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On Some Nonlinear Elliptic Problems with Large Monotonocity in Musielak–Orlicz–Sobolev Spaces

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In this paper, we study the existence of an entropy solution for some nonlinear elliptic problems of Leray-Lions type associated to the equation $-\operatorname{div} a(x,u,\nabla u) = f(x) - \operatorname{div} F(u)$ in Ω with a large monotonicity condition in the setting of Musielak–Orlicz–Sobolev spaces and where the right-hand side f belongs to $L^1(\Omega)$ and $F = (F_1,\dots,F_N)$ satisfies $F \in (C^0(\mathbb{R}))^N$.

Key words: elliptic problem, entropy solutions, Musielak-Orlicz-Sobolev spaces, compact imbedding, Δ_2-condition

Mathematical Subject Classification 2010: 35J62, 35J25

1. Introduction

In this work, we will prove the existence of an entropy solution for an elliptic problem modeled by

$$\begin{cases} A(u) = f(x) - \operatorname{div} F(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega), F \in (C^0(\mathbb{R}))^N$ and $A(u) = -\operatorname{div} a(x,u,\nabla u)$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$.

Note that no growth hypothesis is assumed on the function F, which implies that the term $\operatorname{div} F(u)$ may be meaningless, even as a distribution. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, and continuous with respect to $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ for almost every $x \in \Omega$) such that for all ξ, ξ' in $\mathbb{R}^N$, $(x,s) \in \Omega\times\mathbb{R}$,

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i\overline{\psi}^{-1}(\varphi(x,c_2|s|)) + K_i(\overline{\varphi}^{-1}\varphi(x,c_1\,|\xi|)), \tag{1.2}$$

$$\big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi-\xi') \geq 0, \tag{1.3}$$

$$a(x,s,\xi)\xi \geq \alpha\varphi(x,\lambda_1|\xi|), \tag{1.4}$$

where c_1, c_2, $\lambda_1, K_i > 0$. Let φ, ψ be two Musielak–Orlicz functions such that $\psi \ll \varphi$. Moreover, $\overline{\varphi}$ and $\overline{\psi}$ are two complementary Musielak–Orlicz functions of φ and ψ, respectively, ϕ_0, $\phi_i \in E_{\overline{\varphi}}(\Omega)$ ($E_{\overline{\varphi}}(\Omega)$ is introduced later),

$$f \in L^1(\Omega), \tag{1.5}$$

and $F = (F_1, \dots, F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N. \tag{1.6}$$

The idea of the entropy solution, initiated by Boccardo in [12], makes sense for a possible solution of problem (1.1).

In [12], the existence and regularity of an entropy solution u of problem (1.1) was proved by Boccardo for p such that $2 - 1/N < p < N$. In [6], the existence and uniqueness of entropy solutions was studied by Benilan et. al, and the same problem, where $f \in L^1(\Omega)$ and $F \in L^{p'}(\Omega)^N$, was treated by Leone and Porretta in [25].

In [26], a similar problem was studied by Lions and Murat, where they used the notions of renormalized solutions introduced by Diperna and Lions [17] to study Boltzmann equations.

In the general framework of weighted Orlicz–Sobolev spaces, in [19], a similar problem having large monotonocity with L^1- and $F \equiv 0$ was treated by El Haji, El Moumni, and Kouhaila. In the framework of weighted Sobolev spaces, Akdim, Azroul, and Rhoudaf proved in [2] the existence of T-solutions for the elliptic problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F \in W^{-1,p'}(\Omega, \omega^*)$, and only large monotonicity is assumed on the Caratheodory function $a(x,u,\nabla u)$. For the case of Orlicz spaces, Gossez and Mustonen studied in [23] the following strongly nonlinear elliptic problem:

$$A(u) + g(x,u) = f \quad \text{in } \Omega. \tag{1.7}$$

They proved the existence and regularity of solutions for the unilateral elliptic problem (1.7) (see also [4] for the anisotropic case and [3] for the case of variable exponent).

