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On Centralizers of Belavin–Drinfeld r-Matrices

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The paper provides a structural result about centralizers of Belavin–Drinfeld r-matrices. This result appears to be useful in computing Belavin–Drinfeld cohomology, which was introduced earlier for classificalion of certain Lie bialgebras and quantum groups.

Key words: Belavin–Drinfeld, quantum group, Lie bialgebra.

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1. Introduction

Belavin–Drinfeld cohomology was introduced in [4] and further studied in [1, 5–8, 10] et. al. It was introduced and used as an instrument to classify Lie bialgebra structures on simple Lie algebras over non-algebraically closed fields via reduction to the algebraically closed case, where, in particular, the Belavin–Drinfeld classification [2] is applicable. The case when the base field is the field of formal Laurent series appears to be closely related to classification of quantum groups, and some partial results in this direction was obtained in the above cited papers.

Belavin–Drinfeld cohomology is defined for an r-matrix r with respect to a certain "gauge group" $\mathbf{G}$ in terms of the centralizer of r in $\mathbf{G}$. In particular, the untwisted Belavin–Drinfeld cohomology is in most important cases fully controlled by the structure of this centralizer.

The purpose of this note is to generalise the results on the structure of centralizers of Belavin–Drinfeld matrices obtained in [7]. In particular, this result shed more light on the structure of Belavin–Drinfeld cohomology.

The paper is organized as follows. After making the setup and recalling necessary definitions and results in Section 2, we present the main result on the centralizer structure, Theorem 3.1, in Section 3. Finally, in Section 4 we provide more details on the centralizer structure via dealing with split simple Lie algebras case by case.

2. Notation and setting

Let $\mathbb{F}$ be a field of characteristic zero. We fix an algebraic closure of $\mathbb{F}$, which will be denoted by $\overline{\mathbb{F}}$. The Galois group of the extension $\overline{\mathbb{F}}/\mathbb{F}$ will be denoted by $\mathcal{G}$.

If $\mathbf{K}$ is an affine algebraic group over $\mathbb{F}$, then the corresponding (non-abelian) étale Galois cohomology will be denoted by $H^1(\mathbb{F},\mathbf{K})$ (see [9] for details). We recall that $H^1(\mathbb{F},\mathbf{K})$ coincides with the usual non-abelian continuous cohomology of the profinite group $\mathcal{G}$ acting naturally on $\mathbf{K}(\overline{\mathbb{F}})$.

Let $\mathfrak{g}$ be a split finite dimensional simple Lie algebra over $\mathbb{F}$. In what follows $\mathbf{G}$ will denote a split connected simple algebraic $\mathbb{F}$-group with the Lie algebra $\mathfrak{g}$.

We fix a Killing couple $(\mathbf{B},\mathbf{H})$ of $\mathbf{G}$, whose corresponding Borel and split Cartan subalgebras will be denoted by $\mathfrak{b}$ and $\mathfrak{h}$ respectively. This Killing couple defines a root system Δ with a fixed set of positive roots Δ_+ and the set of simple roots $\Gamma=\{\alpha_1,\dots,\alpha_n\}$.

Let us recall the structure of the Belavin–Drinfeld r-matrices [2]. With respect to $(\mathfrak{b},\mathfrak{h})$, any Belavin–Drinfeld r-matrix depends on a discrete and a continuous parameter. The discrete parameter is an *admissible triple* (Γ_1,Γ_2,τ). Namely, $\Gamma_1,\Gamma_2\subset\Gamma$, and $\tau:\Gamma_1\to\Gamma_2$ is an isometry such that for any $\alpha\in\Gamma_1$ there exists $k\in\mathbb{N}$ satisfying $\tau^k(\alpha)\notin\Gamma_1$. The continuous parameter is a tensor $r_0\in\mathfrak{h}\otimes_{\mathbb{F}}\mathfrak{h}$ satisfying $r_0+r_0^{21}=\Omega_0$ and $(\tau(\alpha)\otimes 1+1\otimes\alpha)(r_0)=0$ for any $\alpha\in\Gamma_1$. Here Ω_0 denotes the Cartan part of the quadratic Casimir element $\Omega\in\mathfrak{g}\otimes_{\mathbb{F}}\mathfrak{g}$. Then the corresponding Belavin–Drinfeld r-matrix is

$$r_{\mathrm{BD}}=r_0+\sum_{\alpha\in\Delta_+}e_\alpha\otimes e_{-\alpha}+\sum_{\alpha\in(\operatorname{Span}\Gamma_1)_+}\sum_{k\geq 1}e_\alpha\wedge e_{-\tau^k(\alpha)}.$$

where e_α and $e_{-\alpha}$ are parts of a fixed Chevalley system of $\mathfrak{g}$ in the sense of [3, Ch. VIII, §2 and §12], $(\operatorname{Span}\Gamma_1)_+$ is the subset of all positive roots in the set of roots generated by Γ_1, and τ is expanded by linearity.

