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Global Existence, Stability and Blow–up of Solutions for p-Biharmonic Hyperbolic Equation with Weak and Strong Damping Terms

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In this paper, we study the initial boundary value problem for the following p-biharmonic hyperbolic equation with weak and strong damping terms:

$$v_{tt} + \Delta_p^2 v - \mu \Delta_m v_t + v_t = \omega |v|^{k-2} v.$$

Under some assumptions on the initial data, the constants p, m and k, we prove the global existence, stability and blow-up results of solutions. The global solution is obtained by using potential well method and the stability based on Komornik's inequality. We also prove that the solution with negative initial energy blows up in finite and in infinite time.

Key words: p-biharmonic equation, damping terms, global existence, stability, blow-up

Mathematical Subject Classification 2020: 35L75, 35A01, 35B35

1. Introduction

In this paper, we study the following p-biharmonic hyperbolic equation with weak and strong damping terms:

$$\begin{cases} v_{tt} + \Delta_p^2 v - \mu \Delta_m v_t + v_t = \omega |v|^{k-2} v, & x \in \Omega,\ t > 0, \\ v(x,t) = \dfrac{\partial}{\partial \eta} v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ denotes the unit outer normal derivative, p, m, k, μ, and ω are positive constants, v_0, v_1 are given functions belonging to suitable spaces, Δ_p^2 is the fourth-order operator called the p-biharmonic operator, which is defined by $\Delta_p^2 v = \Delta\left(|\Delta v|^{p-2}\Delta v\right)$, and the operator $\Delta_p v$ is the classical p-Laplacian given by $\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)$.

Fourth-order differential equations arise in the study of deflections of elastic beams on nonlinear elastic foundations. Therefore, they have important applications in engineering and physical sciences [11, 18, 19]. In recent years, a great attention has been focused on the study of fourth-order differential problems involving biharmonic and p-biharmonic operators.

Nonlinear elliptic equations of p-biharmonic type have been studied by many authors, especially on the existence of ground state solutions, positive solutions, infinitely many solutions, sign-changing solutions and multiplicity of standing wave solutions (see, for instance, [3,5,6,9,14,15,21,24,27] and references therein).

For parabolic and hyperbolc problems involving the p-biharmonic operator, there are few papers that studied the existence, asymptotic behavior and blow-up of solutions, see [7,8,12,16,17,25] and references therein. Liu and Guo [12] considered the following p-biharmonic parabolic equation:

$$v_t + \Delta_p^2 v + \lambda |v|^{p-2} v = 0, \tag{1.2}$$

where $\lambda > 0$ and $p > 2$. Under some assumptions on the initial value, they established the existence of weak solutions by the discrete-time method. The asymptotic behavior and the finite speed of propagation of perturbations of solutions were also discussed. Hao and Zhou [8] investigated the blow-up, extinction and non-extinction of the solutions for the following p-biharmonic parabolic equation:

$$v_t + \Delta_p^2 v = |v|^q - \frac{1}{|\Omega|} \int_\Omega |v|^q \, dx, \tag{1.3}$$

where $\max\left\{1, \frac{2n}{n+4}\right\} < p \leq 2$ and $q > 0$. Liu and Li [16] considered the p-biharmonic parabolic equation with logarithmic nonlinearity

$$v_t + \Delta_p^2 v = \lambda |v|^{q-2} \log\left(|v|\right), \tag{1.4}$$

where $\lambda > 0$, $p > q > \frac{p}{2} + 1$ and $p > \frac{n}{2}$. They established the well-posedness of local weak solution and proved the long-time behavior and the propagation of perturbations, based on the methods of difference and variation. Liu and Fang [17] studied (1.4) with strong damping term (Δv_t) and $\max\left\{1, \frac{2N}{N+4}\right\} < p \leq q < p\left(1 + \frac{N}{4}\right)$. They established the local and global existence of solutions by using the Galerkin approximation combined with the potential well method. They also proved the blow-ups and growth rate of weak solvability, infinite- and finite-time blow-up phenomena of weak solutions in different energy levels. Moreover, they obtained the growth, lifespan and extinction phenomenon of the solutions. Recently, Ferreira et al. [7] studied the nonlinear beam equation with a strong damping and the $p(x)$-biharmonic operator

$$v_t + \Delta_{p(x)}^2 v - \Delta v_t = f(x, t, v) \tag{1.5}$$

and proved the existence of local solutions by using the Faedo–Galerkin method and the decay of energy based on the method of Nakao under assumptions on the variable exponent $p(\cdot)$.

Particularly, when $p = 2$, problem (1.1) is reduced to the Petrovsky equation, which has already been discussed by many authors. For example, Messaoudi [20] studied the following problem:

$$v_{tt} + \Delta^2 v + |v_t|^{m-2} v_t = |v|^{p-2} v, \tag{1.6}$$

and established an existence result and showed that the solution continues to exist globally if $m \geq p$ and blows up in finite time if $m < p$ and the initial energy is negative. Wu and Tsai [26] proved the global existence and blow-up of the solution to problem (1.6). Chen and Zhou [4] extended the blow-up result of [20, 23] to the solution with positive initial energy. Li et al. [13] studied (1.6) with strong damping term (Δv_t) and proved the global existence of the solution under conditions without any relation between m and p and established the exponential decay rate. Pişkin and Polat [23] proved the decay estimate of solutions by using Nakao's inequality to the problem considered in [13].

