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Lie Groups of Dimension Four and Almost Hypercomplex Manifolds with Hermitian–Norden Metrics

Hristo Manev

In the paper, almost hypercomplex manifolds with Hermitian–Norden metrics of the lowest dimension are studied. The considered manifolds are constructed on 4-dimensional Lie groups. A relation between the classes of the classification of 4-dimensional indecomposable real Lie algebras and the classification of the manifolds under study is established. The basic geometrical characteristics of the constructed manifolds are studied in the frame of the mentioned classification of Lie algebras.

Key words: almost hypercomplex structure, Hermitian metric, Norden metric, Lie group, Lie algebra

Mathematical Subject Classification 2010: 53C15, 53C50, 22E60, 22E15, 53C55

1. Introduction

A triad of anticommuting almost complex structures such that each of them is a composition of the other two structures is called an almost hypercomplex structure H on a $4n$-dimensional smooth manifold $\mathcal{M}$.

If the almost hypercomplex manifold is equipped with a Hermitian metric, it is known that the derived metric structure is hyper-Hermitian, i.e., it consists of the given Hermitian metric with respect to the three almost complex structures and the three associated Kähler forms [1]. Almost hypercomplex structures with Hermitian metrics have been widely studied (e.g., [4, 15]).

An object of our interest in this work is a metric structure on $(\mathcal{M}, H)$ derived by a Norden metric. Then the existence of a Norden metric with respect to one of the three almost complex structures implies the existence of one more Norden metric and a Hermitian metric with respect to the other two almost complex structures. Such a metric is called a Hermitian–Norden metric on an almost hypercomplex manifold. The considered type of manifolds is the only possible way to involve Norden-type metrics on almost hypercomplex manifolds. The structure H can be equipped with a metric structure of Hermitian–Norden type generated by a pseudo-Riemannian metric g of neutral signature [8, 9]. In this case, in each tangent fibre, one of the almost complex structures of H acts as

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an isometry and the other two act as anti-isometries with respect to g. The metric g is Hermitian with respect to one of almost complex structures of H and g is a Norden metric regarding the other two. Then we have three associated (0,2)-tensors to the metric g - a Kähler form and two Norden metrics.

The manifold $\mathcal{M}$, equipped with the considered structures, is called an almost hypercomplex manifold with Hermitian-Norden metrics. The same manifolds were studied in [8, 9] under the name almost hypercomplex pseudo-Hermitian manifolds and in [12, 13] as almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics.

Almost hypercomplex manifolds with Hermitian-Norden metrics can be constructed on Lie groups. In this work, we use the classification of 4-dimensional indecomposable Lie algebras known from [6]. The goal of this paper is to find a relation between the classes in this classification and the corresponding manifolds to the classifications given in [7] and [5], which are derived by the tensor structures and metrics of the respective manifolds. This correspondence would provide a wide horizon of combining the results derived for the different structures. Moreover, the present work gives the basic geometrical characteristics of the considered manifolds in each case.

The author's intention with this article is to complete the considered problem for all classes of the mentioned classification and thus to generalize the results from [10] and [11].

Smooth manifolds with similar structures on Lie groups were studied in [3, 14, 15, 17, 21].

2. Almost hypercomplex manifolds with Hermitian-Norden metrics

The subject of our study are *almost hypercomplex manifolds with Hermitian-Norden metrics* ([9]). A differentiable manifold $\mathcal{M}$ of this type has dimension $4n$ and it is denoted by $(\mathcal{M}, H, G)$, where (H, G) is an *almost hypercomplex structure with Hermitian-Norden metrics*. More precisely, the almost hypercomplex structure $H = (J_1, J_2, J_3)$ has the following properties:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \qquad J_\alpha^2 = -I,$$

for all cyclic permutations (α, β, γ) of $(1, 2, 3)$ and the identity I. The quadruplet $G = (g, g_1, g_2, g_3)$ consists of a neutral metric g, associated 2-form g_1 and associated neutral metrics g_2 and g_3 on $(\mathcal{M}, H)$ having the properties

$$g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot), \tag{2.1}$$

$$g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot). \tag{2.2}$$

where

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1 \\ -1, & \alpha = 2, 3 \end{cases}.$$

Here and further, α will run over the range $\{1, 2, 3\}$ unless otherwise is stated.

Let us remark that the considered type of manifolds is the only possible way to involve Norden-type metrics on almost hypercomplex manifolds.

The following three tensors of type $(0,3)$ are the fundamental tensors of the almost hypercomplex manifold with Hermitian–Norden metrics ([9]),

$$F_\alpha(x,y,z) = g\big((\nabla_x J_\alpha)\,y,z\big) = \big(\nabla_x g_\alpha\big)\,(y,z)\,, \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. These tensors have the properties

$$F_\alpha(x,y,z) = -\varepsilon_\alpha F_\alpha(x,z,y) = -\varepsilon_\alpha F_\alpha(x,J_\alpha y,J_\alpha z) \tag{2.4}$$

and they are related to each other as follows:

$$\begin{aligned}
F_1(x,y,z) &= F_2(x,J_3y,z) + F_3(x,y,J_2z),\\
F_2(x,y,z) &= F_3(x,J_1y,z) + F_1(x,y,J_3z),\\
F_3(x,y,z) &= F_1(x,J_2y,z) - F_2(x,y,J_1z).
\end{aligned}$$

The corresponding 1-forms θ_α of F_α, known as Lee forms, are determined by

$$\theta_\alpha(\cdot) = g^{ij}F_\alpha(e_i,e_j,\cdot), \tag{2.5}$$

where $\{e_1,e_2,\dots,e_{4n}\}$ is an arbitrary basis of $T_p\mathcal{M}$, $p\in\mathcal{M}$ and g^{ij} are the corresponding components of the inverse matrix of g.

According to (2.1), $(\mathcal{M},J_1,g)$ is an almost Hermitian manifold whereas the manifolds $(\mathcal{M},J_2,g)$ and $(\mathcal{M},J_3,g)$ are almost complex manifolds with Norden metric. These two types of manifolds are classified in [7] and [5], respectively. In the case of the lowest dimension 4, the four basic classes of almost Hermitian manifolds with respect to J_1 are restricted to two classes—the class $\mathcal{AK}$ of almost Kähler manifolds and the class $\mathcal{H}$ of Hermitian manifolds:

$$\begin{aligned}
\mathcal{AK}:\ & \mathop{\mathfrak{S}}_{x,y,z}\big\{F_1(x,y,z)\big\} = 0;\\
\mathcal{H}:\ & F_1(x,y,z) = \frac{1}{2}\left\{g(x,y)\theta_1(z) - g(x,J_1y)\theta_1(J_1z)\right.\\
&\qquad\qquad\left. -g(x,z)\theta_1(y) + g(x,J_1z)\theta_1(J_1y)\right\},
\end{aligned} \tag{2.6}$$

where $\mathfrak{S}$ is the cyclic sum by three arguments. In the 4-dimensional case, the basic classes of almost Norden manifolds ($\alpha = 2$ or 3) are determined as follows:

$$\begin{aligned}
\mathcal{W}_1(J_\alpha):\ & F_\alpha(x,y,z) = \frac{1}{4}\big\{g(x,y)\theta_\alpha(z) + g(x,J_\alpha y)\theta_\alpha(J_\alpha z)\\
&\qquad\qquad +g(x,z)\theta_\alpha(y) + g(x,J_\alpha z)\theta_\alpha(J_\alpha y)\big\};\\
\mathcal{W}_2(J_\alpha):\ & \mathop{\mathfrak{S}}_{x,y,z}\big\{F_\alpha(x,y,J_\alpha z)\big\} = 0, \qquad \theta_\alpha = 0;\\
\mathcal{W}_3(J_\alpha):\ & \mathop{\mathfrak{S}}_{x,y,z}\big\{F_\alpha(x,y,z)\big\} = 0.
\end{aligned} \tag{2.7}$$

