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Semi-Symmetric Curvature Properties of Robertson–Walker Spacetimes

Uday Chand De, Young Jin Suh, and Sudhakar K. Chaubey

The aim of the present paper is to characterize *Robertson–Walker* (RW) spacetimes satisfying certain curvature conditions. A necessary and sufficient condition for a RW spacetime to be Ricci semisymmetric is given. We prove that a four-dimensional *Ricci symmetric* RW spacetime is vacuum. We also study the properties of projective collineation and matter collineation within the framework of a four-dimensional Ricci symmetric RW spacetime. Among others, it is proved that a Lorentzian manifold of dimension $n \geq 3$ is a RW spacetime if and only if the spacetime is of quasi-constant curvature. Finally, some new characteristics of RW spacetimes are obtained.

Key words: Lorentzian manifolds, symmetric spaces, Robertson–Walker spacetimes, generalized Robertson–Walker spacetimes, perfect fluid spacetimes, spacetime of quasi-constant curvature, projective and conformal curvature tensors

Mathematical Subject Classification 2010: 53B30, 53B50, 53C15

1. Introduction

A semi-Riemannian manifold of dimension n is a smooth n-dimensional differentiable manifold equipped with a semi-Riemannian metric of signature $(\mathfrak{p}, \mathfrak{q})$, where $n = \mathfrak{p} + \mathfrak{q}$. A Lorentzian manifold is a subclass of the semi-Riemannian manifold, that is, a semi-Riemannian manifold M of dimension $n \geq 2$ equipped with a semi-Riemannian metric g of signature $(1, n-1)$ or $(n-1, 1)$ is a Lorentzian manifold [34]. Lorentzian manifolds have many applications to general relativity and cosmology. A spacetime is the stage of present modeling of the physical world: a time oriented Lorentzian manifold.

To describe the gravity of the universe, the curvature tensor R^k_{hij}, the Ricci tensor R_{ij} and the scalar curvature R play an important role. In cosmology, the observation that the space is isotropic and homogeneous on the universe in the large scale chooses the Robertson–Walker (RW) metric. In 1995, Alías, Romero and Sánchez [1] generalized the notion of RW metric to generalized Robertson–Walker (GRW) metric. A Lorentzian manifold M of dimension $n \geq 3$ endowed with the Lorentzian metric g defined by

$$ds^2 = g_{ab}dx^a dx^b = -(dt)^2 + \varphi(t)^2 g^*_{lm}(x)dx^l dx^m, \tag{1.1}$$

where t is the time and $g^*_{lm}(x)$ is the metric tensor of a Riemannian manifold M^*, is a GRW spacetime. In other words, a GRW spacetime is the warped product $-I \times \varphi^2 M^*$, where I is an open interval of the real line, φ is a smooth warping function or scale factor such that $\varphi > 0$ and M^* is an $(n-1)$-dimensional Riemannian manifold. In particular, if M^* is an $(n-1)$-dimensional Riemannian space of constant sectional curvature, then the warped product $-I \times \varphi^2 M^*$ is said to be a RW spacetime. A RW spacetime complies the cosmological principle, that is, the spacetime is locally spatially isotropic and locally spatially homogeneous, although the GRW spacetime is not necessarily spatially homogeneous [14]. In [5], Brozos–Vázquez, Garcia–Rio and Vázquez–Lorenzo bridged the gap between RW spacetime and GRW spacetime by providing the following result [5]. A GRW spacetime is conformally flat if and only if it is a RW spacetime. It is noticed that the GRW spacetimes include the Friedmann cosmological models, the Lorentz–Minkowski spacetime, the Einstein–de Sitter spacetimes, the static Einstein spacetime and the de Sitter spacetimes. For more details of (GRW) spacetimes, we call [2, 9–11, 29, 32] and their references.

A spacetime M of dimension $n \geq 3$ is said to be a perfect fluid spacetime if the non-vanishing Ricci tensor R_{ab} of M satisfies the relation

$$R_{ab} = \alpha u_a u_b + \beta g_{ab}, \tag{1.2}$$

where α and β are scalar fields, g_{ab} is the Lorentzian metric and u_a is a 1-form associated with the unit timelike vector field u^a such that $u_a = g_{ab}u^b$. The expression (1.2) can be obtained from the following Einstein's field equation without cosmological constant:

$$R_{ab} - \frac{R}{2} g_{ab} = \kappa T_{ab}, \tag{1.3}$$

where κ is a non-zero gravitational constant and T_{ab} denotes the energy momentum tensor of the spacetime. For a perfect fluid spacetime, the energy momentum tensor T_{ab} assumes the form