In the present paper, we are concerned with the global existence, stability, and blow-up results to the initial boundary value problem for the p-biharmonic hyperbolic equation with weak and strong damping terms.

The plan of the paper is as follows. In Section 2, we introduce the Lebesgue-Sobolev spaces and give some notations and preliminary lemmas. In Section 3, we establish the global existence of the solution. Section 4 is aimed to state and prove the stability result. In Section 5, we prove the blow-up of solutions.

2. Preliminaries

In this section, we give some notations, assumptions, and lemmas which will be used throughout this paper. We denote by $\|v\|_q$ and $(\cdot,\cdot)$ the usual $L^q(\Omega)$ norm and the inner product in $l^2(\Omega)$, respectively. Moreover, we also denote

$$(v_1, v_2)_\mu = \int_\Omega (\mu \nabla v_1 \nabla v_2 + v_1 v_2)\, dx,$$

and the norm induced by the product $(v_1, v_2)_\mu$ is

$$\|v\|_\mu^2 = (v, v)_\mu.$$

Then $\|v\|_\mu$ is an equivalent eccentric module over $H_0^1(\Omega)$ due to $\mu > 0$.

For the Sobolev spaces norms, we use the notations

$$\|v\|_{1,q} := \|v\|_{W_0^{1,q}} = \|\nabla v\|_q \text{ and } \|v\|_{2,q} := \|v\|_{W_0^{2,q}} = \|\Delta v\|_q, \ \ 1 < q < +\infty.$$

Let c_q and c_* be the optimal constants of Sobolev embedding which satisfies the inequalities

$$\|v\|_q \leq c_q \|\nabla v\|_2, \quad v \in H_0^1(\Omega), \tag{2.1}$$

and

$$\|\nabla v\|_q \leq c_* \|\Delta v\|_2, \quad v \in H_0^2(\Omega). \tag{2.2}$$

To state and prove our results, we need the following assumptions:

($\mathbf{H}_1$) $2 < p < m < k$;

($\mathbf{H}_2$) $2 < p < k$;

($\mathbf{H}_3$) $p < k < \dfrac{2p}{n-p}$, $n \geq p$.

Now we are ready to state the local existence of problem (1.1), whose proof can be found in [[1, 2, 22].

Theorem 2.1. *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_3$) *hold. Then, for every* $v_0 \in W_0^{2,p}(\Omega)$ *and* $v_1 \in L^2(\Omega)$, *problem* (1.1) *admits a unique local solution in the class*

$$v \in L^\infty \left([0,T]; W_0^{2,p}(\Omega)\right), \quad v_t \in L^\infty \left([0,T]; L^2(\Omega)\right) \cap L^m \left([0,T]; W_0^{1,m}(\Omega)\right).$$

We define the energy function associated with problem (1.1) as follows:

$$E(t) = \frac{1}{2}\|v_t\|_2^2 + \frac{1}{p}\|\Delta v\|_p^p - \frac{\omega}{k}\|v\|_k^k. \tag{2.3}$$

Lemma 2.2. *The functional* $E(t)$ *defined in* (2.3) *satisfies*

$$E'(t) \leq -\mu\|\nabla v_t\|_m^m - \|v_t\|_2^2 \leq 0. \tag{2.4}$$

Proof. Multiplying the first equation in (1.1) by $v_t(t)$ and integrating over Ω, we get (2.4). □

Lemma 2.3 (Komornik, [10])**.** *Let* $E : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function, assume that there are constants* $\alpha, \zeta > 0$ *such that*

$$\int_S^{+\infty} E^{\alpha+1}(t)dt \leq \lambda E(S), \quad S > 0.$$

Then

$$E(t) \leq \begin{cases} \dfrac{\lambda E(0)}{(1+t)^{1/\alpha}}, & t > 0, \ \text{if } \alpha > 0, \\ \lambda E(0)e^{-\zeta t}, & t > 0, \ \text{if } \alpha = 0, \end{cases}$$

where k *and* ζ *are positive constants independent of the initial energy* $E(0)$.