Recently, much attention has been paid to the existence of solutions for elliptic and parabolic problems under various assumptions (see, e.g., [11,15,16,18,20,30–33,35–38] and the references therein).

A particular feature of this paper is that treated is a class of problems for which the classical monotone operator methods developed by Minty [28], Browder [14], Brézis [13], and Lions [27] in $W_0^{1,p}(\Omega)$ case are not applied. The reason for this is that $a(x,u,\nabla u)$ does not need to satisfy the strict monotonicity condition

$$\big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi - \xi') > 0 \quad \text{for all } \xi, \xi' \in \mathbb{R}^N \ (\xi \neq \xi')$$

of a typical Leray–Lions operator, but only a large monotonicity

$$\big(a(x,s,\xi)-a(x,s,\xi')\big)(\xi-\xi')\geq 0 \quad \text{for all } \xi,\xi'\in\mathbb{R}^N.$$

The purpose of this note is to show the existence of solutions for (1.1) under a weaker assumption of large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations since this is impossible to prove in our setting. The main tool of our proof is a version of Minty's Lemma. But the techniques we used in the proof differ from those used in [7,8].

The paper is organized as follows. In Section 2, we present some basic definitions and properties in the setting of Musielak–Orlicz–Sobolev spaces and we prepare some auxiliary results which are needed to show our existence result. In the final Section 3, we prove the result desired.

2. Preliminary

Here we give some definitions and properties that concern Musielak–Orlicz spaces (see [29]). Let Ω be an open subset of $\mathbb{R}^n$. Then a Musielak–Orlicz function φ is a real-valued function defined in $\Omega\times\mathbb{R}^+$ such that

(a) $\varphi(x,\cdot)$ is an N-function for all $x\in\Omega$, i.e., convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$ and $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=\infty$.

(b) $\varphi(\cdot,t)$ is a measurable function for all $t\geq 0$.

For a Musielak–Orlicz function φ, let $\varphi_x(t)=\varphi(x,t)$ and let φ_x^{-1} be a nonnegative reciprocal function with respect to t, i.e., the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t))=\varphi\left(x,\varphi_x^{-1}(t)\right)=t.$$

A Musielak–Orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$ and a nonnegative function h, integrable in Ω, we have

$$\varphi(x,2t)\leq k\varphi(x,t)+h(x) \quad \text{for all } x\in\Omega \text{ and } t\geq 0. \tag{2.1}$$

If (2.1) holds only for $t\geq t_0>0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak–Orlicz functions. We say that φ dominate γ and we write $\gamma\prec\varphi$ near infinity (respectively, globally) if there exist two positive constants c and t_0 such that for a.e. $x\in\Omega$:

$$\gamma(x,t)\leq\varphi(x,ct) \text{ for all } t\geq t_0, \ \text{(respectively, for all } t\geq 0, \text{ i.e., } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (respectively, near infinity) and we write $\gamma\prec\prec\varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad \left(\text{respectively, } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0\right).$$

For a Musielak–Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \rho_{\varphi,\Omega}(u) < \infty\}$ is called as the Musielak–Orlicz class or the generalized Orlicz class. The Musielak–Orlicz space (the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega) = \left\{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0\right\}.$$

For a Musielak–Orlicz function φ, we put

$$\psi(x, s) = \sup_{t>0}\{st - \varphi(x, t)\}.$$

Note that ψ is the Musielak–Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$, we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 \,\middle|\, \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \le 1\right\},$$

which are the Luxemburg norm and the so-called Orlicz norm, by

$$\|u\|_{\varphi,\Omega} = \sup_{\|v\|_\psi \le 1} \int_\Omega |u(x)v(x)|\, dx,$$

where ψ is the Musielak–Orlicz function complementary to φ. These two norms are equivalent (see [29]). The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, it is a separable space (see [29, Theorem 7.10]).

