Journal of Mathematical Physics, Analysis, Geometry, **21** (2025), No. 2, 179–202
doi: https://doi.org/10.15407/mag21.02.03

A Logarithmic Extensibility Criterion for a Keller–Segel–Navier–Stokes System in a Bounded Domain

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We consider a Keller–Segel–Navier–Stokes system in a three-dimensional (3D) bounded domain and prove a logarithmic blow-up criterion of the local strong solutions. The L^p method, L^∞-method and the maximal regularity estimates of the parabolic equation are used.

Key words: Keller–Segel–Navier–Stokes, blow-up criterion, bounded domain

Mathematical Subject Classification 2020: 22E46, 53C35, 57S20, 35Q30

1. Introduction

We consider a 3D Keller–Segel–Navier–Stokes system [16]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u + n \nabla \phi = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla p) - \nabla \cdot (n \nabla q), \tag{1.3}$$

$$\partial_t p + u \cdot \nabla p - \Delta p = -np, \tag{1.4}$$

$$\partial_t q + u \cdot \nabla q - \Delta q + q = n \qquad \text{in } \Omega \times (0, \infty), \tag{1.5}$$

$$u \cdot \nu = 0, \ \operatorname{rot} u \times \nu = 0, \ \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = \frac{\partial q}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.6}$$

$$(u, n, p, q)(\cdot, 0) = (u_0, n_0, p_0, q_0)(\cdot) \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{1.7}$$

Here u denotes the velocity of the fluid and π is the pressure, n, p and q denote the density of amoebae, oxygen and chemical attractant, respectively. The function ϕ is a potential function. Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, ν is the unit outward normal vector to $\partial\Omega$. We denote the vorticity by $\omega := \operatorname{rot} u$.

Regarding how amoebae influence the fluid flow, besides their known consumption of oxygen and secretion of chemical attractants, their movements in the environment indeed exert significant impacts on the fluid. Their pseudopodial motility not only alters their own positions but also generates complex fluid dynamic effects through interactions with the surrounding fluid. These effects include fluid perturbations, vortex formation, and redistribution of the flow field. Such influences are crucial factors that should be taken into account when establishing the Keller–Segel–Navier–Stokes system model. Concerning the physical laws that ensure the validity of our model, it is based on the Navier–Stokes equation and the mass transport equation. The Navier–Stokes equation describes the motion of viscous fluids, while the mass transport equation simulates the diffusion and transport processes of the chemical attractants secreted by amoebae in the fluid. These equations have been widely applied in fluid mechanics and biology and are generally recognized as effective tools for describing the behavior of such systems. By coupling these equations, we establish a mathematical model capable of describing the interactions between amoebae and the fluid environment.

When $\phi = 0$, (1.1) and (1.2) reduce to the well-known incompressible Navier–Stokes system. Fan–Zhou [7] (see also [19, 20]) showed an extensibility criterion

$$\int_0^T \frac{\|\omega\|_{BMO}}{\log(e + \|\omega\|_{BMO})} < \infty. \tag{1.8}$$

Here, BMO is the space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO} := \|f\|_{L^2} + [f]_{BMO}$$

with

$$[f]_{BMO} := \sup_{\substack{x\in\Omega \\ r\in(0,d)}} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| \, \mathrm{d}y,$$

$$f_{\Omega_r(x)} := \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) \, \mathrm{d}y,$$

where $\Omega_r(x) := B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω. $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

Definition 1.1. Let $f \in L^{p,q}$ be such that

$$\left(\frac{p}{q} \int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty,$$

where $f^*(t)$ is the nonincreasing function equimeasurable with $|f|$ on $(0, \infty)$. We say that f belongs to the Lorentz space $L^{p,\infty} \equiv L^p_w$ if

$$\mathrm{mes}\{x \in \Omega : |f(x)| > \alpha\} \le A\alpha^{-p} \quad \text{for all } \alpha > 0.$$

On the other hand, when $u = 0$, (1.3), (1.4) and (1.5) reduce to the Keller–Segel system [12–14], which was studied in [5, 6, 10, 11, 22, 27, 28].

In [16], the authors showed the existence and uniqueness of mild solutions. In [8], Fan and Zhao proved some bolw-up criteria when $q = 0$.

We suppose

$$\nabla p \in L^{\frac{2r}{r-3}}(0, T; L^r_w), \nabla q \in L^{\frac{2s}{s-3}}(0, T; L^s_w) \quad \text{with } 3 < r, s \le \infty. \tag{1.9}$$

Here we point out that the blow-up criteria (1.8) and (1.9) can be used in the numerical simulation of the biological model.

The aim of this paper is to prove a regularity criterion of local strong solutions to problem (1.1)–(1.7). We prove the following theorem.

Theorem 1.1. *Let $u_0 \in H^3(\Omega)$, $n_0, p_0, q_0 \in H^2(\Omega)$, $\operatorname{div} u_0 = 0$, $n_0, p_0, q_0 \geq 0$ in Ω, $u_0 \cdot \nu = 0$, $\operatorname{rot} u_0 \times \nu = 0$, $\frac{\partial n_0}{\partial \nu} = \frac{\partial p_0}{\partial \nu} = \frac{\partial q_0}{\partial \nu} = 0$ on $\partial\Omega$. Suppose that $\phi = \phi(x)$ is a smooth function. Let (u, π, n, p, q) be a local strong solution to problem (1.1)–(1.7). If (1.8) and (1.9) hold true for some $0 < T < \infty$, then the solution can be extended beyond $T > 0$.*

Remark 1.1. Compared with [16], the novelty of Theorem 1.1 is that we consider the problem in a bounded domain.

We observe that (1.8) and (1.9) are optimal from the point of view of scaling invariance. Indeed, when neglecting the linear lower order term q in (1.5), we notice that (1.1)–(1.5) is invariant under the scaling transform:

$$\begin{aligned} u &\to u_\lambda := \lambda u(\lambda^2 t, \lambda x), & \pi &\to \pi_\lambda := \lambda^2 \pi(\lambda^2 t, \lambda x), \\ n &\to n_\lambda := \lambda^2 n(\lambda^2 t, \lambda x), & p &\to p_\lambda := p(\lambda^2 t, \lambda x), \\ q &\to q_\lambda := q(\lambda^2 t, \lambda x), & \phi &\to \phi_\lambda := \phi(\lambda^2 t, \lambda x). \end{aligned}$$

This implies that (1.8) and (1.9) are optimal in this sense.

In the following proofs, we use the Gagliardo–Nirenberg inequality [15],

$$\|u\|_{L^{\frac{2r}{r-2},2}} \leq C\|u\|_{L^2}^{1-\frac{3}{r}} \|u\|_{H^1}^{\frac{3}{r}} \quad \text{with } 3 < r < \infty, \tag{1.10}$$

and the generalized Hölder inequality [23],

$$\|uv\|_{L^{p,q}} \leq C\|u\|_{L^{p_1,q_1}} \|v\|_{L^{p_2,q_2}} \tag{1.11}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

2. Preliminaries

In this section we collect some lemmas which will be used in the proof.

Lemma 2.1 (Poincaré inequality)**.** *Let Ω be a bounded domain with smooth boundary, and let w be a smooth vector satisfying $w \cdot \nu = 0$ or $w \times \nu = 0$ on the boundary $\partial\Omega$. Then*

$$\|w\|_{L^p} \leq C\|\nabla w\|_{L^p} \tag{2.1}$$

holds for $2 \leq p < \infty$.

Proof. If $p = 2$, then the proof was given in Lions [17, (6.47), page 75]. We assume $2 < p < \infty$. Using the Gagliardo–Nirenberg inequality and the case $p = 2$, we see that

$$\begin{aligned} \|w\|_{L^p} &\leq C\|w\|_{L^2}^{1-\theta} \|\nabla w\|_{L^p}^{\theta} + C\|w\|_{L^2} \\ &\leq C\|\nabla w\|_{L^2}^{1-\theta} \|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^2} \\ &\leq C\|\nabla w\|_{L^p}^{1-\theta} \|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^p} \leq C\|\nabla w\|_{L^p}. \end{aligned}$$

This completes the proof. □

Lemma 2.2 ([26]). *There holds*

$$\|\nabla w\|_{L^p} \le C(\|\mathrm{div}\, w\|_{L^p} + \|\mathrm{rot}\, w\|_{L^p}) \tag{2.2}$$

for any smooth vector w satisfying $w \cdot \nu = 0$ or $w \times \nu = 0$ on $\partial\Omega$ and $1 < p < \infty$.

Lemma 2.3 ([25]). *There holds*

$$\begin{aligned} &-\int_\Omega \Delta f \cdot f|f|^{p-2}\mathrm{d}x \\ &= \int_\Omega |f|^{p-2}|\nabla f|^2\mathrm{d}x + 4\frac{p-2}{p^2}\int_\Omega |\nabla |f|^{\frac{p}{2}}|^2\mathrm{d}x - \int_{\partial\Omega} |f|^{p-2}(\nu\cdot\nabla)f\cdot f\mathrm{d}S \end{aligned} \tag{2.3}$$

for any smooth vector f and $1 < p < \infty$.

