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(Sub)critical Operators and Spectral Capacities of Rational Frequency Approximants

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We consider one-frequency quasi-periodic Schrödinger operators with analytic potentials. Denoting by S_+ the union of the spectra taken over the phase, we study continuity of logarithmic capacities of S_+ upon rational approximation. We show that if the Lyapunov exponent is zero on the spectrum, then the capacity of the spectrum at irrational frequency is approximated by the capacities of S_+ at its rational approximants.

Key words: logarithmic capacity, quasi-periodic Schrödinger operators, almost Mathieu operator, Lyapunov exponent

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This article is devoted to the memory of I.V.Ostrovskiy. Both authors have been influenced by him in their formative stages. BH did his undergraduate and M.S. mathematics studies at Bilkent University, where IV worked as a professor from 1993 to 2010 and made great contributions to the Department of Mathematics. SJ frequently interacted with IV, who she called Uncle Yuzik, throughout her entire childhood. A good friend of her parents, Valentina Borok and Iakov Zhitomirskiy, he was the one who documented her first steps in an 8mm film. He also introduced her and her family to the joy of canoe trips. Particularly memorable is her very first trip, as a pre-teen, when she shared a canoe with him, leading to many exciting conversations.

1. Introduction

For an irrational $\alpha \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, the quasi-periodic Schrödinger operator

$$H_{\alpha,\theta} : \; l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$$

with an analytic potential $v \in \mathcal{C}^\omega(\mathbb{T}, \mathbb{R})$ is given by

$$(H_{\alpha,\theta}\psi)_n = \psi_{n-1} + \psi_{n+1} + v(\alpha n + \theta)\psi_n, \tag{1.1}$$

where $\theta \in \mathbb{T}$ and fixed $\alpha \in \mathbb{T}$ are called the phase and frequency, respectively. The spectrum of $H_{\alpha,\theta}$ for a fixed phase θ is denoted by $\sigma(\alpha, \theta)$, and is independent of θ for any irrational frequency α. Therefore we use the notation

$$\Sigma(\alpha) := \sigma(\alpha, \theta), \qquad \theta \in \mathbb{T}, \tag{1.2}$$

for irrational α.

When the frequency is rational, $H_{p/q,\theta}$ becomes a periodic operator for any phase, with a purely absolutely continuous spectrum and a band-gap structure. In this case, $\sigma(p/q, \theta)$ depends on θ, so we introduce the union spectrum

$$S_+(p/q) = \bigcup_{\theta \in \mathbb{T}} \sigma(p/q, \theta) \tag{1.3}$$

following [4, 14].

Quasi-periodic Schrödinger operators appear in the modeling of the influence of an external magnetic field on a crystal layer. Their study gained prominence with the works on the almost Mathieu operator, see [19] which is a special case with potential $v(x) = 2\lambda \cos(2\pi x)$. The almost Mathieu operator is classified as subcritical, critical, and supercritical depending on λ, with $\lambda = 1$ being critical. This classification of energies was generalized to quasi-periodic Schrödinger operators with analytic potentials by Avila [3], see Section 2 for details.

In experiments and numerics irrational α are usually replaced by their rational approximants, so corresponding convergence questions become important. When $p_n/q_n \to \alpha$, α is irrational, the sets $S_+(p_n/q_n)$ do converge to $\Sigma(\alpha)$ in Hausdorff metric and also in measure [14]. This alone, however, does not, in general, imply convergence of logarithmic capacities [15]. The purpose of this note is to set up the stage for the study of convergence of capacities and prove that convergence of logarithmic capacities also does hold when the Lyapunov exponent is zero on $\Sigma(\alpha)$, as a corollary of Avila's global theory [3]. We expect the result to also hold in higher generality, but this will be addressed in a future publication.

Capacity is a set function that arises in potential theory as an analogue of the physical concept of the electrostatic capacity. The capacity of a set in the complex plane is called logarithmic capacity, as in two-dimensions the potential function becomes logarithm.

The capacity of a set in the complex plane is obtained by considering the infimum of the logarithmic energy over the Borel probability measures supported on this set. The extremal nature of the logarithmic capacity is observed by two characterizations: one reflects the geometry of the set, and the other depends on the extremal polynomials on this set.

The nth diameter of a compact set K in the complex plane is defined as the maximum of the geometric mean of the distances between all pairs of n points, where the maximum is taken over subset of K with cardinality n. The limit of the nth diameter of K as n tends to ∞ is called the transfinite diameter of K, which is same as its logarithmic capacity.

The nth Chebyshev polynomial of a compact set K in the complex plane is defined as the unique nth degree monic polynomial minimizing the supremum norm on K over monic polynomials of the same degree. That minimizing norm value is the nth Chebyshev number of K. The limit of the nth root of the nth Chebyshev number of K is again nothing but the logarithmic capacity of K.

Our main result (Theorem 4.6) says that if the rational sequence $\{p_n/q_n\}_n$ converges to the irrational α, and the Lyapunov exponent of $H_{\alpha,\theta}$ is zero on $\Sigma(\alpha)$, then logarithmic capacities of $S_+(p_n/q_n)$ converge to the logarithmic capacity of $\Sigma(\alpha)$.

The paper is organized as follows.

Section 2 provides preliminaries from logarithmic potential theory and spectral theory required for our proofs.

Section 3 discusses a proof of the main result for the special case of the almost Mathieu operator, which differs from the main result by explicit computation of the approximating capacities.

Section 4 includes the statement and the proof of our main result.