Let us notice that $\mathcal{K}$ and $\mathcal{W}_0$ are the denotations of the classes of the Kähler type manifolds in the Hermitian case and the Norden case, respectively. Moreover, for the considered lowest dimension 4, the class $\mathcal{H}$ is the only basic class with integrable almost complex structure. As a counterpart, in terms of almost Norden manifolds, this integrable class is $\mathcal{W}_1 \oplus \mathcal{W}_2$ ([5,7]).

The observed integrable class of almost hypercomplex manifolds with Hermitian–Norden metrics, $\mathcal{H}(J_1)\cap\mathcal{W}_1(J_2)\oplus\mathcal{W}_2(J_2)\cap\mathcal{W}_1(J_3)\oplus\mathcal{W}_2(J_3)$, is known as the class of hypercomplex manifolds with Hermitian–Norden metrics ([8]).

On the other hand, for dimension 4, the basic classes of non-integrable manifolds in the both cases are $\mathcal{AK}$ and $\mathcal{W}_3$ ([5,7]).

The curvature (1,3)-tensor of ∇ is defined as usual by $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$. The corresponding curvature (0,4)-tensor with respect to g is denoted by the same letter, i.e.,

$$R(x, y, z, w) = g(R(x, y)z, w), \tag{2.8}$$

and it has the following well-known properties:

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z), \tag{2.9}$$

$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \tag{2.10}$$

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities ρ^*, τ^*_α and τ^{**}_α are defined by

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \qquad \rho^*_\alpha(y, z) = g^{ij} R(e_i, y, z, J_\alpha e_j),$$
$$\tau = g^{ij}\rho(e_i, e_j), \qquad \tau^*_\alpha = g^{ij}\rho^*_\alpha(e_i, e_j), \qquad \tau^{**}_\alpha = g^{ij}\rho^*_\alpha(e_i, J_\alpha e_j).$$

The following properties for ρ and ρ^*_α are valid:

$$\rho_{jk} = \rho_{kj}, \quad (\rho^*_\alpha)_{jk} = -\varepsilon_\alpha(\rho^*_\alpha)_{kj}, \tag{2.11}$$

where $\rho_{jk} = \rho(e_j, e_k)$ and $(\rho^*_\alpha)_{jk} = \rho^*_\alpha(e_j, e_k)$ are the basic components of ρ and ρ^*_α.

Let μ be a non-degenerate 2-plane with a basis $\{x, y\}$ in $T_p\mathcal{M}$, $p \in \mathcal{M}$. The sectional curvature of μ with respect to g and R is defined by

$$k(\mu; p) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}.$$

A 2-plane μ is called *holomorphic* (resp., *totally real*) if the condition $\mu = J_\alpha\mu$ (resp., $\mu \perp J_\alpha\mu \neq \mu$ with respect to g) holds. The sectional curvature of a holomorphic (resp., totally real) 2-plane is called *holomorphic* (resp., *totally real*) *sectional curvature*. The 2-plane μ and its sectional curvature $k(\mu; p)$ are called a *basic 2-plane* and a *basic sectional curvature*, respectively, if μ has a basis $\{e_i, e_j\}$ $(i, j \in \{1, 2, \ldots, 4n\}, i \neq j)$ for a basis $\{e_1, e_2, \ldots, e_{4n}\}$ of $T_p\mathcal{M}$. In the latter case we denote k_{ij}.

3. Four-dimensional indecomposable real Lie algebras

Different authors have studied real 4-dimensional indecomposable Lie algebras. Firstly, a classification was given in [16], which can be easily found in [18] and [6]. The object of investigation in [2] were 4-dimensional solvable real Lie algebras. The authors of this work establish the one-to-one correspondence between their classification and the classifications in [16] and [18]. In all of the cited works, the basic classes are described by the non-zero Lie brackets with respect to a basis $\{e_1, e_2, e_3, e_4\}$. In Table 3.1, the correspondence between the mentioned classifications is shown.

In the present work, we use the notation of the classes from [6], namely

$$
\begin{array}{llll}
\mathfrak{g}_{4,1}: & [e_2,e_4]=e_1, & [e_3,e_4]=e_2; & \\
\mathfrak{g}_{4,2}: & [e_1,e_4]=me_1, & [e_2,e_4]=e_2, & \\
 & [e_3,e_4]=e_2+e_3, & & (m\neq 0); \\
\mathfrak{g}_{4,3}: & [e_1,e_4]=e_1, & [e_3,e_4]=e_2; & \\
\mathfrak{g}_{4,4}: & [e_1,e_4]=e_1, & [e_2,e_4]=e_1+e_2, & \\
 & [e_3,e_4]=e_2+e_3; & & \\
\mathfrak{g}_{4,5}: & [e_1,e_4]=e_1, & [e_2,e_4]=a_1e_2, & \\
 & [e_3,e_4]=a_2e_3, & & (a_1\neq 0, a_2\neq 0); \\
\mathfrak{g}_{4,6}: & [e_1,e_4]=b_1e_1, & [e_2,e_4]=b_2e_2-e_3, & \\
 & [e_3,e_4]=e_2+b_2e_3, & & (b_1\neq 0, b_2\geq 0); \\
\mathfrak{g}_{4,7}: & [e_1,e_4]=2e_1, & [e_2,e_3]=e_1, & \\
 & [e_2,e_4]=e_2, & [e_3,e_4]=e_2+e_3; & \\
\mathfrak{g}_{4,8}: & [e_2,e_3]=e_1, & [e_2,e_4]=e_2, & \\
 & [e_3,e_4]=-e_3; & & \\
\mathfrak{g}_{4,9}: & [e_1,e_4]=(p+1)e_1, & [e_2,e_3]=e_1, & \\
 & [e_2,e_4]=e_2, & [e_3,e_4]=pe_3, & (-1<p\leq 1); \\
\mathfrak{g}_{4,10}: & [e_2,e_3]=e_1, & [e_2,e_4]=-e_3, & \\
 & [e_3,e_4]=e_2; & & \\
\mathfrak{g}_{4,11}: & [e_1,e_4]=2qe_1, & [e_2,e_3]=e_1, & \\
 & [e_2,e_4]=qe_2-e_3, & [e_3,e_4]=e_2+qe_3, & (q>0); \\
\mathfrak{g}_{4,12}: & [e_1,e_3]=e_1, & [e_1,e_4]=-e_2, & \\
 & [e_2,e_3]=e_2, & [e_2,e_4]=e_1, &
\end{array}
\tag{3.1}
$$

where a_1, a_2, b_1, b_2, m, p, $q \in \mathbb{R}$.

