)$, i.e., $\mu\{(0, 0)\} = a \geq 0$, $\mu\{(0, 1)\} = b \geq 0$, where $a + b = 1$. Then the characteristic function $\hat{\mu}(s, l)$ is of the form

$$\hat{\mu}(s, l) = \begin{cases} 1 & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa & \text{if } s \in \mathbb{R},\ l = 1, \end{cases} \tag{1.2}$$

where $\kappa = a - b$. In particular, the characteristic function of the Haar distribution $m_{\mathbb{Z}(2)}$, is of the form

$$\widehat{m}_{\mathbb{Z}(2)}(s, l) = \begin{cases} 1 & \text{if } s \in \mathbb{R},\ l = 0, \\ 0 & \text{if } s \in \mathbb{R},\ l = 1. \end{cases} \tag{1.3}$$

Denote by $\Gamma(\mathbb{R})$ the set of Gaussian distributions on the group $\mathbb{R}$.

2. Class Θ

Let μ be a distribution on the group $\mathbb{R} \times \mathbb{Z}(2)$ such that $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$, i.e., $\mu = \gamma * \omega$, where $\gamma \in \Gamma(\mathbb{R})$, $\omega \in \mathrm{M}^1(\mathbb{Z}(2))$, and the groups $\mathbb{R}$ and $\mathbb{Z}(2)$ are considered as subgroups of the group $\mathbb{R} \times \mathbb{Z}(2)$. It is easy to see that the characteristic function $\hat{\mu}(s, l)$ is of the form

$$\hat{\mu}(s, l) = \begin{cases} \exp\{-\sigma s^2 + i\beta s\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa \exp\{-\sigma s^2 + i\beta s\} & \text{if } s \in \mathbb{R},\ l = 1, \end{cases}$$

where $\sigma \geq 0$, β and κ are real numbers, $|\kappa| \leq 1$. Let us introduce a class of distributions on the group $\mathbb{R} \times \mathbb{Z}(2)$ which is much broader than the class $\Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. For this purpose we need the following assertion proved in [1], see also [3, Lemma 11.1]. For the sake of completeness, we present here its proof.

Lemma 2.1. *Consider the group $\mathbb{R} \times \mathbb{Z}(2)$. Let $f(s,l)$ be a function on the character group of the group $\mathbb{R} \times \mathbb{Z}(2)$ of the form*

$$f(s,l) = \begin{cases} \exp\{-\sigma s^2 + i\beta s\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa \exp\{-\sigma' s^2 + i\beta' s\} & \text{if } s \in \mathbb{R},\ l = 1, \end{cases} \tag{2.1}$$

where $\sigma \geq 0$, $\sigma' \geq 0$ and β, β', κ are real numbers. Then $f(s,l)$ is the characteristic function of a signed measure μ on the group $\mathbb{R} \times \mathbb{Z}(2)$. The signed measure μ is a measure if and only if either

$$0 < \sigma' < \sigma, \quad 0 < |\kappa| \leq \sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(\beta - \beta')^2}{4(\sigma - \sigma')}\right\}, \tag{2.2}$$

or

$$\sigma = \sigma', \quad \beta = \beta', \quad |\kappa| \leq 1. \tag{2.3}$$

Moreover, if (2.3) is fulfilled, then $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$.

Proof. Let $\kappa = 0$. Then $f(s,l)$ is the characteristic function of the distribution μ of the form $\mu = \gamma * m_{\mathbb{Z}(2)}$, where $\gamma \in \Gamma(\mathbb{R})$. Therefore, we can assume that $\kappa \neq 0$. Multiplying, if necessary, the function $f(s,l)$ by a suitable character of the group $\mathbb{Z}(2)$, we can suppose, without loss of generality, that $\kappa > 0$. Take a number $a > 0$ and denote by γ_a a Gaussian distribution on the group $\mathbb{R}$ with the density

$$\rho_a(t) = \frac{1}{2\sqrt{\pi a}} \exp\left\{-\frac{t^2}{4a}\right\}, \quad t \in \mathbb{R}. \tag{2.4}$$