2. Preliminaries

2.1. Logarithmic potential theory. Our result discusses convergence of logarithmic capacities for rational frequency approximants, so let us start by reviewing some basic properties of logarithmic capacity, which can be found, e.g., in [18].

If μ is a finite Borel measure, supported on a compact subset of the complex plane, then the *logarithmic potential* of μ is the function $U^\mu : \mathbb{C} \to (-\infty, \infty]$, defined by

$$U^\mu(z) := \int \log \frac{1}{|z-w|}\, d\mu(w) \tag{2.1}$$

and the *logarithmic energy* of μ is $I(\mu) \in (-\infty, \infty]$, defined by

$$I(\mu) := \int U^\mu(z) d\mu(z) = \int\int \log \frac{1}{|z-w|}\, d\mu(w)\, d\mu(z). \tag{2.2}$$

Letting K be a compact subset of $\mathbb{C}$ and $M(K)$ be the set of Borel probability measures compactly supported inside K, we define the *equilibrium measure* for K, $\mu_K \in M(K)$ as

$$I(\mu_K) = \inf_{\mu \in M(K)} I(\mu). \tag{2.3}$$

Now we are ready to define the logarithmic capacity. The *logarithmic capacity* of a subset E of the complex plane is given by

$$\operatorname{Cap}(E) := \sup_{\mu \in M(E)} \exp\big(-I(\mu)\big). \tag{2.4}$$

In particular if K is compact with equilibrium measure μ_K, then

$$\operatorname{Cap}(K) = \exp\big(-I(\mu_K)\big). \tag{2.5}$$

A Borel set $E \in \mathbb{C}$ is *polar* if $\operatorname{Cap}(E) = 0$, i.e., $I(\mu) = +\infty$ for every $\mu \in M_K$ with $\operatorname{supp}(\mu) \subseteq E$. A property is said to hold *nearly(or quasi) everywhere* if it holds up to a polar set. Two basic, but important examples are $\mathrm{Cap}([-1,1]) = 1/2$ and $\mathrm{Cap}(\overline{\mathbb{D}}) = 1$. Some elementary properties of the logarithmic capacity (Theorem 5.1.2 and Theorem 5.1.3 in [18]) are as follows:

(1) If $E_1 \subseteq E_2$, then $\mathrm{Cap}(E_1) \leq \mathrm{Cap}(E_2)$.
(2) If $E \subset \mathbb{C}$, then $\mathrm{Cap}(E) = \sup\{\mathrm{Cap}(K) \mid \textit{compact } K \subset E\}$.
(3) If $E \subset \mathbb{C}$, then $\mathrm{Cap}(aE + b) = |\alpha| \,\mathrm{Cap}(E)$ for $a, b \in \mathbb{C}$.
(4) If K is a compact subset of $\mathbb{C}$, then $\mathrm{Cap}(K) = \mathrm{Cap}(\partial_e K)$, where ∂_e denotes the exterior boundary.
(5) If $K_1 \supset K_2 \supset K_3 \supset \cdots$ are compact subsets of $\mathbb{C}$ and $K = \cap_n K_n$, then

$$\mathrm{Cap}(K) = \lim_{n\to\infty} \mathrm{Cap}(K_n). \tag{2.6}$$

(6) If $B_1 \subset B_2 \subset B_3 \subset \cdots$ are Borel subsets of $\mathbb{C}$ and $B = \cup_n B_n$, then

$$\mathrm{Cap}(B) = \lim_{n\to\infty} \mathrm{Cap}(B_n). \tag{2.7}$$

A characterization of the logarithmic capacity is given by the transfinite diameter that tells more about its nature. The *nth diameter* of a compact set K, denoted by $\delta_n(K)$, is the supremum of the geometric mean of distances between pairs among n-tuples of K, i.e.,

$$\delta_n(K) := \sup_{z_1,\ldots,z_n\in K} \left\{ \prod_{j<k} |z_j - z_k|^{\frac{2}{n(n-1)}} \right\}. \tag{2.8}$$

An n-tuple $z_1, \ldots, z_n \in K$ for which the supremum is attained is called a *Fekete n-tuple* for K. A classical theorem going back to Fekete and Szegö (Theorem 5.5.2 in [18]) says that the sequence $\{\delta_n(K)\}_{n\geq 2}$ is decreasing and

$$\lim_{n\to\infty} \delta_n(K) = \mathrm{Cap}(K). \tag{2.9}$$

Another characterization of the logarithmic capacity is given by the Chebyshev polynomials. This characterization will be crucial in our proofs.

Definition 2.1 (nth Chebyshev polynomial)**.** Let K be a compact subset of $\mathbb{C}$ and $T_{n,K}$ be the monic polynomial of degree n such that

$$\|T_{n,K}\|_K \leq \|P\|_K \tag{2.10}$$

for any monic polynomial P of degree n, where $\|\cdot\|_K$ denotes the uniform norm over K. Then $T_{n,K}$ is called the *nth Chebyshev polynomial on K*.

Definition 2.2 (nth Chebyshev number)**.** Let K be a compact subset of $\mathbb{C}$ and $T_{n,K}$ be its nth Chebyshev polynomial. The uniform norm of $T_{n,K}$ over K, i.e., $\|T_{n,K}\|_K$ is called the *nth Chebyshev number of* K, denoted by $t_n(K)$.