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer m, we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) \mid \forall |\alpha| \le m \;\; D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) \mid \forall |\alpha| \le m \;\; D^\alpha u \in E_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called a Musielak–Orlicz–Sobolev space. Let for $u \in W^m L_\varphi(\Omega)$:

$$\bar\rho_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \rho_{\varphi,\Omega}(D^\alpha u) \;\text{ and }\; \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 \,\middle|\, \bar\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

These functionals are a convex modular and a norm on $W^m L_M(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\cdot\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see [29]):

$$\exists c_0 > 0 \quad \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\Pi L_\varphi(\Omega)$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed.

The space $W^m_0 L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, and the space $W^m_0 E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \,\middle|\, f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\}$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \,\middle|\, f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

We recall that

$$\varphi(x,t) \leq t\psi^{-1}(\varphi(x,t)) \leq 2\varphi(x,t) \quad \text{ for all } t \geq 0. \tag{2.3}$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see [29]):

$$ts \leq \varphi(x,t) + \psi(x,s) \quad \text{for all } t, s \geq 0 \text{ a.e. } x \in \Omega. \tag{2.4}$$

This inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$, we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

and

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak–Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$. Then we have the Hölder inequality (see [29]):

$$\left| \int_\Omega u(x)v(x)\,dx \right| \leq \|u\|_{\varphi,\Omega} \|v\|_{\psi,\Omega}. \tag{2.8}$$

Definition 2.1. A Musielak function φ is called locally integrable on Ω if

$$\int_E \varphi(x,t)\,dx = \int_\Omega \varphi\left(x, t\chi_E(x)\right)dx < +\infty$$

for all $t \geq 0$ and all measurable sets $E \subset \Omega$ with $\operatorname{mes}(E) < +\infty$.

Remark 2.2. If $P \prec\!\prec \varphi$ near infinity such that P is locally integrable on Ω, then for all $c > 0$ there exists a nonnegative integrable function h such that

$$P(x,t) \leq \varphi(x,ct) + h(x) \quad \text{for all } t \geq 0 \text{ and for a.e. } x \in \Omega.$$

Definition 2.3. A Musielak function φ satisfies the log-Hölder continuity condition on Ω if there exists a constant $A > 0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{A\left(\log\left(\frac{1}{|x-y|}\right)\right)^{-1}}$$

for all $t \geq 1$ and for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$.

2.1. Some technical lemmas. We will use the following technical lemmas.

Lemma 2.4 ([5])**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder continuity such that*

$$\bar{\varphi}(x,1) \leq c_1 \quad \text{a.e in } \Omega$$

for some $c_1 > 0$. Then $D(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Remark 2.5. Note that if

$$\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty,$$

then Lemma 2.4 holds.

Example 2.6. Let $p \in P(\Omega)$ be a bounded variable exponent on Ω such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)}.$$

We can verify that the Musielak function defined by $\varphi(x,t) = t^{p(x)}\log(1+t)$ satisfies the conditions of Lemma 2.4.

Lemma 2.7 (Poincare's inequality: Integral form [5])**.** *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the conditions of Lemma 2.4. Then there exist positive constants β, η and λ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|v|)\,dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla v|)\,dx \quad \text{for all } v \in W_0^1 L_\varphi(\Omega). \tag{2.9}$$

Lemma 2.8 (Poincare's inequality [5]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the same conditions of Lemma 2.7. Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \text{for all } v \in W_0^1 L_\varphi(\Omega).$$

Lemma 2.9 ([34]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$.*

Moreover, if the set D of discontinuity points of F' is finite, we have

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e in } \{x \in \Omega \mid u(x) \in D\} \\ 0 & \text{a.e in } \{x \in \Omega \mid u(x) \notin D\} \end{cases}.$$

Lemma 2.10 ([9]). *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \quad \text{for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 2.11 ([24]). *Let (f_n), $f \in L^1(\Omega)$ such that*

i) $f_n \geq 0$ *a.e in* Ω,

ii) $f_n \to f$ *a.e in* Ω,

iii) $\displaystyle\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx.$

Then $f_n \to f$ strongly in $L^1(\Omega)$.