4. Lie groups as almost hypercomplex manifolds with Hermitian–Norden metrics

Let $\mathcal{L}$ be a simply connected 4-dimensional real Lie group with corresponding Lie algebra $\mathfrak{l}$. A standard hypercomplex structure on $\mathfrak{l}$ for its basis $\{e_1, e_2, e_3, e_4\}$ is defined as in [19]:

$$
\begin{array}{llll}
J_1e_1=e_2, & J_1e_2=-e_1, & J_1e_3=-e_4, & J_1e_4=e_3; \\
J_2e_1=e_3, & J_2e_2=e_4, & J_2e_3=-e_1, & J_2e_4=-e_2;
\end{array}
$$

[6]	[2]	[16]	[18]
$\mathfrak{g}_{4,1}$	$\mathfrak{n}_4$	$\mathfrak{g}_{4,1}$	$A_{4,1}$
$\mathfrak{g}_{4,2}$	$\mathfrak{r}_{4,a}$	$\mathfrak{g}_{4,2}$	$A^a_{4,2}$
$\mathfrak{g}_{4,3}$	$\mathfrak{r}_{4,0}$	$\mathfrak{g}_{4,3}$	$A_{4,3}$
$\mathfrak{g}_{4,4}$	$\mathfrak{r}_4$	$\mathfrak{g}_{4,4}$	$A_{4,4}$
$\mathfrak{g}_{4,5}$	$\mathfrak{r}_{4,a,b}$	$\mathfrak{g}_{4,5}$	$A^{a,b}_{4,5}$
$\mathfrak{g}_{4,6}$	$\mathfrak{r}'_{4,a,b}$	$\mathfrak{g}_{4,6}$	$A^{a,b}_{4,6}$
$\mathfrak{g}_{4,7}$	$\mathfrak{h}_4$	$\mathfrak{g}_{4,7}$	$A_{4,7}$
$\mathfrak{g}_{4,8}$	$\mathfrak{d}_4$	$\mathfrak{g}_{4,8(-1)}$	$A_{4,8}$
$\mathfrak{g}_{4,9}$	$\mathfrak{d}_{4,1/1+b}$	$\mathfrak{g}_{4,8}$	$A^b_{4,9}$
$\mathfrak{g}_{4,10}$	$\mathfrak{d}'_{4,0}$	$\mathfrak{g}_{4,9(0)}$	$A_{4,10}$
$\mathfrak{g}_{4,11}$	$\mathfrak{d}'_{4,a}$	$\mathfrak{g}_{4,9}$	$A^a_{4,11}$
$\mathfrak{g}_{4,12}$	$\mathfrak{aff}(\mathbb{C})$	$\mathfrak{g}_{4,10}$	$A_{4,12}$

Table 3.1: Correspondence between some classifications of Lie algebras

$$J_3e_1 = -e_4, \qquad J_3e_2 = e_3, \qquad J_3e_3 = -e_2, \qquad J_3e_4 = e_1. \tag{4.1}$$

Let g be a pseudo-Riemannian metric of neutral signature for $x(x^1, x^2, x^3, x^4)$, $y(y^1, y^2, y^3, y^4) \in \mathfrak{l}$ defined by

$$g(x, y) = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4.$$

Bearing in mind the latter equality, it is valid that

$$\begin{aligned} g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1,\\ g(e_i, e_j) = 0, \quad i \neq j \in \{1, 2, 3, 4\}. \end{aligned} \tag{4.2}$$

Let us note that further the indices i, j, k, l run over the range $\{1, 2, 3, 4\}$. According to (2.1) and (2.2), the metric g generates an almost hypercomplex structure with Hermitian–Norden metrics on $\mathfrak{l}$. Then $(\mathcal{L}, H, G)$ is an almost hypercomplex manifold with Hermitian–Norden metrics.

Theorem 4.1. *Let $(\mathcal{L}, H, G)$ be a 4-dimensional almost hypercomplex manifold with Hermitian–Norden metrics. Then the manifold $(\mathcal{L}, H, G)$, which is corresponding to the different classes of 4-dimensional Lie algebras $\mathfrak{g}_{4,i}$, $(i = 1, \dots, 12)$, belongs to a certain class regarding J_α given in Table 4.1, where we denote for brevity $\mathcal{W}_i \oplus \mathcal{W}_j$ and $\mathcal{W}_i \oplus \mathcal{W}_j \oplus \mathcal{W}_k$ by $\mathcal{W}_{ij}$ and $\mathcal{W}_{ijk}$, respectively.*

Moreover, we have:

- *for each $a_1 \neq 0$ and $a_2 \neq 0$, $(\mathcal{L}, H, G)$ does not belong to neither of $\mathcal{K}$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_3$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_3$ for J_3;*
- *for each $b_1 \neq 0$, $b_2 \geq 0$, $(\mathcal{L}, H, G)$ does not belong to neither of $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{H}$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ for J_3;*
- *for each $m \neq 0$, $(\mathcal{L}, H, G)$ does not belong to neither of $\mathcal{K}$, $\mathcal{AK}$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_3;*

Lie algebra	Parameters	J_1	J_2	J_3
$\mathfrak{g}_{4,1}$	–	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,2}$	$m=1$	$\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
	$m\neq 0$; $m\neq 1$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,3}$	–	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,4}$	–	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,5}$	$a_1=-1$, $a_2=1$	$\mathcal{AK}$	$\mathcal{W}_2$	$\mathcal{W}_{123}$
	$a_1=-1$, $a_2=-1$	$\mathcal{AK}$	$\mathcal{W}_{123}$	$\mathcal{W}_2$
	$a_1=-1$, $a_2\neq\pm1$	$\mathcal{AK}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
	$a_1=1$, $a_2=1$	$\mathcal{H}$	$\mathcal{W}_1$	$\mathcal{W}_{12}$
	$a_1=1$, $a_2=-3$	$\mathcal{H}$	$\mathcal{W}_{23}$	$\mathcal{W}_{23}$
	$a_1=-1$, $a_2\neq\{-3,1\}$	$\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
	$a_1\neq\pm1$, $a_2=1$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{12}$	$\mathcal{W}_{123}$
	$a_1=-\frac{1}{3}$, $a_2=-\frac{1}{3}$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{23}$	$\mathcal{W}_{12}$
	$a_1=-\frac{1}{2}(a_2+1)$, $a_2\neq\{-3,-\frac{1}{3},1\}$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{23}$	$\mathcal{W}_{123}$
	$a_1=a_2$, $a_2\neq\{\pm1,-\frac{1}{3}\}$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{12}$
	$a_1=-a_2-2$, $a_2\neq\{-3,-1\}$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{23}$
	$a_1\neq 0$, $a_2\neq 0$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,6}$	$b_1\neq 0$, $b_2\geq 0$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{12}$
$\mathfrak{g}_{4,7}$	–	$\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,8}$	–	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_3$
$\mathfrak{g}_{4,9}$	$p=1$	$\mathcal{H}$	$\mathcal{W}_{12}$	$\mathcal{W}_{12}$
	$-1<p<1$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{12}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,10}$	–	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$
$\mathfrak{g}_{4,11}$	$q>0$	$\mathcal{AK}\oplus\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{12}$
$\mathfrak{g}_{4,12}$	–	$\mathcal{H}$	$\mathcal{W}_{123}$	$\mathcal{W}_{123}$