The sequence $\{t_n(K)\}_n$ is not necessarily convergent. However, subadditivity of logarithms of Chebyshev numbers imply the existence of $\lim_{n\to\infty} t_n^{1/n}(K)$. This limit is called the Chebyshev number of K and another classical result of Szegö (Corollary 5.5.5 in [18]) says that it is nothing but the logarithmic capacity of K, i.e.,

$$\lim_{n\to\infty} t_n(K)^{\frac{1}{n}} = \inf_{n\geq 1} t_n(K)^{\frac{1}{n}} = \mathrm{Cap}(K). \tag{2.11}$$

The alternation theorem will allow us to make the connection with the set $S_+(p/q)$ and its qth Chebyshev polynomial. If P is a real polynomial of degree n, then P has an *alternating set* in a compact set $K \subset \mathbb{R}$ if there exists $\{x_k\}_{k=0}^n$ in K satisfying $x_0 < x_1 < \cdots < x_n$ such that

$$P(x_k) = (-1)^{n-k}\|P\|_K. \tag{2.12}$$

The alternation theorem (Theorem 1.1 in [9]) says that $T_{n,K}$ has an alternating set in K for a compact $K \subset \mathbb{R}$ and conversely, any monic polynomial with an alternating set in K is the Chebyshev polynomial on K. The following result of Peherstorfer and Totik helps us to understand the logarithmic capacity of the spectrum with a rational frequency and estimate the logarithmic capacity of $S_+(p/q)$.

Theorem 2.3 ([22, Theorem 1] and [17, Proposition 1.1])**.** *Let $K = \cup_{j=1}^{l}[a_j, b_j]$. Also, let $T_{n,K}$ and $t_n(K)$ denote the nth Chebyshev polynomial of K and nth Chebyshev number of K, respectively, i.e.,*

$$t_n(K) := \|T_{n,K}(z)\|_K.$$

For a natural number $n \geq 1$ the following are pairwise equivalent.

(a) $\dfrac{t_n(K)}{\mathrm{Cap}(K)^n} = 2.$

(b) *$T_{n,K}$ has $n + l$ extreme points on K.*

(c) $K = \{z \mid T_{n;K}(z) \in [-t_n(K), t_n(K)]\}$.

(d) *If μ_K denotes the equilibrium measure of K, then each $\mu_K([a_j, b_j])$, $j = 1, 2, \ldots, l$, is of the form $\dfrac{q_j}{n}$ with integer q_j ($q_j + 1$ is the number of extreme points on $[a_j, b_j]$).*

(e) *With $\pi(x) = \prod_{j=1}^{l}(x - a_j)(x - b_j)$ the equation*

$$P_n^2(x) - \pi(x)Q_{n-l}^2(x) = c$$

is solvable for the polynomials P_n and Q_{n-l} of degree n and $n-l$, respectively, where c is a positive constant.

The ratio in item (a) is called the Widom Factor [13], denoted by $W_n(K)$, and the study of its estimates and asymptotics is still active [1, 9, 10].

2.2. Spectral theory. On the spectral theory side we have two different pictures depending on whether we have a rational or irrational frequency α, i.e., a periodic or quasi-periodic operator, respectively. All we review below can be found in textbooks or survey papers, e.g., [16, 21] for periodic and [11, 12] for quasi-periodic Schrödonger operators.

For rational α, the operator $H_{\alpha,\theta}$ is periodic, so using standard results of Floquet–Bloch theory one gets basic spectral properties of the spectrum, listed below in Proposition 2.4. These properties are given in terms of the discriminant $t_{p/q}(\theta, E)$ satisfying

$$\sigma(p/q, \theta) = t_{p/q}(\theta, \cdot)^{-1}\big([-2, 2]\big), \tag{2.13}$$

and defined as

$$t_{p/q}(\theta, E) := \operatorname{tr}\left\{ \prod_{j=n-1}^{0} A^E\Big(j\frac{p}{q} + \theta\Big) \right\}, \tag{2.14}$$

where

$$A^E(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.15}$$

Proposition 2.4 ([14, Fact 4.1])**.** *Let p/q be a rational number such that $(p, q) = 1$. Then for any $\theta \in \mathbb{T}$ we have*

(1) *The discriminant $t_{p/q}(\theta, E)$ is a monic polynomial in E of degree q.*
(2) *The discriminant $t_{p/q}(\theta, E)$ splits over the real line with q distinct roots.*
(3) *The discriminant $t_{p/q}(\theta, E)$ is greater than or equal to 2 at all its local maxima and less than or equal to -2 at all its local minima.*
(4) *The spectrum $\sigma(p/q, \theta)$ is the inverse image of $[-2, 2]$ under $t_{p/q}(\theta, E)$ on the real line, consists of q possibly touching but not overlapping bands and is purely absolutely continuous.*

For irrational α, a main tool in our proofs will be the Lyapunov exponent, which characterizes dynamical properties of solutions to the second order eigenvalue difference equation

$$H_{\alpha,\theta}\psi = E\psi \tag{2.16}$$

over $\mathbb{C}^{\mathbb{Z}}$. For fixed $\alpha \in \mathbb{T}$ and $E \in \mathbb{R}$, we define $A^E : \mathbb{T} \to SL(2, \mathbb{C})$ as in (2.15). We call the pair (α, A^E) an analytic Schrödinger cocycle and define it as a linear skew-map on $\mathbb{T} \times \mathbb{C}^2$

$$\big(\alpha, A^E\big)(\theta, v) := (\theta + \alpha, A^E(\theta)v), \qquad \theta \in \mathbb{T},\ v \in \mathbb{C}^2. \tag{2.17}$$