Lemma 2.4 ([24, Lemma 2.2]). *Assume that u is sufficiently smooth, satisfying the boundary condition (1.3) on $\partial\Omega$. Then the following identity for $\omega :=$ $\mathrm{rot}\, u$ holds:*

$$-\frac{\partial\omega}{\partial\nu}\cdot\omega = (\epsilon_{1jk}\epsilon_{1\beta\gamma} + \epsilon_{2jk}\epsilon_{2\beta\gamma} + \epsilon_{3jk}\epsilon_{3\beta\gamma})\omega_j\omega_\beta\partial_k\nu_\gamma \tag{2.4}$$

on $\partial\Omega$, where ϵ_{ijk} denotes the totally anti-symmetric tensor such that $(a\times b)_i = \epsilon_{ijk}a_jb_k$, and $\omega := (\omega_j)$, $\nu := (\nu_j)$.

Lemma 2.5 ([1, Lemma 7.44], [18, Corollary 1.7]). *There holds*

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{2.5}$$

for any smooth f and $1 < p < \infty$.

Lemma 2.6 ([3]). *There holds*

$$\|f\|_{L^p(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{q}{p}}\|f\|_{BMO(\Omega)}^{1-\frac{q}{p}} \tag{2.6}$$

for any smooth f and $1 \le q < p < \infty$.

Lemma 2.7 ([21]). *Let $1 \le q, r \le \infty$ and $0 \le j < m$. Then the following inequalities hold:*

$$\|D^j u\|_{L^p} \le C_1\|D^m u\|_{L^r}^a\|u\|_{L^q}^{1-a} + C_2\|u\|_{L^s}, \tag{2.7}$$

for any function $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \quad \frac{j}{m} \le a \le 1, \tag{2.8}$$

$s > 0$ is arbitrary, and C_1 and C_2 depend only on Ω, m and n.

Lemma 2.8 ([7]). *Let u be a solution to the Stokes system*

$$-\Delta u + \nabla \pi = f \text{ and } \operatorname{div} u = 0 \quad \text{in } \Omega \tag{2.9}$$

with the boundary condition

$$u \cdot \nu = 0, \quad \operatorname{rot} u \times \nu = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

Then it holds

$$\|u\|_{H^2} \leq C\|f\|_{L^2}. \tag{2.11}$$

□

Lemma 2.9 ([4,9]). *Let s be a non-negative real number. If u belonging to $H^2(\Omega)$ is such that $\Delta u \in H^s(\Omega)$ and such that*

$$\operatorname{div} u = 0 \text{ in } \Omega \quad \text{and} \quad u \times \nu = 0 \text{ on } \partial\Omega$$

or such that

$$u \cdot \nu = 0, \quad \operatorname{rot} u \times \nu = 0 \quad \text{on } \partial\Omega,$$

then

$$\|u\|_{H^{s+2}(\Omega)} \leq C\left(\|\Delta u\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{2.12}$$

Using Lemma 2.9, we have

$$\begin{aligned}\|u\|_{H^3} &\leq C\left(\|\Delta u\|_{H^1} + \|u\|_{L^2}\right) = C\left(\|\operatorname{rot} \omega\|_{H^1} + \|u\|_{L^2}\right)\\ &\leq C\left(\|\omega\|_{H^2} + \|u\|_{L^2}\right) \leq C\left(\|\Delta\omega\|_{L^2} + \|\omega\|_{L^2} + \|u\|_{L^2}\right),\end{aligned}$$

which will be used in proving (3.30).

Lemma 2.10. *Let $\operatorname{div} u = 0$, $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ and $q_0 \in L^\infty(\Omega)$ and $q \geq 0$ be a weak solution to the following problem:*

$$\partial_t q + u \cdot \nabla q - \Delta q + q = n + \operatorname{div} g \qquad \text{in } \Omega \times (0,T), \tag{2.13}$$

$$\frac{\partial q}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{2.14}$$

$$q(\cdot,0) = q_0 \qquad \text{in } \Omega. \tag{2.15}$$

Then it holds that

$$\|q\|_{L^\infty(\Omega\times(0,T))} \leq C\left(\|q_0\|_{L^\infty} + \|n\|_{L^k(0,T;L^k)} + \|g\|_{L^m(0,T;L^m)}\right) \tag{2.16}$$

with $k > 3$ and $m > 5$.

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since local existence results can be proved by using standard arguments (see Appendix), we only deal with the a priori estimates.

First of all, from the equations of n,p, and q and the maximum principle, we easily find that

$$n,p,q\geq 0,\quad p\leq C,\quad \int n\mathrm{d}x=\int n_0\mathrm{d}x. \tag{3.1}$$

For any $k\geq 2$, testing (1.3) by n^{k-1}, using (1.10), (1.11), the boundary and incompressibility conditions, and denoting $w:=n^{\frac{k}{2}}$, we derive

$$\begin{aligned}
\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int w^2\mathrm{d}x+\frac{4(k-1)}{k}\int|\nabla w|^2\mathrm{d}x&=C\int w(\nabla p+\nabla q)\cdot\nabla w\mathrm{d}x\\
&\leq C\|\nabla p\|_{L^r_w}\|w\|_{L^{\frac{2r}{r-2},2}}\|\nabla w\|_{L^2}+C\|\nabla q\|_{L^s_w}\|w\|_{L^{\frac{2s}{s-2},2}}\|\nabla w\|_{L^2}\\
&\leq C\|\nabla p\|_{L^r_w}\left(\|w\|_{L^2}^{1-\frac{3}{r}}\|\nabla w\|_{L^2}^{1+\frac{3}{r}}+\|w\|_{L^2}\|\nabla w\|_{L^2}\right)\\
&\quad+C\|\nabla q\|_{L^s_w}\left(\|w\|_{L^2}^{1-\frac{3}{s}}\|\nabla w\|_{L^2}^{1+\frac{3}{s}}+\|w\|_{L^2}\|\nabla w\|_{L^2}\right)\\
&\leq\frac{k-1}{k^2}\|\nabla w\|_{L^2}^2+C\left(1+\|\nabla p\|_{L^r_w}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s_w}^{\frac{2s}{s-3}}\right)\|w\|_{L^2}^2,
\end{aligned}$$

which yields

$$\|n\|_{L^2(0,T;H^1)}+\|n\|_{L^\infty(0,T;L^k)}\leq C\quad\text{for any } k\geq 2. \tag{3.2}$$

Testing (1.1) by u, using (1.2) and (2.2), we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x+\int|\nabla u|^2\mathrm{d}x=-\int n(\nabla\phi)u\mathrm{d}x\leq\|n\|_{L^2}\|\nabla\phi\|_{L^\infty}\|u\|_{L^2}\leq C\|u\|_{L^2},$$

which yields

$$\|u\|_{L^\infty(0,T;L^2)}+\|u\|_{L^2(0,T;H^1)}\leq C. \tag{3.3}$$

Testing (1.4) by p and using (1.2), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int p^2\mathrm{d}x+\int|\nabla p|^2\mathrm{d}x+\int np^2\mathrm{d}x=0,$$

which yields

$$\|p\|_{L^2(0,T;H^1)}\leq C. \tag{3.4}$$

Applying the standard L^∞-estimate of heat equations (Lemma 2.10), it follows from (1.2), (1.5), and (3.2) that

$$\|q\|_{L^\infty(0,T;L^\infty)}\leq C. \tag{3.5}$$

Testing (1.5) by q and using (1.2) and (3.2), we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int q^2\mathrm{d}x+\int|\nabla q|^2\mathrm{d}x+\int q^2\mathrm{d}x=\int nq\mathrm{d}x\leq\|n\|_{L^2}\|q\|_{L^2}\leq C\|q\|_{L^2},$$

which implies

$$\|q\|_{L^2(0,T;H^1)} \leq C. \tag{3.6}$$

Taking rot to (1.1) using (1.2), we get the well-known equation

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - \Delta \omega = -\mathrm{rot}\,(n \nabla \phi). \tag{3.7}$$