It is obvious that

$$\hat{\gamma}_a(s) = \exp\{-as^2\}, \quad s \in \mathbb{R}.$$

Let μ be the signed measure on the group $\mathbb{R} \times \mathbb{Z}(2)$ which is defined by the following way

$$\mu(B \times \{k\}) = \begin{cases} \frac{1}{2}(\gamma_\sigma * E_\beta + \kappa\gamma_{\sigma'} * E_{\beta'})(B) & \text{if } k = 0, \\ \frac{1}{2}(\gamma_\sigma * E_\beta - \kappa\gamma_{\sigma'} * E_{\beta'})(B) & \text{if } k = 1, \end{cases}$$

where B is a Borel subset of $\mathbb{R}$. Put

$$\lambda_0 = \frac{1}{2}(\gamma_\sigma * E_\beta + \kappa\gamma_{\sigma'} * E_{\beta'}), \quad \lambda_1 = \frac{1}{2}(\gamma_\sigma * E_\beta - \kappa\gamma_{\sigma'} * E_{\beta'}).$$

Taking into account that

$$\hat{\lambda}_0(s) + \hat{\lambda}_1(s) = \hat{\gamma}_\sigma(s) e^{i\beta s}$$

and

$$\hat{\lambda}_0(s) - \hat{\lambda}_1(s) = \kappa \hat{\gamma}_{\sigma'}(s) e^{i\beta' s},$$

we have

$$\hat{\mu}(s,l) = \int_{\mathbb{R}\times\mathbb{Z}(2)} e^{its}(-1)^{kl} d\mu(t,k) = \int_{\mathbb{R}\times\{0\}} e^{its} d\mu(t,0) + \int_{\mathbb{R}\times\{1\}} e^{its}(-1)^l d\mu(t,1) = f(s,l).$$

Thus, $f(s,l)$ is the characteristic function of the signed measure μ. Moreover, the signed measure μ is a measure if and only if the signed measure λ_1 is a measure. It is obvious that if the signed measure λ_1 is a measure, then either $\sigma > 0$ and $\sigma' > 0$ or $\sigma = \sigma' = 0$. It is clear that if $\sigma = \sigma' = 0$, then the signed measure μ is a measure if and only if $\beta = \beta'$ and $\kappa \leq 1$. In this case the lemma is proved.

Let $\sigma > 0$ and $\sigma' > 0$. In view of (2.4), the signed measure λ_1 is a measure if and only if the equality

$$\frac{1}{2\sqrt{\pi\sigma}} \exp\left\{-\frac{(t-\beta)^2}{4\sigma}\right\} - \frac{\kappa}{2\sqrt{\pi\sigma'}} \exp\left\{-\frac{(t-\beta')^2}{4\sigma'}\right\} \geq 0$$

holds for all $t \in \mathbb{R}$. This inequality is equivalent to the following

$$\kappa \leq \sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(t-\beta)^2}{4\sigma} + \frac{(t-\beta')^2}{4\sigma'}\right\}, \quad t \in \mathbb{R}. \tag{2.5}$$

Suppose that $\sigma = \sigma'$. Then it follows from (2.5) that $\beta = \beta'$ and $\kappa \leq 1$.

Let $\sigma \neq \sigma'$. Inasmuch as $\kappa > 0$, we have $\sigma' < \sigma$. The minimum of the function on the right side of inequality (2.5) is reached at the point

$$t_0 = \frac{\sigma\beta' - \sigma'\beta}{\sigma - \sigma'},$$

and it is equal to

$$\sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(\beta-\beta')^2}{4(\sigma-\sigma')}\right\}.$$

It follows from the above that the signed measure λ_1, and hence the signed measure μ is a measure if and only if either (2.2) or (2.3) is fulfilled. It is also obvious that if (2.3) holds, then $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. □

Definition 2.2. We say that a distribution μ on the group $\mathbb{R} \times \mathbb{Z}(2)$ belongs to the class Θ if $\hat{\mu}(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and either (2.2) or (2.3) holds.

Since the product of characteristic functions corresponds to the convolution of distributions, it follows from the Lemma 2.1 that the class Θ is a convolution semigroup. The purpose of this note is to answer the main questions that arise in the study of the arithmetic of the semigroup Θ. Namely, we describe in Θ the class of infinitely divisible distributions, the class of indecomposable distributions, and the class of distributions which have no indecomposable factors.

Remark 2.3. Heyde's theorem on characterization of the Gaussian distribution on the real line by the symmetry of the conditional distribution of one linear form of independent random variables given another is well known ([5, § 13.4.1]). The class of distributions Θ arises in connection with the study of an analogue of this theorem for the group $\mathbb{R}\times\mathbb{Z}(2)$.