3. Global existence

The aim of this section is to prove the global existence of solutions for problem (1.1). For this goal, we put the following functionals:

$$I(t) = \|\Delta v\|_p^p - \omega\|v\|_k^k, \tag{3.1}$$

$$J(t) = \frac{1}{p}\|\Delta v\|_p^p - \frac{\omega}{k}\|v\|_k^k. \tag{3.2}$$

Then we have

$$E(t) = \frac{1}{2}\|v_t\|_2^2 + J(t). \tag{3.3}$$

Lemma 3.1. *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_3$) *hold. For any* $u_0 \in W_0^{2,p}(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *such that*

$$I(0) > 0 \quad \text{and} \quad \gamma = \omega c_k^k c_*^k c^p \left[\frac{kp}{(k-p)}E(0)\right]^{\frac{k-p}{p}} < 1, \tag{3.4}$$

we have

$$I(t) > 0, \quad t > 0. \tag{3.5}$$

Proof. By continuity of $v(t)$, there exists a time $T_* < T$ such that

$$I(t) \geq 0, \quad t \in [0, T_*]. \tag{3.6}$$

From (3.1), (3.2) and (3.4), we have

$$J(t) = \frac{1}{k} I(t) + \frac{(k-p)}{kp} \|\Delta v\|_p^p \geq \frac{(k-p)}{kp} \|\Delta v\|_p^p. \tag{3.7}$$

By using (3.7) and Lemma 2.2, we obtain

$$\|\Delta v\|_p^p \leq \frac{kp}{k-p} J(t) \leq \frac{kp}{k-p} E(t) \leq \frac{kp}{k-p} E(0). \tag{3.8}$$

Exploiting (2.1), (3.2), **(H$_1$)**, and (3.4), we obtain

$$\begin{aligned} \omega \|v\|_k^k &\leq \omega c_k^k \|\nabla v\|_2^k \leq \omega c_k^k c_*^k \|\Delta v\|_2^k \leq \omega c_k^k c_*^k c^p \|\Delta v\|_p^k \\ &= \omega c_k^k c_*^k c^p \|\Delta v\|_p^p \|\Delta v\|_p^{k-p} \leq \omega c_k^k c_*^k c^p \left[\frac{kp}{(k-p)} E(0) \right]^{\frac{k-p}{p}} \|\Delta v\|_p^p \\ &= \gamma \|\Delta v\|_p^p < \|\Delta v\|_p^p, \quad t \in [0, T^*]. \end{aligned} \tag{3.9}$$

Therefore, we conclude that

$$I(t) > 0, \ \ \forall \, t \in [0, T_*].$$

By repeating the procedure, T_* is extended to T. □

Remark 3.2. According to (3.3), (3.7), **(H$_1$)**, and Lemma 3.1, we deduce that $E(t)$ is positive.

Theorem 3.3. *Assume that the conditions of Lemma* 3.1 *hold, then the solution of* (1.1) *is global and bounded.*

Proof. By virtue of (2.4), (3.3) and (3.7), we obtain

$$E(0) \geq E(t) = \frac{1}{2} \|v_t\|_2^2 + J(t) \geq \frac{1}{2} \|v_t\|_2^2 + \frac{(k-p)}{kp} \|\Delta v\|_p^p, \tag{3.10}$$

which means

$$\|u_t\|_2^2 + \|\Delta v\|_p^p \leq K E(0), \tag{3.11}$$

and this shows that the local solution is global and bounded. □

4. Stability

In this section, we state and prove the stability result of solution to problem (1.1) by using Komornik's method.

Theorem 4.1. *Assume that the conditions of Lemma 3.1 hold. Then there exist two positive constants* λ *and* ζ *such that*

$$\begin{cases} E(t) \leq \dfrac{\lambda}{(1+t)^{1/\sigma}}, & t>0, \ \text{if } \sigma>0, \\ E(t) \leq \lambda e^{-\zeta t}, & t>0, \ \text{if } \sigma=0. \end{cases}$$

Proof. Multiplying the first equation in (1.1) by $v_t E^\sigma(t)$ $(\sigma > 0)$ and integrating over $\Omega \times (S,T)$, we have

$$\int_S^T E^\sigma(t) \int_\Omega \left[v_{tt} - \Delta_p^2 v - \mu \Delta_m v_t + v_t\right] v \, dx \, dt = \sigma\omega \int_S^T E^\sigma(t) \int_\Omega |v|^k \, dx \, dt, \quad (4.1)$$

which gives

$$\int_S^T E^\sigma(t) \int_\Omega \left[\frac{d}{dt}(v_t v) - |v_t|^2 + |\Delta v|^p + \mu |\nabla v_t|^{m-2} \nabla v_t \nabla v + v_t v - \omega |v|^k\right] dx \, dt = 0. \quad (4.2)$$

Using the definition of $E(t)$, we see that

$$p \int_S^T E^\sigma(t) \, dt = \int_S^T E^\sigma(t) \int_\Omega \Big[-\frac{d}{dt}(v_t v) + \left(\frac{p}{2}+1\right) |v_t|^2 - \mu |\nabla v_t|^{m-2} \nabla v_t \nabla v - v_t v + \omega \left(1 - \frac{p}{k}\right) |v|^k \Big] dx \, dt. \quad (4.3)$$

On the other hand, we have

$$\frac{d}{dt}\left[E^\sigma(t) \int_\Omega v_t v \, dx\right] = \sigma E'(t) E^{\sigma-1}(t) \int_\Omega v_t v \, dx + E^\sigma(t) \frac{d}{dt}\left[\int_\Omega v_t v \, dx\right].$$