Testing the above equation by $|\omega|^{p-2}\omega$ $(2 \leq p < \infty)$, using (1.1), (2.3)–(2.6), and (3.2), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\omega|^p \mathrm{d}x + \int_\Omega |\omega|^{p-2}|\nabla\omega|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int_\Omega |\nabla|\omega|^{\frac{p}{2}}|^2 \mathrm{d}x \\
&\quad = \int_{\partial\Omega} |\omega|^{p-2}(\nu\cdot\nabla)\omega\cdot\omega \mathrm{d}S + \int_\Omega (\omega\cdot\nabla)u\cdot|\omega|^{p-2}\omega\,\mathrm{d}x \\
&\qquad - \int n\nabla\phi\,\mathrm{rot}\left(|\omega|^{p-2}\omega\right)\mathrm{d}x \\
&\quad = -\int_{\partial\Omega} |\omega|^{p-2}\sum_i \epsilon_{ijk}\epsilon_{i\beta\gamma}\omega_j\omega_\beta\partial_k\nu_\gamma\,\mathrm{d}S \\
&\qquad + \int_\Omega (\omega\cdot\nabla)u\cdot|\omega|^{p-2}\omega\,\mathrm{d}x - \int n\nabla\phi\,\mathrm{rot}\,(|\omega|^{p-2}\omega)\,\mathrm{d}x \\
&\quad \leq C\int_{\partial\Omega}|\omega|^p\,\mathrm{d}S + \|\omega\|_{L^{p+1}}^p\|\nabla u\|_{L^{p+1}} + C\|\nabla\phi\|_{L^\infty}\int|\omega|^{p-2}|\nabla\omega|n\,\mathrm{d}x \\
&\quad \leq C\int_{\partial\Omega} f^2\,\mathrm{d}S + C\|\omega\|_{L^{p+1}}^{p+1} + C\int|\omega|^{p-2}|\nabla\omega|n\,\mathrm{d}x \quad (f := |\omega|^{\frac{p}{2}}) \\
&\quad \leq C\|f\|_{L^2(\Omega)}\|f\|_{H^1(\Omega)} + C\|\omega\|_{L^{p+1}}^{p+1} + C\int|\omega|^{p-2}|\nabla\omega|n\,\mathrm{d}x \\
&\quad \leq 2\frac{p-2}{p^2}\int_\Omega|\nabla f|^2\,\mathrm{d}x + C\|\omega\|_{L^p}^p + C\|\omega\|_{BMO}\|\omega\|_{L^p}^p + \frac{1}{2}\int|\omega|^{p-2}|\nabla\omega|^2\,\mathrm{d}x,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p} &\leq C_0\|\omega\|_{L^p}\left(1+\|\omega\|_{BMO}\right) \\
&\leq C_0\|\omega\|_{L^p}\log\left(e+\|\omega\|_{BMO}\right)\frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)} \\
&\leq C_0\|\omega\|_{L^p}\log\left(e+\|u\|_{H^3}\right)\frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\|\omega(t_0)\|_{L^p}}^{\|\omega(t)\|_{L^p}} \mathrm{d}\log\|\omega\|_{L^p} &\leq C_0\log(e+y)\int_{t_0}^t \frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)}\,\mathrm{d}s \\
&\leq \log(e+y)^{C_0\epsilon},
\end{aligned}$$

which implies

$$\int |\omega|^p\,\mathrm{d}x \leq C(e+y)^{C_0\epsilon} \tag{3.8}$$

provided that

$$\int_{t_0}^{t} \frac{1 + \|\omega\|_{BMO}}{\log\left(e + \|\omega\|_{BMO}\right)} \, \mathrm{d}s \le \epsilon \ll 1 \tag{3.9}$$

and $y(t) := \sup_{[t_0,t]} \|u\|_{H^3}$ for any $0 < t_0 \le t \le T$ and C_0 is an absolute constant.

Here, the estimate

$$\|\omega(\cdot, t_0)\|_{L^p} \le C$$

is used the because $0 < t < T$ and T is the possible first blow-up time.

Testing (1.1) by $\partial_t u$, using (1.1), (3.2), and (3.8), we derive

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, u|^2 \,\mathrm{d}x + \int |\partial_t u|^2 \,\mathrm{d}x &= -\int (u\cdot\nabla)u \cdot \partial_t u \,\mathrm{d}x - \int n\nabla\phi \partial_t u \,\mathrm{d}x \\
&\le \|u\|_{L^6}\|\nabla u\|_{L^3}\|\partial_t u\|_{L^2} + \|n\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\partial_t u\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\|\partial_t u\|_{L^2} + C\|\partial_t u\|_{L^2} \\
&\le \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}^2 + C \\
&\le \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|\omega\|_{L^3}^4 + C,
\end{aligned}$$

which implies

$$\int_{t_0}^{t} \|\partial_t u\|_{L^2}^2 \mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{3.10}$$

Here we use the facts that

$$\int \nabla\pi \cdot \partial_t u \,\mathrm{d}x = 0,$$

and

$$-\int \Delta u \cdot \partial_t u \,\mathrm{d}x = \int \mathrm{rot}^{\,2} u \cdot \partial_t u \,\mathrm{d}x = \int \mathrm{rot}\, u \cdot \mathrm{rot}\,\partial_t u \,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, u|^2 \,\mathrm{d}x,$$

and

$$-\Delta u = \mathrm{rot}^{\,2} u$$

since $\mathrm{div}\, u = 0$.

Applying ∂_t to (1.1), we have

$$\partial_t^2 u + u\cdot\nabla\partial_t u + \nabla\partial_t\pi - \Delta\partial_t u = -\partial_t u\cdot\nabla u - \partial_t n\nabla\phi. \tag{3.11}$$

Testing (3.11) by $\partial_t u$, using (1.2), (1.3), (3.8), and (3.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t u|^2 \,\mathrm{d}x + \int |\mathrm{rot}\,\partial_t u|^2 \,\mathrm{d}x \\
&\quad = -\int \partial_t u\cdot\nabla u\cdot\partial_t u \,\mathrm{d}x + \int \partial_t u \nabla\phi\nabla\cdot(n\nabla p + n\nabla q + un - \nabla n)\,\mathrm{d}x \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{3.12}$$

We bound I_1 as follows:

$$\begin{aligned}|I_1| &\le \|\nabla u\|_{L^6}\|\partial_t u\|_{L^2}\|\partial_t u\|_{L^3} \le C\|\omega\|_{L^6}\|\partial_t u\|_{L^2}^{\frac{3}{2}}\|\mathrm{rot}\,\partial_t u\|_{L^2}^{\frac{1}{2}}\\ &\le \frac{1}{4}\|\mathrm{rot}\,\partial_t u\|_{L^2}^2 + C\|\omega\|_{L^6}^{\frac{4}{3}}\|\partial_t u\|_{L^2}^2,\end{aligned}$$

We use (3.8), (3.2) and (3.10) to bound I_2,

$$\begin{aligned}|I_2| &\le \left|\int (n\nabla p + n\nabla q + un - \nabla n)\cdot\nabla(\partial_t u\nabla\phi)\,\mathrm{d}x\right|\\ &\le C(\|\nabla p\|_{L^r_w}\|n\|_{L^{\frac{2r}{r-2},2}} + \|\nabla q\|_{L^s_w}\|n\|_{L^{\frac{2s}{s-2},2}}\\ &+ \|u\|_{L^6}\|n\|_{L^3} + \|\nabla n\|_{L^2})\|\nabla\partial_t u\|_{L^2}\\ &\le C\left(\|\nabla p\|_{L^r_w} + \|\nabla q\|_{L^s_w} + \|u\|_{L^6} + \|\nabla n\|_{L^2}\right)\|\mathrm{rot}\,\partial_t u\|_{L^2}\\ &\le \frac{1}{4}\|\mathrm{rot}\,\partial_t u\|_{L^2}^2 + C\|\nabla p\|_{L^r_w}^2 + C\|\nabla q\|_{L^s_w}^2 + C\|u\|_{L^6}^2 + C\|\nabla n\|_{L^2}^2.\end{aligned}$$

Inserting the above estimates into (3.12) and integrating the results, we have

$$\int |\partial_t u|^2\,\mathrm{d}x + \int_{t_0}^t\int |\mathrm{rot}\,\partial_t u|^2\mathrm{d}x\,\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{3.13}$$

Here we use the fact that

$$-\int \Delta\partial_t u\cdot\partial_t u\,\mathrm{d}x = \int \mathrm{rot}^{\,2}\partial_t u\cdot\partial_t u\,\mathrm{d}x = \int |\mathrm{rot}\,\partial_t u|^2\,\mathrm{d}x.$$

On the other hand, due to the H^2-theory of the Stokes system (see Lemma 2.8), it follows from (1.1), (3.2), (3.8), and (3.13) that

$$\begin{aligned}\|u\|_{H^2} &\le C\|-\Delta u + \nabla\pi\|_{L^2} \le C\|\partial_t u + u\cdot\nabla u + n\nabla\phi\|_{L^2}\\ &\le C\|\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C \le C\|\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3} + C\\ &\le C\|\partial_t u\|_{L^2} + C\|\omega\|_{L^2}\|\omega\|_{L^3} + C \le C(e+y)^{C_0\epsilon}.\end{aligned} \tag{3.14}$$

Testing (1.4) by $-\Delta p$, using (3.1), (3.2), (3.4) and (3.8), we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla p|^2\,\mathrm{d}x + \int |\Delta p|^2\,\mathrm{d}x &= \int (u\cdot\nabla p + np)\Delta p\,\mathrm{d}x\\ &\le (\|u\|_{L^6}\|\nabla p\|_{L^3} + \|n\|_{L^2}\|p\|_{L^\infty})\,\|\Delta p\|_{L^2}\\ &\le C(\|\nabla u\|_{L^2}\|\nabla p\|_{L^2}^{\frac{1}{2}}\|\nabla^2 p\|_{L^2}^{\frac{1}{2}} + 1)\|\Delta p\|_{L^2}\\ &\le \frac{1}{2}\|\Delta p\|_{L^2}^2 + C\|\omega\|_{L^2}^4\|\nabla p\|_{L^2}^2 + C,\end{aligned}$$

which gives

$$\int |\nabla p|^2\,\mathrm{d}x + \int_0^t\int |\Delta p|^2\,\mathrm{d}x\,\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{3.15}$$

