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On controllability problems for the wave equation on a half-plane

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Necessary and sufficient conditions for null-controllability and approximate null-controllability are obtained for the wave equation on a half-plane. Controls solving these problems are found explicitly. Moreover bang-bang controls solving the approximate null-controllability problem are constructed with the aid of the Markov power moment problem.

0. Introduction

Controllability problems for hyperbolic partial differential equation were investigated in a number of papers (see, e.g., the references in [1]).

One of the most generally accepted ways to study control systems with distributed parameters is their interpretation in the form

$$\frac{d\mathrm{w}}{dt} = A\mathrm{w} + Bu, \qquad t \in (0,T), \tag{0.1}$$

where $T > 0$, $\mathrm{w} : (0,T) \longrightarrow \mathcal{H}$ is an unknown function, $u : (0,T) \longrightarrow H$ is a control, $\mathcal{H}$, H are Banach spaces, A is an infinitesimal operator in $\mathcal{H}$, $B : H \longrightarrow \mathcal{H}$ is a linear bounded operator. An important advantage of this approach is a possibility to employ ideas and technique of the semigroup operator theory. At the same time it should be noticed that the most substantial and important for

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applications results on operator semigroups deal with the case when the semigroup generator A has a discrete spectrum or a compact resolvent and therefore the semigroup may be treated by means of eigenelements of A. These assumptions correspond to differential equations in bounded domains only.

In this paper we consider the wave equation on a half-plane. We should note that most of papers studied controllability problems for the wave equation dealt with this equation on bounded domains and controllability problems considered in context of L^2-controllability or, more generally, L^p-controllability $(2 \le p < +\infty)$ [2–6]. But only L^∞-controls can be realized practically. Moreover, such controls should be bounded by a hard constant (like in restriction (0.4)) for practical purposes. Furthermore classical control theory started precisely from this point view as switching controls are the ones realized in a concrete system. That is why we build also bang-bang controls solving approximate null-controllability problem in this paper.

Controllability problems for the wave equation on a half-axis in context of bounded of a hard constant controls were investigated in [9, 10].

Consider the wave equation on a half-plane

$$\frac{\partial^2 w}{\partial t^2} = \Delta w, \qquad x_1 \in \mathbb{R},\ x_2 > 0,\ t \in (0,T), \tag{0.2}$$

controlled by the boundary condition

$$w(x_1,0,t) = \delta(x_1)u(t), \qquad x_1 \in \mathbb{R},\ t \in (0,T), \tag{0.3}$$

where $T > 0$. We also assume that the control u satisfies the restriction

$$u \in \mathcal{B}(0,T) = \left\{ v \in L^2(0,T) \mid |v(t)| \le 1 \text{ almost everywhere on } (0,T) \right\}. \tag{0.4}$$

All functions appearing in the equation (0.2) are defined for $x_1 \in \mathbb{R}$, $x_2 \ge 0$. Further, we assume everywhere that they are defined for $x \in \mathbb{R}^2$ and vanish for $x_2 < 0$.

Let us give definitions of the spaces used in our work. Let $\mathcal{S}$ be the Schwartz space [7]

$$\begin{aligned} \mathcal{S} = \Big\{ \varphi \in C^\infty(\mathbb{R}^n) \mid \forall m \in \mathbb{N} \\ \forall l \in \mathbb{N}\ \sup \Big\{ \Big| D^\alpha \varphi(x) \Big| \left(1 + |x|^2\right)^l \mid x \in \mathbb{R}^n \wedge |\alpha| \le m \Big\} < +\infty \Big\}, \\ \mathcal{S}_+ = \{ \varphi \in \mathcal{S} \mid \operatorname{supp} \varphi \in \mathbb{R} \times (0, +\infty) \} \end{aligned}$$

and let $\mathcal{S}'$, $\mathcal{S}'_+$ be the dual spaces, here $D = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$, $\alpha = (\alpha_1, \ldots \alpha_n)$ is multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\cdot|$ is the Euclidean norm.

Denote by H_l^s the following Sobolev spaces:

$$\begin{aligned} H_l^s &= \left\{\varphi \in \mathcal{S}' \mid \left(1+|x|^2\right)^{l/2}\left(1+|D|^2\right)^{s/2}\varphi \in L^2\left(\mathbb{R}^n\right)\right\}, \\ \|\varphi\|_l^s &= \left(\int\limits_{\mathbb{R}^n}\left|\left(1+|x|^2\right)^{l/2}\left(1+|D|^2\right)^{s/2}\varphi(x)\right|^2 dx\right)^{1/2}. \end{aligned}$$

Let $\mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$ be the Fourier transform operator. For $\varphi \in \mathcal{S}$ we have

$$(\mathcal{F}\varphi)(\sigma) = (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} e^{-i\langle x,\sigma\rangle}\varphi(x)\,dx,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$ corresponding to the Euclidean norm. It is well known [8, Ch. 1] that $\mathcal{F}H_0^s = H_s^0$ and $\|\varphi\|_0^s = \|\mathcal{F}\varphi\|_s^0$, if $\varphi \in H_0^s$.