Lemma 2.12 ([10]). *If a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \rightharpoonup g$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.*

Lemma 2.13 ([10]). *Let u_n, $u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Lemma 2.14 ([21]). *If $P \prec \varphi$ and $u_n \to u$ for the modular convergence in $L_\varphi(\Omega)$, then $u_n \to u$ strongly in $E_P(\Omega)$.*

Lemma 2.15 (Jensen inequality [39]). *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and let $g : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\varphi\left(\int_\Omega g\,d\mu\right) \leq \int_\Omega \varphi \circ g\,d\mu.$$

Lemma 2.16 (The Nemytskii Operator). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|),$$

where k_1 and k_2 are real positive constants and $c(\cdot) \in E_\psi(\Omega)$. Then the Nemytski operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod \left\{u \in L_M(\Omega) : d\left(u, E_M(\Omega)\right) < \frac{1}{k_2}\right\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore, if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\!\prec \psi$, then N_f is strongly continuous from $\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$.

Definition 2.17 (Segment property [1])**.** A domain Ω is said to satisfy the segment property if there exists a finite open covering $\{\theta\}_{i=1}^k$ of $\bar{\Omega}$ and the corresponding nonzero vectors $z_i \in R^N$ such that $\left(\bar{\Omega} \cap \theta_i\right) + tz_i \subset \Omega$ for all $t \in (0, 1)$ and $i = 1, \ldots, k$.

Lemma 2.18 ([22])**.** *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then there exists a sequence $u_n \in D(\Omega)$ such that*

$$u_n \to u \quad \text{for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, then $\|u_n\|_\infty \leq (N + 1)\|u\|_\infty$.

Lemma 2.19. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$, then $\int_\Omega \operatorname{div}(u)\, dx = 0$.*

Proof. Fix a vector $u = \left(u^1, \ldots, u^N\right) \in \left(W_0^1 L_\varphi(\Omega)\right)^N$. Since $W_0^1 L_\varphi(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1 L_\varphi(\Omega)$, then each term u^i can be approximated by a suitable sequence $u_k^i \in D(\Omega)$ such that u_k^i converges to u^i in $W_0^1 L_\varphi(\Omega)$. Moreover, due to the fact that $u_k^i \in C_0^\infty(\Omega)$, the Green formula gives

$$\int_\Omega \frac{\partial u_k^i}{\partial x_i}\, dx = \int_{\partial\Omega} u_k^i \vec{n}\, ds = 0. \tag{2.10}$$

On the other hand, $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L_\varphi(\Omega)$. Thus $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L^1(\Omega)$, which gives in view of (2.10) that

$$\int_\Omega \operatorname{div}(u)\, dx = 0. \qquad \square$$

Throughout the paper, T_k denotes the truncation function at height $k \geq 0$,

$$T_k(s) = \max(-k, \min(k, s)).$$

3. Main results

Let Y be a closed subspace of $W^1 L_\varphi(\Omega)$ for $\sigma(\prod L_\varphi, \prod E_{\overline{\varphi}})$ and let

$$Y_0 = Y \cap W^1 L_\varphi(\Omega)$$

such that Y is the closure of Y_0 for $\sigma(\prod L_\varphi, \prod E_{\overline{\varphi}})$. Next we consider the complementary system (Y, Y_0, Z, Z_0) generated by Y, i.e., Y_0^* can be identified to Z and Z_0^* can be identified to Y by means of $\langle \cdot, \cdot \rangle$. Let the mapping T (associated to the operator A) be defined from

$$D(T) = \{u \in Y \mid a_0(x, u, \nabla u) \in L_{\overline{\varphi}}(\Omega), a_i(x, u, \nabla u) \in L_{\overline{\varphi}}(\Omega)\}$$

into Z by the formula

$$a(u, v) = \int_\Omega a_0(x, u, \nabla u) v(x)\, dx + \sum_{1 \le i \le N} \int_\Omega a_i(x, u, \nabla u) \frac{\partial v(x)}{\partial x_i}\, dx, \quad v \in Y_0.$$