Here we use the Gagliardo–Nirenberg inequality

$$\|\nabla p\|_{L^3}^2 \le C\|\nabla p\|_{L^2}\|\nabla^2 p\|_{L^2}$$

and the H^2-regularity of Poisson equation

$$\|p\|_{H^2} \leq C\|\Delta p\|_{L^2} + C\|p\|_{L^2}.$$

Now we decompose p as $p := p_1 + p_2$, where p_1 and p_2 satisfy

$$\begin{cases} \partial_t p_1 - \Delta p_1 = -\nabla \cdot (up) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial p_1}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ p_1(\cdot, 0) = 0 & \text{in } \Omega \end{cases} \tag{3.16}$$

and

$$\begin{cases} \partial_t p_2 - \Delta p_2 = -np & \text{in } \Omega \times (0,T), \\ \dfrac{\partial p_2}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ p_2(\cdot, 0) = p_0 & \text{in } \Omega. \end{cases} \tag{3.17}$$

Using (3.1), (3.2), (3.8) and the classical theory of heat equation [2], we obtain

$$\|\nabla p_1\|_{L^m(\Omega\times(0,t))} \leq C\|up\|_{L^m(\Omega\times(0,t))} \leq C(e+y)^{C_0\epsilon}, \tag{3.18}$$

$$\|p_2\|_{L^m(0,t;W^{2,m})} \leq C \tag{3.19}$$

for any $m > 3$, and thus

$$\|\nabla p\|_{L^m(\Omega\times(0,t))} \leq C(e+y)^{C_0\epsilon}, \tag{3.20}$$

which gives

$$\begin{aligned} \|u \cdot \nabla p\|_{L^m(\Omega\times(0,t))} \leq C\|u\|_{L^\infty(\Omega\times(0,t))}\|\nabla p\|_{L^m(\Omega\times(0,t))} \\ Q \leq C(e+y)^{C_0\epsilon}. \end{aligned} \tag{3.21}$$

Now, using the $W_m^{2,1}$-theory of heat equations to (1.4), we arrive at

$$\|p\|_{L^m(0,t;W^{2,m})} \leq C(e+y)^{C_0\epsilon}. \tag{3.22}$$

Then, in a similar way as in (3.15) and (3.22), we get

$$\int |\nabla q|^2 \, dx + \int_0^t \int |\Delta q|^2 \, dx \, ds \leq C(e+y)^{C_0\epsilon}, \tag{3.23}$$

$$\|q\|_{L^m(0,t;W^{2,m})} \leq C. \tag{3.24}$$

Equation (1.3) can be rewritten as

$$\partial_t n - \Delta n = f := -\mathrm{div}\,(un + n\nabla p + n\nabla q). \tag{3.25}$$

Using (3.2), (3.8), (3.22), (3.24) and similarly to (3.18) and (3.19), we have

$$\|\nabla n\|_{L^m(0,t;L^m)} \leq C(e+y)^{C_0\epsilon}. \tag{3.26}$$

Hence, it is easy to deduce that

$$\|f\|^2_{L^2(0,t;L^2)} \le C(e+y)^{C_0\epsilon}. \tag{3.27}$$

By parabolic regularity, this implies that

$$\int |\nabla n|^2 \,\mathrm{d}x + \int_0^t \int |\nabla^2 n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int |n_t|^2 \,\mathrm{d}x\,\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{3.28}$$

Testing (3.11) by $-\Delta\partial_t u + \nabla\partial_t\pi$, using (1.2), (3.13), (3.14), and (3.28), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\,\partial_t u|^2 \,\mathrm{d}x + \int |-\Delta\partial_t u + \nabla\partial_t\pi|^2 \,\mathrm{d}x \\
&\quad = \int (-\partial_t u\cdot\nabla u - u\cdot\nabla\partial_t u - \partial_t n\nabla\phi)(-\Delta\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x \\
&\quad \le C\big(\|\nabla u\|_{L^3}\|\partial_t u\|_{L^6} + \|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2} + \|\partial_t n\|_{L^2}\big)\|-\Delta\partial_t u + \nabla\partial_t\pi\|_{L^2} \\
&\quad \le C\big(\|u\|_{H^2}\|\mathrm{rot}\,\partial_t u\|_{L^2} + \|\partial_t n\|_{L^2}\big)\|-\Delta\partial_t u + \nabla\partial_t\pi\|_{L^2} \\
&\quad \le \frac{1}{2}\|-\Delta\partial_t u + \nabla\partial_t\pi\|^2_{L^2} + C\|u\|^2_{H^2}\|\mathrm{rot}\,\partial_t u\|^2_{L^2} + C\|\partial_t n\|^2_{L^2},
\end{aligned}$$

which leads to

$$\int |\mathrm{rot}\,\partial_t u|^2 \,\mathrm{d}x + \int_{t_0}^t \|\partial_t u\|^2_{H^2}\,\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{3.29}$$

Here we use the fact that

$$\begin{aligned}
\int \partial_t^2 u(-\Delta\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x &= \int \partial_t^2 u(\mathrm{rot}\,^2\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x \\
&= \int \partial_t^2 u\cdot\mathrm{rot}\,^2\partial_t u\,\mathrm{d}x = \int \mathrm{rot}\,\partial_t^2 u\cdot\mathrm{rot}\,\partial_t u\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\,\partial_t u|^2\,\mathrm{d}x
\end{aligned}$$

due to

$$\int \partial_t^2 u\cdot\nabla\partial_t\pi\mathrm{d}x = 0.$$

We also use the fact that

$$\|\mathrm{rot}\,\partial_t u(\cdot,t_0)\|_{L^2} \le C$$

because $t_0 < T$ and T is the possible fist blow-up time.

On the other hand, it follows from (2.12), (3.7), (3.8), (3.14), (3.28), and (3.29) that

$$\begin{aligned}
\|u\|_{H^3} &\le C(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\Delta\omega\|_{L^2}) \\
&\le C\big(1 + \|\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u + \sqrt{n}\nabla\phi\|_{L^2}\big) \\
&\le C + C\|\partial_t\omega\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\omega\|_{L^2} + C\|\omega\|_{L^4}\|\nabla u\|_{L^4} + C\|\nabla n\|_{L^2}
\end{aligned}$$

$$\leq C + C\|\partial_t\omega\|_{L^2} + C\|u\|_{H^2}^2 + C\|\nabla n\|_{L^2}$$
$$\leq C(e+y)^{C_0\epsilon} \leq C(e+y)^{\frac{1}{2}} \leq \frac{1}{2}y + C,$$

by taking $C_0\epsilon \leq \frac{1}{2}$, which gives

$$\|u\|_{L^\infty(0,T;H^3)} \leq C. \tag{3.30}$$

Now, by the standard energy method, it is easy to deduce that

$$\begin{aligned}\|(\partial_t u, \partial_t n, \partial_t p, \partial_t q)\|_{L^\infty(0,T;L^2)} + \|(\partial_t u, \partial_t n, \partial_t p, \partial_t q)\|_{L^2(0,T;H^1)} \leq C,\\ \|(n,p,q)\|_{L^\infty(0,T;H^2)} + \|(n,p,q)\|_{L^2(0,T;H^3)} \leq C.\end{aligned} \tag{3.31}$$

In fact, we can prove it as follows. First, taking ∂_t to (1.3), testing by $\partial_t n$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t n|^2\,\mathrm{d}x + \int |\nabla\partial_t n|^2\,\mathrm{d}x\\
&= -\int \partial_t u\cdot\nabla n\cdot\partial_t n\,\mathrm{d}x + \int \partial_t n\nabla p\nabla\partial_t n\,\mathrm{d}x + \int n\nabla\partial_t p\nabla\partial_t n\,\mathrm{d}x\\
&\quad + \int \partial_t n\nabla q\nabla\partial_t n\,\mathrm{d}x + \int n\nabla\partial_t q\nabla\partial_t n\,\mathrm{d}x\\
&= \int n\nabla\partial_t n\cdot\partial_t u\,\mathrm{d}x + \int \partial_t n\left(\nabla p + \nabla q\right)\nabla\partial_t n\,\mathrm{d}x\\
&\quad + \int n\left(\nabla\partial_t p + \nabla\partial_t q\right)\nabla\partial_t n\,\mathrm{d}x\\
&\leq \|n\|_{L^\infty}\|\nabla\partial_t n\|_{L^2}\|\partial_t u\|_{L^2}\\
&\quad + \|\partial_t n\|_{L^3}\|\nabla(p+q)\|_{L^6}\|\nabla\partial_t n\|_{L^2}\\
&\quad + \|n\|_{L^\infty}\|\nabla\partial_t(p+q)\|_{L^2}\|\nabla\partial_t n\|_{L^2}\\
&\leq C\|\partial_t u\|_{L^2}^2 + C\|\partial_t n\|_{L^2}^2 + C + \frac{1}{8}\|\nabla\partial_t n\|_{L^2}^2\\
&\quad + C\|\nabla(p+q)\|_{L^6}^4\|\partial_t n\|_{L^2}^2 + C_1\|\nabla\partial_t(p+q)\|_{L^2}^2.
\end{aligned} \tag{3.32}$$

Here we use the facts that

$$\|n\|_{L^\infty(0,T;L^\infty)} \leq C, \quad \|\partial_t u\|_{L^\infty(0,T;L^2)} \leq C. \tag{3.33}$$