$$T_{ab} = (p + \mu) u_a u_b + p g_{ab}, \tag{1.4}$$

where μ and p are the energy density and the isotropic pressure of the fluid. A perfect fluid spacetime with $p = p(\mu)$ is an isentropic fluid [20, p. 70]. A RW spacetime is a perfect fluid spacetime [34]. A four-dimensional GRW spacetime is a perfect fluid spacetime if and only if it is a RW spacetime [23]. If the energy-matter content of spacetime is a perfect fluid with fluid velocity u^a, then the Einstein field equations show that the Ricci tensor R_{ab} assumes the form (1.2) and the scalars α and β are linearly related to the pressure p and the energy density μ measured in the locally comoving inertial frame [30]. Shepley and Taub [38] considered a four-dimensional perfect fluid spacetime with divergence free Weyl curvature tensor ($C^a_{bcd,a} = 0$) and the equation of state $p = p(\mu)$ and proved that the spacetime is conformally flat ($C_{abcd} = 0$), the flow is shear-free, geodesic and irrotational, and the metric is RW. In [37], Sharma studied a perfect fluid spacetime. He proved that if a four-dimensional perfect fluid spacetime admits a proper conformal Killing vector field ($X_{b,a} + X_{a,b} = 2\rho g_{ab}$) and the Weyl conformal curvature tensor is divergence free, then the spacetime is conformally flat. Guilfoyle and Nolan [22] showed that a four-dimensional perfect fluid spacetime with $p + \mu \neq 0$ is a RW spacetime if and only if it is a Yang pure spacetime ($C^a_{bcde,a} = 0$, $R_{,a} = 0$). If a perfect fluid spacetime satisfying the Einstein field equations with $p = p(\mu)$, $p + \mu \neq 0$ and a proper conformal Killing vector field is parallel to the fluid four-velocity, then it is locally a RW spacetime [16]. In [17], De and Ghosh proved that a conformally flat perfect fluid spacetime with closed u_a possesses a concircular vector field. Mantica, Molinari and De [30] proved that if a perfect fluid spacetime of dimension $n > 3$ admits an irrotational vector field and divergence free Weyl conformal curvature tensor, then it is a generalized RW spacetime with Einstein fiber. Recently, Chaubey [13] characterized the perfect fluid spacetime with gradient η-Ricci soliton and gradient Einstein solitons. De et al. [18] studied the properties of perfect fluid spacetime with Yamabe solitons. The properties of the perfect fluid spacetimes have been noticed in [28, 30, 33].

In [25], the algebraic restrictions on the Ricci tensor in a Ricci recurrent spacetime are determined. The restriction imposed on the Petrov type of the Weyl tensor are also given.

The above results motivate us to study some curvature properties of RW spacetimes. In Section 2, we give some known basic results and definitions. In the next sections, we prove several results:

Theorem 1.1. *A* RW *spacetime is Ricci semisymmetric if and only if* $u_{a,cd} = u_{a,dc}$.

Theorem 1.2. *A Ricci semisymmetric* RW *spacetime obeying the Einstein field equations without cosmological constant is vacuum, and the equation of state is given by* $p = -\mu + \frac{2\alpha\mu}{R}$.

Theorem 1.3. *A spacetime of dimension* $n > 3$ *is a* RW *spacetime if and only if the manifold is of quasi-constant curvature.*

Theorem 1.4. *A four-dimensional* RW *spacetime satisfying the Einstein field equations without cosmological constant is a Yang pure spacetime.*

2. Preliminaries

Let C^d_{abc} denote the conformal curvature tensor on an n-dimensional Lorentzian manifold M with $n \geq 3$, then it is defined as

$$C^d_{abc} = R^d_{abc} - \frac{1}{n-2}\{R^d_c g_{ab} - R^d_b g_{ac} + R_{ab}\delta^d_c - R_{ac}\delta^d_b\} + \frac{R}{(n-1)(n-2)}\{g_{ab}\delta^d_c - g_{ac}\delta^d_b\}, \tag{2.1}$$

where $R = R_{ab}g^{ab}$ [40]. The Lorentzian manifold M is conformally flat if and only if $C^d_{abc} = 0$ for $n > 3$.

The Weyl projective curvature tensor P^d_{abc} [41] on an n-dimensional Lorentzian manifold with $n \geq 3$ is given by

$$P^d_{abc} = R^d_{abc} - \frac{1}{n-1}\{R_{ab}\delta^d_b - R_{bc}\delta^d_a\}. \tag{2.2}$$

A Lorentzian manifold M is projectively flat if and only if $P^d_{abc} = 0$ for $n > 2$.

In 1972, Chen and Yano [15] introduced the notion of a Riemannian manifold with quasi-constant curvature. If the curvature tensor R_{abcd} of the Lorentzian manifold M of dimension n satisfies the following condition:

$$R_{abcd} = \iota_1(g_{bc}g_{ad} - g_{bd}g_{ac}) + \iota_2(g_{bc}u_a u_d + g_{ad}u_b u_c - g_{bd}u_a u_c - g_{ac}u_b u_d)$$

for some smooth functions ι_1 and ι_2, then M is called a manifold of quasi-constant curvature.