We consider the complementary system

$$(Y, Y_0, Z, Z_0) = (W_0^1 L_\varphi(\Omega), W_0^1 E_\varphi(\Omega), W^{-1} E_{\overline{\varphi}}(\Omega), W^{-1} L_{\overline{\varphi}}(\Omega)).$$

As in [12], we define an entropy solution of our problem.

Definition 3.1. An entropy solution of the problem (1.1) is a measurable function u such that $T_k(u) \in W_0^1 L_\varphi(\Omega)$ for every $k > 0$ and such that

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - \phi)\, dx \le \int_\Omega f T_k(u - \phi)\, dx + \int_\Omega F(u) \nabla T_k(u - \phi)\, dx$$

for every $\phi \in W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega)$.

Our main results are collected in the following theorem.

Theorem 3.2. *Under assumptions (1.2)–(1.6), there exists an entropy solution u of the problem (1.1).*

3.1. Main Lemma

Lemma 3.3. *Let u be a mesurable function such that $T_k(u)$ belongs to $W_0^1 L_\varphi(\Omega)$ for every $k > 0$. Then*

$$\int_\Omega a(x, u, \nabla \phi) \nabla T_k(u - \phi)\, dx \le \int_\Omega f\ T_k(u - \phi)\, dx + \int_\Omega F(u) \nabla T_k(u - \phi)\, dx \quad (3.1)$$

is equivalent to

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - \phi)\, dx = \int_\Omega f\ T_k(u - \phi)\, dx + \int_\Omega F(u) \nabla T_k(u - \phi)\, dx \quad (3.2)$$

for every $\phi \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ and for every $k > 0$.

Proof of Lemma 3.3. In fact, (3.2) implies (3.1), which can be easily proved. Indeed, by adding and subtracting the term

$$\int_\Omega a(x, u, \nabla \phi) \nabla T_k(u - \phi)\, dx$$

in (3.2) and then using assumption (1.3), we obtain (3.1).

Thus, it remains to prove that (3.1) implies (3.2). Let h and k be positive real numbers, let $\lambda \in]-1,1[$ and $\Psi \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$.

Choosing $\phi = T_h(u - \lambda T_k(u-\Psi)) \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ as a test function in (3.1), we have

$$I_{hk} \le J_{hk} \tag{3.3}$$

with

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u-\lambda T_k(u-\Psi)))\,dx$$

and

$$J_{hk} = \int_\Omega f\, T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx + \int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Put

$$A_{hk} = \{x \in \Omega \mid |u - T_h(u-\lambda T_k(u-\Psi))| \le k\},$$

and

$$B_{hk} = \{x \in \Omega \mid |u - \lambda T_k(u-\Psi)| \le h\}.$$

Then we obtain

$$\begin{aligned} I_{hk} = &\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ &+ \int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ &+ \int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx. \end{aligned}$$

Since $\nabla T_k(u - T_h(u-\lambda T_k(u-\Psi)))$ is different from zero only on A_{kh}, we have

$$\int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \tag{3.4}$$

Moreover, if $x \in B_{hk}^C$, we have $\nabla T_h(u-\lambda T_k(u-\Psi)) = 0$. Using (1.4), we deduce that

$$\begin{aligned} &\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ &\qquad = \int_{A_{kh}\cap B_{hk}^C} a(x,u,0)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$I_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Letting $h \to +\infty$, $|\lambda| \le 1$, we have

$$A_{kh} \to \{x \mid |\lambda||T_k(u-\Psi)| \le h\} = \Omega \quad \text{and} \quad B_{hk} \to \Omega, \tag{3.6}$$

which implies $A_{kh} \cap B_{hk} \to \Omega$. By using the Lebesgue theorem, we may conclude that

$$\lim_{h\to+\infty} \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$= \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \tag{3.7}$$