A distribution $f \in \mathcal{S}'$ is said to be *odd* if $(f, \varphi(\xi)) = -(f, \varphi(-\xi))$, $\varphi \in \mathcal{S}$.

Further, we assume throughout the paper that $s \le 0$ and use the spaces

$$\begin{aligned} \mathcal{H}^s &= \left\{\varphi \in H_0^s \times H_0^{s-1} \mid \varphi \in \mathcal{S}'_+ \wedge \exists\varphi(+0) \in \mathbb{R}\right\}, \\ \widetilde{H}^s &= \left\{\varphi \in H_0^s \times H_0^{s-1} \mid \varphi \text{ is odd with resp. to } x_2\right\} \end{aligned}$$

with the norm $|||\varphi|||^s = \left(\left(\|\varphi_0\|_0^s\right)^2 + \left(\|\varphi_1\|_0^{s-1}\right)^2\right)^{1/2}$ and also the space

$$\widehat{H}_s = \left\{\varphi \in H_s^0 \times H_{s-1}^0 \mid \varphi \text{ is odd with resp. to } \sigma_2\right\}$$

with the norm $[\![\varphi]\!]_s = \left(\left(\|\varphi_0\|_s^0\right)^2 + \left(\|\varphi_1\|_{s-1}^0\right)^2\right)^{1/2}$.

Denote by A the following operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \qquad A : \widetilde{H}^{s-2} \longrightarrow \widetilde{H}^{s-2}, \qquad D(A) = \widetilde{H}^s \tag{0.5}$$

and by B the operator

$$B = \begin{pmatrix} 0 \\ -2\delta(x_1)\delta'(x_2) \end{pmatrix}, \qquad B : \mathbb{R} \longrightarrow \widetilde{H}^{s-2}, \qquad D(B) = \mathbb{R}, \tag{0.6}$$

where δ is the Dirac function. Then the system (0.2), (0.3) is reduced to the form (0.1) with these operators A and B.

In Section 1 we obtain necessary and sufficient conditions for null-controllability and approximate null-controllability for the system (0.2), (0.3) with restrictions (0.4) on the control. Controls solving the problems of null-controllability and approximate null-controllability are found explicitly. But these controls may have a rather complicated form.

The main goal of the Section 2 is to build bang-bang controls solving the approximate null-controllability problem. We show that this problem can be reduced to a system of Markov power moment problems. They may be solved by the method given in [9]. Further, we prove that solutions of the Markov power moment problems give us solutions of the approximate null-controllability problem (Theorems 2.3, 2.4).

In Sections 3 and 4 some auxiliary statements are proved.

1. Null-controllability problems

Consider the control system (0.2), (0.3) with the initial conditions

$$\begin{cases} w(x,0) &= w_0^0(x) \\ \partial w(x,0)/\partial t &= w_1^0(x) \end{cases}, \qquad x_1 \in \mathbb{R},\ x_2 > 0, \tag{1.1}$$

and the steering conditions

$$\begin{cases} w(x,T) &= w_0^T(x) \\ \partial w(x,T)/\partial t &= w_1^T(x) \end{cases}, \qquad x_1 \in \mathbb{R},\ x_2 > 0, \tag{1.2}$$

where $w^0 = \begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix} \in \mathcal{H}^s$, $w^T = \begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix} \in \mathcal{H}^s$. We consider solutions of the problem (0.2), (0.3) in the space $\mathcal{H}^s$.

Let $T > 0$, $w^0 \in \mathcal{H}^s$. Denote by $\mathcal{R}_T(w^0)$ the set of states $w^T \in \mathcal{H}^s$ for which there exists a control $u \in \mathcal{B}(0,T)$ such that the problem (0.2), (0.3), (1.1), (1.2) has a unique solution.

Definition 1.1. *A state $w^0 \in \mathcal{H}^s$ is called null-controllable at a given time $T > 0$ if 0 belongs to $\mathcal{R}_T(w^0)$ and approximately null-controllable at a given time $T > 0$ if 0 belongs to the closure of $\mathcal{R}_T(w^0)$ in $\mathcal{H}^s$.*

Let $\mathrm{w}^0 = \Omega_2 w^0$, $\mathrm{w}^T = \Omega_2 w^T$, $\mathrm{w}(\cdot,t) = \Omega_2 \begin{pmatrix} w(\cdot,t) \\ \partial w(\cdot,t)/\partial t \end{pmatrix}$, where Ω_2 is the odd-extension operator with respect to x_2. Evidently, $\mathrm{w}^0 \in \widetilde{H}^s$, $\mathrm{w}^T \in \widetilde{H}^s$, $\mathrm{w}(\cdot,t) \in \widetilde{H}^s$ $(t \in (0,T))$. It is easy to see that control problem (0.2), (0.3), (1.1), (1.2) is equivalent to the following problem for system (0.1):

$$\mathrm{w}(x,0) = \mathrm{w}^0, \tag{1.3}$$

$$\mathrm{w}(x,T) = \mathrm{w}^T. \tag{1.4}$$

Let us investigate this new problem. First we analyze the following auxiliary Cauchy problem: system (0.1) with an arbitrary parameter $u \in \mathcal{B}(0,T)$ under initial condition (1.3).