Table 4.1: Correspondence between different classes of Lie algebras and the classes of almost hypercomplex manifold with Hermitian–Norden metrics

- *for each* $-1<p\leq 1$, $(\mathcal{L},H,G)$ *does not belong to neither of* $\mathcal{K}$, $\mathcal{AK}$ *for* J_1*;* $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ *for* J_2*;* $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1\oplus\mathcal{W}_3$, $\mathcal{W}_2\oplus\mathcal{W}_3$ *for* J_3*;*
- *for each* $q>0$, $(\mathcal{L},H,G)$ *does not belong to neither of* $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{H}$ *for* J_1*;* $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1\oplus\mathcal{W}_2$, $\mathcal{W}_1\oplus\mathcal{W}_3$, $\mathcal{W}_2\oplus\mathcal{W}_3$ *for* J_2*;* $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ *for* J_3.

Proof. Now we give our arguments for the case when the corresponding Lie algebra of $(\mathcal{L},H,G)$ is from $\mathfrak{g}_{4,1}$. Then, using (2.1), (3.1), (4.1) and the well-known Koszul equality

$$2g\left(\nabla_{e_i}e_j,e_k\right)=g\left([e_i,e_j],e_k\right)+g\left([e_k,e_i],e_j\right)+g\left([e_k,e_j],e_i\right),$$

we obtain the components of the Levi-Civita connection ∇ for the considered basis. The non-zeros of them are:

$$\begin{aligned}
&\nabla_{e_1}e_2=\nabla_{e_2}e_1=\nabla_{e_2}e_3=\nabla_{e_3}e_2=\frac{1}{2}e_4,\\
&\nabla_{e_1}e_4=\nabla_{e_3}e_4=\nabla_{e_4}e_1=-\nabla_{e_4}e_3=\frac{1}{2}e_2,
\end{aligned}$$

$$\nabla_{e_2} e_4 = \frac{1}{2}(e_1 - e_3), \quad \nabla_{e_4} e_2 = -\frac{1}{2}(e_1 + e_3). \tag{4.3}$$

Then we obtain the basic components $(F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k)$ of F_α by virtue of (2.3), (4.1), (4) and (4.3). The non-zeros of them are determined by the following ones and properties (2.4):

$$\begin{aligned}(F_1)_{141} &= (F_1)_{213} = (F_1)_{341} = (F_1)_{413} = (F_2)_{212} = (F_2)_{223} = (F_2)_{414} \\ &= -(F_2)_{412} = \frac{1}{2}(F_2)_{122} = \frac{1}{2}(F_2)_{322} = (F_3)_{134} = -(F_3)_{213} \\ &= (F_3)_{334} = (F_3)_{413} = -\frac{1}{2}(F_3)_{211} = -\frac{1}{2}(F_3)_{422} = \frac{1}{2}.\end{aligned} \tag{4.4}$$

Using (2.5) and (4.4), we establish the basic components $(\theta_\alpha)_i = (\theta_\alpha)(e_i)$ of the corresponding Lee forms and the non-zeros are

$$(\theta_1)_2 = (\theta_2)_3 = -(\theta_3)_2 = -(\theta_3)_4 = 1.$$

After that, bearing in mind the classification conditions (2.6) and (2.7) for dimension 4, we conclude that in this case the manifold $(\mathcal{L}, H, G)$ belongs to

$$(\mathcal{AK} \oplus \mathcal{H})(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3)(J_3).$$

The proofs for the cases of the classes $\mathfrak{g}_{4,2}$, $\mathfrak{g}_{4,5}$, $\mathfrak{g}_{4,6}$, $\mathfrak{g}_{4,9}$, and $\mathfrak{g}_{4,11}$ are given in [10] and [11].

In a similar way, we prove the assertions for the other classes using the following results for each case:

$\mathfrak{g}_{4,3}$:

$$\begin{aligned}&\nabla_{e_1} e_1 = 2\nabla_{e_2} e_3 = 2\nabla_{e_3} e_2 = e_4, \quad \nabla_{e_1} e_4 = e_1, \quad \nabla_{e_2} e_4 = \nabla_{e_4} e_2 = -\frac{1}{2} e_3, \\ &\nabla_{e_3} e_4 = -\nabla_{e_4} e_3 = \frac{1}{2} e_2; \quad (F_1)_{113} = -2(F_1)_{314} = 2(F_1)_{413} = 1, \\ &(F_2)_{112} = (F_2)_{322} = 2(F_2)_{223} = -2(F_2)_{412} = 1, \\ &\frac{1}{2}(F_3)_{111} = 2(F_3)_{213} = 2(F_3)_{312} = (F_3)_{422} = -1; \\ &(\theta_1)_2 = (\theta_1)_3 = (\theta_2)_2 = (\theta_2)_3 = -\frac{1}{2}(\theta_3)_1 = -(\theta_3)_4 = 1;\end{aligned}$$

$\mathfrak{g}_{4,4}$:

$$\begin{aligned}&\nabla_{e_1} e_1 = 2\nabla_{e_1} e_2 = 2\nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 2\nabla_{e_2} e_3 = 2\nabla_{e_3} e_2 = -\nabla_{e_3} e_3 = e_4, \\ &\nabla_{e_1} e_4 = e_1 + \frac{1}{2} e_2, \quad \nabla_{e_2} e_4 = \frac{1}{2} e_1 + e_2 - \frac{1}{2} e_3, \quad \nabla_{e_3} e_4 = \frac{1}{2} e_2 + e_3, \\ &\nabla_{e_4} e_1 = -\nabla_{e_4} e_3 = \frac{1}{2} e_2, \quad \nabla_{e_4} e_2 = -\frac{1}{2} e_1 - \frac{1}{2} e_3; \\ &\frac{1}{2}(F_1)_{113} = -(F_1)_{114} = (F_1)_{213} = -\frac{1}{2}(F_1)_{214} = -(F_1)_{314} = (F_1)_{413} = \frac{1}{2}, \\ &(F_2)_{112} = (F_2)_{122} = 2(F_2)_{212} = -2(F_2)_{214} = \frac{1}{2}(F_2)_{222}\end{aligned}$$

$$= (F_2)_{314} = (F_2)_{322} = -2(F_2)_{412} = 2(F_2)_{414} = 1,$$

$$\frac{1}{2}(F_3)_{111} = 2(F_3)_{112} = (F_3)_{211} = (F_3)_{212} = 2(F_3)_{213}$$

$$= 2(F_3)_{312} = -(F_3)_{313} = -2(F_3)_{413} = (F_3)_{422} = -1;$$

$$2(\theta_1)_2 = (\theta_1)_3 = \frac{1}{2}(\theta_2)_2 = 2(\theta_2)_3 = -\frac{1}{2}(\theta_3)_1 = -2(\theta_3)_2 = -2(\theta_3)_4 = 2;$$