Thus,

$$\lim_{h\to+\infty} I_{hk} = \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \tag{3.8}$$

On the other hand, we have

$$J_{hk} = \int_\Omega f\, T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx + \int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Then

$$\lim_{h\to+\infty} \int_\Omega f\, T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$+ \int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$= \lambda \int_\Omega f T_k(u-\Psi)dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx,$$

i.e.,

$$\lim_{h\to+\infty} J_{hk} = \lambda \int_\Omega f T_k(u-\Psi)\,dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{3.9}$$

After using (3.8), (3.9) and passing to the limit in (3.3), we obtain

$$\lambda \left(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx \right)$$
$$\le \lambda \left(\int_\Omega f T_k(u-\Psi)dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx \right)$$

for every $\Psi \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ and for every $k > 0$. Choosing $\lambda > 0$, dividing by λ and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx \le \int_\Omega f T_k(u-\Psi)\,dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{3.10}$$

For $\lambda < 0$, dividing by λ and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx \ge \int_\Omega f T_k(u-\Psi)\,dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{3.11}$$

Combining (3.10) and (3.11), we can write the following equality:

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx = \int_\Omega f T_k(u-\Psi)\,dx + \int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{3.12}$$

This completes the proof of Lemma 3.3. □

3.2. Proof of Theorem 3.2

3.2.1. Approximate problem and a priori estimate. For $n \in \mathbb{N}$, define $f_n := T_n(f)$, $F_n = F(T_n)$. Let u_n be a solution, in $W_0^1 L_\varphi(\Omega)$, of the problem

$$\begin{cases} -\operatorname{div}(a(x, u_n, \nabla u_n)) = f_n - \operatorname{div} F_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

which exists due to [23, Proposition 1, Remark 2]. Choosing $T_k(u_n)$ as a test function in (3.13), we have

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx + \int_\Omega F_n(u_n) \nabla T_k(u_n)\, dx,$$

We claim that

$$\int_\Omega F_n(u_n) \nabla T_k(u_n)\, dx = 0. \tag{3.14}$$

Using $\nabla T_k(u_n) = \nabla u_{n\chi\{|u_n| \le k\}}$, define

$$\Theta(t) = F_n(t)\chi\{t| \le k\} \quad \text{and} \quad \tilde{\Theta}(t) = \int_0^t \Theta(\tau)\ d\tau.$$

We have by Lemma 2.19, $\tilde{\Theta}(u_n) \in (W_0^1 L_\varphi(\Omega))^N$,

$$\begin{aligned} \int_\Omega F_n(u_n) \nabla T_k(u_n)\, dx &= \int_\Omega F_n(u_n) \chi\{|u_n| \le k\} \nabla u_n\, dx \\ &= \int_\Omega \Theta(u_n) \nabla u_n\, dx = \int_\Omega \operatorname{div}(\tilde{\Theta}(u_n))\, dx = 0, \end{aligned} \tag{3.15}$$

(by 2.19) which proves the claim.

Now, thanks to assumption (1.4), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n)\, dx \ge \int_\Omega \varphi(x, \lambda_1 |\nabla T_k(u_n)|)\, dx.$$

Then

$$\int_\Omega \varphi(x, \lambda_1 |\nabla T_k(u_n)|)\, dx \le C_1 k, \tag{3.16}$$

where C_1 is a constant independent of n.