Applying the Fourier transform with respect to x to problem (0.1), (1.3), we obtain the following Cauchy problem in $\widehat{H}_s$:

$$\frac{d\mathrm{v}}{dt} = \begin{pmatrix} 0 & 1 \\ -|\sigma|^2 & 0 \end{pmatrix} \mathrm{v} - \frac{i\sigma_2}{\pi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad t \in (0,T), \tag{1.5}$$

$$\mathrm{v}(\cdot,0) = \mathrm{v}^0, \tag{1.6}$$

where $\mathrm{v}(\cdot,t) = \mathcal{F}\mathrm{w}(\cdot,t)$, $t \in [0,T]$, $\mathrm{v}^0 = \mathcal{F}\mathrm{w}^0$. Then the function

$$\mathrm{v}(\sigma,t) = \Sigma(|\sigma|,t) \left(\mathrm{v}^0(\sigma) - \frac{i\sigma_2}{\pi} \int_0^t \Sigma(|\sigma|,-\tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(\tau)\, d\tau \right), \; t \in [0,T], \tag{1.7}$$

where

$$\Sigma(\rho,t) \equiv \begin{pmatrix} \cos(\rho t) & \dfrac{\sin(\rho t)}{\rho} \\ -\rho \sin(\rho t) & \cos(\rho t) \end{pmatrix} \equiv \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \frac{\sin(\rho t)}{\rho}$$

is a unique solution of (1.5), (1.6) in $\widehat{H}_s$.

Put $E(|x|,t) = \mathcal{F}_\sigma^{-1}\Sigma(|\sigma|,t)/(2\pi)$. It is well known that

$$F^{-1}\left[\frac{\sin(|\sigma|t)}{|\sigma|}\right](x) = \frac{\operatorname{sign} t \; H\left(|t| - |x|\right)}{\sqrt{t^2 - |x|^2}}, \tag{1.8}$$

where H is the Heaviside function: $H(\xi) = 1$ if $\xi \geq 0$ and $H(\xi) = 0$ otherwise. Then we have

$$E(r,t) = \frac{1}{2\pi} \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \frac{\operatorname{sign} t \; H\left(|t| - |x|\right)}{\sqrt{t^2 - |x|^2}}.$$

It follows from (1.7) that

$$\mathrm{w}(x,T) = E(|x|,T) * \left[\mathrm{w}^0(x) - \frac{1}{\pi}\frac{\partial}{\partial x_2} \mathcal{F}^{-1} \left(\int_0^T \begin{pmatrix} -\dfrac{\sin(|\sigma|t)}{|\sigma|} \\ \cos(|\sigma|t) \end{pmatrix} u(t)\, dt \right) \right]. \tag{1.9}$$

Here and further $*$ is the convolution with respect to x. With regard to Lemma 4.1 we get

$$\mathrm{w}(x,T) = E(|x|,T) * \left[\mathrm{w}^0(x) - \frac{1}{\sqrt{2\pi}} \frac{x_2}{|x|} \Phi \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} (|x|) \right], \tag{1.10}$$

where $\mathcal{U}(t) = u(t)\left(H(t) - H(t-T)\right)$, $t \in \mathbb{R}$.

Denote for $\mathrm{w}^0 \in \widetilde{H}^s$

$$R_T(\mathrm{w}^0) = \left\{ E(|x|,T) * \left[\mathrm{w}^0(x) - \frac{1}{\sqrt{2\pi}} \frac{x_2}{|x|} \Phi \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} (|x|) \right] \mid u \in \mathcal{B}(0,T) \right\}.$$

Then Definition 1.1 is equivalent to

Definition 1.2. *A state* $\mathrm{w}^0 \in \widetilde{H}^s$ *is called null-controllable at a given time* $T > 0$ *if 0 belongs to* $R_T(\mathrm{w}^0)$ *and approximately null-controllable at a given time* $T > 0$ *if 0 belongs to the closure of* $R_T(\mathrm{w}^0)$ *in* $\widetilde{H}^s$.

Obviously, the following two statements are true.

Statement 1.1. *A state* $w_0 \in \mathcal{H}^s$ *is null-controllable at a given time* $T > 0$ *iff the state* $\mathrm{w}^0 = \Omega_2 w^0$ *is null-controllable at this time.*

Statement 1.2. *A state* $w_0 \in \mathcal{H}^s$ *is approximately null-controllable at a given time* $T > 0$ *iff the state* $\mathrm{w}^0 = \Omega_2 w^0$ *is approximately null-controllable at this time.*

Further we consider the (approximate) null-controllability problem for the system (0.1) where w^0 is an odd function with respect to x_2.

The following theorem give us sufficient conditions for (approximate) null-controllability.