Let a be a topological automorphism of the group $\mathbb{R}\times\mathbb{Z}(2)$. It is obvious that a is of the form $a(t,k)=(c_a t,k)$, where $c_a\in\mathbb{R}$, $c_a\neq 0$. We identify a and c_a, i.e., we write $a(t,k)=(at,k)$ and assume that $a\in\mathbb{R}$, $a\neq 0$. The following group analogue of Heyde's theorem for the group $\mathbb{R}\times\mathbb{Z}(2)$ was proved in [2], see also [3, Theorem 11.6]).

Consider the group $\mathbb{R}\times\mathbb{Z}(2)$ and let a_j, b_j, $j=1,2,\dots,n$, $n\geq 2$, be topological automorphisms of $\mathbb{R}\times\mathbb{Z}(2)$ satisfying the conditions $b_i a_i^{-1}+b_j a_j^{-1}\neq 0$ for all i,j. Let ξ_j be independent random variables with values in the group $\mathbb{R}\times\mathbb{Z}(2)$ and distributions μ_j with nonvanishing characteristic functions. If the conditional distribution of the linear form $L_2=b_1\xi_1+\dots+b_n\xi_n$ given $L_1=a_1\xi_1+\dots+a_n\xi_n$ is symmetric, then all distributions μ_j belong to the class Θ.

The class of distributions Θ also arises in connection with the study of an analogue of Heyde's theorem on $\boldsymbol{a}$-adic solenoids containing an element of order 2 ([1], see also [3, Theorem 11.20]). Note also that some problems related to independent random variables with values in the group $\mathbb{R}\times\mathbb{Z}(2)$ were studied in [4, 10, 11].

3. Arithmetic of the semigroup Θ

The proof of the main theorem is based on the following lemma.

Lemma 3.1. *Let $\mu\in\Theta$ and $\hat\mu(s,l)=f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and*

$$0<\sigma'<\sigma,\quad |\kappa|=\sqrt{\frac{\sigma'}{\sigma}}\exp\left\{-\frac{(\beta-\beta')^2}{4(\sigma-\sigma')}\right\} \tag{3.1}$$

is fulfilled. Then μ is an indecomposable distribution.

Proof. We break the proof into several steps.

1 Assume $\mu=\mu_1*\mu_2$, where $\mu_j\in \mathrm{M}^1(\mathbb{R}\times\mathbb{Z}(2))$ and μ_j are nondegenerate distributions. We have

$$\hat\mu(s,l)=\hat\mu_1(s,l)\hat\mu_2(s,l),\quad s\in\mathbb{R},\ l\in\mathbb{Z}(2). \tag{3.2}$$

Substituting $l=0$ in (3.2), we obtain

$$\exp\{-\sigma s^2+i\beta s\}=\hat\mu_1(s,0)\hat\mu_2(s,0),\quad s\in\mathbb{R}.$$

By Cramér's theorem on decomposition of the Gaussian distribution on the real line,

$$\hat\mu_j(s,0)=\exp\{-\sigma_j s^2+i\beta_j s\},\quad s\in\mathbb{R},$$

where $\sigma_j \geq 0$, $\beta_j \in \mathbb{R}$, $j = 1, 2$. It follows from definition of the characteristic function that $\hat{\mu}_j(s, 1)$ is an entire function and

$$\max_{s\in\mathbb{C},\ |s|\leq r} |\hat{\mu}_j(s,1)| \leq \max_{s\in\mathbb{C},\ |s|\leq r} |\exp\{-\sigma_j s^2 + i\beta_j s\}|. \tag{3.3}$$

It follows from (3.2) that the functions $\hat{\mu}_j(s, 1)$ do not vanish in the complex plane $\mathbb{C}$. In view of (3.3), the entire functions $\hat{\mu}_j(s, 1)$ are of at most order 2 and type σ_j. Taking into account that $\hat{\mu}_j(-s, 1) = \overline{\hat{\mu}_j(s, 1)}$, by the Hadamard theorem on the representation of an entire function of finite order and Lemma 2.1, we obtain that the characteristic functions $\hat{\mu}_j(s, l)$ can be written in the form

$$\hat{\mu}_j(s,l) = \begin{cases} \exp\{-\sigma_j s^2 + i\beta_j s\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa_j \exp\{-\sigma'_j s^2 + i\beta'_j s\} & \text{if } s \in \mathbb{R},\ l = 1, \end{cases} \tag{3.4}$$

where either

$$0 < \sigma'_j < \sigma_j, \quad 0 < |\kappa_j| \leq \sqrt{\frac{\sigma'_j}{\sigma_j}} \exp\left\{-\frac{(\beta_j - \beta'_j)^2}{4(\sigma_j - \sigma'_j)}\right\},\ j = 1, 2,$$

or

$$0 \leq \sigma_j = \sigma'_j, \quad \beta_j = \beta'_j, \quad 0 < |\kappa_j| \leq 1,\ j = 1, 2.$$

Moreover, $\mu_j \in \Theta$.