$\mathfrak{g}_{4,7}$:

$$\frac{1}{2}\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = -\nabla_{e_3}e_3 = e_4, \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\nabla_{e_4}e_2 = \frac{1}{2}e_3,$$

$$\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\nabla_{e_4}e_3 = \frac{1}{2}e_2, \quad \nabla_{e_1}e_4 = 2e_1, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_4,$$

$$\nabla_{e_2}e_4 = e_2 - \frac{1}{2}e_3, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_4, \quad \nabla_{e_3}e_4 = \frac{1}{2}e_2 + e_3;$$

$$(F_1)_{113} = -(F_1)_{214} = -3(F_1)_{314} = 3(F_1)_{413} = \frac{3}{2},$$

$$\frac{2}{5}(F_2)_{112} = (F_2)_{211} = -2(F_2)_{214} = \frac{1}{2}(F_2)_{222}$$

$$= \frac{2}{3}(F_2)_{314} = (F_2)_{322} = -2(F_2)_{412} = 1,$$

$$\frac{1}{4}(F_3)_{111} = -(F_3)_{122} = 2(F_3)_{212} = 2(F_3)_{213}$$

$$= 2(F_3)_{312} = -\frac{2}{3}(F_3)_{313} = (F_3)_{422} = -1;$$

$$(\theta_1)_2 = \frac{1}{3}(\theta_1)_3 = \frac{1}{6}(\theta_2)_2 = (\theta_2)_3 = -\frac{1}{4}(\theta_3)_1 = -(\theta_3)_4 = 1;$$

$\mathfrak{g}_{4,8}$:

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}\nabla_{e_3}e_4 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = \frac{1}{2}\nabla_{e_2}e_4 = \nabla_{e_3}e_1 = \frac{1}{2}e_2,$$

$$\nabla_{e_2}e_2 = \nabla_{e_3}e_3 = e_4, \quad \nabla_{e_2}e_3 = -\nabla_{e_3}e_2 = \frac{1}{2}e_1;$$

$$(F_1)_{113} = \frac{1}{3}(F_1)_{214} = -\frac{1}{2}, \quad (F_3)_{122} = -2(F_3)_{212} = -2(F_3)_{313} = 1,$$

$$2(F_2)_{112} = (F_2)_{211} = \frac{1}{2}(F_2)_{222} = -2(F_2)_{314} = 1; \quad (\theta_1)_3 = \frac{1}{2}(\theta_2)_2 = 1;$$

$\mathfrak{g}_{4,10}$:

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}\nabla_{e_2}e_4 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}\nabla_{e_3}e_4 = \frac{1}{2}e_2,$$

$$\nabla_{e_2}e_3 = \frac{1}{2}e_1 + e_4, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_1 + e_4;$$

$$(F_1)_{113} = (F_1)_{214} = \frac{1}{2}(F_1)_{314} = -\frac{1}{2},$$

$$(F_2)_{112} = \frac{1}{2}(F_2)_{211} = -\frac{1}{2}(F_2)_{214} = (F_2)_{314} = \frac{1}{4}(F_2)_{322} = \frac{1}{2},$$

$$(F_3)_{122} = 2(F_3)_{212} = -(F_3)_{213} = -(F_3)_{312} = 2(F_3)_{313} = 1;$$

$$(\theta_1)_2 = (\theta_2)_2 = (\theta_2)_3 = -\frac{1}{2}(\theta_3)_4 = 1;$$

$\mathfrak{g}_{4,12}:$

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_1}e_3 = -\nabla_{e_4}e_2 = e_1, \quad \nabla_{e_2}e_3 = \nabla_{e_4}e_1 = e_2,$$

$$(F_1)_{114} = \frac{3}{2}(F_1)_{213} = -\frac{3}{2}, \quad \frac{1}{2}(F_2)_{111} = (F_2)_{212} = (F_2)_{414} = 1,$$

$$(F_3)_{112} = \frac{1}{2}(F_3)_{222} = (F_3)_{413} = 1, \quad (\theta_1)_4 = -(\theta_2)_1 = -(\theta_3)_2 = -2.$$

The theorem is proved. □

5. Curvature properties of the manifolds under study

In this section, we determine some geometric characteristics of the manifolds $(\mathcal{L}, H, G)$ in all the classes considered in the previous section. The focus of the considerations in [10] and [11] are the classes of the classification of 4-dimensional indecomposable real Lie algebras, given in (3.1), depending on real parameters. Actually, these five classes are the families of manifolds whose properties are functions of the parameters. The curvature properties of the considered manifolds are summarized in the following

Theorem 5.1. *Let $(\mathcal{L}, H, G)$ be a 4-dimensional almost hypercomplex manifold with Hermitian–Norden metrics, and let the corresponding Lie algebra $\mathfrak{l}$ of L be from the class $\mathfrak{g}_{4,i}$, $(i = 1, \dots, 12)$ given in (3.1). Then the following propositions are valid:*

1. *Every $(\mathcal{L}, H, G)$ is non-flat;*
2. *An $(\mathcal{L}, H, G)$ is an Einstein manifold if and only if $\mathfrak{l}$ belongs to:*
   a) $\mathfrak{g}_{4,5}$ $\big(a_1 = a_2 = 1\big)$,
   b) $\mathfrak{g}_{4,6}$ $\big(b_1 = -2b_2 = -\frac{2\sqrt{3}}{3}\big)$;
3. *An $(\mathcal{L}, H, G)$ is scalar flat if and only if $\mathfrak{l}$ belongs to:*
   a) $\mathfrak{g}_{4,1}$,
   b) $\mathfrak{g}_{4,6}$ $\big(b_1 = -b_2 \pm \sqrt{1 - 2b_2^2},\ 0 \leq b_2 \leq \frac{\sqrt{2}}{2},\ b_2 \neq \frac{\sqrt{3}}{3}\big)$,
   c) $\mathfrak{g}_{4,11}$ $\big(q = \frac{\sqrt{3}}{6}\big)$;
4. *An $(\mathcal{L}, H, G)$ has positive scalar curvature if and only if $\mathfrak{l}$ belongs to:*
   a) $\mathfrak{g}_{4,i}$ $\big(i = 2, 3, 4, 5, 7, 8, 9, 11, 12\big)$,
   b) $\mathfrak{g}_{4,6}$ $\big(b_1 \neq 0,\ b_2 > \frac{\sqrt{2}}{2}\big)$,
   c) $\mathfrak{g}_{4,11}$ $\big(q > \frac{\sqrt{3}}{6}\big)$;
5. *An $(\mathcal{L}, H, G)$ has negative scalar curvature if and only if $\mathfrak{l}$ belongs to:*
   a) $\mathfrak{g}_{4,10}$,
   b) $\mathfrak{g}_{4,11}$ $\big(0 < q < \frac{\sqrt{3}}{6}\big)$;