Then inequality (4.3) becomes

$$\begin{aligned} p \int_S^T E^{\sigma+1}(t) \, dt &= \sigma \int_S^T E^{\sigma-1}(t) E'(t) \int_\Omega v_t v \, dx - \int_S^T \frac{d}{dt}\left[E^\sigma(t) \int_\Omega v_t v \, dx\right] dt \\ &+ \left(\frac{p}{2}+1\right) \int_S^T E^\sigma(t) \int_\Omega |v_t|^2 \, dx \, dt + \omega \left(1-\frac{p}{k}\right) \int_S^T E^\sigma(t) \int_\Omega |v|^k \, dx \, dt \\ &- \mu \int_S^T E^\sigma(t) \int_\Omega |\nabla v_t|^{m-2} \nabla v_t \nabla v \, dx \, dt - \int_S^T E^\sigma(t) \int_\Omega v_t v \, dx \, dt. \end{aligned} \quad (4.4)$$

In what follows, we will estimate the right-hand side terms in (4.4). Exploiting (2.4), (3.3), (3.8), and (3.9), we obtain

$$\begin{aligned} \int_S^T E^\sigma(t) \|v_t\|_2^2 \, dt &\leq \int_S^T E^\sigma(t) \left(-E'(t)\right) dt \leq c_1 \left[E^{\sigma+1}(S) - E^{\sigma+1}(T)\right] \\ &\leq c_1 E^{\sigma+1}(S) \leq c_1 E^\sigma(0) E(S) \end{aligned} \quad (4.5)$$

and

$$\omega \int_S^T E^\sigma(t) \|v\|_k^k \, dt \leq \gamma \int_S^T E^\sigma(t) \|\Delta v\|_p^p \, dt \leq \gamma \frac{kp}{k-p} \int_S^T E^{\sigma+1}(t) \, dt. \quad (4.6)$$

By using Young's inequality, (4.5) and (4.6), we get

$$\begin{aligned}\int_S^T E^\sigma(t)\int_\Omega v_t v\,dx\,dt &\le \int_S^T E^\sigma(t)\left[\varepsilon\|v\|_k^k + c_\varepsilon\|v_t\|_{k/(k-1)}^{k/(k-1)}\right]dt\\ &\le \int_S^T E^\sigma(t)\left[\varepsilon\|v\|_k^k + c_\varepsilon c_\varrho\|v_t\|_2^2\right]dt\\ &\le \varepsilon c_2\int_S^T E^{\sigma+1}(t)dt + c_\varepsilon c_\varrho\int_S^T E^\sigma(t)\left(-E'(t)\right)^{k/2(k-1)}dt\\ &\le (\varepsilon + c_\varepsilon)c_2\int_S^T E^{\sigma+1}E(t)\,dt + c_\varepsilon c_3 E(S), \end{aligned} \tag{4.7}$$

where c_ϱ is the best embedding constant of $L^2(\Omega)\hookrightarrow L^{k/(k-1)}(\Omega)$.

Analogously to (4.7), we have

$$\begin{aligned}&\sigma\int_S^T E^{\sigma-1}(t)E'(t)\int_\Omega v_t v\,dx\,dt\\ &\qquad\le \sigma\int_S^T E^{\sigma-1}(t)\left(-E'(t)\right)\left[\varepsilon\|v\|_k^k + c_\varrho c_\varepsilon\|v_t\|_{k/2(k-1)}^2\right]dt\\ &\qquad\le (\varepsilon + c_\varepsilon)\,c_3\int_S^T \left[E^\sigma(t) + E^{\sigma+k/2(k-1)-1}(t)\right]\left(-E'(t)\right)dt\\ &\qquad\le (\varepsilon + c_\varepsilon)\,c_3 E(S) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&-\int_S^T \frac{d}{dt}\left[E^\sigma(t)\int_\Omega v_t v\,dx\right]dt\\ &\quad= E^\sigma(S)\int_\Omega v_t(S)v(S)dx - E^\sigma(T)\int_\Omega v_t(T)v(T)\,dx\\ &\quad\le E^\sigma(S)\int_\Omega |v_t(S)v(S)|\,dx + E^\sigma(T)\int_\Omega |v_t(T)v(T)|\,dx\\ &\quad\le (\varepsilon + c_\varrho c_\varepsilon)\left[E^{\sigma+1}(S) + E^{\sigma+1}(T)\right]\\ &\quad\le (\varepsilon + c_\varepsilon)\,c_4 E(S) \le (\varepsilon + c_\varepsilon)\,c_4 E(S). \end{aligned} \tag{4.9}$$