Note that in this discussion, we used only that $\mu \in \Theta$, i.e., $\hat{\mu}(s, l) = f(s, l)$, where the function $f(s, l)$ is represented in the form (2.1) and either (2.2) or (2.3) holds and the fact that the characteristic function $\hat{\mu}(s, l)$ does not vanish, i.e., in (2.1) $\kappa \neq 0$.

2. It follows from (2.1) and (3.4) that

$$\kappa = \kappa_1\kappa_2, \quad \sigma = \sigma_1 + \sigma_2, \quad \sigma' = \sigma'_1 + \sigma'_2, \quad \beta = \beta_1 + \beta_2, \quad \beta' = \beta'_1 + \beta'_2. \tag{3.5}$$

Since $0 < \sigma' < \sigma$, it follows from (3.5) that for at least one j, say for $j = 1$, the inequality $\sigma'_1 < \sigma_1$ holds, i.e., $\mu_1 \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Hence the inequalities

$$0 < \sigma'_1 < \sigma_1, \quad 0 < |\kappa_1| \leq \sqrt{\frac{\sigma'_1}{\sigma_1}} \exp\left\{-\frac{(\beta_1 - \beta'_1)^2}{4(\sigma_1 - \sigma'_1)}\right\} \tag{3.6}$$

are fulfilled.

There are two possibilities for the distribution μ_2: either $\mu_2 \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$ or $\mu_2 \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$.

3. Let $\mu_2 \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Then we have $0 < \sigma_2 = \sigma'_2$, $\beta_2 = \beta'_2$ and $0 < |\kappa_2| \leq 1$. Moreover, the equality $\kappa = \kappa_1\kappa_2$ implies that $|\kappa| \leq |\kappa_1|$. In view of (3.1) and (3.6), it follows from this that

$$|\kappa| = \sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(\beta - \beta')^2}{4(\sigma - \sigma')}\right\} = \sqrt{\frac{\sigma'_1 + \sigma_2}{\sigma_1 + \sigma_2}} \exp\left\{-\frac{(\beta_1 - \beta'_1)^2}{4(\sigma_1 - \sigma'_1)}\right\}$$

$$\leq |\kappa_1| \leq \sqrt{\frac{\sigma'_1}{\sigma_1}} \exp\left\{-\frac{(\beta_1-\beta'_1)^2}{4(\sigma_1-\sigma'_1)}\right\}.$$

Hence,

$$\sqrt{\frac{\sigma'_1+\sigma_2}{\sigma_1+\sigma_2}} \leq \sqrt{\frac{\sigma'_1}{\sigma_1}},$$

which is obviously impossible because $0<\sigma'_1<\sigma_1$ and $\sigma_2>0$.

4. Let $\mu_2 \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Then the inequalities

$$0<\sigma'_2<\sigma_2, \quad 0<|\kappa_2| \leq \sqrt{\frac{\sigma'_2}{\sigma_2}} \exp\left\{-\frac{(\beta_2-\beta'_2)^2}{4(\sigma_2-\sigma'_2)}\right\} \tag{3.7}$$

hold. Taking into account (3.5)–(3.7), we obtain

$$|\kappa| = \sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(\beta-\beta')^2}{4(\sigma-\sigma')}\right\} = \sqrt{\frac{\sigma'_1+\sigma'_2}{\sigma_1+\sigma_2}} \exp\left\{-\frac{(\beta_1+\beta_2-\beta'_1-\beta'_2)^2}{4(\sigma_1+\sigma_2-\sigma'_1-\sigma'_2)}\right\}$$
$$= |\kappa_1\kappa_2| \leq \sqrt{\frac{\sigma'_1}{\sigma_1}} \exp\left\{-\frac{(\beta_1-\beta'_1)^2}{4(\sigma_1-\sigma'_1)}\right\} \sqrt{\frac{\sigma'_2}{\sigma_2}} \exp\left\{-\frac{(\beta_2-\beta'_2)^2}{4(\sigma_2-\sigma'_2)}\right\}.$$