By a *string* of an admissible triple (Γ_1,Γ_2,τ) (or the corresponding r-matrix r_{BD}) we mean a subset of Γ of the form $\{\alpha,\tau(\alpha),\dots,\tau^{l-1}(\alpha)\}$, where $\alpha\notin\Gamma_2$ and $\tau^{l-1}(\alpha)\notin\Gamma_1$. The number l is the *lengths* of the string. If $\alpha\notin\Gamma_1\cup\Gamma_2$ then we have a string $\{\alpha\}$ of lengths 1. Thus, Γ is a disjoint union of strings.

Let r_{BD} be a Belavin–Drinfeld r-matrix. We denote by $\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})$ the centralizer of r_{BD} in $\mathbf{G}$ under the adjoint action. I.e., if R is a commutative ring extension of $\mathbb{F}$ then

$$\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})(R)=\{X\in\mathbf{G}(R):\operatorname{Ad}_X(r_{\mathrm{BD}})=r_{\mathrm{BD}}\}.$$

It was shown in [5, Theorem 1] that $\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})$ is a closed subgroup of $\mathbf{H}$. Moreover, by [5, Theorem 2] the centralizer $\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})$ can be described as follows: for any commutative ring extension $R\supset\mathbb{F}$ and $h\in\mathbf{H}(R)$, we have $h\in\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})(R)$ if and only if for any string of r_{BD} the corresponding characters of $\mathbf{H}$ take the same values on h.

We also recall the definition of untwisted Belavin–Drinfeld cohomology. Let r_{BD} be a Belavin–Drinfeld r-matrix. An element $X\in\mathbf{G}(\overline{\mathbb{F}})$ is called a *Belavin–Drinfeld cocycle* associated to $\mathbf{G}$ and r_{BD} if for any $\gamma\in\mathcal{G}$ we have $X^{-1}\gamma(X)\in\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})(\overline{\mathbb{F}})$. Denote by $Z(\mathbf{G},r_{\mathrm{BD}})$ the set of all Belavin–Drinfeld cocycles associated to $\mathbf{G}$ and r_{BD}. Two cocycles $X_1,X_2\in Z(\mathbf{G},r_{\mathrm{BD}})$ are called *equivalent* if there exists $Q\in\mathbf{G}(\mathbb{F})$ and $C\in\mathbf{C}(\mathbf{G},r_{\mathrm{BD}})(\overline{\mathbb{F}})$ such that $X_1=QX_2C$. The set of equivalence classes of cocycles in $Z(\mathbf{G},r_{\mathrm{BD}})$ is called the *untwisted Belavin–Drinfeld cohomology* associated to $\mathbf{G}$ and r_{BD} and is denoted by $H(\mathbf{G},r_{\mathrm{BD}})$.

3. Main result

Let $Q \subset P$ be the root and weight lattices of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\chi(\mathbf{H})$ be the group of (algebraic) characters of the torus $\mathbf{H}$. The map $\lambda \mapsto \mathrm{d}\lambda$, where d is the differential at the identity, is an isomorphism of $\chi(\mathbf{H})$ onto a lattice X with $Q \subset X \subset P$.

Let $\gamma_1, \dots, \gamma_n$ be a $\mathbb{Z}$-basis of X, $t_1, \dots, t_n \in \chi(\mathbf{H})$ the corresponding characters. Then the map $h \mapsto (t_1(h), \dots, t_n(h))$ defines an isomorphism $\mathbf{H} \to (\mathbb{G}_\mathrm{m})^n$ of algebraic tori.

Denote by μ_m the finite multiplicative $\mathbb{F}$-group of m-roots of unity.