Theorem 1.1. *For a state* $\mathrm{w}^0 \in \widetilde{H}^s$ *assume that there exists* $\overline{\mathrm{w}}^0 \in \mathcal{S}'$ *such that following conditions hold:*

$$\mathrm{w}^0 = \frac{x_2}{|x|}\overline{\mathrm{w}}^0(|x|) \quad \textit{in } H_0^s \times H_0^{s-1}, \tag{1.11}$$

$$\operatorname{supp}\overline{\mathrm{w}}^0 \subset [0,T], \tag{1.12}$$

$$\left|\overline{\mathrm{w}}_0^0(r)\right| \le \frac{T}{\pi r\sqrt{T^2-r^2}} \quad \textit{a.e. on } (0,T), \tag{1.13}$$

$$\overline{\mathrm{w}}_1^0(r) = \frac{d}{dr}\left[\overline{\mathrm{w}}_0^0(r) + \int_{-\infty}^{\infty} \overline{\mathrm{w}}_0^0(\xi)k(\xi,r)\,d\xi\right], \tag{1.14}$$

where $k(\xi,r) = \dfrac{2}{\pi}H\left(\xi(\xi-r)\right)\displaystyle\int_0^{\pi/2}\frac{\sin^2\alpha\,d\alpha}{\sqrt{\xi^2\sin^2\alpha + r^2\cos^2\alpha}}$. *Then the state* w^0 *is null-controllable at the time* T. *Moreover, the solution of the null-controllability problem (the control* u*) is unique and*

$$u(t) = 2t\int_t^T \frac{\overline{\mathrm{w}}_0^0(r)\,dr}{\sqrt{r^2-t^2}} \quad \textit{a.e. on } (0,T).$$

P r o o f. Put

$$\mathcal{U} = \frac{1}{\sqrt{2\pi}} \Phi \overline{\mathrm{w}}_0^0. \tag{1.15}$$

It follows from (1.12) and Lemma 3.2 that $\operatorname{supp}\mathcal{U} \subset (0,T)$ and

$$\mathcal{U}(t) = 2t \int_t^T \frac{\overline{\mathrm{w}}_0^0(r)\,dr}{\sqrt{r^2 - t^2}} \quad \text{a.e. on } (0,T).$$

Denote $u(t) = \mathcal{U}(t)$, $t \in (0,T)$. Due to (1.13) we obtain $|u(t)| \le 1$ a.e. on $(0,T)$. Applying Lemma 4.2 and (1.15), we have

$$\overline{\mathrm{w}}_1^0 = \frac{d}{dr}\left[\overline{\mathrm{w}}_0^0 + \int_{-\infty}^{\infty} \overline{\mathrm{w}}_0^0(\xi) k(\xi,\cdot)\,d\xi\right] = \Phi \frac{d}{dt} \Phi^{-1} \overline{\mathrm{w}}_0^0 = \frac{1}{\sqrt{2\pi}} \Phi \mathcal{U}'.$$

Finally, taking into account (1.10), (1.11), (1.15), we get that $\mathrm{w}(x,T) = 0$ for the found control u where w is a solution of the Cauchy problem (0.1), (1.3). Invertibility of the operator Φ (see Sect. 4) implies uniqueness of the control u solving the null-controllability problem.

Thus the state w^0 is null-controllable at the time T that was to be proved.

The following theorem asserts that conditions (1.11)—(1.14) are not only sufficient but also necessary for (approximate) null-controllability.

Theorem 1.2. *If a state* $\mathrm{w}^0 \in \widetilde{H}^s$ *is approximately null controllable at a given time* $T > 0$ *then there exists* $\overline{\mathrm{w}}^0 \in \mathcal{S}'$ *such that conditions (1.11)—(1.14) hold.*

P r o o f. For each $n \in \mathbb{N}$ there exists a state $\mathrm{w}^n \in R_T(\mathrm{w}^0)$ such that $|\!|\!|\mathrm{w}^n|\!|\!|^s < 1/n$. With regard to (1.10) for some $u_n \in \mathcal{B}(0,T)$ we have

$$\mathrm{w}^n(x) = E(|x|,T) * \left[\mathrm{w}^0(x) - \frac{1}{\sqrt{2\pi}} \frac{x_2}{|x|} \Phi \begin{pmatrix} \mathcal{U}_n \\ \mathcal{U}_n' \end{pmatrix} (|x|)\right], \qquad t \in \mathbb{R},$$

where $\mathcal{U}_n(t) = u_n(t)\,(H(t) - H(t-T))$. Using Lemma 4.4, we obtain

$$\frac{1}{\sqrt{2\pi}} \frac{x_2}{|x|} \Phi \begin{pmatrix} \mathcal{U}_n \\ \mathcal{U}_n' \end{pmatrix} (|x|) \longrightarrow \mathrm{w}^0 \qquad \text{as } n \longrightarrow \infty \text{ in } \widetilde{H}^s. \tag{1.16}$$