Hence,

$$\sqrt{\frac{\sigma'_1+\sigma'_2}{\sigma_1+\sigma_2}} \exp\left\{-\frac{(\beta_1+\beta_2-\beta'_1-\beta'_2)^2}{4(\sigma_1+\sigma_2-\sigma'_1-\sigma'_2)}\right\}$$
$$\leq \sqrt{\frac{\sigma'_1}{\sigma_1}} \exp\left\{-\frac{(\beta_1-\beta'_1)^2}{4(\sigma_1-\sigma'_1)}\right\} \sqrt{\frac{\sigma'_2}{\sigma_2}} \exp\left\{-\frac{(\beta_2-\beta'_2)^2}{4(\sigma_2-\sigma'_2)}\right\}. \tag{3.8}$$

It is easy to see that the inequalities $0<\sigma'_1<\sigma_1$ and $0<\sigma'_2<\sigma_2$ imply the inequality

$$\sqrt{\frac{\sigma'_1}{\sigma_1}}\sqrt{\frac{\sigma'_2}{\sigma_2}} < \sqrt{\frac{\sigma'_1+\sigma'_2}{\sigma_1+\sigma_2}}. \tag{3.9}$$

Note that if $a, b \in \mathbb{R}$, $c>0$, $d>0$, then the inequality

$$\frac{(a+b)^2}{c+d} \leq \frac{a^2}{c} + \frac{b^2}{d} \tag{3.10}$$

is fulfilled. Substituting $a=\beta_1-\beta'_1$, $b=\beta_2-\beta'_2$, $c=\sigma_1-\sigma'_1$, $d=\sigma_2-\sigma'_2$ in (3.10), we get from the obtained inequality

$$\exp\left\{-\frac{(\beta_1-\beta'_1)^2}{4(\sigma_1-\sigma'_1)}\right\} \exp\left\{-\frac{(\beta_2-\beta'_2)^2}{4(\sigma_2-\sigma'_2)}\right\}$$
$$\leq \exp\left\{-\frac{(\beta_1+\beta_2-\beta'_1-\beta'_2)^2}{4(\sigma_1+\sigma_2-\sigma'_1-\sigma'_2)}\right\}. \tag{3.11}$$

Note that the inequality (3.8) contradicts the inequality that results when we multiply (3.9) and (3.11).

We assumed that both distributions μ_1 and μ_2 are nondegenerate and came to a contradiction. Thus, at least one of the distributions μ_j is degenerate, i.e., μ is an indecomposable distribution. □

Remark 3.2. Consider the group $\mathbb{R}\times\mathbb{Z}(2)$ and let $\mu\in\Theta$. If μ is an infinitely divisible distribution in the semigroup $\mathrm{M}^1(\mathbb{R}\times\mathbb{Z}(2))$, then μ is an infinitely divisible distribution in the semigroup Θ, i.e., distributions μ_n in the definition of an infinitely divisible distribution belong to the semigroup Θ. Indeed, if the characteristic function of μ does not vanish, it follows from the proof of item 1 of Lemma 3.1. If the characteristic function of μ vanishes, then μ can be represented in the form $\mu=\gamma*m_{\mathbb{Z}(2)}$, where $\gamma\in\Gamma(\mathbb{R})$. In this case, the statement is obviously true.

Note that if the characteristic function of μ vanishes, then μ is decomposable in the semigroup Θ. Hence if μ is indecomposable in Θ, then the characteristic function of μ does not vanish. Then it follows from the proof of item 1 of Lemma 3.1 that μ is also indecomposable in the semigroup $\mathrm{M}^1(\mathbb{R}\times\mathbb{Z}(2))$.

As proven in [7, Chapter IV, Theorem 11.3], the following assertion holds.

Let X be a second countable locally compact Abelian group and $\mu\in\mathrm{M}^1(X)$. Assume that μ has no factors of the form m_K, where K is a nonzero compact subgroup of the group X. Then the distribution μ can be represented as a convolution of a finite or countable number of indecomposable distributions and a distribution that has no indecomposable factors.

Using Lemma 3.1, for distributions belonging to the semigroup Θ this assertion can be considerably strengthened. Note that if $\mu\in\Theta$ and the Haar distribution $m_{\mathbb{Z}(2)}$ is not a factor of μ, then the characteristic function $\hat\mu(s,l)$ does not vanish. Note also that by the classical Cramér theorem, Gaussian distributions on the group $\mathbb{R}$ have no indecomposable factors.