For the last term of (4.4), by using Young's inequality, (2.2), $(\mathbf{H}_1)$, (2.3), (3.11), and Lemma 2.2, we obtain

$$\begin{aligned}\mu\int_S^T E^\sigma(t)\int_\Omega |\nabla v_t|^{m-2}\nabla v_t\nabla v\,dx\,dt &\le \mu\int_S^T E^\sigma(t)\left[\varepsilon\|\nabla v\|_m^m + c_\varepsilon\|\nabla v_t\|_m^m\right]dt\\ &\le \mu\int_S^T E^\sigma(t)\left[\varepsilon c_*^m c^m\|\Delta v\|_p^m + c_\varepsilon\|\nabla v_t\|_m^m\right]dt\\ &= \varepsilon\mu c_*^m c^m\int_S^T E^\sigma(t)\|\Delta v\|_p^{m-p}\|\Delta v\|_p^p\,dt + c_\varepsilon\mu\int_S^T E^\sigma(t)\|\nabla v_t\|_m^m\,dt\\ &\le \varepsilon\mu c_*^m c^m\left(KE(0)\right)^{m-p}\int_S^T E^\sigma(t)\|\Delta v\|_p^p\,dt + c_\varepsilon\int_S^T E^\sigma(t)\left(-E'(t)\right)dt \end{aligned}$$

$$= \varepsilon c_5 \int_S^T E^{\sigma+1}(t)\,dt + c_\varepsilon \int_S^T E^\sigma(t)\big(-E'(t)\big)\,dt$$
$$\leq \varepsilon c_5 \int_S^T E^{\sigma+1}(t)\,dt + c_\varepsilon c_6 E^\sigma(0)E(S). \tag{4.10}$$

Inserting (4.5)–(4.10) into (4.4), we get

$$\{p(1-\gamma) - (\varepsilon + c_\varepsilon)c_7\} \int_S^T E^{\sigma+1}(t)\,dt \leq c_8 E(S). \tag{4.11}$$

Since $0 < \gamma < 1$, then $p(1-\gamma) > 0$. We choose ε small enough such that

$$p(1-\gamma) - \varepsilon c_7 > 0.$$

Then inequality (4.8) becomes

$$\int_S^T E^{\sigma+1}(t)\,dt \leq \lambda E(S).$$

By taking $T \to \infty$, we get

$$\int_S^\infty E^{\sigma+1}(t)\,dt \leq \lambda E(S).$$

Thus Komornik's lemma provides the desired result. □

5. Blow-up

5.1. Finite-time blow-up: case $m > 2$

Theorem 5.1. *Let* ($\mathbf{H}_1$), ($\mathbf{H}_3$) *and* $E(0) < 0$ *hold. Then the solution of problem* (1.1) *blows up in finite time* T^*.

Proof. Set
$$H(t) = -E(t). \tag{5.1}$$

Then (2.3) and (2.4) give us

$$H'(t) = -E'(t) \geq \mu\|\nabla v_t\|_m^m + \|v_t\|_2^2 \geq 0 \tag{5.2}$$

and
$$0 < H(0) \leq H(t) \leq \frac{\omega}{k}\|v\|_k^k,\ t \in [0, T). \tag{5.3}$$

Next, we define
$$\Gamma(t) = H^{1-\theta}(t) + \varepsilon \int_\Omega v v_t\,dx, \tag{5.4}$$

where $\varepsilon > 0$ is a small constant that will be chosen later and $0 < \theta < \frac{1}{2}$. Differentiating (5.4) with respect to t and using (1.1), we have

$$\Gamma'(t) = (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon\|v_t\|_2^2 - \varepsilon\|\Delta u\|_p^p + \varepsilon\omega\|v\|_k^k$$

$$- \mu \int_\Omega |\nabla v_t|^{m-2} \nabla v_t \nabla v \, dx \quad - \int_\Omega v_t v \, dx. \tag{5.5}$$

Similarly to (4.7) and (4.10), for any $\eta > 0$, we obtain

$$\begin{aligned} \mu \int_\Omega |\nabla v_t|^{m-2} \nabla v_t \nabla v \, dx &\le \mu \frac{\eta_1^m}{m} \|\nabla v\|_m^m + \mu \frac{m-1}{m} \eta_1^{-\frac{m}{m-1}} \|\nabla v_t\|_m^m \\ &\le \mu c_1 \frac{\eta_1^m}{m} \|\Delta v\|_p^p + \frac{m-1}{m} \eta_1^{-\frac{m}{m-1}} H'(t) \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \int_\Omega v_t v \, dx &\le \frac{\eta_2^k}{k} \|v\|_k^k + \frac{k-1}{k} \eta_2^{-\frac{k}{k-1}} c_\varrho \|v_t\|_2^2 \\ &\le \frac{\eta_2^k}{k} \|v\|_k^k + \frac{k-1}{k} \eta_2^{-\frac{k}{k-1}} c_\varrho c_0 (H'(t) + H(t)). \end{aligned} \tag{5.7}$$