In [29], Mantica and Molinari proved that a spacetime is a GRW spacetime if and only if there exists a timelike unit torse-forming vector field u^a ($u^2 = u^a \cdot u_a = -1$) that is also an eigenvector of the Ricci operator $R^a_b = R_{bc}g^{ac}$. Also, Mantica and Molinari [31] obtained the expression for the Ricci tensor in a GRW spacetime,

$$R_{ab} = \frac{R - n\xi}{(n-1)}u_a u_b + \frac{R-\xi}{(n-1)}g_{ab} - (n-2)C_{cabd}u^c u^d, \tag{2.3}$$

and $R_{ab}u^b = \xi u_a \implies \xi$ is an eigenvalue. We consider a conformally flat GRW spacetime, which implies that the spacetime is a RW spacetime. Therefore (2.3) reduces to

$$R_{ab} = \frac{R - n\xi}{(n-1)}u_a u_b + \frac{R-\xi}{(n-1)}g_{ab}. \tag{2.4}$$

The above equation can be expressed as equation (1.2), where $\alpha = \frac{R-n\xi}{(n-1)}$ and $\beta = \frac{R-\xi}{(n-1)}$, which implies that the RW spacetime is a perfect fluid spacetime. Also, Mantica and Molinari [29] expressed the curvature tensor in a RW spacetime as

$$\begin{aligned} R_{abcd} = {} & \frac{2\xi - R}{(n-1)(n-2)}(g_{bc}g_{ad} - g_{bd}g_{ac}) \\ & + \frac{R - n\xi}{(n-1)(n-2)}(g_{bc}u_a u_d + g_{ad}u_b u_c - g_{bd}u_a u_c - g_{ac}u_b u_d). \end{aligned} \tag{2.5}$$

The above expression tells us that a RW spacetime is a spacetime of quasi-constant curvature [15].

An n-dimensional Lorentzian manifold M with $n \geq 3$ is said to be:

(i) Ricci symmetric [21] if $R_{ab,c} = 0$,
(ii) semisymmetric [39] if $R_{abcd,ef} - R_{abcd,fe} = 0$,
(iii) Ricci semisymmetric [39] if $R_{ab,cd} - R_{ab,dc} = 0$,
(iv) recurrent [36] if $R_{abcd,e} = \alpha_e R_{abcd}$, where α_e is a non-zero 1-form.

Let M be a Lorentzian manifold with a Levi-Civita connection ∇. A continuous group of local diffeomorphism of M is said to be projective collineation (PC) [4] if it maps geodesics into geodesics, and the generator of this group is called a projective vector field. A vector filed V is a PC if and only if

$$L_V \Gamma^a_{bc} = \delta^a_b \mathsf{p}_c + \delta^a_c \mathsf{p}_b,$$

where L_V denotes the Lie derivative operator along V, $\mathsf{p}_b = \mathsf{p}_{,b}$ and p is a 1-form. Thus p_b is locally an exact form. In particular, if $L_V \Gamma^a_{bc} = 0$, then the projective collineation reduces to the affine collineation or affine motion. When the manifold is flat, the affine collineation satisfies the equation

$$V_{c,ab} = 0,$$

and its solution is $V_c = \mathcal{A}_{ac} x^a + \mathcal{B}_c$, where $\mathcal{A}_{ac}$ and $\mathcal{B}_c$ are constants and x^a denotes a local coordinate system. The maximum dimension of the projective algebra of M is $n^2 + n$ for which M is projectively flat. Also recall that the projective vector field V satisfies

$$L_V R^d_{abc} = \delta^d_c \mathsf{p}_{a,b} - \delta^d_b \mathsf{p}_{a,c}, \tag{2.6}$$

$$L_V R_{ab} = (1-n)\mathsf{p}_{a,b} \tag{2.7}$$

$$L_V P^d_{abc} = 0. \tag{2.8}$$

3. Ricci semisymmetric RW spacetime

In [25], Hall characterized Ricci recurrent spacetimes. It is well known that the Ricci semisymmetry is weaker than the Ricci recurrent spacetime. Thus, we are interested in studying the Ricci semisymmetric RW spacetimes in this section.

Proof of Theorem 1.1. Suppose that a Lorentzian manifold of dimension n is Ricci semisymmetric, that is, the Ricci tensor R_{ab} satisfies the condition

$$R_{ab,cd} - R_{ab,dc} = 0. \tag{3.1}$$

Taking covariant differentiation of (1.2) twice, we get

$$R_{ab,cd} - R_{ab,dc} = \alpha\{(u_{a,cd} - u_{a,dc})u_b + (u_{b,cd} - u_{b,dc})u_a\}. \tag{3.2}$$

Suppose that the RW spacetime is Ricci semisymmetric. Then, from (3.1) and (3.2), we infer

$$(u_{a,cd} - u_{a,dc})u_b + (u_{b,cd} - u_{b,dc})u_a = 0$$

since $\alpha \neq 0$. Transvecting the above equation with u^b, we get

$$u_{a,cd} - u_{a,dc} = 0$$

since $u^b u_{b,cd} = 0$. Conversely, if $u_{a,cd} - u_{a,dc} = 0$, then from (3.2) it follows that $R_{ab,cd} - R_{ab,dc} = 0$. Hence Theorem 1.1 is proved. □

It is well known that the Ricci semisymmetric spacetimes are a natural generalization of the semisymmetric, Ricci symmetric and recurrent spacetimes. These facts along with Theorem 1.1 state the following:

Corollary 3.1. *If a* RW *spacetime is semisymmetric, then*

$$u_{a,cd} - u_{a,dc} = 0.$$

Corollary 3.2. *If a* RW *spacetime is Ricci symmetric, then*

$$u_{a,cd} - u_{a,dc} = 0.$$

Corollary 3.3. *If a* RW *spacetime is recurrent, then*

$$u_{a,cd} - u_{a,dc} = 0.$$

Proof of Theorem 1.2. We suppose that a four-dimensional RW spacetime is Ricci semisymmetric. Then $u_{a,cd} - u_{a,dc} = 0$, and the Ricci identity ($u_{a,cd} - u_{a,dc} = u_b R^b_{acd}$) gives

$$u_b R^b_{acd} = 0 \;\Rightarrow\; u_b R^b_d = 0 \;\Rightarrow\; \xi = 0. \tag{3.3}$$

Hence equation (1.2) infers $\alpha = \beta$ and

$$R_{ab} = \alpha\{u_a u_b + g_{ab}\}. \tag{3.4}$$

In view of equations (1.3), (1.4), and (3.4), we conclude that

$$\kappa p = \alpha - \frac{R}{2}, \quad \kappa\mu = \frac{R}{2}, \quad \text{and} \quad p = -\mu + \frac{2\alpha\mu}{R}. \tag{3.5}$$

From equations (2.4) and (3.3), we lead $\alpha = \frac{R}{3}$ and hence equation (3.5) gives $p = -\frac{\mu}{3}$, that is, $p = p(\mu)$.

Let us suppose that a four-dimensional RW spacetime is Ricci symmetric ($R_{ab,c} = 0$) and therefore $u_{a,cd} - u_{a,dc} = 0$. Thus equation (3.3) holds, that is, $u_b R^b_d = 0$. Taking the covariant derivative of this equation, we have

$$u_{b,a} R^b_d + u_a R^b_{d,a} = 0,$$

which gives

$$u_{b,a} R^b_d = 0 \tag{3.6}$$

because $R^b_{d,a} = 0$.

In [29], Mantica and Molinari proved that a GRW spacetime admits the unit timelike torse-forming vector field, $u_{b,a} = \varphi\{u_a u_b + g_{ab}\}$, which is also an eigenvector of the Ricci tensor R_{ab}, that is, $R_{ab}u^b = \xi u_a$. Here φ denotes the non-vanishing smooth function.

The above discussions along with equation (3.6) lead to

$$\varphi\{u_a u_b + g_{ab}\} R^b_d = 0, \tag{3.7}$$

which implies that $R^b_d = 0$. This shows that the Ricci symmetric RW spacetime is Ricci flat and hence it is vacuum. Thus the proof of Theorem 1.2 is completed. □

Remark 3.4. For a perfect fluid spacetime, the equation of state ω is given by

$$\omega = \frac{p}{\mu}.$$

From (3.5), we infer that $\omega = \frac{p}{\mu} = -\frac{1}{3}$, which gives the condition for late-time accelerating universe [20]. Also, in a four-dimensional Ricci symmetric RW spacetime $p = p(\mu)$ and hence the fluid is isentropic [26].

Now we consider the projective collineation in a Ricci symmetric RW spacetime. Since equation (3.7) reflects that the Ricci symmetric RW spacetime is Ricci flat, therefore from (2.7) we infer $\mathsf{p}_{a,b} = 0$. If $\mathsf{p}_a \neq 0$, then the projective collineation is proper and, for the case of $n = 4$, the metric must be either flat or a *pp*-wave [24]. If $\mathsf{p}_a = 0$, then V generates an affine collineation, and for the spacetime the metric is decomposable, or is a *pp*-wave, or V is a homothetic Killing vector field. Thus, we summarize the results as:

Theorem 3.5. *Let a Ricci symmetric RW spacetime admit a projective collineation V. Then*

(i) *the projective collineation is proper and, for $n = 4$, the metric is either flat or a pp-wave, provided $\mathsf{p}_a \neq 0$.*

(ii) *V generates an affine collineation and the metric is decomposable, or a pp-wave, or a homothetic Killing vector field, provided $\mathsf{p}_a = 0$.*

Remark 3.6. To our knowledge, the projective collineation has not been studied in a RW spacetime, which is a perfect fluid spacetime. In Theorem 3.5, we characterize PC in a Ricci symmetric RW spacetime.

If a (non trivial) given symmetry vector field V of M leaves matter tensor invariant ($L_V T_{ab} = 0$), then we say that M admits a matter collineation. Well known examples are Killing and homothetic symmetries.

If we assume that a Ricci symmetric RW spacetime satisfies the Einstein field equations, then (1.3) holds. Since a Ricci symmetric RW spacetime is Ricci flat, equation (1.3) turns into $\kappa T_{ab} = 0$, which infers $T_{ab} = 0$. Thus, we conclude that a Ricci symmetric RW spacetime admits matter collineation. It should be mentioned that Caret et al. [7] obtained the examples of matter collineation in dust fluids, included Szekeres's space-time: $ds^2 = -dt^2 + e^{\lambda} dr^2 + e^{\varrho}(dx^2 + dy^2)$ for smooth functions λ and ϱ. Hence we state:

Theorem 3.7. *Every Ricci symmetric RW spacetime admits matter collineation.*

4. Lorentzian manifold of quasi-constant curvature

It can be easily proved that a Lorentzian manifold of dimension $n > 3$ is of quasi-constant curvature if and only if

(i) the manifold is conformally flat,

(ii) the Ricci tensor has the form

$$R_{ab} = \frac{R - n\sigma}{(n-1)} u_a u_b + \frac{R - \sigma}{(n-1)} g_{ab},$$

where $R_{ab}u^b = \sigma u_a$. Since a RW spacetime is of quasi-constant curvature, then a natural question arises whether a Lorentzian manifold of quasi-constant curvature is a RW spacetime.