Proposition 3.3. *Let $\mu\in\Theta$ and μ be a nondegenerate distribution. Then either $\mu\in\Gamma(\mathbb{R})*\mathrm{M}^1(\mathbb{Z}(2))$, or μ is an indecomposable distribution, or $\mu=\nu*\gamma$, where $\nu\in\Theta$, ν is an indecomposable distribution, and γ is a nondegenerate Gaussian distribution on the group $\mathbb{R}$.*

Proof. Let $\mu\notin\Gamma(\mathbb{R})*\mathrm{M}^1(\mathbb{Z}(2))$. Then $\hat\mu(s,l)=f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and inequalities (2.2) hold. If (3.1) is fulfilled, then by Lemma 3.1, μ is an indecomposable distribution. Assume that (3.1) is false. Then the inequalities

$$0<|\kappa|<\sqrt{\frac{\sigma'}{\sigma}}\exp\left\{-\frac{(\beta-\beta')^2}{4(\sigma-\sigma')}\right\} \tag{3.12}$$

hold. Put

$$b=\exp\left\{-\frac{(\beta-\beta')^2}{4(\sigma-\sigma')}\right\}. \tag{3.13}$$

We have
$$0 < |\kappa| < \sqrt{\frac{\sigma'}{\sigma}}b. \tag{3.14}$$
Put
$$a = \frac{\sigma\kappa^2 - \sigma' b^2}{\kappa^2 - b^2}. \tag{3.15}$$
It follows from (3.14) that $0 < a < \sigma'$, and (3.15) implies that
$$|\kappa| = \sqrt{\frac{\sigma' - a}{\sigma - a}}b. \tag{3.16}$$
By Lemma 2.1, we get from (3.13) and (3.16) that there is a distribution $\nu \in \Theta$ with the characteristic function of the form
$$\hat{\nu}(s,l) = \begin{cases} \exp\{-(\sigma - a)s^2 + i\beta s\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa\exp\{-(\sigma' - a)s^2 + i\beta' s\} & \text{if } s \in \mathbb{R},\ l = 1. \end{cases} \tag{3.17}$$
In view of (3.13) and (3.16), by Lemma 3.1, ν is an indecomposable distribution.

Denote by γ the Gaussian distribution on the group $\mathbb{R}$ with the characteristic function $\hat{\gamma}(s) = \exp\{-as^2\}$. If we consider γ as a distribution on the group $\mathbb{R} \times \mathbb{Z}(2)$, then
$$\hat{\gamma}(s,l) = \begin{cases} \exp\{-as^2\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \exp\{-as^2\} & \text{if } s \in \mathbb{R},\ l = 1. \end{cases} \tag{3.18}$$
In view of $\hat{\mu}(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1), it follows from (3.17) and (3.18) that
$$\hat{\mu}(s,l) = \hat{\nu}(s,l)\hat{\gamma}(s,l), \quad s \in \mathbb{R},\ l \in \mathbb{Z}(2).$$
Hence, $\mu = \nu * \gamma$.

It is easy to see that γ is the maximal, in the natural sense, Gaussian factor of the distribution μ. □

Corollary 3.4. *Let $\mu \in \Theta$ and $\mu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Then the distribution μ has an indecomposable factor.*

We will complement Proposition 3.3 with the following assertion.

Proposition 3.5. *Let $\mu \in \Theta$, $\mu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$, and μ be a decomposable distribution. Then for each natural n there are indecomposable distributions $\mu_j \in \Theta$, $j = 1, 2, \dots, n$, and a nondegenerate Gaussian distribution γ_n on the group $\mathbb{R}$ such that*
$$\mu = \mu_1 * \mu_2 * \cdots * \mu_n * \gamma_n.$$

Proof. In view of Proposition 3.3, it suffices to prove that the distribution μ can be represented in the form $\mu = \mu_1 * \nu$, where $\mu_1, \nu \in \Theta$, μ_1 is an indecomposable distribution, and $\nu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$ and ν is a decomposable distribution.