Theorem 3.1. *Write $X/Q = \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_s\mathbb{Z}$. Let r_{BD} be a Belavin–Drinfeld r-matrix. Then $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}}) = \mathbf{T} \times \mathbf{C}'$, where $\mathbf{T}$ is a split torus over $\mathbb{F}$ and $\mathbf{C}'$ is a finite $\mathbb{F}$-group isomorphic to a subgroup of $\mu_{m_1} \times \ldots \times \mu_{m_s}$.*

Proof. Since $\mathbf{C} = \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is a closed subgroup of $\mathbf{H}$, it is of the form $\mathbf{C} = \mathbf{T} \times \mathbf{C}'$, where $\mathbf{T}$ is a split torus over $\mathbb{F}$ and $\mathbf{C}'$ is a finite commutative $\mathbb{F}$-group.

Let $\mathbf{K}$ be the torus corresponding to Q. We have a quotient map $\pi : \mathbf{H} \to \mathbf{K}$ induced by the inclusion $Q \subset X$, and $\operatorname{Ker} \pi \simeq \mu_{m_1} \times \ldots \times \mu_{m_s}$.

Let $q_1, \dots, q_n \in \chi(\mathbf{K})$ be the characters corresponding to the $\mathbb{Z}$-basis $\alpha_1, \dots, \alpha_n$ of Q, where α_i are simple roots. Applying [5, Theorem 2], we see that $\pi(\mathbf{C}) \subset \mathbf{K} \simeq (\mathbb{G}_\mathrm{m})^n$ is defined by equations of the form $q_{i_1} = \ldots = q_{i_k}$ for any string

$$\{\alpha_{i_1},\, \alpha_{i_2} = \tau(\alpha_{i_1}),\, \dots,\, \alpha_{i_k} = \tau^{k-1}(\alpha_{i_1})\}$$

of the r-matrix r_{BD}. Therefore, $\pi(\mathbf{C}) \simeq (\mathbb{G}_\mathrm{m})^{n(r_{\mathrm{BD}})}$, where $n(r_{\mathrm{BD}})$ is the number of strings of r_{BD}. In particular, $\pi(\mathbf{C})$ is connected, and thus $\mathbf{C}' \subset \operatorname{Ker} \pi$. □

As a direct corollary, we recover the statement of [7, Proposition 6.1]:

Corollary 3.2. *Let $X = Q$, i.e. the group $\mathbf{G}$ is of adjoint type. Then $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is connected for any Belavin–Drinfeld r-matrix r_{BD}.*

Remark 3.3. Write $\alpha_i = \sum_j n_{ij}\gamma_j$ with $n_{ij} \in \mathbb{Z}$. Let $h = (h_1, \dots, h_n) \in \mathbf{H}(R)$ for a commutative ring extension $R \supset \mathbb{F}$. According to [5, Theorem 2], we have $h \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(R)$ if and only if it satisfies the system of equations

$$\prod_j h_j^{n_{ij}} = \prod_l h_l^{n_{kl}}, \tag{3.1}$$

where $\alpha_i \in \Gamma_1$ and $\tau(\alpha_i) = \alpha_k$.

Write $\mathbf{C}' = \mu_{n_1} \times \ldots \times \mu_{n_s}$. Then

$$H^1(\mathbb{F}, \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})) = \mathbb{F}^\times/(\mathbb{F}^\times)^{n_1} \times \ldots \times \mathbb{F}^\times/(\mathbb{F}^\times)^{n_s}.$$

Assume $\mathbb{F}$ is of cohomological dimension 1. Then, by [8, Corollary 4.13] we have $H(\mathbf{G}, r_{\mathrm{BD}}) = H^1(\mathbb{F}, \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}}))$. I.e., in this case we have

$$H(\mathbf{G}, r_{\mathrm{BD}}) = \mathbb{F}^\times/(\mathbb{F}^\times)^{n_1} \times \ldots \times \mathbb{F}^\times/(\mathbb{F}^\times)^{n_s}.$$

In particular, if $\mathbb{F} = \mathbb{C}((t))$ then we get

$$H(\mathbf{G}, r_{\mathrm{BD}}) = \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_s\mathbb{Z}.$$

4. Examples

Assume that $X = P$, i.e., $\mathbf{G}$ is simply connected. Then the fundamental weights $\varpi_1, \dots, \varpi_n$ can be taken as a $\mathbb{Z}$-basis of P, and $\alpha_i = \sum_j a_{ij}\varpi_j$, where a_{ij} are the entries of the corresponding Cartan matrix.