Theorem 1.3 gives an affirmative answer to this question.

Proof of Theorem 1.3. Suppose that the Lorentzian manifold possesses quasi-constant curvature. Then it is conformally flat and hence $C^a_{bcd,a} = 0$. Also, a RW spacetime is a perfect fluid spacetime with $p + \mu \neq 0$. Mantica, Molinari and De [30] proved that a perfect fluid spacetime of dimension $n \geq 4$ with $p + \mu \neq 0$ and $C^a_{bcd,a} = 0$ is a GRW spacetime. In fact, the above discussions tell us that a Lorentzian manifold of quasi-constant curvature is a GRW spacetime. Again, since a manifold is conformally flat, then a GRW spacetime becomes a RW spacetime, and thus Theorem 1.3 is proved. □

Next, from (2.4), (3.4), and (1.3), we infer

$$\kappa p = \frac{R - \xi}{(n-1)} - \frac{R}{2}, \quad p + \mu = \frac{R - n\xi}{\kappa(n-1)} \neq 0, \tag{4.1}$$

from which it follows that

$$\mu = \frac{R}{2\kappa} - \frac{\xi}{\kappa}. \tag{4.2}$$

By equations (4.1) and (4.2), we obtain

$$p = \frac{\mu}{n-1} - \frac{(n-2)}{2\kappa(n-1)} R.$$

Thus we can state:

Proposition 4.1. *In a* RW *spacetime obeying the Einstein field equations without cosmological constant, the equation of state is* $p = \frac{\mu}{n-1} - \frac{(n-2)}{2\kappa(n-1)} R$.

Proof of Theorem 1.4. For a four-dimensional RW spacetime, we get $p + \mu \neq 0$ and $p = \frac{1}{3}\mu - \frac{R}{3\kappa}$. In [22], Guilfoyle and Nolan named "Young pure spacetime" a four-dimensional Lorentzian manifold (M, g) whose metric tensor solves Yang's equations $R_{ab,c} = R_{ac,b}$, which implies that the scalar curvature $R = \text{const}$.

In the same paper [22], Guilfoyle and Nolan proved that a four-dimensional perfect fluid spacetime (M, g) with $p + \mu \neq 0$ is a Yang pure spacetime if and only if (M, g) is a RW spacetime with $p = \frac{1}{3}\mu + c$ for some constant c.

Here, we consider a four-dimensional RW spacetime which is a perfect fluid. Also the state equation $p = \frac{1}{3}\mu + c$ holds. Thus our Theorem 1.4 is proved. □

5. Characterizations of RW spacetimes

5.1. In [35], Prvanović introduced the notion of extended recurrent manifold defined as

$$R_{abcd,\,e} = A_e R_{abcd} + (\beta - \psi) A_e G_{abcd} + \frac{\beta}{2}[A_a G_{ebcd} + A_b G_{aecd} + A_c G_{abed} + A_d G_{abcd}],$$

where A_e is a closed covector, ψ and β are scalar functions with $\psi_{,a} = A_a\beta$ and $G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$. In [29], Mantica and Molinari proved that an extended recurrent spacetime with a timelike vector field A_a is a RW spacetime.

5.2. A Lorentzian manifold is said to be a pseudosymmetric manifold in the sense of Chaki [8] if the curvature tensor R_{abcd} satisfies the condition

$$R_{abcd,e} = 2A_e R_{abcd} + A_a R_{ebcd} + A_b R_{aecd} + A_c R_{abed} + A_d R_{abce},$$

where A_e is a non-zero covector. Such a manifold is denoted by $(PS)_n$. It is known that in a conformally flat $(PS)_n$ $(n \geq 3)$,

$$(n-1)A_a R_{bc} - (n-1)A_b R_{ac} - RA_a g_{bc} + RA_b g_{ac} + B_a g_{bc} - B_b g_{ac} = 0, \quad (5.1)$$

where $B_a = R_{ac}A^c$ [8]. Transvecting the above equation by A^c, we obtain

$$(n-2)A_a B_b - (n-2)A_b B_a + RA_b A_a = 0 \quad (5.2)$$

In its turn, transvecting (5.2) with A_a, we infer

$$B_b = -\left(t + \frac{R}{n-2}\right) A_b, \quad (5.3)$$

where $t = B_a A^a$ and $A_a A^a = -1$. After using (5.3) in (5.1) and then transvecting by A^a, it follows that

$$R_{bc} = \left(\frac{t}{n-1} + \frac{R}{n-2}\right) g_{bc} + \left(\frac{t}{n-1} + \frac{2R}{n-2}\right) A_b A_c,$$

which implies that a conformally flat pseudosymmetric manifold represents a perfect fluid spacetime. Since the manifold is conformally flat, then $C^a_{bcd,a} = 0$. Also, in our case $p + \mu = \frac{(n-2)t + 2(n-1)R}{(n-1)(n-2)k} \neq 0$, therefore such a spacetime becomes a RW spacetime.