6. *Every* $(\mathcal{L}, H, G)$ *is* $*$*-scalar flat w.r.t.* J_1 *and* J_2*;*
7. *An* $(\mathcal{L}, H, G)$ *is* $*$*-scalar flat w.r.t.* J_3 *if and only if* $\mathfrak{l}$ *belongs to:*
   a) $\mathfrak{g}_{4,i}$ $\big(i = 1, 5, 8, 9, 10, 12\big)$,
   b) $\mathfrak{g}_{4,2}$ $\big(m = -2\big)$,
   c) $\mathfrak{g}_{4,6}$ $\big(b_1 = -2b_2\big)$;
8. *An* $(\mathcal{L}, H, G)$ *is* $**$*-scalar flat w.r.t.* J_1 *if and only if* $\mathfrak{l}$ *belongs to:*
   a) $\mathfrak{g}_{4,2}$ $\big(m = -\frac{1}{4}\big)$,
   b) $\mathfrak{g}_{4,5}$ $\big(a_1 = -a_2^2\big)$,
   c) $\mathfrak{g}_{4,6}$ $\big(b_1 = b_2^{-1} - b_2\big)$,
   d) $\mathfrak{g}_{4,11}$ $\big(q = \frac{\sqrt{15}}{6}\big)$;
9. *An* $(\mathcal{L}, H, G)$ *is* $**$*-scalar flat w.r.t.* J_2 *if and only if* $\mathfrak{l}$ *belongs to:*
   a) $\mathfrak{g}_{4,2}$ $\big(m = -\frac{5}{4}\big)$,
   b) $\mathfrak{g}_{4,5}$ $\big(a_2 = -a_1^2\big)$,
   c) $\mathfrak{g}_{4,6}$ $\big(b_1 = b_2^{-1} - b_2\big)$,
   d) $\mathfrak{g}_{4,11}$ $\big(q = \frac{\sqrt{15}}{6}\big)$;
10. *An* $(\mathcal{L}, H, G)$ *is* $**$*-scalar flat w.r.t.* J_3 *if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,1}$,
    b) $\mathfrak{g}_{4,9}$ $\big(p = \frac{\sqrt{2}-3}{2}\big)$;
11. *An* $(\mathcal{L}, H, G)$ *has positive basic holomorphic sectional curvatures w.r.t.* J_1 *(i.e.,* k_{12} *and* k_{34}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 4, 7\big)$,
    b) $\mathfrak{g}_{4,2}$ $\big(m > 0\big)$,
    c) $\mathfrak{g}_{4,5}$ $\big(a_1 > 0\big)$,
    d) $\mathfrak{g}_{4,6}$ $\big(b_1 > 0,\ b_2 > 1\big)$,
    e) $\mathfrak{g}_{4,9}$ $\big(-\frac{3}{4} < p \leq 1,\ p \neq 0\big)$,
    f) $\mathfrak{g}_{4,11}$ $\big(q > 1\big)$;
12. *An* $(\mathcal{L}, H, G)$ *has positive basic holomorphic sectional curvatures w.r.t.* J_2 *(i.e.,* k_{13} *and* k_{24}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 4, 7\big)$,
    b) $\mathfrak{g}_{4,2}$ $\big(m > 0\big)$,
    c) $\mathfrak{g}_{4,5}$ $\big(a_2 > 0\big)$,
    d) $\mathfrak{g}_{4,6}$ $\big(b_1 > 0,\ b_2 > 1\big)$,
    e) $\mathfrak{g}_{4,9}$ $\big(\frac{\sqrt{2}-1}{2} < p \leq 1\big)$,
    f) $\mathfrak{g}_{4,11}$ $\big(q > 1\big)$;

13. *An* $(\mathcal{L}, H, G)$ *has positive basic holomorphic sectional curvatures w.r.t.* J_3 *(i.e.,* k_{14} *and* k_{23}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 2, 3, 4, 6, 7, 11\big)$,
    b) $\mathfrak{g}_{4,5}$ $\big(a_1 a_2 > 0\big)$,
    c) $\mathfrak{g}_{4,9}$ $\big(-\frac{3}{4} < p \leq 1,\ p \neq 0\big)$*;*
14. *An* $(\mathcal{L}, H, G)$ *has negative basic holomorphic sectional curvatures w.r.t.* J_1 *(i.e.,* k_{12} *and* k_{34}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,1}$,
    b) $\mathfrak{g}_{4,6}$ $\big(b_1 < 0,\ 0 < b_2 < 1\big)$,
    c) $\mathfrak{g}_{4,11}$ $\big(0 < q < \frac{\sqrt{2}}{4}\big)$*;*
15. *An* $(\mathcal{L}, H, G)$ *has negative basic holomorphic sectional curvatures w.r.t.* J_2 *(i.e.,* k_{13} *and* k_{24}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,6}$ $\big(b_1 < 0,\ 0 < b_2 < 1\big)$,
    b) $\mathfrak{g}_{4,11}$ $\big(0 < q < \frac{\sqrt{2}}{4}\big)$*;*
16. *Every* $(\mathcal{L}, H, G)$ *has non-negative basic holomorphic sectional curvatures w.r.t.* J_3 *(i.e.,* k_{14} *and* k_{23}*);*
17. *An* $(\mathcal{L}, H, G)$ *has positive basic totally real sectional curvatures w.r.t.* J_1 *(i.e.,* k_{13}*,* k_{14}*,* k_{23} *and* k_{24}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 4, 7\big)$,
    b) $\mathfrak{g}_{4,2}$ $\big(m > 0\big)$,
    c) $\mathfrak{g}_{4,5}$ $\big(a_1 > 0,\ a_2 > 0\big)$,
    d) $\mathfrak{g}_{4,6}$ $\big(b_1 > 0,\ b_2 > 1\big)$,
    e) $\mathfrak{g}_{4,9}$ $\big(\frac{\sqrt{2}-1}{2} < p \leq 1\big)$,
    f) $\mathfrak{g}_{4,11}$ $\big(q > 1\big)$*;*
18. *An* $(\mathcal{L}, H, G)$ *has positive basic totally real sectional curvatures w.r.t.* J_2 *(i.e.,* k_{12}*,* k_{14}*,* k_{23} *and* k_{34}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 4, 7\big)$,
    b) $\mathfrak{g}_{4,2}$ $\big(m > 0\big)$,
    c) $\mathfrak{g}_{4,5}$ $\big(a_1 > 0,\ a_2 > 0\big)$,
    d) $\mathfrak{g}_{4,6}$ $\big(b_1 > 0,\ b_2 > 1\big)$,
    e) $\mathfrak{g}_{4,9}$ $\big(-\frac{3}{4} < p \leq 1,\ p \neq 0\big)$,
    f) $\mathfrak{g}_{4,11}$ $\big(q > 1\big)$*;*
19. *An* $(\mathcal{L}, H, G)$ *has positive basic totally real sectional curvatures w.r.t.* J_3 *(i.e.,* k_{12}*,* k_{13}*,* k_{24} *and* k_{34}*) if and only if* $\mathfrak{l}$ *belongs to:*
    a) $\mathfrak{g}_{4,i}$ $\big(i = 4, 7\big)$,
    b) $\mathfrak{g}_{4,2}$ $\big(m > 0\big)$,

c) $\mathfrak{g}_{4,5}$ $\big(a_1 > 0,\ a_2 > 0\big)$,

d) $\mathfrak{g}_{4,6}$ $\big(b_1 > 0,\ b_2 > 1\big)$,

e) $\mathfrak{g}_{4,9}$ $\big(\frac{\sqrt{2}-1}{2} < p \le 1\big)$,

f) $\mathfrak{g}_{4,11}$ $\big(q > 1\big)$;