Analogously, it follows from (1.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\partial_t p)^2\,\mathrm{d}x + \int|\nabla\partial_t p|^2\,\mathrm{d}x + \int n(\partial_t p)^2\,\mathrm{d}x\\
&= -\int \partial_t u\cdot\nabla p\cdot\partial_t p\,\mathrm{d}x - \int \partial_t n\, p\,\partial_t p\,\mathrm{d}x\\
&\leq \|\partial_t u\|_{L^6}\|\nabla p\|_{L^3}\|\partial_t p\|_{L^2} + \|p\|_{L^\infty}\|\partial_t n\|_{L^2}\|\partial_t p\|_{L^2}\\
&\leq C\|\nabla p\|_{L^3}\|\partial_t p\|_{L^2} + C\|\partial_t n\|_{L^2}\|\partial_t p\|_{L^2}\\
&\leq C\|\nabla p\|_{L^3}^2 + C\|\partial_t n\|_{L^2}^2 + C\|\partial_t p\|_{L^2}^2
\end{aligned} \tag{3.34}$$

due to

$$\|p\|_{L^\infty(\Omega\times(0,T))} \le C, \quad \|\nabla\partial_t u\|_{L^\infty(0,T;L^2)} \le C. \tag{3.35}$$

Naturally, it follows from (1.5) and (3.35) that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_t q)^2\,\mathrm{d}x + \int |\nabla\partial_t q|^2\,\mathrm{d}x + \int (\partial_t q)^2\,\mathrm{d}x \\
&= -\int \partial_t u\cdot\nabla q\partial_t q\,\mathrm{d}x + \int \partial_t n\partial_t q\,\mathrm{d}x \\
&\le \|\partial_t u\|_{L^6}\|\nabla q\|_{L^3}\|\partial_t q\|_{L^2} + \|\partial_t n\|_{L^2}\|\partial_t q\|_{L^2} \\
&\le C\|\nabla q\|_{L^3}\|\partial_t q\|_{L^2} + \|\partial_t n\|_{L^2}\|\partial_t q\|_{L^2} \\
&\le C\|\partial_t n\|_{L^2}^2 + C\|\partial_t q\|_{L^2}^2 + C\|\nabla q\|_{L^3}^2.
\end{aligned} \tag{3.36}$$

Summing up (3.32) + $4C_1\times$ (3.34) + $4C_1\times$ (3.36), we get

$$\partial_t(n,p,q)\|_{L^\infty(0,T;L^2)} + \|\partial_t(n,p,q)\|_{L^2(0,T;H^1)} \le C. \tag{3.37}$$

Using (1.4) and (3.37), we find that

$$\begin{aligned}
\|p\|_{H^2} &\le C\|\partial_t p + u\cdot\nabla p + np\|_{L^2} \\
&\le C\|\partial_t p\|_{L^2} + C\|u\|_{L^\infty}\|\nabla p\|_{L^2} + C\|n\|_{L^2}\|p\|_{L^\infty} \\
&\le C + C\|\nabla p\|_{L^2} \le C. \qquad\qquad (3.38)\\
\|p\|_{H^3} &\le C\|\partial_t p + u\cdot\nabla p + np\|_{H^1} \\
&\le C\|\partial_t p\|_{H^1} + C\|u\|_{L^\infty}\|p\|_{H^2} + C\|\nabla u\|_{L^\infty}\|\nabla p\|_{L^2} \\
&\quad + C\|n\|_{H^1}\|p\|_{L^\infty} + C\|n\|_{L^\infty}\|p\|_{H^1} \\
&\le C\|\partial_t p\|_{H^1} + C,
\end{aligned}$$

and thus

$$\|p\|_{L^2(0,T;H^3)} \le C. \tag{3.39}$$

In a similar way, one can get

$$\begin{aligned}
\|q\|_{H^2} &\le C\|\partial_t q + u\nabla q + q - n\|_{L^2} \\
&\le C + C\|u\|_{L^\infty}\|\nabla q\|_{L^2} \le C, \qquad\qquad (3.40)\\
\|q\|_{H^3} &\le C\|\partial_t q + u\nabla q + q - n\|_{H^1} \\
&\le C\|\partial_t q\|_{H^1} + C\|u\|_{L^\infty}\|\nabla q\|_{H^1} + C\|\nabla u\|_{L^\infty}\|\nabla q\|_{L^2} \\
&\quad + C\|q-n\|_{H^1} \\
&\le C\|\partial_t q\|_{H^1} + C,
\end{aligned}$$

and thus

$$\|q\|_{L^2(0,T;H^3)} \le C. \tag{3.41}$$

Using (1.3) and (3.37)–(3.41), we deduce that

$$\begin{aligned}
\|n\|_{H^2} &\le C\|\partial_t n + u\cdot\nabla n + \nabla\cdot(n\nabla p) + \nabla\cdot(n\nabla q)\|_{L^2} \\
&\le C\|\partial_t n\|_{L^2} + C\|u\|_{L^\infty}\|\nabla n\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C\|n\|_{L^\infty}\left\|\nabla^2 p\right\|_{L^2}+C\|\nabla n\|_{L^4}\|\nabla p\|_{L^4}\\
&+C\|n\|_{L^\infty}\left\|\nabla^2 q\right\|_{L^2}+C\|\nabla n\|_{L^4}\|\nabla q\|_{L^4}\\
\leq\ & C+C\|\nabla n\|_{L^4}\leq\frac{1}{2}\|n\|_{H^2}+C,
\end{aligned}
$$

which gives

$$
\|n\|_{H^2}\leq C, \tag{3.42}
$$

On the other hand, one has

$$
\begin{aligned}
\|n\|_{H^3}\leq\ & C\left\|\partial_t n+u\cdot\nabla n+\nabla\cdot(n\nabla p)+\nabla\cdot(n\nabla q)\right\|_{H^1}\\
\leq\ & C\left\|\partial_t n\right\|_{H^1}+C\|u\|_{L^\infty}\|n\|_{H^2}+C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2}\\
&+C\|n\|_{L^\infty}\|p\|_{H^3}+C\|\nabla p\|_{L^\infty}\|n\|_{H^2}\\
&+C\|n\|_{L^\infty}\|q\|_{H^3}+C\|\nabla q\|_{L^\infty}\|n\|_{H^2}\\
\leq\ & C\left\|\partial_t n\right\|_{H^1}+C+C\|p\|_{H^3}+C\|q\|_{H^3},
\end{aligned}
$$

and therefore

$$
\|n\|_{L^2(0,T;H^3)}\leq C. \tag{3.43}
$$

4. Appendix: Local well-posedness of strong solutions

First, we give a definition of strong solutions.

Definition 4.1. Let $u_0\in H^3$, $(n_0,p_0,q_0)\in H^2$ with $\operatorname{div}u_0=0$, and let $n_0,p_0,q_0\geqslant 0$ in Ω and $u_0\cdot\nu=0$, $\operatorname{rot}u_0\times\nu=0$,

$$
\frac{\partial n_0}{\partial\nu}=\frac{\partial p_0}{\partial\nu}=\frac{\partial q_0}{\partial\nu}=0\quad\text{on}\quad\Omega. \tag{4.1}
$$

Let also $u\in L^\infty\left(0,T;H^3\right)$, $\partial_t u\in L^\infty\left(0,T;H^1\right)\cap L^2\left(0,T;H^2\right)$,

$$
\begin{aligned}
(n,p,q)&\in L^\infty\left(0,T;H^2\right)\cap L^2\left(0,T;H^3\right),\\
\partial_t(n,p,q)&\in L^\infty\left(0,T;L^2\right)\cap L^2\left(0,T;H^1\right),
\end{aligned}\tag{4.2}
$$

and equations (1.1)–(1.5) hold almost everywhere in $\Omega\times(0,T)$. Then (u,n,p,q) is a strong solution to problem (1.1)–(1.7).

In this section, we will prove

Theorem 4.1. *Let (4.1) hold true. Then problem (1.1)–(1.7) has a unique strong solution (u,n,p,q) satisfying (4.2) for some $0<T\ll 1$.*

To prove Theorem 4.1, we will use the Banach fixed point theorem. To this end, we define a nonempty set

$$
\mathscr{A}:=\left\{\tilde n\in\mathscr{A};\tilde n(\cdot,0)=n_0,\tilde n\geqslant 0,\frac{\partial\tilde n}{\partial\nu}=0\text{ on }\partial\Omega\times(0,T),\|\tilde n\|_{\mathscr{A}}\leq R\right\},
$$

where

$$\|\tilde n\|_{\mathscr A} := \|\tilde n\|_{L^\infty(0,T;H^2)} + \|\tilde n\|_{L^2(0,T;H^3)} + \|\partial_t \tilde n\|_{L^\infty(0,T;L^2)} + \|\partial_t \tilde n\|_{L^2(0,T;H^1)} \tag{4.3}$$

and R is a positive constant which is to be determined.