5.3. The $f(R,T)$ gravity, introduced by Harko et al. [27], is a natural extension of general relativity and $f(R)$ gravity. Here, R and T are the scalar curvature and the trace of stress energy tensor. They considered that the gravitational Lagrangian is an arbitrary function of R and T, and the field equations for this theory can be derived by the action of Hilbert-Einstein type variational

$$A = \frac{1}{16\pi}\int [\mathcal{L}_m + f(R,T)]\sqrt{(-g)}d^4x,$$

where $\mathcal{L}_m$ denotes the matter Lagrangian density. The field equations for $f(R,T)$ gravity take the form

$$\begin{aligned}(g_{ab}\nabla_k\nabla^k - \nabla_a\nabla_b)f_R(R,T) + f_R(R,T)R_{ab} - \frac{1}{2}f(R,T)g_{ab}\\ = (8\pi - f_T(R,T))T_{ab} - f_T(R,T)\Theta_{ab},\end{aligned} \tag{5.4}$$

where $f_R(R,T)$, $f_T(R,T)$ represent the partial derivatives of f with respect to R and T, and ∇_a is the covariant derivative. The stress energy tensor T_{ab} of matter is defined as

$$T_{ab} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g})\mathcal{L}_m}{\delta^{ab}},$$

where the variation of stress energy is

$$\Theta_{ab} = g_{ab}\mathcal{L}_m - 2T_{ab} - 2g^{lk}\frac{\partial^2\mathcal{L}_m}{\partial g^{ab}\partial g^{lk}}.$$

Let the spacetime be conformally flat with pressure p, energy density μ and four-velocity u^a such that $u_a u^a = -1$, and hence $u^a u_{a,a} = 0$. We also suppose that the vector field u^a is irrotational, that is, $u_{a,b} - u_{b,a} = 0$. Since there is no unique definition of the matter Lagrangian, we choose $\mathcal{L}_m = -p$ [12]. Let the stress energy tensor T_{ab} and the variation of stress energy for the perfect fluid take the form

$$T_{ab} = (p+\mu)u_a u_b + p g_{ab} \tag{5.5}$$

and

$$\Theta_{ab} = -2T_{ab} - p g_{ab}. \tag{5.6}$$

It is observed that the field equations depend on the nature of matter. Thus, each choice of $f(R,T)$ gives a theoretical model. In particular, if we take $f(R,T) = f(R)$, then the $f(R,T)$ gravity reduces to the $f(R)$ gravity [6].

If we choose $f(R,T) = R + 2f(T)$, then equations (5.4), (5.5), and (5.6) lead to

$$R_{ab} = A u_a u_b + B g_{ab},$$

where $A = [8\pi + 2f'(T)]\,(p+\mu)$ and $B = \left[\frac{R}{2} - f(T) + 8\pi p + 4pf'(T)\right]$, which shows that the spacetime is a conformally flat perfect fluid spacetime. In [30], Mantica et al. proved the following:

Let M be a spacetime of dimension $n \geq 3$, with the Ricci tensor $R_{ab} = Au_a u_b + Bgab$, where A and B are scalar fields, $A \neq 0$, and u^a is a unit timelike vector field. If $C^d_{abc,d} = 0$ and $u_{a,b} = u_{b,a}$, then M is a GRW spacetime with $u^a C_{bcda} = 0$ [24, Theorem 2.1].

Since the spacetime is conformally flat, then $C^d_{abc,d} = 0$. Thus, the conformally flat $f(R,T)$ gravity is a RW spacetime.

6. Examples

6.1. Let $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $\mathbb{R}^n$ stands for an n-dimensional real number space. Then we define a Lorentzian metric g on $\mathbb{R}^4$ as

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+1}(dx^1)^2 + e^{x^1}\{(dx^2)^2 + (dx^3)^2 - (dx^4)^2\}.$$

The only non-vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensors are:

$$\Gamma^1_{11} = \frac{1}{2},\ \Gamma^1_{22} = \Gamma^1_{33} = -\frac{1}{2e},\ \Gamma^1_{44} = \frac{1}{2e},\ \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2},$$
$$R_{2332} = \frac{1}{4}e^{x^1-1},\ R_{2442} = R_{3443} = -\frac{1}{4}e^{x^1-1},\ R_{22} = R_{33} = -R_{44} = \frac{1}{2e}.$$

The scalar curvature is $r = \frac{3}{2e^{x^1+1}}$. Here,

$$R_{22,1} = -\frac{1}{2e},\ R_{22,2} = R_{22,3} = R_{22,4} = 0,\ R_{33,1} = -\frac{1}{2e},$$
$$R_{33,2} = R_{33,3} = R_{33,4} = 0,\ R_{44,1} = \frac{1}{2e},\ R_{44,2} = R_{44,3} = R_{44,4} = 0. \tag{6.1}$$

We can also derive that

$$R_{22,12} = R_{22,21} = R_{22,13} = R_{22,31} = 0. \tag{6.2}$$

In a similar way, we can calculate other second-order derivatives which are zero. From the non-vanishing components of curvature tensors, we observe that

$$R_{2332,12} - R_{2332,21} \neq 0,\ R_{2332,23} - R_{2332,32} \neq 0.$$

From the above calculations, we obtain $R_{ij,lm} - R_{ij,ml} = 0$, but $R_{hijk,lm} - R_{hijk,ml} \neq 0$. Thus, the spacetime of dimension four is Ricci semisymmetric but it is not semisymmetric.