3.2.2. Locally convergence of u_n in measure. Taking $\lambda |T_k(u_n)|$ in (3.13) and using (3.16), one has

$$\int_\Omega \varphi\left(x, \lambda_1 \frac{|\nabla T_k(u_n)|}{\lambda}\right) dx \le \int_\Omega \varphi(x, \lambda_1 |\nabla T_k(u_n)|)\, dx \le C_1 k. \tag{3.17}$$

Then, by using (3.17), we deduce that

$$\operatorname{meas}\{|u_n| > k\} \le \frac{1}{\inf_k \varphi\left(x, \frac{k}{\lambda}\right)} \int_{\{|u_n| > k\}} \varphi\left(x, \frac{|u_n(x)|}{\lambda}\right) dx$$

$$\leq \frac{1}{\inf_k \varphi\left(x, \frac{k}{\lambda}\right)} \int_\Omega \varphi\left(x, \frac{1}{\lambda}|T_k(u_n)|\right) dx \tag{3.18}$$

$$\leq \frac{C_1 k}{\inf_k \varphi\left(x, \frac{k}{\lambda}\right)} \quad \text{for all } n,\ k \geq 0. \tag{3.19}$$

For any $\beta > 0$, we have

$$\begin{aligned}\operatorname{meas}\{|u_n - u_m| > \beta\} \leq \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} \\ + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \beta\},\end{aligned}$$

and thus

$$\operatorname{meas}\{|u_n - u_m| > \beta\} \leq \frac{2C_1 k}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{3.20}$$

By using (3.16) and the Poincaré inequality in Musielak–Orlicz–Sobolev spaces (Lemma 2.8), we deduce that $(T_k(u_n))$ is bounded in $W_0^1 L_\varphi(\Omega)$. Hence there exists $\omega_k \in W_0^1 L_\varphi(\Omega)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$, strongly in $E_{\overline{\varphi}}(\Omega)$ and a.e. in Ω. Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$. Then, by (3.20) and the fact that

$$\frac{2C_1 k}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} \to 0 \quad \text{as} \quad k \to +\infty,$$

there exists some $k = k(\varepsilon) > 0$ such that

$$\operatorname{meas}\{|u_n - u_m| > \lambda\} < \varepsilon \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

This proves that u_n is a Cauchy sequence in measure, and thus u_n converges almost everywhere to some measurable function u. Finally, there exists a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}}), \\ u_n &\to u \quad \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{aligned}$$

3.2.3. An intermediate inequality. In this step, we shall prove that for $\phi \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}&\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \\ &\qquad \leq \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega F_n \nabla T_k(u_n - \phi)\, dx.\end{aligned} \tag{3.21}$$

We choose now $T_k(u_n - \phi)$ as a test function in (3.13), with ϕ in $W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, to obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)\, dx$$

$$= \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega F_n \nabla T_k(u_n - \phi)\, dx.$$

Adding and substracting the term

$$\int_\Omega a(x, u_n, \nabla\phi), \nabla T_k(u_n - \phi)\, dx$$

give us

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)\, dx + \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \\ - \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \\ = \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega F_n \nabla T_k(u_n - \phi)\, dx. \tag{3.22}$$

Thanks to assumption (1.3) and the definition of the truncation function, we have

$$\int_\Omega (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla\phi))\nabla T_k(u_n - \phi)\, dx \geq 0. \tag{3.23}$$

Combining (3.22) and (3.23), we obtain (3.21).

3.2.4. Passing to the limit. We shall prove that for $\phi \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \leq \int_\Omega f T_k(u - \phi) dx + \int_\Omega F \nabla T_k(u - \phi)\, dx.$$

Firstly, we claim that

$$\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \to \int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \quad \text{as } n \to +\infty.$$

Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1 L_\varphi(\Omega)$, with $M = k + \|\phi\|_\infty$, then

$$T_k(u_n - \phi) \rightharpoonup T_k(u - \phi) \text{ in } W_0^1 L_\varphi(\Omega), \tag{3.24}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n - \phi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(u - \phi) \text{ weakly in } L_\varphi(\Omega), \quad i = 1, \ldots, N. \tag{3.25}$$