Therefore $\mathrm{w}^0 = \dfrac{x_2}{|x|} \overline{\mathrm{w}}^0(|x|)$. According to the Lemma 3.2 $\operatorname{supp} \overline{\mathrm{w}}_0^0 \subset [0,T]$. Thus (1.11), (1.12) are true. Denote $\Phi \mathcal{U}_n = h_0^n$, $\Phi \mathcal{U}_n' = h_1^n$. Taking into account Lemma 4.3, we obtain

$$|h_0^n| \le \frac{T}{\pi r \sqrt{T^2 - r^2}}, \qquad r \in (0,T). \tag{1.17}$$

Let an arbitrary $\varepsilon > 0$ be fixed, $V(\varepsilon) = \left\{x \in \mathbb{R}^2 \mid |x| < \varepsilon\right\}$. It follows from (1.16) that

$$h^n(|x|) \longrightarrow \overline{\mathrm{w}}^0(|x|) \qquad \text{as } n \longrightarrow \infty \text{ in } \mathcal{S}'. \tag{1.18}$$

Since $h_0^n(|x|) \in L^2\left(\mathbb{R}^2 \backslash V(\varepsilon)\right)$ and $\mathcal{S}$ is dense in $L^2\left(\mathbb{R}^2\right)$ we obtain

$$h_0^n(|x|) \longrightarrow \overline{\mathrm{w}}_0^0(|x|) \qquad \text{as } n \longrightarrow \infty \text{ in } \left(L^2\left(\mathbb{R}^2 \backslash V(\varepsilon)\right)\right)'.$$

By the Riesz theorem we conclude that $\overline{\mathrm{w}}_0^0(|x|) \in L^2\left(\mathbb{R}^2 \backslash V(\varepsilon)\right)$ and $\overline{\mathrm{w}}_0^0 \in L^2(\varepsilon, +\infty)$. Taking into account arbitrariness of $\varepsilon > 0$ and (1.17), we get (1.13). We have $h_1^n = \Phi \dfrac{d}{dt} \Phi^{-1} h_0^n$. Due to Lemmas 3.1, 4.2 and (1.17) we get (1.14). The theorem is proved.

2. Bang-bang controls and the Markov power moment problem

The solution of the null-controllability problem (i.e., the control) found in Sect. 1 may be too complicated for the practical purposes. In this section we find bang-bang controls solving the approximate null-controllability problem.We consider a system of Markov power moment problems and show that their bang-bang solutions are solutions of the approximate null-controllability problem.

Consider control system (0.1), (1.3) and assume that for $T > 0$ and $\mathrm{w}^0 \in \widetilde{H}^s$ conditions (1.11)–(1.14) hold. According to Theorem 1.1 there exists $\widetilde{u} \in \mathcal{B}(0, T)$ such that

$$\overline{\mathrm{w}}^0 = \frac{1}{\sqrt{2\pi}} \left(\Phi \widetilde{\mathcal{U}}\right)(r), \tag{2.1}$$

where $\widetilde{\mathcal{U}}(t) = \widetilde{u}(t)\left[H(t) - H(t - T)\right]$. With regard to Lemma 4.1 and (1.11) we get

$$\mathrm{v}^0(\sigma) = \frac{1}{\pi} i\sigma_2 \int_0^T \begin{pmatrix} -\dfrac{\sin(|\sigma| t)}{|\sigma|} \\ \cos(|\sigma| t) \end{pmatrix} \widetilde{u}(t)\, dt,$$

where $\mathrm{v}^0 = \mathcal{F}\Omega_2 w^0$. Put

$$h(\rho, u) = \frac{1}{\pi} \int_0^T \begin{pmatrix} -\dfrac{\sin(\rho t)}{\rho} \\ \cos(\rho t) \end{pmatrix} \left(\widetilde{u}(t) - u(t)\right)\, dt. \tag{2.2}$$

Then for system (1.5), (1.6) we get

$$\mathrm{v}(\sigma, T) = \Sigma(|\sigma|, T) i\sigma_2 h(|\sigma|, u).$$

With regard to (1.7) and Lemma 4.4 we conclude that

$$[\![\mathrm{v}(\sigma, T)]\!]_s \le \sqrt{4T^2 + 6}\, [\![i\sigma_2 h(|\sigma|, u)]\!]_s .$$

We have

$$\left(\|i\sigma_2 h_j(|\sigma|,u)\|^0_{s-j}\right)^2 = \pi \int_0^\infty \left(1+\rho^2\right)^{s-j} |h_j(|\sigma|,u)|^2 \rho^3\, d\rho, \qquad j=0,1.$$

Hence

$$[\![\mathrm{v}(\sigma,T)]\!]_s \le \sqrt{4T^2+6}\,\pi \left(\sum_{j=0}^{1} \int_0^\infty \left(1+\rho^2\right)^{s-j} |h_j(|\sigma|,u)|^2 \rho^3\, d\rho\right)^{1/2}. \tag{2.3}$$