Iteration of (α, A^E) produces the solution of (2.16) with initial condition $(\psi_0, \psi_{-1})^{\mathrm{T}}$ as

$$\big(\alpha, A^E(\theta)\big)^n \left(\theta, \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix}\right) = \left(\theta + n\alpha, \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}\right), \qquad n \in \mathbb{N}. \tag{2.18}$$

The *Lyapunov exponent* of the cocycle (α, A^E) (and hence the discrete Schrödinger operator $H_{\alpha,\theta}$) is defined by

$$L\big(\alpha, A^E\big) := \inf_{n\in\mathbb{N}} \frac{1}{n} \int_{\mathbb{T}} \log \left\| \prod_{j=n-1}^{0} A^E\Big(j\alpha + \theta\Big) \right\| d\theta, \tag{2.19}$$

where $\|\cdot\|$ denotes any matrix norm. By Kingman's sub-additive ergodic theorem, if α is irrational we have

$$L\big(\alpha, A^E\big) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{j=n-1}^{0} A^E\Big(j\alpha + \theta\Big) \right\| \tag{2.20}$$

for a.e. $\theta \in \mathbb{T}$ and if $\alpha = p/q$ is rational with $(p, q) = 1$ we have

$$L\big(p/q, A^E\big) = \frac{1}{q} \int_{\mathbb{T}} \log \rho \left(\prod_{j=q-1}^{0} A^E\Big(j\frac{p}{q} + \theta\Big) \right) d\theta, \tag{2.21}$$

where $\rho(M)$ is the spectral radius of a matrix $M \in M_2(\mathbb{C})$. For fixed irrational α, as a function of $E \in \mathbb{C}$, the Lyapunov exponent has the following properties:

(1) $L\big(\alpha, A^E\big) \geq 0$ for $E \in \mathbb{C}$.
(2) $L\big(\alpha, A^E\big) > 0$ for $E \in \mathbb{C} \setminus \Sigma(\alpha)$.
(3) $L\big(\alpha, A^E\big)$ is a subharmonic function.
(4) $L\big(\alpha, A^E\big)$ is harmonic on $\mathbb{C} \setminus \Sigma(\alpha)$.
(5) $L\big(\alpha, A^E\big) = \log|E| + \mathcal{O}(1/|E|)$ as $|E| \to \infty$.

The Lyapunov exponent also provides a characterization of the spectra. The spectral nature of the almost Mathieu operator, $H_{\alpha,\theta}$ with the potential $v(x) = 2\lambda\cos(2\pi x)$, is characterized by the coupling constant λ as follows:

- *subcritical:* $0 < |\lambda| < 1$ - purely absolutely continuous spectrum,
- *critical:* $|\lambda| = 1$ - purely singular continuous spectrum,
- *supercritical:* $|\lambda| > 1$ - pure point spectrum with exponentially localized eigenfunctions (Anderson localization) for a.e. phase θ.

Recalling that for the almost Mathieu operator

$$L\big(\alpha, A^E\big) = \max\{0, \log|\lambda|\}$$

on the spectrum, we see that the Lyapunov exponent is zero for the subcritical and critical cases, and positive for the supercritical case. Avila obtained spectral characterizations for analytic potentials using complexified Lyapunov exponents [3]. Avila considers $L(\alpha, A^E)$ with α and

$$A^E(x + i\epsilon) = \begin{pmatrix} E - v(x + i\epsilon) & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.22}$$

which is obtained by complexifying the phase θ in (2.15) using analyticity of the potential v. We denote this complexified Lyapunov exponent by $L(\alpha, A^E_\epsilon)$. Then the spectrum of $H_{\alpha,\theta}$ with an analytic potential decomposes into three mutually disjoint sets as follows:

- The energy E is *subcritical* if $L(\alpha, A_\epsilon^E)$ is zero in a neighborhood of $\epsilon = 0$.
- The energy E is *critical* if $L(\alpha, A_\epsilon^E)$ is zero at $\epsilon = 0$, but E is not subcritical.
- The energy E is *supercritical* if $L(\alpha, A_\epsilon^E)$ is positive at $\epsilon = 0$.

For $H_{\alpha,\theta}$ without supercritical energies, the logarithmic capacity of the spectrum is provided by a result of Simon, which was given for ergodic families of orthogonal polynomials on the real line [20]. Let us state this result in our setting of quasi-periodic Schrödinger operators.

Theorem 2.5 ([20, Theorem 1.15])**.** *Let α be irrational and, $\Sigma(\alpha)$ and $L\big(\alpha, A^E\big)$ be the spectrum and the Lyapunov exponent of $H_{\alpha,\theta}$ respectively. Also let $\mu_{\Sigma(\alpha)}$ denote the equilibrium measure of $\Sigma(\alpha)$. Then the following are equivalent:*

(a) $L(\alpha, A^E) = 0$ *for* $\mu_{\Sigma(\alpha)}$*-a.e.* E.

(b) $\mathrm{Cap}\,\big(\Sigma(\alpha)\big) = 1$.

3. Almost Mathieu operator as a prototype

We start by providing our result in the special case of the almost Mathieu operator. We have three reasons for dealing with it separately: first, it reflects the core ideas of the proof for our main result; second, we can explicitly compute capacities of $S_+(p_n/q_n)$ rather than approximating them; third, using Aubry duality [2,5], we get the result also for the supercritical almost Mathieu operator.