The condition of the proposition implies that $\hat{\mu}(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and inequalities (3.12) hold. Take numbers σ_1 and σ_1' such that $0 < \sigma_1' < \sigma_1$. Put $\kappa_1 = \sqrt{\frac{\sigma_1'}{\sigma_1}}$. By Lemma 2.1, there is a distribution $\mu_1 \in \Theta$ such that its characteristic function is of the form

$$\hat{\mu}_1(s,l) = \begin{cases} \exp\{-\sigma_1 s^2\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \kappa_1 \exp\{-\sigma_1' s^2\} & \text{if } s \in \mathbb{R},\ l = 1. \end{cases}$$

By Lemma 3.1, μ_1 is an indecomposable distribution.

Put $\tau = \sigma - \sigma_1$, $\tau' = \sigma' - \sigma_1'$, $\varsigma = \kappa/\kappa_1$. We can assume that σ_1 is arbitrarily small and κ_1 is arbitrarily close to 1. Then (3.12) implies that the inequalities

$$0 < |\varsigma| < \sqrt{\frac{\tau'}{\tau}} \exp\left\{-\frac{(\beta - \beta')^2}{4(\tau - \tau')}\right\} \tag{3.19}$$

are valid. By Lemma 2.1, it follows from this that there is a distribution $\nu \in \Theta$ with the characteristic function

$$\hat{\nu}(s,l) = \begin{cases} \exp\{-\tau s^2 + i\beta s\} & \text{if } s \in \mathbb{R},\ l = 0, \\ \varsigma \exp\{-\tau' s^2 + i\beta' s\} & \text{if } s \in \mathbb{R},\ l = 1. \end{cases} \tag{3.20}$$

Since

$$\hat{\mu}(s,l) = \hat{\mu}_1(s,l)\hat{\nu}(s,l), \quad s \in \mathbb{R},\ l \in \mathbb{Z}(2),$$

then $\mu = \mu_1 * \nu$. It is clear that $\nu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Since (3.19) and (3.20) hold, the above reasoning shows that ν is a decomposable distribution. ☐

Remark 3.6. Let a distribution μ on the group $\mathbb{R} \times \mathbb{Z}(2)$ belong to the semigroup Θ, and let $\hat{\mu}(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1), and the inequalities $0 < \sigma' < \sigma$ and (3.12) are fulfilled. Using Lemma 3.1 and representation (1.2) of the characteristic functions of distributions belonging to the semigroup $\mathrm{M}^1(\mathbb{Z}(2))$, it is easy to check that the distribution μ can be represented as $\mu = \lambda * \pi$, where $\lambda \in \Theta$ and λ is an indecomposable distribution, and $\pi \in \mathrm{M}^1(\mathbb{Z}(2))$.

Note that any distribution belonging to the semigroup $\mathrm{M}^1(\mathbb{Z}(2))$ has no indecomposable factors.

Let us now prove the main result of the note. Denote by $\mathrm{I}(\Theta)$ the class of infinitely divisible distributions, by $\mathrm{I}_0(\Theta)$ the class of distributions which have no indecomposable factors, and by $\mathrm{Ind}(\Theta)$ the class of indecomposable distributions in the semigroup Θ.

Theorem 3.7. *The following statements are true:*

1. $\mathrm{I}(\Theta) = \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$;
2. $\mu \in \mathrm{I}_0(\Theta)$ *if and only if* $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$ *and* μ *can not be represented as* $\mu = \gamma * m_{\mathbb{Z}(2)}$, *where* γ *is a nondegenerate Gaussian distribution on the group* $\mathbb{R}$;

3. $\mu \in \mathrm{Ind}(\Theta)$ *if and only if the characteristic function* $\hat\mu(s,l)$ *is represented in the form*

$$\hat\mu(s,l) = \begin{cases} \exp\{-\sigma s^2 + i\beta s\} & \textit{if } s \in \mathbb{R},\ l = 0, \\ \kappa \exp\{-\sigma' s^2 + i\beta' s\} & \textit{if } s \in \mathbb{R},\ l = 1, \end{cases}$$

and (3.1) *is fulfilled.*

Proof. 1. It is easy to see that all distributions belonging to the semigroup $\mathrm{M}^1(\mathbb{Z}(2))$ are infinitely divisible. Therefore, if $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$, then μ is an infinitely divisible distribution. Assume that $\mu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Then $\hat\mu(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and inequalities (2.2) are satisfied. Suppose that μ is an infinitely divisible distribution. Then for each natural n there is a distribution $\mu_n \in \Theta$ and an element $(t_n, k_n) \in \mathbb{R} \times \mathbb{Z}(2)$ such that $\mu = \mu_n^{*n} * E_{(t_n,k_n)}$. Hence,

$$\hat\mu(s,l) = (\hat\mu_n(s,l))^n e^{it_n s}(-1)^{k_n l}, \quad s \in \mathbb{R},\ l \in \mathbb{Z}(2).$$