Inserting (5.6) and (5.7) into (5.5) and using (5.1), we get

$$\begin{aligned} \Gamma'(t) \ge & \left\{ (1-\theta) H^{-\theta}(t) - \varepsilon \frac{m-1}{m} \eta_1^{-\frac{m}{m-1}} - \varepsilon c_\varrho \frac{k-1}{k} \eta_2^{-\frac{k}{k-1}} \right\} H'(t) \\ & + \varepsilon \left\{ \frac{\xi}{2} + 1 \right\} \|v_t\|_2^2 + \varepsilon \left\{ \frac{\xi}{p} - 1 \right\} \|\Delta v\|_p^p + \varepsilon \omega \left\{ 1 - \frac{\xi}{k} \right\} \|v\|_k^k \\ & - \mu c_1 \frac{\eta_1^m}{m} \|\Delta v\|_p^p - \frac{\eta_2^k}{k} \|v\|_k^k + \varepsilon \left(\xi - c_4 \eta_2^{-k/(k-1)} \right) H(t) \end{aligned} \tag{5.8}$$

for any $\xi > 0$.

By taking

$$\eta_1 = \left[\kappa_1 \frac{m}{m-1} H^{-\theta}(t) \right]^{-\frac{m-1}{m}} \quad \text{and} \quad \eta_2 = \left[\kappa_2 \frac{k}{c_\varrho (k-1)} H^{-\theta}(t) \right]^{-\frac{k-1}{k}},$$

where κ_1 and κ_2 are positive constants to be specified later, we see that

$$\begin{aligned} \Gamma'(t) \ge & \left\{ (1-\theta) - \varepsilon (\kappa_1 + \kappa_2) \right\} H^{-\theta}(t) H'(t) + \varepsilon \left\{ \frac{\xi}{2} + 1 \right\} \|v_t\|_2^2 \\ & + \varepsilon \left\{ \frac{\xi}{p} - 1 \right\} \|\Delta v\|_p^p + \varepsilon \omega \left\{ 1 - \frac{\xi}{k} \right\} \|v\|_k^k - c_2 \kappa_1^{1-m} H^{\theta(m-1)}(t) \|\Delta v\|_p^p \\ & - c_3 \kappa_2^{1-k} H^{\theta(k-1)}(t) \|v\|_k^k + \varepsilon \left(\xi - c_5 \kappa_2^{1-k} \right) H(t), \end{aligned} \tag{5.9}$$

where c_2 and c_3 are positive constants, which depend only on m and k, respectively.

Exploiting (5.3), (3.9) and (3.11), we get

$$\begin{aligned} c_2 H^{\theta(m-1)}(t) &\le c_2 \left(\frac{\omega}{k} \|v\|_k^k \right)^{\theta(m-1)} \le c_2 \left(\frac{\gamma}{k} \|\Delta v\|_p^p \right)^{\theta(m-1)} \\ &\le c_2 \left(\frac{\gamma}{k} K E(0) \right)^{\theta(m-1)} := c_m. \end{aligned} \tag{5.10}$$

In a similar way, we obtain

$$c_3 H^{\theta(k-1)}(t) \leq c_3 \left(\frac{\gamma}{k} KE(0)\right)^{\theta(k-1)} := c_k. \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.9), we have

$$\begin{aligned}\Gamma'(t) &\geq \{(1-\theta) - \varepsilon(\kappa_1+\kappa_2)\} H^{-\theta}(t) H'(t) \\ &\quad + \varepsilon\left(\xi - c_5 \kappa_2^{1-k}\right) H(t) + \varepsilon \left\{\frac{\xi}{2} + 1\right\} \|v_t\|_2^2 \\ &\quad + \varepsilon \left\{\frac{\xi}{p} - 1 - c_m \kappa_1^{1-m}\right\} \|\Delta v\|_p^p + \varepsilon \left\{\omega\left(1 - \frac{\xi}{k}\right) - c_k \kappa_2^{1-k}\right\} \|v\|_k^k. \end{aligned} \tag{5.12}$$

At this point, we choose our constant carefully. First, we choose $p < \xi < k$ such that

$$\frac{\xi}{p} - 1 > 0 \quad \text{and} \quad 1 - \frac{\xi}{k} > 0.$$

For any fixed ξ, we choose κ_1 and κ_2 so large that

$$\frac{\xi}{p} - 1 - c_m \kappa_1^{1-m} > 0 \quad \text{and} \quad \omega\left(1 - \frac{\xi}{k}\right) - c_k \kappa_2^{1-k} > 0.$$

Once k_1 and k_2 are fixed, we choose $\varepsilon > 0$ small enough such that

$$(1-\theta) - \varepsilon(\kappa_1+\kappa_2) > 0 \quad \text{and} \quad \Gamma(0) = H^{1-\theta}(0) + \varepsilon \int_\Omega v_0 v_1 \, dx.$$

Then inequality (5.12) becomes

$$\Gamma'(t) \geq \gamma \left(\|v_t\|_2^2 + \|\Delta v\|_p^p + \|v\|_k^k + H(t)\right), \tag{5.13}$$

where γ is a positive constant.