Theorem 3.1. *Let $H_{\alpha,\theta}$ be the almost Mathieu operator (AMO) with potential $v(x) = 2\lambda\cos(2\pi x)$ and $\alpha \in \mathbb{T}$ be irrational such that $p_n/q_n \to \alpha$ and $(p_n, q_n) = 1$. Then*

$$\lim_{p_n/q_n \to \alpha} \mathrm{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) = \mathrm{Cap}\,\big(\Sigma(\alpha)\big). \tag{3.1}$$

Proof. For a given rational number p/q satisfying $(p, q) = 1$, the discriminant is a q-periodic function, so we have

$$t_{\frac{p}{q}}(\theta, E) = \sum_{k\in\mathbb{Z}} a_{q,k}(E) e^{2\pi i q k\theta}. \tag{3.2}$$

Since $v(x) = 2\lambda\cos(2\pi x)$ is a trigonometric polynomial of degree 1, for the AMO, equation (3.2) turns to the famous Chambers' formula [8] (see also [6] for a proof):

$$t_{p/q}(\theta, E) = a_{q,0}(E) - 2\lambda^q \cos(2\pi q\theta). \tag{3.3}$$

Therefore for any fixed $\theta \in \mathbb{T}$ we have

$$\sigma\left(\frac{p}{q}, \theta\right) = t_{\frac{p}{q}}(\theta, \cdot)^{-1}[-2, 2] = a_{q,0}(E)^{-1}\big[-2 + 2\lambda^q\cos(2\pi q\theta), 2 + 2\lambda^q\cos(2\pi q\theta)\big] \tag{3.4}$$

and hence

$$S_+\left(\frac{p}{q}\right) = \bigcup_{\theta\in\mathbb{T}} \sigma\left(\frac{p}{q},\theta\right) = a_{q,0}(E)^{-1}\big[-2-2|\lambda|^q,\, 2+2|\lambda|^q\big]. \tag{3.5}$$

Noting item (3) of Proposition 2.4, by (3.3) absolute value of the critical values of $a_{q,0}$ are greater than or equal to $2(1+|\lambda|^q)$. This observation, combined with (3.5) and the fact that $a_{q,0}$ is a monic polynomial of degree q imply that $a_{q,0}$ is the qth Chebyshev polynomial of $S_+(p/q)$. Then by items (c) and (a) of Theorem 2.3 and recalling that $t_n(K)$ denotes the nth Chebyshev number of the set K, we have

$$t_q\left(S_+\left(\frac{p}{q}\right)\right) = 2+2|\lambda|^q \tag{3.6}$$

and

$$2 = \frac{t_q\left(S_+\left(\frac{p}{q}\right)\right)}{\mathrm{Cap}^q\left(S_+\left(\frac{p}{q}\right)\right)} = \frac{2+2|\lambda|^q}{\mathrm{Cap}^q\left(S_+\left(\frac{p}{q}\right)\right)} \tag{3.7}$$

respectively. Therefore we get the logarithmic capacity of $S_+(p/q)$ as

$$\mathrm{Cap}\left(S_+\left(\frac{p}{q}\right)\right) = \left(1+|\lambda|^q\right)^{1/q}. \tag{3.8}$$

On the other hand for irrational frequency $\alpha\in\mathbb{T}$, it is known that (equation (1.37) of [20])

$$\mathrm{Cap}(\Sigma(\alpha)) = \exp\Big(\int L(\alpha, A^E)\, d\mu_{\Sigma(\alpha)}(E)\Big), \tag{3.9}$$

where $L(\alpha, A^E)$ and $\mu_{\Sigma(\alpha)}$ denote the Lyapunov exponent and the equilibrium measure of $\Sigma(\alpha)$ respectively. For the almost Mathieu operator, Aubry duality [2, 5] and the continuity of the Lyapunov exponent [7] imply that

$$L(\alpha, A^E) = \max\big\{0, \log|\lambda|\big\} \tag{3.10}$$

on the spectrum $\Sigma(\alpha)$, so by (3.9) we get the logarithmic capacity of $\Sigma(\alpha)$ as

$$\mathrm{Cap}(\Sigma(\alpha)) = \max\big\{1, |\lambda|\big\}. \tag{3.11}$$

Now for any irrational frequency α satisfying $p_n/q_n\to\alpha$ with $(p_n,q_n)=1$, we can use (3.8) and (3.11) to get the desired result as

$$\begin{aligned}\lim_{p_n/q_n\to\alpha} \mathrm{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) &= \lim_{n\to\infty}\left(1+|\lambda|^{q_n}\right)^{1/q_n}\\ &= \begin{cases}1 & \text{if } |\lambda|\le 1\\ |\lambda| & \text{if } |\lambda|>1\end{cases} = \mathrm{Cap}(\Sigma(\alpha)).\end{aligned} \tag{3.12}$$

The theorem is proved. □

4. Analytic quasi-periodic Schrödinger operators without supercritical energies

We consider analytic quasi-periodic Schrödinger operators without supercritical energies according to the classification of the energies that we discussed in Section 2. Our main approach will be to obtain lower and upper estimates on logarithmic capacities of $S_+(p_n/q_n)$ and then to show that in the limit they agree with the logarithmic capacity of $\Sigma(\alpha)$.

Obtaining the lower bound will be easier as we use Chebyshev polynomial properties of the discriminant and connections with the logarithmic capacity.