Thus we have proved

Theorem 2.1. *Assume that* $T>0$ *and for a state* $\mathrm{w}^0 \in \widetilde{H}^s$ *conditions (1.11)–(1.14) are fulfilled. Then the following two assertions hold:*

i. w^0 *is null-controllable at the time* T *iff there exists* $u \in \mathcal{B}(0,T)$ *such that* $h(\rho,u) \equiv 0$ *on* $\mathbb{R}$*;*

ii. w^0 *is approximately null-controllable at the time* T *iff for each* $\varepsilon > 0$ *there exists* $u_\varepsilon \in \mathcal{B}(0,T)$ *such that*

$$\int_0^\infty \left(1+\rho^2\right)^{s-j} |h_j(|\sigma|,u_\varepsilon)|^2 \rho^3\, d\rho < \varepsilon^2, \qquad j=0,1. \tag{2.4}$$

Moreover, if estimate (2.4) is true then

$$|\!|\!|\mathrm{w}(\cdot,T)|\!|\!|^s = [\![\mathrm{v}(\cdot,T)]\!]_s \le \pi\varepsilon\sqrt{4T^2+6}, \tag{2.5}$$

where w *and* v *are solutions of (0.1), (1.3) and (1.5), (1.6), respectively.*

Due to the Wiener–Paley theorem we conclude that $h(\rho,u)$ is an entire function with respect to ρ. Let us expand it in the Taylor series. To do this we calculate $h^{(m)}(0,u)$ (we consider the derivatives with respect to ρ). Put

$$\widetilde{\mathrm{v}}^0(\rho) = \frac{1}{\pi}\int_0^T \begin{pmatrix} -\dfrac{\sin(\rho t)}{\rho} \\ \cos(\rho t) \end{pmatrix} \widetilde{u}(t)\, dt, \qquad \widetilde{\mathrm{w}}^0(|x|) = \mathcal{F}^{-1}\widetilde{\mathrm{v}}^0(|\sigma|). \tag{2.6}$$

Evidently $\widetilde{\mathrm{v}}^0$ is also entire. With regard to (1.11) and Lemma 4.1 we get

$$\overline{\mathrm{w}}^0 = \widetilde{\mathrm{w}}^{0\prime}. \tag{2.7}$$

According to (1.11), (1.12) and (1.14), we conclude that

$$\widetilde{\mathrm{w}}_0^0(r) = (H(r) - H(r-T)) \int_r^T \frac{\widetilde{u}(t)\,dt}{\sqrt{t^2 - r^2}}, \tag{2.8}$$

$$\widetilde{\mathrm{w}}_1^0(r) = \overline{\mathrm{w}}_0^0(r) + \int_{-\infty}^{\infty} \overline{\mathrm{w}}_0^0(\xi) k(\xi, r)\,d\xi. \tag{2.9}$$

Obviously, $\operatorname{supp} \widetilde{\mathrm{w}}_0^0 \subset [0, T]$. It follows from (1.12) that $\operatorname{supp} \widetilde{\mathrm{w}}_1^0 \subset [0, T]$. Taking into account

$$\frac{2T}{\pi} \int_r^T \frac{1}{\xi\sqrt{T^2 - \xi^2}} \int_0^{\pi/2} \frac{\sin^2\alpha\,d\alpha}{\sqrt{\xi^2 \sin^2\alpha + r^2\cos^2\alpha}}\,d\xi$$

$$\le \frac{T}{r} \int_r^T \frac{1}{\xi\sqrt{T^2 - \xi^2}} = \frac{1}{2r} \ln\left|\frac{T + \sqrt{T^2 - r^2}}{T - \sqrt{T^2 - r^2}}\right|, \quad r \in (0, T), \tag{2.10}$$

and (1.13), (1.14), we get

$$\left|\widetilde{\mathrm{w}}_0^0(r)\right| \le \int_r^T \frac{dt}{\sqrt{t^2 - r^2}} = -\ln\left(\frac{T}{r} - \sqrt{\left(\frac{T}{r}\right)^2 - 1}\right), \quad r \in (0, T), \tag{2.11}$$

$$\begin{aligned} \left|\widetilde{\mathrm{w}}_1^0(r)\right| &\le \frac{T}{\pi r\sqrt{T^2 - r^2}} + \frac{1}{\pi} \int_r^T \frac{T}{\pi\xi\sqrt{T^2 - \xi^2}} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{\xi^2\sin^2\alpha + r^2\cos^2\alpha}}\,d\xi \\ &= \frac{T}{\pi r\sqrt{T^2 - r^2}} + \frac{1}{2r} \ln\left|\frac{T + \sqrt{T^2 - r^2}}{T - \sqrt{T^2 - r^2}}\right|, \quad r \in (0, T). \end{aligned} \tag{2.12}$$