20. *Every $(\mathcal{L}, H, G)$ has non-negative basic totally real sectional curvatures w.r.t. J_1 (i.e., k_{13}, k_{14}, k_{23} and k_{24}) and J_2 (i.e., k_{12}, k_{14}, k_{23} and k_{34});*
21. *An $(\mathcal{L}, H, G)$ has negative basic totally real sectional curvatures w.r.t. J_3 (i.e., k_{12}, k_{13}, k_{24} and k_{34}) if and only if $\mathfrak{l}$ belongs to:*

a) $\mathfrak{g}_{4,6}$ $\big(b_1 < 0,\ 0 < b_2 < 1\big)$,

b) $\mathfrak{g}_{4,11}$ $\big(0 < q < \frac{\sqrt{2}}{4}\big)$.

Proof. Firstly, we present our proof for the case when the corresponding Lie algebra of $\mathcal{L}$ belongs to $\mathfrak{g}_{4,1}$.

Using (2.1), (2.8), (4.1), and the definition of $\mathfrak{g}_{4,1}$ in (3.1), we calculate the basic components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ of R. The non-zeros of them are determined by the following ones and properties (2.9):

$$R_{1212} = -R_{1223} = -R_{1414} = R_{1434} = R_{2323} = \frac{1}{4}R_{2424} = \frac{1}{3}R_{3434} = \frac{1}{4}. \qquad (5.1)$$

Bearing in mind the latter equalities, (2.1), (4.1), and (4), we obtain the basic components $\rho_{jk} = \rho(e_j, e_k)$, $(\rho^*_\alpha)_{jk} = \rho^*_\alpha(e_j, e_k)$, as well as the values of τ, τ^*_α, τ^{**}_α and $k_{ij} = k(e_i, e_j)$. Having in mind properties (2.11), the non-zeros of them are determined by

$$\begin{gathered}
\rho_{11} = -\frac{1}{2}\rho_{22} = -\rho_{33} = -\frac{1}{2}, \quad (\rho^*_1)_{12} = (\rho^*_1)_{14} = -(\rho^*_1)_{23} = \frac{1}{3}(\rho^*_1)_{34} = -\frac{1}{4}, \\
(\rho^*_2)_{22} = -\frac{1}{2}(\rho^*_2)_{24} = (\rho^*_2)_{44} = -\frac{1}{2}, \quad (\rho^*_3)_{12} = (\rho^*_3)_{14} = (\rho^*_3)_{23} = \frac{1}{4}, \\
\tau^{**}_1 = -\tau^{**}_2 = -2, \quad k_{12} = k_{14} = -k_{23} = -\frac{1}{4}k_{24} = \frac{1}{3}k_{34} = -\frac{1}{4}. \qquad (5.2)
\end{gathered}$$

By virtue of (5.1) and (5.2), we establish the truthfullness of the statements for the case of $\mathfrak{g}_{4,1}$.

The results for the cases of the classes $\mathfrak{g}_{4,2}$, $\mathfrak{g}_{4,5}$, $\mathfrak{g}_{4,6}$, $\mathfrak{g}_{4,9}$ and $\mathfrak{g}_{4,11}$, which are summarized here, are given in [10] and [11].

In a similar way as for $\mathfrak{g}_{4,1}$, we obtain the following results for $(\mathcal{L}, H, G)$ in the other cases and we prove the respective assertions:

$\mathfrak{g}_{4,3}$:

$$\begin{aligned}
&-\frac{1}{2}R_{1213} = \frac{1}{4}R_{1414} = R_{2323} = R_{2424} = \frac{1}{3}R_{3434} = \frac{1}{4}; \\
&\frac{1}{2}\rho_{11} = \rho_{22} = \rho_{23} = \rho_{33} = -\rho_{44} = \frac{1}{2}, \\
&-\frac{1}{3}(\rho^*_1)_{34} = -\frac{1}{2}(\rho^*_2)_{12} = (\rho^*_2)_{24} = \frac{1}{4}(\rho^*_3)_{11} = -\frac{1}{4}(\rho^*_3)_{14} = (\rho^*_3)_{23} = \frac{1}{4},
\end{aligned}$$

$$\tau=\frac{3}{2}\tau_3^*=-\tau_1^{**}=3\tau_2^{**}=5\tau_3^{**}=\frac{3}{2},\quad \frac{1}{4}k_{14}=k_{23}=k_{24}=-\frac{1}{3}k_{34}=\frac{1}{4};$$

$\mathfrak{g}_{4,4}$:

$$\begin{aligned}-\frac{4}{3}R_{1212}&=-2R_{1213}=-4R_{1223}=R_{1313}=2R_{1323}=\frac{4}{3}R_{1414}\\&=R_{1424}=4R_{1434}=\frac{4}{5}R_{2323}=\frac{1}{2}R_{2424}=R_{2434}=-4R_{3434}=1;\end{aligned}$$

$$\frac{3}{5}\rho_{11}=\rho_{12}=\frac{3}{8}\rho_{22}=\rho_{23}=-\frac{3}{5}\rho_{33}=-\frac{1}{2}\rho_{44}=\frac{3}{2},$$

$$\frac{1}{3}(\rho_1^*)_{12}=\frac{1}{2}(\rho_1^*)_{13}=-(\rho_1^*)_{14}=(\rho_1^*)_{23}=-\frac{1}{4}(\rho_1^*)_{24}=(\rho_1^*)_{34}=\frac{1}{4},$$

$$\begin{aligned}(\rho_2^*)_{12}&=-\frac{1}{2}(\rho_2^*)_{13}=-\frac{1}{2}(\rho_2^*)_{14}=(\rho_2^*)_{22}=-(\rho_2^*)_{23}\\&=-\frac{1}{4}(\rho_2^*)_{24}=-\frac{1}{2}(\rho_2^*)_{34}=(\rho_2^*)_{44}=-\frac{1}{2},\end{aligned}$$

$$\begin{aligned}(\rho_3^*)_{11}&=4(\rho_3^*)_{12}=2(\rho_3^*)_{13}=-\frac{4}{3}(\rho_3^*)_{14}=\frac{4}{5}(\rho_3^*)_{23}\\&=-(\rho_3^*)_{24}=-4(\rho_3^*)_{34}=-\frac{1}{2}(\rho_3^*)_{44}=1,\end{aligned}$$

$$\tau=4\tau_3^*=6\tau_1^{**}=2\tau_2^{**}=3\tau_3^{**}=12,$$

$$k_{12}=\frac{4}{3}k_{13}=k_{14}=\frac{3}{5}k_{23}=\frac{3}{8}k_{24}=3k_{34}=\frac{3}{4};$$