Proposition 4.1. *Let p/q be rational such that $(p,q)=1$. Then for any $\theta \in \mathbb{T}$ we have*

$$1 = \operatorname{Cap}\left(\sigma\left(\frac{p}{q},\theta\right)\right) \le \operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right).$$

Proof. The discriminant $t_{\frac{p}{q}}(\theta,E)$ is the qth Chebyshev polynomial of $\sigma\left(\frac{p}{q},\theta\right)$ for any rational p/q and $\theta \in \mathbb{T}$ by the alternation theorem. We also know that

$$\left\| t_{\frac{p}{q}}(\theta,\cdot)\right\|_{\sigma\left(\frac{p}{q},\theta\right)} = 2 \tag{4.1}$$

and

$$\sigma\left(\frac{p}{q},\theta\right) = t_{\frac{p}{q}}(\theta,\cdot)^{-1}[-2,2] \tag{4.2}$$

consists of q bands. Therefore by Theorem 2.3

$$\frac{\left\| t_{\frac{p}{q}}(\theta,\cdot)\right\|_{\sigma\left(\frac{p}{q},\theta\right)}}{\operatorname{Cap}\left(\sigma\left(\frac{p}{q},\theta\right)\right)} = 2 \tag{4.3}$$

and hence by (4.3) and monotonicity of the logarithmic capacity we get

$$1 = \operatorname{Cap}\left(\sigma\left(\frac{p}{q},\theta\right)\right) \le \operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right). \tag{4.4}$$

□

Next, we get the upper estimate in several steps. In addition to seeing the discriminant as a Chebyshev polynomial and making connections with the logarithmic capacity, we will use some elements of Avila's global theory [3] to estimate Fourier expansion of the discriminant in the phase and relate it to the Lyapunov exponent.

The first step is again using the relation between the logarithmic capacity and Chebyshev polynomials.

Lemma 4.2. *Let p/q be rational such that $(p,q)=1$. Then there exist $\theta_1 \in \mathbb{T}$ and $E_1 \in S_+(p/q)$ such that*

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \le \left(\frac{\left|t_{\frac{p}{q}}(\theta_1,E_1)\right|}{2}\right)^{1/q}.$$

Proof. Let's consider the set

$$S_q := T^{-1}_{S_+\left(\frac{p}{q}\right),q}\left[-\left\|T_{S_+\left(\frac{p}{q}\right),q}\right\|_{S_+\left(\frac{p}{q}\right)}, \left\|T_{S_+\left(\frac{p}{q}\right),q}\right\|_{S_+\left(\frac{p}{q}\right)}\right],$$

where $T_{S_+(p/q),q}$ denotes the qth Chebyshev polynomial for $S_+(p/q)$. Note that by definition $S_+(p/q)$ is a subset of S_q, so

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \leq \operatorname{Cap}(S_q). \tag{4.5}$$

Also $\|T_{S_+(p/q),q}\|_{S_+(p/q)}$ is the qth Chebyshev number for S_q by definition, which makes S_q satisfy item (c) of Theorem 2.3. Therefore S_q satisfies item (a) of the same theorem and we get

$$\left\|T_{S_+(p/q),q}\right\|_{S_+(p/q)} = \left\|T_{S_q,q}\right\|_{S_q} = 2Cap^q(S_q). \tag{4.6}$$

Combining (4.5) and (4.6) we get

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \leq \operatorname{Cap}(S_q) \leq \left(\frac{\|T_{S_q,q}\|_{S_q}}{2}\right)^{1/q} = \left(\frac{\|T_{S_+(p/q),q}\|_{S_+(p/q)}}{2}\right)^{1/q}. \tag{4.7}$$

Now let $E_1 \in S_+(p/q)$ satisfies the sup-norm of $T_{S_+(p/q),q}$ on $S_+(p/q)$. Since the set $S_+(p/q)$ is defined as

$$S_+\left(\frac{p}{q}\right) = \bigcup_{\theta\in\mathbb{T}} \sigma\left(\frac{p}{q},\theta\right) = \bigcup_{\theta\in\mathbb{T}} t_{\frac{p}{q}}(\theta,\cdot)^{-1}[-2,2],$$

there exists $\theta_1 \in \mathbb{T}$ such that

$$\left|t_{\frac{p}{q}}(\theta_1,E_1)\right| = \max_{E\in S_+(p/q)} \left|t_{\frac{p}{q}}(\theta_1,E)\right|. \tag{4.8}$$

We know that $t_{\frac{p}{q}}(\theta_1,E)$ is a qth degree monic polynomial of E, so $T_{S_+(p/q),q}$ being the qth Chebyshev polynomial for $S_+(p/q)$ and (4.8) imply

$$\left\|T_{S_+(p/q),q}\right\|_{S_+(p/q)} \leq \left|t_{\frac{p}{q}}(\theta_1,E_1)\right|. \tag{4.9}$$

Finally combining (4.7) and (4.9) we get the desired result

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \leq \left(\frac{\left\|T_{S_+(p/q),q}\right\|_{S_+(p/q)}}{2}\right)^{1/q} \leq \left(\frac{\left|t_{\frac{p}{q}}(\theta_1,E_1)\right|}{2}\right)^{1/q}. \tag{4.10}$$

The lemma is proved. □

The second step is getting uniform estimates for the Fourier coefficients of the discriminants.