We define the nonlinear map

$$\mathscr T : \tilde n \in \mathscr A \longrightarrow n \in \mathscr A \tag{4.4}$$

as follows:

$$\partial_t u + u\cdot\nabla u + \nabla\pi - \Delta u = -\tilde n \nabla\phi, \tag{4.5a}$$
$$\operatorname{div}\, u = 0, \tag{4.5b}$$
$$u\cdot\nu = 0, \quad \operatorname{rot}\, u\times\nu = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.5c}$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{4.5d}$$

$$\partial_t q + u\cdot\nabla q - \Delta q + q = \tilde n, \tag{4.6a}$$
$$\frac{\partial q}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.6b}$$
$$q(\cdot,0) = q_0 \quad \text{in } \Omega, \tag{4.6c}$$

$$\partial_t p + u\cdot\nabla p - \Delta p = -\tilde n p, \tag{4.7a}$$
$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.7b}$$
$$p(\cdot,0) = p_0 \quad \text{in } \Omega. \tag{4.7c}$$

$$\partial_t n + u\cdot\nabla n - \Delta n = -\nabla\cdot(n\nabla p) - \nabla\cdot(n\nabla q), \tag{4.8a}$$
$$\frac{\partial n}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.8b}$$
$$n(\cdot,0) = n_0 \text{ in } \Omega. \tag{4.8c}$$

Lemma 4.1. *Let $\tilde n \in \mathscr A$ be given. Then problem (4.5) has a unique solution u satisfying*

$$\|u\|_{L^\infty(0,T;H^3)} + \|\partial_t u\|_{L^\infty(0,T;H^1)} + \|\partial_t u\|_{L^2(0,T;H^2)} \le C \tag{4.9}$$

for some $0 < T \ll 1$ and C independent of $R > 0$.

Proof. The existence and the uniqueness are well-known, we only need to show the estimates.

First, we see that

$$\frac{\mathrm d}{\mathrm dt}\int \tilde n^2 + |\nabla\tilde n|^2\,\mathrm dx = 2\int(\tilde n - \Delta\tilde n)\partial_t\tilde n\,\mathrm dx \le 2\|\tilde n\|_{H^2}\,\|\partial_t\tilde n\|_{L^2} \le CR^2,$$

which gives

$$\|\tilde n\|_{H^1}^2 = \|\tilde n_0\|_{H^1}^2 + \int_0^T CR^2\,\mathrm dt$$

and

$$\|\tilde{n}\|_{H^1} \leq C \tag{4.10}$$

if $T \leq 1$ and $R^2 T \leq 1$.

Testing (4.5a) by u and using $\operatorname{div} u = 0$ and (4.10), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 \,\mathrm{d}x + \int |\operatorname{rot} u|^2 \,\mathrm{d}x = -\int \tilde{n}\nabla\phi\cdot u \,\mathrm{d}x \leq \|\tilde{n}\|_{L^2}\|\nabla\phi\|_{L^\infty}\|u\|_{L^2} \leq C\|u\|_{L^2},$$

which leads to

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C \tag{4.11}$$

for some $0 < T \leq 1$.

We have (3.7) with n replaced by $\tilde{n}$. Testing it by ω, we infer that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\omega|^2 \,\mathrm{d}x + \int |\operatorname{rot}\omega|^2 \,\mathrm{d}x &= \int \omega\cdot\nabla u\cdot\omega \,\mathrm{d}x - \int (\nabla\tilde{n}\times\nabla\phi)\cdot\omega \,\mathrm{d}x \\
&\leq \|\omega\|_{L^3}^2\|\nabla u\|_{L^3} + \|\nabla\phi\|_{L^\infty}\|\nabla\tilde{n}\|_{L^2}\|\omega\|_{L^2} \\
&\leq C\|\omega\|_{L^2}^{\frac{3}{2}}\|\operatorname{rot}\omega\|_{L^2}^{\frac{3}{2}} + C\|\omega\|_{L^2} \\
&\leq \frac{1}{2}\|\operatorname{rot}\omega\|_{L^2}^2 + C\|\omega\|_{L^2}^6 + C,
\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;H^1)} \leq C \tag{4.12}$$

for some $0 < T \leq 1$.

We have (3.11) with $\partial_t n$ replaced by $\partial_t\tilde{n}$. Using (4.12), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t u|^2 \,\mathrm{d}x &+ \int |\operatorname{rot}\partial_t u|^2 \,\mathrm{d}x \\
&= -\int (\partial_t u\cdot\nabla u)\cdot\partial_t u \,\mathrm{d}x - \int \partial_t\tilde{n}\cdot\nabla\phi\cdot\partial_t u \,\mathrm{d}x \\
&\leq \|\nabla u\|_{L^2}\|\partial_t u\|_{L^4}^2 + \|\nabla\phi\|_{L^\infty}\|\partial_t\tilde{n}\|_{L^2}\|\partial_t u\|_{L^2} \\
&\leq C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\operatorname{rot}\partial_t u\|_{L^2}^{\frac{3}{2}} + CR\|\partial_t u\|_{L^2} \\
&\leq \frac{1}{2}\|\operatorname{rot}\partial_t u\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2 + CR^2,
\end{aligned}$$

which yields

$$\|\partial_t u\|_{L^\infty(0,T;L^2)} + \|\partial_t u\|_{L^2(0,T;H^1)} \leq C \tag{4.13}$$

if $T \leq 1$ and $R^2 T \leq 1$.

On the other hand, using (4.10), (4.12), and (4.13), we have

$$\begin{aligned}
\|u\|_{H^2} &\leq C\|\partial_t u + u\cdot\nabla u + \tilde{n}\nabla\phi\|_{L^2} \\
&\leq C\|\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\tilde{n}\|_{L^2}\|\nabla\phi\|_{L^\infty} \\
&\leq C + C\|\nabla u\|_{L^3} \leq C + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}} \leq \frac{1}{2}\|u\|_{H^2} + C,
\end{aligned}$$

and therefore

$$\|u\|_{L^\infty(0,T;H^2)} \le C. \tag{4.14}$$

Testing (3.11) by $-\Delta_t u + \nabla\partial_t\pi$, using (4.14), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\,\partial_t u|^2\,\mathrm{d}x + \int |-\Delta\partial_t u + \nabla\partial_t\pi|^2\,\mathrm{d}x \\
&= \int (-\partial_t u\cdot\nabla u - u\cdot\nabla\partial_t u - \partial_t\tilde{n}\nabla\phi)(-\Delta\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x \\
&\le \big(\|\partial_t u\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2} \\
&\quad + \|\partial_t\tilde{n}\|_{L^2}\|\nabla\phi\|_{L^\infty}\big)\|-\Delta\partial_t u + \nabla\partial_t\pi\|_{L^2} \\
&\le C(\|\mathrm{rot}\,\partial_t u\|_{L^2} + R)\|-\Delta\partial_t u + \nabla\partial_t\pi\|_{L^2} \\
&\le \frac{1}{2}\|-\Delta\partial_t u + \nabla\partial_t\pi\|_{L^2}^2 + C\|\mathrm{rot}\,\partial_t u\|_{L^2}^2 + CR^2,
\end{aligned}$$

which yields

$$\|\partial_t u\|_{L^\infty(0,T;H^1)} + \|\partial_t u\|_{L^2(0,T;H^2)} \le C. \tag{4.15}$$

Similarly to (3.30), by (4.15), one has

$$\begin{aligned}
\|u\|_{H^3} &\le C(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\Delta\omega\|_{L^2}) \\
&\le C + C\|\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u + \nabla\tilde{n}\times\nabla\phi\|_{L^2} \\
&\le C + C\|\partial_t\omega\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\omega\|_{L^2} + C\|\omega\|_{L^4}\|\nabla u\|_{L^4} \\
&\quad + C\|\nabla\tilde{n}\|_{L^2}\|\nabla\phi\|_{L^\infty} \le C.
\end{aligned} \tag{4.16}$$

This completes the proof. □

Lemma 4.2. *Let $\tilde{n}\in\mathscr{A}$ be given and u be given by Lemma 4.1. Then problem* (4.6) *has a unique solution q satisfying*

$$\|q\|_{L^\infty(0,T;H^2)} + \|q\|_{L^2(0,T;H^3)} + \|\partial_t q\|_{L^\infty(0,T;L^2)} + \|\partial_t q\|_{L^2(0,T;H^1)} \le C, \tag{4.17}$$

$$\|\nabla q\|_{L^m(0,T;L^m)} \le C \tag{4.18}$$

for any $m>3$.