$\mathfrak{g}_{4,7}$:

$$\begin{aligned}-\frac{4}{7}R_{1212}&=-R_{1213}=4R_{1224}=-2R_{1234}=\frac{4}{7}R_{1313}=2R_{1324}=4R_{1334}\\&=\frac{1}{4}R_{1414}=R_{1423}=\frac{1}{2}R_{2323}=\frac{4}{5}R_{2424}=R_{2434}=-4R_{3434}=1;\end{aligned}$$

$$\frac{2}{15}\rho_{11}=\frac{1}{5}\rho_{22}=\frac{1}{2}\rho_{23}=-\frac{1}{4}\rho_{33}=-\frac{2}{11}\rho_{44}=1,$$

$$\frac{1}{3}(\rho_1^*)_{12}=(\rho_1^*)_{13}=-\frac{3}{5}(\rho_1^*)_{24}=(\rho_1^*)_{34}=\frac{3}{4},$$

$$-\frac{4}{5}(\rho_2^*)_{12}=\frac{4}{13}(\rho_2^*)_{13}=\frac{4}{11}(\rho_2^*)_{24}=\frac{4}{3}(\rho_2^*)_{34}=1,$$

$$(\rho_3^*)_{11}=-\frac{1}{2}(\rho_3^*)_{14}=-4(\rho_3^*)_{22}=(\rho_3^*)_{23}=-4(\rho_3^*)_{33}=-(\rho_3^*)_{44}=2,$$

$$\frac{6}{11}\tau=3\tau_3^*=3\tau_1^{**}=2\tau_2^{**}=\tau_3^{**}=12,$$

$$k_{12}=k_{13}=\frac{7}{16}k_{14}=\frac{7}{8}k_{23}=\frac{7}{5}k_{24}=7k_{34}=\frac{7}{4};$$

$\mathfrak{g}_{4,8}$:

$$4R_{1212}=2R_{1234}=-4R_{1313}=2R_{1324}=-4R_{2323}=R_{2424}=-R_{3434}=1;$$

$$\rho_{11}=-\rho_{22}=\rho_{33}=\frac{1}{4}\rho_{44}=-\frac{1}{2},$$

$$(\rho_1^*)_{12}=-\frac{3}{2}(\rho_1^*)_{34}=(\rho_2^*)_{13}=-\frac{3}{2}(\rho_2^*)_{24}=-\frac{3}{4}(\rho_3^*)_{14}=\frac{3}{5}(\rho_3^*)_{23}=-\frac{3}{4},$$

$$\tau=\frac{5}{3}\tau_1^{**}=\frac{5}{3}\tau_2^{**}=-5\tau_3^{**}=\frac{5}{2},\quad k_{12}=k_{13}=k_{23}=-\frac{1}{4}k_{24}=-\frac{1}{4}k_{34}=-\frac{1}{4};$$

$\mathfrak{g}_{4,10}$:

$$4R_{1212}=2R_{1224}=-4R_{1313}=2R_{1334}=\frac{4}{7}R_{2323}=-R_{2424}=R_{3434}=1;$$

$$\rho_{11}=-\rho_{22}=\rho_{33}=-\frac{1}{4}\rho_{44}=-\frac{1}{2},\quad 4(\rho_1^*)_{12}=2(\rho_1^*)_{13}=2(\rho_1^*)_{24}=(\rho_1^*)_{34}=-1,$$

$$2(\rho_2^*)_{12}=4(\rho_2^*)_{13}=(\rho_2^*)_{24}=2(\rho_2^*)_{34}=-1,\quad (\rho_3^*)_{22}=-\frac{4}{7}(\rho_3^*)_{23}=(\rho_3^*)_{33}=-1,$$

$$\tau=\frac{7}{10}\tau_1^{**}=\frac{7}{10}\tau_2^{**}=-\frac{1}{2}\tau_3^{**}=-\frac{7}{4},\quad k_{12}=k_{13}=-\frac{1}{7}k_{23}=\frac{1}{4}k_{24}=\frac{1}{4}k_{34}=-\frac{1}{4};$$

$\mathfrak{g}_{4,12}$:

$$R_{1212}=-R_{1313}=-R_{2323}=-1;\quad \rho_{11}=\rho_{22}=-\rho_{33}=2,$$

$$(\rho_1^*)_{12}=(\rho_2^*)_{13}=(\rho_3^*)_{23}=1,\quad \frac{1}{3}\tau=\tau_1^{**}=\tau_2^{**}=\tau_3^{**}=2,\quad k_{12}=k_{13}=k_{23}=1.$$

The theorem is proved. □

Let us remark that the author of [20] considers a 4-parametric family of Lie algebras:

$$\begin{aligned} &[e_1,e_2]=\lambda_1e_1+\lambda_2e_2, && [e_2,e_3]=\lambda_4e_2-\lambda_1e_3,\\ &[e_1,e_3]=\lambda_4e_1+\lambda_2e_3, && [e_2,e_4]=\lambda_3e_2-\lambda_1e_4,\\ &[e_1,e_4]=\lambda_3e_1+\lambda_2e_4, && [e_3,e_4]=\lambda_3e_3-\lambda_4e_4, \end{aligned} \tag{5.3}$$

where $\lambda_i\in\mathbb{R}$ $(i=1,2,3,4)$. It is proved that the corresponding Lie groups of these Lie algebras, equipped with almost complex structure and Norden metric, form an almost Norden manifold from the class $\mathcal{W}_1$. Moreover, it is shown that the constructed manifold is Einstein. Considering (5.3) for $\lambda_3=1,\lambda_1=\lambda_2=\lambda_4=0$, we get the class $\mathfrak{g}_{4,5}$ in the case when $a_1=a_2=1$. Therefore, the almost complex structure studied in [20] corresponds to J_2 in the almost hypercomplex structure $H=(J_1,J_2,J_3)$. The results obtained in [20] confirm the assertions given in Theorem 4.1 and Theorem 5.1 for J_2 in the case of $\mathfrak{g}_{4,5}$ $(a_1=a_2=1)$.

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Hristo Manev,

Medical University – Plovdiv, Faculty of Pharmacy, Department of Medical Physics and Biophysics, 15-A Vasil Aprilov Blvd., Plovdiv 4002, Bulgaria,

E-mail: hristo.manev@mu-plovdiv.bg

Чотиривимірні групи Лі і майже гіперкомплексні многовиди з ермітовою метрикою Нордена

Hristo Manev

У цій роботі вивчено майже гіперкомплексні многовиди з ермітовими метриками Нордена найменшої розмірності. Зазначені многовиди побудовано на чотиривимірних групах Лі. Установлено зв'язок між класами класифікації нерозкладних чотиривимірних дійсних алгебр Лі і класифікації многовидів, що досліджуються. У рамках зазначеної класифікації алгебр Лі вивчено основні геометричні характеристики побудованих многовидів.

Ключові слова: майже гіперкомплексна структура, ермітова метрика, метрика Нордена, група Лі, алгебра Лі