Lemma 4.3. *Let $\delta > 0$ such that $v(\theta)$ extends holomorphically to a neighborhood of $|Im(\theta + iy)| \leq \delta$. Let $\alpha \in \mathbb{T}$ be irrational, $p_n/q_n \to \alpha$ with $(p_n, q_n) = 1$,*

$$S := \left(\bigcup_{n=1}^{\infty} S_+ \left(\frac{p_n}{q_n} \right) \right) \bigcup \Sigma(\alpha)$$

and

$$t_{\frac{p_n}{q_n}}(\theta, E) = \sum_{k \in \mathbb{Z}} a_{q_n,k}(E) e^{2\pi i q_n k \theta}.$$

Then for any $0 < \delta_1 < \delta$, there exists $K \in \mathbb{N} \cup \{0\}$ such that

$$\max_{\theta \in \mathbb{T}} \left| \sum_{|k| > K} a_{q_n,k}(E) e^{2\pi i q_n k(\theta + i\epsilon)} \right| \leq C_{\delta_1} e^{-2\pi q_n \delta_1} \tag{4.11}$$

for all $0 \leq |\epsilon| \leq \delta_1$ and for all $E \in S$, where C_{δ_1} is independent of q_n and

$$t_{\frac{p_n}{q_n}}(\theta + i\epsilon, E) = \sum_{k \in \mathbb{Z}} a_{q_n,k}(E) e^{2\pi i q_n k(\theta + i\epsilon)}.$$

Proof. Let us recall that

$$A^E(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$t_{\frac{p}{q}}(\theta, E) = \operatorname{tr} \left(\prod_{j=q-1}^{0} A^E\Big(j\frac{p}{q} + \theta\Big) \right).$$

Since the set S is bounded and V extends holomorphically to a neighborhood of $|\operatorname{Im}(z)| \leq \delta$ we have

$$\left\| A^E \right\|_\delta \leq C(1 + \|v\|_\delta) \tag{4.12}$$

for $E \in S$, where

$$\left\| A^E \right\|_\delta := \sup_{|\operatorname{Im}(z)| \leq \delta} \left\| A^E(z) \right\| \quad \text{and} \quad \|v\|_\delta := \sup_{|\operatorname{Im}(z)| \leq \delta} \|v(z)\|.$$

Let's call the uniform upper bound in (4.12) $C_\delta := C(1 + \|v\|_\delta)$. Now writing

$$t_{\frac{p_n}{q_n}}(x + i\epsilon, E) = \sum_{k \in \mathbb{Z}} a_{q_n,k}(E) e^{2\pi i q_n k(x + i\epsilon)},$$

analyticity in a neighborhood of $|\operatorname{Im}(z)| \leq \delta$ provides the following decay of Fourier coefficients

$$|a_{q_n,k}(E)| \leq 2C_\delta^{q_n} e^{-2\pi |k| q_n \delta} \tag{4.13}$$

for $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Choosing $K \in \mathbb{N}$ sufficiently large such that

$$2\pi\delta K > \log C_\delta, \tag{4.14}$$

we get exponential decay of $a_{q_n,k}(E)$ in (4.13) independent of n for $|k| > K$. Therefore for any fixed $0 < \delta_1 < \delta$, there exist constants K (depending only on C_δ and δ_1) and C_{δ_1} (depending on δ_1) such that

$$\max_{x\in\mathbb{T}} \left| \sum_{|k|>K} a_{q_n,k}(E) e^{2\pi i q_n k(x+i\epsilon)} \right| \leq C_{\delta_1} e^{-2\pi q_n \delta_1} \tag{4.15}$$

for all $0 \leq |\epsilon| \leq \delta_1$ and for all $E \in S$. □

The third step is to connect the Fourier coefficients of the approximating discriminants with the Lyapunov exponent of $H_{\alpha,\theta}$. For this, we use part of Lemma 3.5 in [14] with a slight change of replacing $\Sigma(\alpha)$ with the set S defined below (4.17). This lemma was proved in Appendix A of [14] and it is a more detailed version of Avila's theorem on quantization of the acceleration, which was introduced in [3]. By changing $\Sigma(\alpha)$ to S we lose no generality since the uniformity of the result on $\Sigma(\alpha)$ follows from

$$\left\| A^E \right\|_\delta \leq C(1 + \|v\|_\delta) \tag{4.16}$$

for $E \in \Sigma(\alpha)$ (see (A.2) in [14]), which is also valid for $E \in S$.

Lemma 4.4 ([14, Lemma 3.5]). *Let $\delta > 0$ such that $v(\theta)$ extends holomorphically to a neighborhood of $|Im(\theta + iy)| \leq \delta$. Let $\alpha \in \mathbb{T}$ be irrational and $p_n/q_n \to \alpha$ with $(p_n, q_n) = 1$ and*

$$S := \left(\bigcup_{n=1}^{\infty} S_+\left(\frac{p_n}{q_n}\right) \right) \bigcup \Sigma(\alpha). \tag{4.17}$$

Then for any $0 < \delta_1 < \delta$, there exists $K \in \mathbb{N} \cup \{0\}$ such that

$$L(\alpha, A^E_\epsilon) = \frac{1}{q_n} \log_+ \left(\max_{-K\leq k\leq K} \left| a_{q_n,k(E)} \right| e^{-2\pi\epsilon k q_n} \right) + o(1) \tag{4.18}$$

uniformly over $0 \leq |\epsilon| \leq \delta_1$ and uniformly over $E \in S$ as $p_n/q_n \to \alpha$.

Now we are ready to prove the upper estimate on the logarithmic capacity of $S_+(p_n/q_n)$ as a subexponentially growing sequence.