If $\mathfrak{g}$ is of the type A_n, i.e., $\mathbf{G} = \mathbf{SL}(n+1)$, then it was shown in [4, Theorem 7] that for $\mathbb{F} = \mathbb{C}((t))$ the number of elements in $H(\mathbf{G}, r_{\mathrm{BD}})$ is always a divisor of $n+1$. Since in this case $P/Q = \mathbb{Z}/(n+1)\mathbb{Z}$, the result is in total accordance with Theorem 3.1.

Now let $\mathfrak{g}$ be of the type B_n, i.e., $\mathbf{G} = \mathrm{Spin}(2n+1)$. In this case $P/Q = \mathbb{Z}/2\mathbb{Z}$. However, we have the following

Proposition 4.1. *Let $\mathbf{G} = \mathrm{Spin}(2n+1)$. Then $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is connected for any Belavin–Drinfeld r-matrix r_{BD}.*

Proof. Enumerate the simple roots in a standard way, so that α_n is a short root. For any admissible triple $(\Gamma_1, \Gamma_2, \tau)$, since τ is an isometry, we see that $\alpha_n \notin \Gamma_1 \cup \Gamma_2$. Write $a_1 = h_1^2 h_2^{-1}$, $a_i = h_{i-1}^{-1} h_i^2 h_{i+1}^{-1}$ for $i = 2, \dots, n-1$. In this notation the system of equations (3.1) that defines $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ can be written as

$$a_{i_1} = a_{j_1}, \; \dots, \; a_{i_m} = a_{j_m},$$

where $i_1 < j_1, \dots, i_m < j_m$ and $2 \le j_1 < \dots < j_m \le n-1$. This system can be solved rationally for $h_{j_1+1}, \dots, h_{j_m+1}$. Thus $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is connected. □

If $\mathfrak{g}$ is of the type C_n, i.e., $\mathbf{G} = \mathrm{Sp}(2n)$, then it was shown in [5, Theorem 4.3] that for $\mathbb{F} = \mathbb{C}((t))$ the Belavin–Drinfeld cohomology $H(\mathbf{G}, r_{\mathrm{BD}})$ is always trivial. Thus the situation in this case is similar to what we have for the B_n case.

For the type D_n the situation is more subtle. In this case $P/Q = \mathbb{Z}/4\mathbb{Z}$ for n odd, and $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for n even. For n even, examples show that if $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is not connected, then both $\mathbf{C}' = \mu_2$ and $\mathbf{C}' = \mu_2 \times \mu_2$ are possible. For n odd, no examples with $\mathbf{C}' = \mu_4$ are known; only the case $\mathbf{C}' = \mu_2$ has been observed.

In the examples below the simple roots are enumerated in a standard way, so that in particular α_{n-1} and α_n are incident to α_{n-2}.

E.g., consider $n = 4$. If we take $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_3\}$, $\tau(\alpha_1) = \alpha_3$, then from (3.1) one easily gets $\mathbf{C}' = \mu_2$. Similarly, if $\Gamma_1 = \{\alpha_1, \alpha_3\}$, $\Gamma_2 = \{\alpha_3, \alpha_4\}$, $\tau(\alpha_1) = \alpha_3$, $\tau(\alpha_3) = \alpha_4$, then $\mathbf{C}' = \mu_2 \times \mu_2$.

Generally, if we consider $\Gamma_1 = \{\alpha_{n-1}\}$, $\Gamma_2 = \{\alpha_n\}$, $\tau(\alpha_{n-1}) = \alpha_n$, then we get $\mathbf{C}' = \mu_2$.

The complete answer in this case is yet unknown and can be a subject of further research.

Finally, the exceptional simple Lie algebras were considered in [7, Appendix B]. For the type E_6, we have $P/Q = \mathbb{Z}/3\mathbb{Z}$. In this case, $\mathbf{C}' = \mu_3$ indeed occurs, and the complete list of the respective r-matrices is given in [7, Appendix B]. For the type E_7, we have $P/Q = \mathbb{Z}/2\mathbb{Z}$. However, in this case $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is always connected. In all other exceptional cases we have $P = Q$, so $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is also always connected.

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Про централізатори r-матриць Белавіна–Дрінфельда

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У статті наводиться структурний результат про централізатори r-матриць Белавіна–Дрінфельда. Цей результат є корисним для обчислення когомологій Белавіна–Дрінфельда, які були введені раніше для класифікації деяких біалгебр Лі та квантових груп.

Ключові слова: Белавін–Дрінфельд, квантова група, біалгебра Лі