Taking into account (2.11), (2.12), (2.6), we obtain $\widetilde{\mathrm{v}}^{0\,(2m+1)}(0) = 0$:

$$\begin{aligned} \widetilde{\mathrm{v}}^{0\,(2m)}(0) &= \frac{1}{\pi} \frac{d^{2m}}{d\rho^{2m}} \int_0^\infty \left(\int_0^\pi e^{-ir\rho\cos\varphi}\,d\varphi\right) \widetilde{\mathrm{w}}^0(r)\,dr \Big|_{\rho=0} \\ &= \frac{(-1)^m}{\pi} \int_0^\infty \left(\int_0^\pi \cos^{2m}\varphi\,d\varphi\right) r^{2m+1} \widetilde{\mathrm{w}}^0(r)\,dr \\ &= \frac{(-1)^m}{\pi} B\left(m + \frac{1}{2}, \frac{1}{2}\right) \int_0^\infty r^{2m+1} \widetilde{\mathrm{w}}^0(r)\,dr, \end{aligned} \tag{2.13}$$

where $B(\cdot,\cdot)$ is the Euler beta-function. Therefore

$$\widetilde{\mathrm{v}}_0^{0\,(2m)}(0) = \frac{(-1)^m}{\pi(2m+2)} B\left(m+\frac{1}{2},\frac{1}{2}\right) \int_0^\infty r^{2m+2}\overline{\mathrm{w}}_0^0(r)\,dr.$$

With regard to (2.9) we have

$$\begin{aligned}\widetilde{\mathrm{v}}_1^{0\,(2m)}(0) &= \frac{(-1)^m}{\pi} B\left(m+\frac{1}{2},\frac{1}{2}\right) \left[\int_0^\infty r^{2m+1}\overline{\mathrm{w}}_0^0(r)\,dr\right. \\ &+ \left.\int_0^\infty r^{2m+1}\int_r^\infty \overline{\mathrm{w}}_0^0(\xi)k(\xi,r)\,d\xi\,dr\right].\end{aligned}$$

Since

$$\int_0^{\pi/2} \sqrt{\xi^2\sin^2\alpha + r^2\cos^2\alpha}\,d\alpha = \int_r^\xi \frac{t^2\,dt}{\sqrt{\xi^2-t^2}\sqrt{t^2-r^2}}$$

then

$$\begin{aligned}\int_0^\xi r^{2m+1}k(\xi,r)\,dr &= -\xi^{2m+1} + \frac{2}{\pi}\frac{1}{\xi}\frac{d}{d\xi}\int_0^\xi r^{2m+1}\int_r^\xi \frac{t^2\,dt}{\sqrt{\xi^2-t^2}\sqrt{t^2-r^2}}\,dr \\ &= -\xi^{2m+1} + \frac{2}{\pi}\frac{1}{\xi}\frac{d}{d\xi}\int_0^{\pi/2}\xi^{2m+3}\sin^{2m+3}\psi\,d\psi\int_0^{\pi/2}\sin^{2m+1}\varphi\,d\varphi \\ &= -\xi^{2m+1} + \frac{2m+3}{2\pi}B\left(m+1,\frac{1}{2}\right)B\left(m+2,\frac{1}{2}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\widetilde{\mathrm{v}}_1^{0\,(2m)}(0) &= \frac{(-1)^m(2m+3)}{2\pi^2} B\left(m+\frac{1}{2},\frac{1}{2}\right)B\left(m+1,\frac{1}{2}\right)B\left(m+2,\frac{1}{2}\right) \\ &\times \int_0^\infty r^{2m+1}\overline{\mathrm{w}}_0^0(r)\,dr.\end{aligned}$$

Put

$$\omega_n = \int_0^\infty r^{n+1}\overline{\mathrm{w}}_0^0(r)\,dr. \tag{2.14}$$

Hence

$$\widetilde{\mathrm{v}}_0^{0(2m)}(0) = (-1)^{m+1}\frac{(2m-1)!!}{(2m+2)!!}, \tag{2.15}$$

$$\widetilde{\mathrm{v}}_1^{0(2m)}(0) = \frac{(-1)^m}{\pi}\frac{(2m+2)!!}{(2m+1)!!}. \tag{2.16}$$

We have

$$h^{(2m+1)}(0,u) = 0, \tag{2.17}$$

and $h^{(2m)}(0,u) = 0$ iff

$$0 = \widetilde{\mathrm{v}}_0^{0(2m)}(0) + \frac{(-1)^m}{2m+1}\int_0^T t^{2m+1}u(t)\,dt, \tag{2.18}$$

$$0 = \widetilde{\mathrm{v}}_1^{0(2m)}(0) - (-1)^m\int_0^T t^{2m}u(t)\,dt. \tag{2.19}$$

Thus

$$h^{(n)}(0,u) = 0, \tag{2.20}$$

iff

$$\int_0^T t^n u(t)\,dt = \overline{\omega}_n, \qquad n = \overline{0,\infty}, \tag{2.21}$$

where

$$\overline{\omega}_{2m} = \frac{(2m+2)!!}{\pi(2m+1)!!}\omega_{2m}, \tag{2.22}$$

$$\overline{\omega}_{2m+1} = \frac{(2m+1)!!}{(2m+2)!!}\omega_{2m+1}. \tag{2.23}$$

According to Theorem 2.1, we obtain that the state w^0 is null-controllable at the time T iff (2.21) is valid.