Proof. The existence and the uniqueness are well-known, we only need to show the a priori estimates. First, it follows from (4.6a) and Lemma 2.10 that

$$0 \le q \le C. \tag{4.19}$$

Testing (4.6a) by $-\Delta q$, using (4.9) and (4.10), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla q|^2\,\mathrm{d}x + \int |\Delta q|^2\mathrm{d}x + \int |\nabla q|^2\,\mathrm{d}x \\
&= -\int (u\cdot\nabla q)\Delta q\mathrm{d}x - \int \tilde{n}\Delta q\,\mathrm{d}x \\
&\le \|u\|_{L^\infty}\|\nabla q\|_{L^2}\|\Delta q\|_{L^2} + \|\tilde{n}\|_{L^2}\|\Delta q\|_{L^2}
\end{aligned}$$

$$\leq C\,\|\nabla q\|_{L^2}\|\,\Delta q\,\|_{L^2} + C\|\,\Delta q\|_{L^2}$$
$$\leq \frac{1}{2}\|\Delta q\|_{L^2}^2 + C\|\nabla q\|_{L^2}^2 + C,$$

which implies

$$\|q\|_{L^\infty(0,T;H^1)} + \|q\|_{L^2(0,T;H^2)} \leq C. \tag{4.20}$$

Similarly to (3.36), using (4.9) and (4.19), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t q|^2\,\mathrm{d}x + \int |\nabla \partial_t q|^2\,\mathrm{d}x + \int |\partial_t q|^2\,\mathrm{d}x \\
&\qquad = -\int (\partial_t u\cdot\nabla q)\,\partial_t q\,\mathrm{d}x + \int \partial_t\tilde{n}\partial_t q\,\mathrm{d}x. \\
&\qquad = \int \partial_t u q\cdot\nabla\partial_t q\,\mathrm{d}x + \int \partial_t\tilde{n}\partial_t q\,\mathrm{d}x \\
&\qquad \leq \|q\|_{L^\infty}\,\|\partial_t u\|_{L^2}\,\|\nabla\partial_t q\|_{L^2} + \|\partial_t\tilde{n}\|_{L^2}\,\|\partial_t q\|_{L^2} \\
&\qquad \leq C\,\|\nabla\partial_t q\|_{L^2} + CR\,\|\partial_t q\|_{L^2} \\
&\qquad \leq \frac{1}{2}\,\|\nabla\partial_t q\|_{L^2}^2 + \frac{1}{2}\,\|\partial_t q\|_{L^2}^2 + CR^2,
\end{aligned}$$

which implies

$$\|\partial_t q\|_{L^\infty(0,T;L^2)} + \|\partial_t q\|_{L^2(0,T;H^1)} \leq C \tag{4.21}$$

if $T \leq 1$ and $R^2 T \leq 1$

On the other hand, using (4.9), (4.10), and (4.21), we obtain

$$\begin{aligned}
\|q\|_{H^2} &\leq C\,\|\partial_t q + u\cdot\nabla q + q - \tilde{n}\|_{L^2} \\
&\leq C\,\|\partial_t q\|_{L^2} + C\|u\|_{L^\infty}\|\nabla q\|_{L^2} + C\|q\|_{L^2} + C\|\tilde{n}\|_{L^2} \leq C,
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\|q\|_{H^3} &\leq C\,\|\partial_t q + u\cdot\nabla q + q - \tilde{n}\|_{H^1} \\
&\leq C\,\|\partial_t q\|_{H^1} + C\|u\|_{L^\infty}\|q\|_{H^2} + C\|\nabla u\|_{L^\infty}\|\nabla q\|_{L^2} + C\|q\|_{H^1} + C\|\tilde{n}\|_{H^1} \\
&\leq C\,\|\partial_t q\|_{H^1} + C,
\end{aligned}$$

and thus

$$\|q\|_{L^2(0,T;H^3)} \leq C. \tag{4.23}$$

Similarly to (3.24), we get (4.18). This completes the proof. □

Lemma 4.3. *Let $\tilde{n} \in \mathscr{A}$ be given and u be given in Lemma 4.1. Then problem (4.7) has a unique solution p satisfying*

$$\|p\|_{L^\infty(0,T;H^2)} + \|p\|_{L^2(0,T;H^3)} + \|\partial_t p\|_{L^\infty(0,T;L^2)} + \|\partial_t p\|_{L^2(0,T;H^1)} \leq C, \tag{4.24}$$

and

$$\|\nabla p\|_{L^m(0,T;H^m)} \leq C \tag{4.25}$$

for any $m > 3$.

Proof. The existence and the uniqueness are standard, we only need to prove the a priori estimates.

First, due to the maximum principle, one has

$$0 \le p \le C. \tag{4.26}$$

Testing (4.7a) by $-\Delta p$, using (4.9), (4.10), and (4.26), we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla p|^2\,dx + \int |\Delta p|^2\,\mathrm{d}x &= \int \tilde{n}p\Delta p\,\mathrm{d}x + \int (u\cdot\nabla p)\Delta p\,\mathrm{d}x\\ &\le \|\tilde{n}\|_{L^2}\|p\|_{L^\infty}\|\Delta p\|_{L^2} + \|u\|_{L^\infty}\|\nabla p\|_{L^2}\|\Delta p\|_{L^2}\\ &\le C\|\Delta p\|_{L^2} + C\|\nabla p\|_{L^2}\|\Delta p\|_{L^2} \le \frac{1}{2}\|\Delta p\|_{L^2}^2 + C + C\|\nabla p\|_{L^2}^2,\end{aligned}$$

which gives

$$\|p\|_{L^\infty(0,T;H^1)} + \|p\|_{L^2(0,T;H^2)} \le C. \tag{4.27}$$

Similarly to (3.34), using (4.9), (4.10), and (4.26), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t p|^2\,\mathrm{d}x + \int |\nabla\partial_t p|^2\,\mathrm{d}x + \int \tilde{n}\,(\partial_t p)^2\,\mathrm{d}x\\ &\qquad = -\int (\partial_t u\cdot\nabla p)\,\partial_t p\,\mathrm{d}x - \int \partial_t\tilde{n}p\partial_t p\,\mathrm{d}x\\ &\qquad = \int \partial_t u p\nabla\partial_t p\,\mathrm{d}x - \int \partial_t\tilde{n}p\partial_t p\,\mathrm{d}x\\ &\qquad \le \|\partial_t u\|_{L^2}\|p\|_{L^\infty}\|\nabla\partial_t p\|_{L^2} + \|\partial_t\tilde{n}\|_{L^2}\|p\|_{L^\infty}\|\partial_t p\|_{L^2}\\ &\qquad \le C\|\nabla\partial_t p\|_{L^2} + CR\|\partial_t p\|_{L^2}\\ &\qquad \le \frac{1}{2}\|\nabla\partial_t p\|_{L^2}^2 + C + C\|\partial_t p\|_{L^2}^2 + CR^2,\end{aligned}$$

which gives

$$\|\partial_t p\|_{L^\infty(0,T;L^2)} + \|\partial_t p\|_{L^2(0,T;H^1)} \le C. \tag{4.28}$$

On the other hand, using (4.9), (4.10), and (4.26)–(4.28), we have

$$\begin{aligned}\|p\|_{H^2} &\le C\,\|\partial_t p + u\cdot\nabla p + \tilde{n}p\|_{L^2}\\ &\le C\,\|\partial_t p\|_{L^2} + C\|u\|_{L^\infty}\|\nabla p\|_{L^2} + C\|\tilde{n}\|_{L^2}\|p\|_{L^\infty} \le C.\\ \|p\|_{H^3} &\le C\,\|\partial_t p + u\cdot\nabla p + \tilde{n}p\|_{H^1}\\ &\le C\,\|\partial_t p\|_{H^1} + C\|u\|_{L^\infty}\|p\|_{H^2} + C\|\nabla u\|_{L^\infty}\|\nabla p\|_{L^2}\\ &\quad + C\|\tilde{n}\|_{H^1}\|p\|_{L^\infty} + C\|\tilde{n}\|_{L^6}\|\nabla p\|_{L^3} \le C\,\|\partial_t p\|_{H^1} + C,\end{aligned} \tag{4.29}$$

and thus

$$\|p\|_{L^2(0,T;H^3)} \le C. \tag{4.30}$$

In the same way as in (3.20), we arrive at (4.25). This completes the proof. □

Lemma 4.4. *Let (u,p,q) be given in Lemma 4.1, Lemma 4.2 and Lemma 4.3, respectively. Then problem (4.8) has a unique solution n satisfying*

$$0 \le n \le C,$$
$$\|n\|_{L^\infty(0,T;H^2)} + \|n\|_{L^2(0,T;H^3)} + \|\partial_t n\|_{L^\infty(0,T;L^2)} + \|\partial_t n\|_{L^2(0,T;H^1)} \le C.$$

Proof. The existence and the uniqueness are standard, we only need to show the a priori estimates. First, in view of the maximum principle, it follows that

$$n \geqslant 0 \text{ in } \Omega \times (0,T).$$

Using (4.18) and (4.25), we obtain (3.2). Using Lemma 2.10, (4.18) and (4.25), we have

$$n \le C \text{ in } \Omega \times (0,T). \tag{4.31}$$

Testing (4.8a) by $-\Delta n$, using (4.9), (4.17), (4.24), and (4.31), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla n|^2 \mathrm{d}x + \int |\Delta n|^2 \mathrm{d}x = \int [u\cdot\nabla n + \nabla\cdot(n\nabla p) + \nabla\cdot(n\nabla q)]\Delta n \mathrm{d}x \\
&\quad \le C(\|u\|_{L^\infty}\|\nabla n\|_{L^2} + \|n\|_{L^\infty}\|p\|_{H^2} + \|\nabla n\|_{L^3}\|\nabla p\|_{L^6} \\
&\qquad + \|n\|_{L^\infty}\|q\|_{H^2} + \|\nabla n\|_{L^3}\|\nabla q\|_{L6})\|\Delta n\|_{L^2} \\
&\quad \le C\left(\|\nabla n\|_{L^2} + C + C\|\nabla n\|_{L^3}\right)\|\Delta n\|_{L^2} \le \frac{1}{2}\|\Delta n\|_{L^2}^2 + c + c\|\nabla n\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|n\|_{L^\infty(0,T;H^1)} + \|n\|_{L^2(0,T;H^2)} \le C. \tag{4.32}$$

We still have (3.32), and thus

$$\|\partial_t n\|_{L^\infty(0,T;L^2)} + \|\partial_t n\|_{L^2(0,T;H^1)} \le C. \tag{4.33}$$

We still have (3.42) and (3.43). This completes the proof. □

By Lemmas 4.1–4.4 and taking $R := 16C + 1$, we arrive at Lemma 4.5.