The above expressions of the Ricci tensors and their covariant derivatives show that the spacetime is Ricci semisymmetric, but it is not Ricci symmetric or Ricci parallel $(Rij, l = 0)$.

From (6.1) and (6.2), it follows that the spacetime is Ricci semisymmetric, but not Ricci recurrent. Since $R_{12} = 0$, but $R_{12,2} = -\frac{1}{4e} \neq \lambda_2 R_{12}$, where λ_2 is a covariant vector. There does not exist any vector λ_2 such that $R_{12,2} = -\frac{1}{4e}$ since $R_{12} = 0$.

6.2. In [3], the authors proved that under certain condition (for instance, if dimension is 3), $R_{hijk,lm} - R_{hijk,ml} = 0$ and $R_{ij,lm} - R_{ij,ml} = 0$ are equivalent. Also, in [19], the authors constructed an example of a hypersurface which is Ricci semisymmetric but not semisymmetric.

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References

[1] L. Alías, A. Romero, and M. Sánchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson–Walker spacetimes*, Gen. Relativ. Gravit. **27** (1995), 71–84.

[2] K. Arslan, R. Deszcz, R. Ezentas, M. Hotloś, and C. Murathan, *On generalized Robertson–Walker spacetimes satisfying some curvature condition*, Turk J. Math **38** (2014), 353–373.

[3] K. Arslan, Y. Ç̣elik, R. Deszcz, and R. Ezentaș, *On the equivalence of Ricci-semisymmetry and semisymmetry*, Colloq. Math. **76** (1998), 279–294.

[4] A. Barnes, *Projective collineations in Einstein spaces*, Classical Quantum Gravity **10** (1993), 1139–1145.

[5] M. Brozos-Vázquez, E. Garcia-Rio, and R. Vázquez-Lorenzo, *Some remarks on locally conformally flat static spacetimes*, J. Math. Phys. **46** (2005), 022501.

[6] S. Capozziello, S. Carloni, and A. Troisi, *Quintessence without scalar fields*, Recent Res. Dev. Astron. Astrophys. **625** (2003), astro-ph/030304.

[7] J. Carot, J. da Costa, and E.G.L.R. Vaz, *Matter collineations: The inverse "symmetry inheritance" problem*, J. Math. Phys. **35** (2014), 4832–4838.

[8] M.C. Chaki, *On pseudo-symmetric manifolds*, An. Sti. Ale Univ., "AL. I.CUZA" Din Iasi **33** (1987), 53–58.

[9] S.K. Chaubey, *Generalized Robertson–Walker space-times with W_1-curvature tensor*, J. Phys. Math. **10** (2019), 1000303.

[10] S.K. Chaubey and Y.J. Suh, *Generalized Ricci recurrent spacetimes and GRW spacetimes*, Int. J. Geom. Methods Mod. Phys. **18** (2021), No. 13, 2150209.

[11] S.K. Chaubey, Y.J. Suh, and U.C. De, *Characterizations of the Lorentzian manifolds admitting a type of semi-symmetric metric connection*, Anal. Math. Phys. **10** (2020), No. 4, 61.

[12] S.K. Chaubey, *Certain results on $N(k)$-quasi Einstein manifolds*, Afr. Mat. **30** (2019), 113–127.

[13] S.K. Chaubey, *Characterization of perfect fluid spacetimes admitting gradient η-Ricci and gradient Einstein solitons*, J. Geom. Phys. **162** (2021), 104069.

[14] B.-Y. Chen, *A simple characterization of generalized Robertson–Walker spacetimes*, Gen Relativ Gravit **46** (2014), 1833.

[15] B.-Y.Chen and K. Yano, *Hypersurfaces of a conformally flat space*, Tensor N. S. **26** (1972), 318–322.

[16] A.A. Coley, *Fluid spacetimes admitting a conformal Killing vector parallel to the velocity vector*, Class. Quantum Grav. **8** (1991), 955–968.

[17] U.C. De and S.K. Ghosh, *On conformally flat pseudo-symmetric spaces*, Balk. J. Geom. Appl. **5** (2000), 61–64.

[18] U.C. De, S.K. Chaubey, and S. Shenawy, *Perfect fluid spacetimes and Yamabe solitons*, J. Math. Phys. **62** (2021), No. 3, 032501.

[19] R. Deszcz and M. Głogowska, *Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces*, Colloq. Math. **94** (2002), 87–101.