On the other hand, we have

$$\Gamma^{\frac{1}{1-\theta}}(t) \leq c_\theta \left\{ H(t) + \left[\int_\Omega v v_t \, dx\right]^{\frac{1}{1-\theta}} \right\}. \tag{5.14}$$

Applying Hölder's and Young's inequalities, we have

$$\left|\int_\Omega v v_t \, dx\right|^{\frac{1}{1-\theta}} \leq c_3 \|v\|_k^{\frac{1}{1-\theta}} \|v_t\|_2^{\frac{1}{1-\theta}} \leq c_3 \left(\|v\|_k^{\frac{\nu}{1-\theta}} + \|u_t\|_2^{\frac{\delta}{1-\theta}}\right).$$

Taking $\delta = 2(1-\theta)$, which gives $\frac{\nu}{1-\theta} = \frac{2}{1-2\theta}$, we deduce that

$$\left|\int_\Omega v v_t \, dx\right|^{\frac{1}{1-\theta}} \leq c_3 \left(\|v\|_k^{\frac{2}{1-2\theta}} + \|v_t\|_2^2\right). \tag{5.15}$$

Exploiting (3.9), (3.11) and (5.3), we obtain

$$\|v\|_k^{\frac{2}{1-2\theta}} \leq \gamma^{\frac{1}{k}} \|\Delta v\|_p^{\frac{2p}{k(1-2\theta)}} \leq \gamma^{\frac{1}{k}} (KE(0))^{\frac{2}{k(1-2\theta)}} \frac{H(t)}{H(0)} \leq c_4 \|v\|_k^k. \tag{5.16}$$

A substitution of (5.15) and (5.16) into (5.14), gives us

$$\Gamma^{\frac{1}{1-\theta}}(t) \leq \lambda \left(\|v_t\|_2^2 + \|v\|_k^k + H(t) \right). \tag{5.17}$$

It follows from (5.13) and (5.17) that

$$\Gamma'(t) \geq \kappa \Gamma^{\frac{1}{1-\theta}}(t), \quad t > 0, \tag{5.18}$$

where κ is a positive constant. A simple integration of (5.18) over $(0,t)$ yields

$$\Gamma^{\frac{\theta}{1-\theta}}(t) \geq \frac{1}{\Gamma^{\frac{1}{1-\theta}}(0) - \frac{\kappa\theta t}{1-\theta}}.$$

Therefore, $\Gamma(t)$ blows up in a finite time T^* and $T^* \leq 1-\theta/\left(\kappa\theta\Gamma^{\frac{\theta}{1-\theta}}(0)\right)$. On the other hand, from the definition of $\Gamma(t)$ and (5.3), it follows that the norm of $\|v\|_k$ of the solution blows up in a finite time. This completes the proof. □

5.2. Infinite-time blow-up: case $m = 2$

Theorem 5.2. *Assume that* $(\mathbf{H}_2)$*–*$(\mathbf{H}_3)$ *hold. For any* $(v_0, v_1) \in W_0^{2,p}(\Omega) \times L^2(\Omega)$*, the solution of problem* (1.1) *blows up as time* t *goes to infinity.*

Proof. Suppose that the solution v is global. Then, for any $T > 0$, we define the following auxiliary function:

$$\Gamma(t) := \Gamma(v) = \|v\|_2^2 + \int_0^t \|v(\tau)\|_\mu^2 \, d\tau + (T-t)\|v_0\|_\mu^2. \tag{5.19}$$

It is clear that $\Gamma(t) > 0$ for all $t \in [0,T]$. By the continuity of $\Gamma(t)$, we obtain that there is $\kappa > 0$ such that

$$\Gamma(t) > \kappa \quad \text{for all } t \in [0,T], \tag{5.20}$$

where κ is independent of T. Taking a derivative of (5.19) with respect to t, using (1.1) and (3.1), we obtain

$$\Gamma'(t) = 2\int_\Omega v v_t \, dx + \left(\|v\|_\mu^2 - \|v_0\|_\mu^2\right) = 2\int_\Omega v v_t \, dx + 2\int_0^t (v(\tau), v_t(\tau))_\mu \, d\tau \tag{5.21}$$

and

$$\begin{aligned} \Gamma''(t) &= 2\|v_t\|_2^2 + 2\langle v_{tt}, v\rangle + 2(v, v_t)_\mu \\ &= 2\|v_t\|_2^2 - 2\|\Delta v\|_p^p + 2\omega\|v\|_k^k = 2\|v_t\|_2^2 - 2I(t). \end{aligned} \tag{5.22}$$

On the other hand, from (5.21), we have

$$\left(\Gamma'(t)\right)^2 = 4\left(\int_\Omega v v_t \, dx\right)^2 + 4\left(\int_0^t (v(\tau), v_t(\tau))_\mu \, d\tau\right)^2$$

$$+\,8\int_\Omega vv_t\,dx\int_0^t (v(\tau),v_t(\tau))_\mu\,d\tau. \tag{5.23}$$

By using Hölder's and Young's inequalities, we obtain

$$\left(\int_\Omega vv_t\,dx\right)^2 \le \|v\|_2^2\|v_t\|_2^2, \tag{5.24}$$

$$\left(\int_0^t (v(\tau),v_t(\tau))_\mu\,d\tau\right)^2 \le \int_0^t \|v(\tau)\|_\mu^2\,d\tau\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau \tag{5.25}$$

and

$$\int_\Omega vv_t\,dx\int_0^t (v(\tau),v_t(\tau))_\mu\,d\tau \le \|v\|_2^2\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau + \|v_t\|_2^2\int_0^t \|v(\tau)\|_\mu^2\,d\tau. \tag{5.26}$$