Show that

$$a(x, T_M(u_n), \nabla\phi) \to a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{\varphi}}(\Omega))^N.$$

By assumption (1.2), we obtain

$$|a_i(x, T_M(u_n), \nabla\phi)| \leq |\phi_i(x)| + K_i \overline{\psi}^{-1}(\varphi(x, c_2 |T_M(u_n)|)) + K_i \overline{\varphi}^{-1}\varphi(x, c_1 |\nabla\phi|)$$

with c_1 and c_2 being positive constants. Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1L_\varphi(\Omega)$ and $W_0^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_{\overline{\varphi}}(\Omega)$, then $T_M(u_n) \rightharpoonup T_M(u)$ strongly in $L_\varphi(\Omega)$ and a.e. in Ω, we obtain

$$|a(x, T_M(u_n), \nabla\phi)| \to |a(x, T_M(u), \nabla\phi)| \quad \text{a.e. in } \Omega$$

and

$$\begin{aligned}&|\phi_i(x)| + K_i\overline{\psi}^{-1}(\varphi(x, c_2|T_M(u_n)|)) + K_i\overline{\varphi}^{-1}\varphi(x, c_1\,|\nabla\phi|)\\ &\qquad \to |\phi_i(x)| + K_i\overline{P}^{-1}(\varphi(x, c_2|T_M(u)|)) + K_i\overline{\varphi}^{-1}\varphi(x, c_1\,|\nabla\phi|) \quad \text{a.e. in } \Omega.\end{aligned}$$

Then, By Vitali's theorem, we deduce that

$$a(x, T_M(u_n), \nabla\phi) \to a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{\varphi}}(\Omega))^N \quad \text{as } n \to \infty. \tag{3.26}$$

Combining (3.25) and (3.26), we obtain

$$\begin{aligned}&\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx\\ &\qquad \to \int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \quad \text{as } n \to +\infty.\end{aligned} \tag{3.27}$$

Secondly, we show that

$$\int_\Omega f_n T_k(u_n - \phi)\, dx \to \int_\Omega f T_k(u - \phi)\, dx \tag{3.28}$$

and

$$\int_\Omega F_n \nabla T_k(u_n - \phi)\, dx \to \int_\Omega F\nabla T_k(u - \phi)\, dx. \tag{3.29}$$

We have $f_nT_k(u_n - \phi) \to fT_k(u - \phi)$ a.e. in Ω and $|fT_k(u_n - \phi)| \le k|f|$, and $F_n\nabla T_k(u_n - \phi) \to F\nabla T_k(u - \phi)$ a.e. in Ω, and $|F\nabla T_k(u_n - \phi)| \le k|F|$. Then, by using Vitali's theorem, we obtain (3.28) and (3.29). By (3.27), (3.28) and (3.29), we can pass to the limit in the inequality (3.21), so that $\forall\phi \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$, and thus we deduce that

$$\int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \le \int_\Omega fT_k(u - \phi)\, dx + \int_\Omega F\nabla T_k(u - \phi)\, dx.$$

In view of the main lemma, we can deduce that u is an entropy solution of the problem (1.1). This completes the proof of our main desired result.

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Про деякі нелінійні еліптичні проблеми з великою монотонністю в просторах Мусйелака–Орлича–Соболєва

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У цій роботі ми вивчаємо існування ентропійного розв'язку деякої нелінійної еліптичної проблеми типу Лерея–Ліонса, пов'язану з рівнянням $-\operatorname{div} a(x,u,\nabla u) = f(x) - \operatorname{div} F(u)$ в Ω з умовою великої монотонності у визначенні просторів Мусйелака–Орлича–Соболєва, де права частина належить $L^1(\Omega)$ і $F = (F_1,\dots,F_N)$ задовольняє умову $F \in (C^0(\mathbb{R}))^N$.

Ключові слова: еліптична проблема, ентропійний розв'язок, простори Мусйелака–Орлича–Соболєва, компактне вкладення, Δ_2-умова