Lemma 4.5. *$\mathscr{T}$ is well defined and maps $\mathscr{A}$ into $\mathscr{A}$.*

Next, we will prove Lemma 4.6.

Lemma 4.6. *If T is small enough, then*

$$\|\mathscr{T}\tilde{n}_1 - \mathscr{T}\tilde{n}_2\|_{L^\infty(0,T;L^2)} \le \frac{1}{2}\|\tilde{n}_1 - \tilde{n}_2\|_{L^\infty(0,T;L^2)}. \tag{4.34}$$

Proof. Let $(u_i, q_i, p_i, n_i, \pi_i)\,(i=1,2)$ be the corresponding solutions to the problem with given $\tilde{n}_i (i=1,2)$.

We denote $(\delta u, \delta q, \delta p, \delta n, \delta\pi) := (u_1 - u_2, q_1 - q_2, p_1 - p_2, n_1 - n_2, \pi_1 - \pi_2)$ and $\delta\tilde{n} := \tilde{n}_1 - \tilde{n}_2$. Then we have

$$\partial_t \delta n + u_1\cdot\nabla\delta n - \Delta\delta n = -\,\delta u\cdot\nabla n_2 - \nabla\cdot(n_1\nabla\delta p + \delta n\nabla p_2)$$

$$- \nabla \cdot \left(n_1 \nabla \delta q + \delta n \nabla q_2 \right), \tag{4.35}$$

$$\partial_t \delta u + u_1 \cdot \nabla \delta u + \nabla \delta \pi - \Delta \delta u = -\delta u \cdot \nabla u_2 - \delta \tilde{n} \nabla \phi, \tag{4.36}$$

$$\partial_t \delta q + u_1 \cdot \nabla \delta q - \Delta \delta q + \delta q = \delta \tilde{n} - \delta u \cdot \nabla q_2, \tag{4.37}$$

$$\partial_t \delta p + u_1 \cdot \nabla \delta p - \Delta \delta p + \tilde{n_1} \delta p = -\delta \tilde{n} p_2 - \delta u \cdot \nabla p_2. \tag{4.38}$$

Testing (4.35) by δn and using Lemma 4.5, we have

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |\delta n|^2 \, \mathrm{d}x + \int |\nabla \delta n|^2 \, \mathrm{d}x &\\
&= \int \delta u \cdot n_2 \nabla \delta n \, \mathrm{d}x + \int \left(n_1 \nabla \delta p + \delta n \nabla p_2 \right) \nabla \delta n \, \mathrm{d}x \\
&\quad + \int \left(n_1 \nabla \delta q + \delta n \nabla q_2 \right) \nabla \delta n \, \mathrm{d}x \\
&\le \|n_2\|_{L^\infty} \|\delta u\|_{L^2} \|\nabla \delta n\|_{L^2} \\
&\quad + \left(\|n_1\|_{L^\infty} \|\nabla \delta p\|_{L^2} + \|\delta n\|_{L^3} \|\nabla p_2\|_{L^6} \right) \|\nabla \delta n\|_{L^2} \\
&\quad + \left(\|n_1\|_{L^\infty} \| \nabla \delta q \|_{L^2} + \| \delta n\|_{L^3} \|\nabla q_2\|_{L^6} \right) \|\nabla \delta n\|_{L^2} \\
&\le C \|\delta u\|_{L^2} \| \nabla \delta n \|_{L^2} + C \left(\|\nabla \delta p\|_{L^2} + \|\delta n\|_{L^3} \right) \| \nabla \delta n\|_{L^2} \\
&\quad + C \|\nabla \delta q\|_{L^2} \|\nabla \delta n\|_{L^2} \\
&\le \frac{1}{2} \|\nabla \delta n\|_{L^2}^2 + C \|\delta u\|_{L^2}^2 + C \|\nabla \delta p\|_{L^2}^2 + C \|\nabla \delta q\|_{L^2}^2 + C \|\delta n\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\delta n\|_{L^2}^2 \le C e^{CT} \int_0^T \left(\|\delta u\|_{L^2}^2 + \|\nabla \delta p\|_{L^2}^2 + \|\nabla \delta q\|_{L^2}^2 \right) \mathrm{d}t. \tag{4.39}$$

Testing (4.36) by δu and using Lemma 4.5, we have

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |\delta u|^2 \, \mathrm{d}x + \int |\mathrm{rot}\, u \delta u|^2 \, \mathrm{d}x &\\
&= - \int (\delta u \cdot \nabla) u_2 \cdot \delta u \mathrm{d}x - \int \delta \tilde{n} \nabla \phi \cdot \delta u \, \mathrm{d}x \\
&\le \|\nabla u_2\|_{L^\infty} \|\delta u\|_{L^2}^2 + \|\nabla \phi\|_{L^\infty} \|\delta \tilde{n}\|_{L^2} \|\delta u\|_{L^2} \\
&\le C \|\delta u\|_{L^2}^2 + C \|\delta \tilde{n}\|_{L^2} \|\delta u\|_{L^2} \le C \|\delta u\|_{L^2}^2 + C \|\delta \tilde{n}\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\delta u\|_{L^2}^2 \le C e^{CT} \int_0^T \|\delta \tilde{n}\|_{L^2}^2 \le C T e^{CT} \|\delta \tilde{n}\|_{L^\infty(0,T;L^2)}^2. \tag{4.40}$$

Testing (4.37) by δq and using Lemma 4.5, we have

$$\begin{aligned}
\frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}t} \int |\delta q|^2 \, \mathrm{d}x + \int |\nabla \delta q|^2 \, \mathrm{d}x + \int |\delta q|^2 \, \mathrm{d}x \right) &\\
&= \int \left(\delta \tilde{n} - \delta u \cdot \nabla q_2 \right) \delta q \, \mathrm{d}x = \int \delta \tilde{n} \delta q \mathrm{d}x + \int \delta u \cdot q_2 \nabla \delta q \, \mathrm{d}x
\end{aligned}$$

$$\leq \|\delta\tilde{n}\|_{L^2}\|\delta q\|_{L^2} + \|q_2\|_{L^\infty}\|\delta u\|_{L^2}\|\nabla\delta q\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla\delta q\|_{L^2}^2 + \frac{1}{2}\|\delta q\|_{L^2}^2 + C\|\delta\tilde{n}\|_{L^2}^2 + C\|\delta u\|_{L^2}^2,$$

which gives

$$\int_0^T\int |\nabla\delta q|^2\,\mathrm{d}x\,\mathrm{d}t \leq CT\left(\|\delta\tilde{n}\|_{L^\infty(0,T;L^2)}^2 + \|\delta u\|_{L^\infty(0,T;L^2)}^2\right). \tag{4.41}$$

Testing (4.38) by δp and using Lemma 4.5, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\delta p|^2\,\mathrm{d}x + \int |\nabla\delta p|^2\,\mathrm{d}x + \int \tilde{n}_1|\delta p|^2\,\mathrm{d}x$$
$$= -\int \delta\tilde{n}p_2\delta p\,\mathrm{d}x + \int \delta u\cdot p_2\nabla\delta p\,\mathrm{d}x$$
$$\leq \|p_2\|_{L^\infty}\|\delta\tilde{n}\|_{L^2}\|\delta p\|_{L^2} + \|p_2\|_{L^\infty}\|\delta u\|_{L^2}\|\nabla\delta p\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla\delta p\|_{L^2}^2 + C\|\delta p\|_{L^2}^2 + C\|\delta\tilde{n}\|_{L^2}^2 + C\|\delta u\|_{L^2}^2,$$

which gives

$$\int_0^T\int |\nabla\delta p|^2\,\mathrm{d}x\,\mathrm{d}t \leq CTe^{CT}\left(\|\delta\tilde{n}\|_{L^\infty(0,T;L^2)}^2 + \|\delta u\|_{L^\infty(0,T;L^2)}^2\right). \tag{4.42}$$

The combining of (4.39) with (4.40)–(4.42) by taking T small enough, conduces that (4.34) holds tune. This completes the proof. □

Proof of Theorem 4.1. By Lemmas 4.5 and 4.6, and using Banach's fixed point theorem, we arrive at Theorem 4.1. This completes the proof. □

Acknowledgments. The authors are indebted to the three referees for some nice suggestions. This paper was supported by the project of National Natural Science Foundation of China (No.12171459, No.12172166). The key project of University Natural Science of Anhui province (No. 2023AH052096).

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Received January 23, 2024, revised April 27, 2024.

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Логарифмічний критерій продовжуваності для системи Келлера–Сеґеля–Нав'є–Стокса в обмеженій області

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Розглянуто систему Келлера–Сеґеля–Нав'є–Стокса в тривимірній обмеженій області, доведено логарифмічний критерій руйнування локальних сильних розв'язків, використано L^p-метод, L^∞-метод та оцінку максимальної регулярності параболічного рівняння.

Ключові слова: система Келлера–Сеґеля–Нав'є–Стокса, критерій розриву, обмежена область