This implies that $|\hat\mu(s,l)| = |\hat\mu_n(s,l)|^n$. Since $\mu_n \in \Theta$, we have

$$|\hat\mu_n(s,l)| = \begin{cases} \exp\{-\frac{\sigma}{n} s^2\} & \text{if } s \in \mathbb{R},\ l = 0, \\ |\kappa|^{\frac{1}{n}} \exp\{-\dfrac{\sigma'}{n} s^2\} & \text{if } s \in \mathbb{R},\ l = 1, \end{cases}$$

and the inequality

$$|\kappa|^{\frac{1}{n}} \le \sqrt{\frac{\sigma'}{\sigma}}$$

holds. Since n is an arbitrary natural number, we get a contradiction. Therefore, μ is not an infinitely divisible distribution.

Thus, the class of infinitely divisible distributions in the semigroup Θ coincides with the class $\Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$.

2. Let $\mu \in \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$ and μ is not represented in the form $\mu = \gamma * m_{\mathbb{Z}(2)}$, where γ is a nondegenerate Gaussian distribution on the group $\mathbb{R}$. Two cases are possible.

2a. The characteristic function $\hat\mu(s,l)$ does not vanish. It follows from the proof of item 1 of Lemma 3.1 and (3.5) that all factors of μ also belong to the class $\Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$, i.e., μ has no indecomposable factors.

2b. The characteristic function $\hat\mu(s,l)$ vanishes. Then $\mu = E_b * m_{\mathbb{Z}(2)}$, where $b \in \mathbb{R}$. It is obvious that μ has no indecomposable factors.

Let $\mu = \gamma * m_{\mathbb{Z}(2)}$, where γ is a nondegenerate Gaussian distribution on the group $\mathbb{R}$. Then the characteristic function $\hat\mu(s,l)$ is of the form

$$\hat\mu(s,l) = \begin{cases} \exp\{-as^2 + ibs\} & \text{if } s \in \mathbb{R},\ l = 0, \\ 0 & \text{if } s \in \mathbb{R},\ l = 1, \end{cases}$$

where $a > 0$, $b \in \mathbb{R}$. It is easy to see that μ has a factor μ_1 such that $\hat\mu_1(s,l) = f(s,l)$, where the function $f(s,l)$ is represented in the form (2.1) and (3.1) is satisfied. By Lemma 3.1, μ_1 is an indecomposable distribution.

If $\mu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$, then by Corollary 3.4, μ has an indecomposable factor.

3. Obviously, all nondegenerate distributions belonging to the class $\Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$ are decomposable. Let μ be a nondegenerate distribution and let $\mu \notin \Gamma(\mathbb{R}) * \mathrm{M}^1(\mathbb{Z}(2))$. Then $\hat{\mu}(s, l) = f(s, l)$, where the function $f(s, l)$ is represented in the form (2.1) and inequalities (2.2) are satisfied. It follows from (2.2) that either (3.1) is true or inequalities (3.12) hold. As follows from the proof of Proposition 3.3, if inequalities (3.12) hold, then the distribution μ is decomposable. Thus, the validity of (3.1), according to Lemma 3.1, is not only a sufficient condition for the distribution μ to be indecomposable, but also a necessary one. $\square$

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Арифметика однієї напівгрупи ймовірнісних розподілів на групі $\mathbb{R} \times \mathbb{Z}(2)$

Gennadiy Feldman

Ми розглядаємо деяку напівгрупу Θ ймовірнісних розподілів відносно згортки на групі $\mathbb{R}\times\mathbb{Z}(2)$, де $\mathbb{R}$ – група дійсних чисел, а $\mathbb{Z}(2)$ — група класів лишків за модулем 2. Ця напівгрупа виникає в зв'язку з однією характерізаційною задачею математичної статистики на $\boldsymbol{a}$-адичних соленоїдах, які містять елемент порядку 2. Ми даємо відповіді на природні питання, що виникають при вивченні арифметики напівгрупи Θ. А саме, даємо повний опис класу безмежно подільних розподілів, класу нерозкладних розподілів та класу розподілів, які не мають нерозкладних дільників.

Ключові слова: безмежно подільний розподіл, нерозкладний розподіл, напівгрупа розподілів