By (5.24)–(5.26), inequality (5.23) becomes

$$\begin{aligned}\left(\Gamma'(t)\right)^2 &\le 4\left[\|v\|_2^2+\int_0^t \|v(\tau)\|_\mu^2\,d\tau\right]\left[\|v_t\|_2^2+\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\right]\\ &\le 4\Gamma(t)\left[\|v_t\|_2^2+\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\right].\end{aligned} \tag{5.27}$$

It follows from (5.19) and (5.27) that

$$\begin{aligned}\Gamma(t)\Gamma''(t)-\left(\Gamma'(t)\right)^2 &\ge 2\Gamma(t)\left[\|v_t\|_2^2 - I(t)\right] - 4\Gamma(t)\left[\|v_t\|_2^2+\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\right]\\ &= \Gamma(t)\left[-2\|v_t\|_2^2-2I(t)-4\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\right] = \Gamma(t)\zeta(t),\end{aligned} \tag{5.28}$$

where

$$\zeta(t) = -2\|v_t\|_2^2-2I(t)-4\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau. \tag{5.29}$$

Using (3.1), (3.2), (3.11), and Lemma 2.2, we obtain

$$\begin{aligned}\zeta(t) &= 4J(t)-4E(t)-2I(t)-4\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\\ &= 2\omega\frac{(k-2)}{k}\|v\|_k^k - 2\frac{p-2}{p}\|\Delta v\|_p^p - 4E(t) - 4\int_0^t \|v_t(\tau)\|_\mu^2\,d\tau\\ &\ge 2\omega\frac{(k-2)}{k}\|v\|_k^k - 2\left(K\frac{p-2}{p}+2\right)E(0).\end{aligned} \tag{5.30}$$

Since k, $p > 2$ and $E(0) < 0$, we conclude that

$$\zeta(t) > \nu > 0. \tag{5.31}$$

Then, by using (5.31), we get

$$\Gamma(t)\Gamma''(t)-\left(\Gamma'(t)\right)^2 \ge \Gamma(t)\nu > 0, \quad t\in[0,T]. \qquad \square \tag{5.32}$$

It is clear that

$$(\ln \Gamma(t))' = \frac{\Gamma'(t)}{\Gamma(t)} \tag{5.33}$$

and

$$(\ln \Gamma(t))'' = \frac{\Gamma(t)\Gamma''(t) - (\Gamma'(t))^2}{(\Gamma(t))^2} > 0. \tag{5.34}$$

From (5.34), we deduce that $(\ln \Gamma(t))'$ is increasing on t. A simple integration of (5.33) over (t_0, t) yields

$$\begin{aligned} \ln \Gamma(t) - \ln \Gamma(t_0) &= \int_{t_0}^{t} (\ln \Gamma(\tau))' \, d\tau \\ &= \int_{t_0}^{t} \frac{\Gamma'(\tau)}{\Gamma(\tau)} d\tau \geq \frac{\Gamma'(t_0)}{\Gamma(t_0)}(t - t_0), \quad 0 \leq t_0 \leq t. \end{aligned} \tag{5.35}$$

Then

$$\Gamma(t) \geq \Gamma(t_0) \exp\left(\frac{\Gamma'(t_0)}{\Gamma(t_0)}(t - t_0)\right). \tag{5.36}$$

If we take t_0 sufficiently small such that $\Gamma'(t_0) > 0$ and $\Gamma(t_0) > 0$, then, from (5.36), we deduce sufficiently large t,

$$\lim_{t \to +\infty} \Gamma(t) = +\infty. \tag{5.37}$$

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Глобальне існування, стійкість та руйнування розв'язків для p-бігармонічного гіперболічного рівняння зі слабкими та сильними демпфувальними членами

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У цій статті ми досліджуємо початково-крайову задачу для p-бігармонічного гіперболічного рівняння зі слабкими та сильними демпфувальними членами:

$$v_{tt} + \Delta_p^2 v - \mu \Delta_m v_t + v_t = \omega |v|^{k-2} v.$$

При деяких припущеннях на початкові дані, сталі p, m та k, ми довели глобальне існування, стійкість та результати стосовно руйнування розв'язків. Глобальний розв'язок одержано методом потенціальної ями, а стійкість ґрунтується на нерівності Коморніка. Також доведено, що розв'язок з від'ємною початковою енергією вибухає за скінченний та за нескінченний час.

Ключові слова: p-бігармонічне рівняння, демпфувальні члени, глобальне існування, стійкість, руйнування