Proposition 4.5. *Let $\alpha \in \mathbb{T}$ be irrational, $p_n/q_n \to \alpha$ with $(p_n, q_n) = 1$ and $H_{\alpha,\theta}$ satisfy following properties:*

- *The potential $v(x)$ is analytic.*
- *The Lyapunov exponent is zero on $\Sigma(\alpha)$ or equivalently $\Sigma(\alpha)$ consists of subcritical or critical energies.*

Then

$$\operatorname{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) \leq \left(\rho(q_n)\right)^{1/q_n},$$

where $\{\rho(q_n)\}_n$ grows subexponentially, i.e.,

$$\lim_{n\to\infty} \frac{\log \rho(q_n)}{q_n} = 0$$

Proof. Lemma 4.2 implies

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \le \left(\frac{\left|t_{\frac{p}{q}}(\theta_1, E_1)\right|}{2}\right)^{1/q} \tag{4.19}$$

and by letting $\epsilon = 0$ in Lemma 4.3 we get

$$\left|t_{\frac{p}{q}}(\theta_1, E_1)\right| \le C_{\delta_1} e^{-2\pi q_n \delta_1} + \left|\sum_{-K \le k \le K} a_{q_n,k}(E_1) e^{2\pi i q_n k \theta_1}\right| \tag{4.20}$$

$$\le C_{\delta_1} e^{-2\pi q_n \delta_1} + (2K+1) \max_{-K \le k \le K} |a_{q_n,k}(E_1)|. \tag{4.21}$$

Now let us recall

$$S := \left(\bigcup_{n=1}^{\infty} S_+\left(\frac{p_n}{q_n}\right)\right) \bigcup \Sigma(\alpha).$$

Then from Lemma 4.4 we have

$$L\big(\alpha, A^E\big) = \frac{1}{q_n} \log_+ \Big(\max_{-K \le k \le K} |a_{q_n,k}(E)|\Big) + o(1) \tag{4.22}$$

uniformly over $E \in S$ as $p_n/q_n \to \alpha$. Note that E_1 may not belong to $\Sigma(\alpha)$, so $L(\alpha, A^{E_1})$ is not necessarily zero, but we assumed that $L(\alpha, A^E) = 0$ for any $E \in \Sigma(\alpha)$. Therefore, using the convergence of $S_+(p_n/q_n)$ to $\Sigma(\alpha)$ in Hausdorff metric [4] and continuity of the Lyapunov exponent $L(\alpha, A^E)$ in $E \in \mathbb{R}$ [7] we have

$$L\big(\alpha, A^{E_1}\big) \to 0 \qquad \text{as } \; p_n/q_n \to \alpha. \tag{4.23}$$

Therefore by (4.22) we have

$$\frac{1}{q_n} \log_+ \Big(\max_{-K \le k \le K} |a_{q_n,k}(E_1)|\Big) \to 0 \qquad \text{as } \; p_n/q_n \to \alpha,$$

i.e., $\max_{-K \le k \le K} |a_{q_n,k}(E_1)|$ grows subexponentially and hence the upper estimate (4.21) grows subexponentially as $p_n/q_n \to \alpha$ (recall that K and C_{δ_1} are independent of n). Finally introducing

$$\rho(q_n) := \frac{1}{2}\Big(C_{\delta_1} e^{-2\pi q_n \delta_1} + (2K+1) \max_{-K \le k \le K} |a_{q_n,k}(E_1)|\Big) \tag{4.24}$$

and recalling Lemma 4.2 we get the desired result as

$$\operatorname{Cap}\left(S_+\left(\frac{p}{q}\right)\right) \le \left(\frac{\left|t_{\frac{p}{q}}(\theta_1, E_1)\right|}{2}\right)^{1/q} \le \big(\rho(q_n)\big)^{1/q_n}, \tag{4.25}$$

where $\{\rho(q_n)\}_n$ grows subexponentially as $p_n/q_n \to \alpha$. □

Finally we are ready to prove our main result.

Theorem 4.6. *Let $\alpha \in \mathbb{T}$ be irrational and $p_n/q_n \to \alpha$ with $(p_n, q_n) = 1$. Also, let $H_{\alpha,\theta}$ be a quasi-periodic Schrödinger operator (1.1) with an analytic potential v. If $L(\alpha, E) = 0$ for $E \in \Sigma(\alpha)$, then*

$$\lim_{p_n/q_n \to \alpha} \operatorname{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) = \operatorname{Cap}\big(\Sigma(\alpha)\big). \tag{4.26}$$

Proof. By Proposition 4.1 and Proposition 4.5 we have

$$1 \leq \operatorname{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) \leq \big(\rho(q_n)\big)^{1/q_n}. \tag{4.27}$$

Since $\{\rho(q_n)\}_n$ grows subexponentially, the right end of (4.27) converges to 1 as p_n/q_n converges to α, so by the squeeze theorem we get

$$\lim_{p_n/q_n \to \alpha} \operatorname{Cap}\left(S_+\left(\frac{p_n}{q_n}\right)\right) = 1. \tag{4.28}$$

On the other hand, our assumption of zero Lyapunov exponent on the spectrum and Theorem 2.5 imply that $\operatorname{Cap}(\Sigma(\alpha)) = 1$, so we get the desired result. □

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(Суб)критичні оператори і спектральна ємність їх раціонально-частотних наближень

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Ми розглядаємо одно-частотні квазіперіодичні оператори Шредингера з аналітичними потенціалами. Позначивши через S_+ об'єднання спектрів, взяте за фазами, ми вивчаємо неперервність логарифмічної ємності відносно раціональних наближень. Ми доводимо, що якщо показник Ляпунова є нульовим на спектрі, то ємність спектра для ірраціональної частоти наближувана ємностями спектра для її раціональних наближень.

Ключові слова: логарифмічна ємність, квазіперіодичні оператори Шредінгера, оператор майже Матьє, показник Ляпунова