[20] L.M. Diaz-Rivera and L.O. Pimentel, *Required conditions for late time acceleration and early time deceleration in generalized scalar-tensor theories*, Int. J. Mod. Phys. A **18** (2003), 651–672.

[21] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata **7** (1978), 259–280.

[22] B.S. Guilfoyle and B.C. Nolan, *Yang's gravitational theory*, Gen. Relativ. Gravit. **30** (1998), 473–495.

[23] M. Gutiérrez and B. Olea, *Global decomposition of a Lorentzian manifold as a generalized Robertson–Walker space*, Differential Geom. Appl. **27** (2009), 146–156.

[24] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in Gravitation: an Introduction to Current Research (Ed. L. Witten), John Wiley and Sons, Inc., New York, 49–101, 1962.

[25] G.S. Hall, *Ricci recurrent space times*, Phys. Lett. A **56** (1976), 17–18.

[26] S.W. Hawking and G.F.R. Ellis, *The large-scale structure of spacetime*, Cambridge Univ. Press, 1973.

[27] T. Harko, F.S.N. Lobo, S. Nojiri, and S. D. Odintsov, $f(R,T)$*-gravity*, Phys. Rev. D **84** (2011), 024020.

[28] C.A. Mantica, U.C. De, Y.J. Suh, and L.G. Molinari, *Perfect fluid spacetimes with harmonic generalized curvature tensor*, Osaka J. Math. **56** (2019), 173–182.

[29] C.A. Mantica and L.G. Molinari, *Generalized Robertson–Walker spacetimes—a survey*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1730001.

[30] C.A. Mantica, L.G. Molinari, and U.C. De, *A condition for a perfect-fluid space-time to be a generalized Robertson–Walker space-time*, J. Math. Phys. **57** (2016), 022508.

[31] C.A. Mantica and L.G. Molinari, *On the Weyl and the Ricci tensors of GRW space-times*, J. Math. Physics **57** (2016), 102502.

[32] C.A. Mantica, Y.J. Suh, and U.C. De, *A note on generalized Robertson–Walker space-times*, Int. J. Geom. Methods Mod. Phys. **13** (2016), 1650079.

[33] C.A. Mantica, L.G. Molinari, Y.J. Suh, and S. Shenawy, *Perfect-fluid, generalized Robertson–Walker space-times, and Gray's decomposition*, J. Math. Phys. **60** (2019), 052506.

[34] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, Inc. 1983.

[35] M. Prvanović, *Extended recurrent manifolds*, Izv. Vyssh. Uchebn. Zaved. Mat. **1999** (1999), 41–50.

[36] H.S. Ruse, *On simply harmonic spaces*, J. Lond. Math. Soc. **21** (1946), 243–247.

[37] R. Sharma, *Proper conformal symmetries of space-times with divergence-free Weyl tensor*, J. Math. Phys. **34** (1993), 3582–3587.

[38] L.C. Shepley and A.H. Taub, *Space-times containing perfect fluids and having a vanishing conformal divergence*, Commun. Math. Phys. **5** (1967), 237–256.

[39] Z.I. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$*. I. The local version*, J. Diff. Geom **17** (1982), 531–582.

[40] H. Weyl, *Reine Infinitesimalgeometrie*, Math. Z. **2** (1918), 384–411.

[41] K. Yano and S. Bochner, *Curvature and Betti numbers*, Ann. Math. Stud. **32**, Princeton University Press, 1953.

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Uday Chand De,
Department of Pure Mathematics, University of Calcutta 35, Ballygaunge Circular Road Kolkata 700019, West Bengal, India,
E-mail: uc_de@yahoo.com

Young Jin Suh,
Department of Mathematics and RIRCM, Kyungpook National University, Daegu 41566, Republic of Korea,
E-mail: yjsuh@knu.ac.kr

Sudhakar K. Chaubey,
Section of Mathematics, Department of Information Technology, University of Technology and Applied Sciences – Shinas, P.O. Box 77, Postal Code 324, Oman,
E-mail: sudhakar.chaubey@shct.edu.om

Властивості напівсиметричності кривини простору-часу Робертсона–Вокера

Uday Chand De, Young Jin Suh, and Sudhakar K. Chaubey

Метою цієї роботи є характеризація просторів-часів Робертсона–Вокера (РВ), що задовольняють деякі умови на кривину. Отримано необхідні та достатні умови того, що РВ простір-час є Річчі напівсиметричним. Доведено, що чотиривимірний Річчі симетричний РВ простір-час є вакуумним. Також ми досліджуємо властивості проективної колінеації та колінеації матерії в рамках чотиривимірного Річчі симетричного РВ простору-часу. Поміж іншого доведено, що лоренцевий многовид розмірності $n \geq 3$ є РВ простором тоді, і лише тоді, коли простір-час має квазісталу кривину. Нарешті, отримано деякі нові характеристики РВ просторів-часів.

Ключові слова: лоренцевий многовид, симетричний простор, простір-час Робертсона–Вокера, узагальнений простір-час Робертсона–Вокера, простір-час ідеальної рідини, простір-час квазісталої кривини, тензори проективної та конформної кривини