The problem of determination of a function $u \in \mathcal{B}(0,T)$ satisfying condition (2.21) for a given $\{\overline{\omega}_n\}_{n=0}^{\infty}$ and $T > 0$ is called a Markov power moment problem on $(0,T)$ for the infinite sequence $\{\overline{\omega}_n\}_{n=0}^{\infty}$.

Uniqueness of the solution of the null-controllability problem yields uniqueness of the solution of the Markov moment problem (2.21) (see Theorem 1.1). Hence $u = \overline{u}$ is the unique solution of this Markov moment problem.

Thus we have proved

Theorem 2.2. *Assume that* $T > 0$ *and for a state* $\mathrm{w}^0 \in \widetilde{H}^s$ *conditions (1.11)–(1.14). Assume also that* $\{\overline{\omega}_n\}_{n=0}^{\infty}$ *is defined by (2.14), (2.22), (2.23). Then Markov power moment problem (2.21) on* $(0, T)$ *for* $\{\overline{\omega}_n\}_{n=0}^{\infty}$ *has a unique solution. Moreover, this solution is a solution of the null-controllability problem for* w^0 *at the time* T*.*

Consider (2.21) for a finite set of n:

$$\int_0^T t^n u(t)\,dt = \overline{\omega}_n, \qquad n = \overline{0, N}. \tag{2.24}$$

The problem of determination of a function $u \in \mathcal{B}(0, T)$ satisfying condition (2.24) for a given $\{\overline{\omega}_n\}_{n=0}^{N}$ and $T > 0$ is called a Markov power moment problem on $(0, T)$ for the finite sequence $\{\overline{\omega}_n\}_{n=0}^{N}$.

Obviously, $u = \overline{u}$ is a solution of this problem, but it is not unique.

Let us show that solutions of moment problem (2.24) for various N give us controls solving the approximate null-controllability problem.

Theorem 2.3. *Let* $T > 0$, $\mathrm{w}^0 \in \widetilde{H}^s$, $s < -1$*. Let also conditions (1.11)–(1.14) be fulfilled and* $\{\overline{\omega}_n\}_{n=0}^{\infty}$ *be defined by (2.14), (2.22), (2.23). Then* $\forall \varepsilon > 0$ *there exists* $N > 0$ *such that for each solution* $u_N \in \mathcal{B}(0, T)$ *of moment problem (2.24) the corresponding solution* w *of control system (0.1), (1.3) satisfies the condition* $|||\mathrm{w}(\cdot, T)|||^s < \varepsilon$*.*

P r o o f. Let $N = 2K + 1$, $u_N \in \mathcal{B}(0, T)$ be a solution of problem (2.24). With regard to (2.20) and (2.21) for the function $h(\rho, u)$ defined by (2.2) we get

$$h^{(n)}(0, u_N) = 0, \qquad n = \overline{0, 2K+1}.$$

By the Taylor formula for $|\rho| < a$ we obtain

$$\left|\rho^{1-j} h_j(\rho, u_N)\right| \le \frac{a^{2K+2}}{(2K+2)!} \sup_{|\xi| \le a} \left| \left(\xi^{1-j} h_j\right)^{(2K+2)} (\xi, u_N) \right|, \qquad j = 0, 1.$$

Taking into account (2.2), we conclude that

$$\left| \left(\xi^{1-j} h_j(\xi, u_N)\right)^{(2K+2)} \right| \le \frac{T^{2K+3}}{\pi(2K+3)}, \qquad j = 0, 1.$$

Hence

$$\left|\rho^{1-j} h_j(\rho, u_N)\right| \le \frac{T}{\pi} \frac{(Ta)^{2K+2}}{(2K+3)!}, \qquad j = 0, 1, \ |\rho| \le a.$$

Then

$$\int_0^a \left(1+\rho^2\right)^{s-j} |h_j(\rho,u_N)|^2 \rho^3\, d\rho \le \frac{a}{\pi}\frac{(Ta)^{2K+3}}{(2K+3)!}, \qquad j=0,1. \tag{2.25}$$

With regard to (2.2) we get

$$\left|\rho^{1-j} h_j(\rho,u_N)\right| \le \frac{T}{\pi}, \qquad j=0,1,\ \rho>0.$$

Therefore

$$\int_a^\infty \left(1+\rho^2\right)^{s-j} |h_j(\rho,u_N)|^2 \rho^3\, d\rho$$

$$\le \frac{T}{\pi}\int_a^\infty \left(1+\rho^2\right)^{s} \rho\, d\rho \le -\frac{Ta^{2(s+1)}}{2\pi(s+1)}, \qquad j=0,1.$$

Taking into account (2.25), we obtain

$$\pi\int_0^\infty \left(1+\rho^2\right)^{s-j} |h_j(\rho,u_N)|^2 \rho^3\, d\rho \le \frac{a(Ta)^{2K+3}}{(2K+3)!} - \frac{Ta^{2(s+1)}}{2(1+s)}, \qquad j=0,1.$$

Due to Theorem 2.1 and (2.3) we conclude that

$$|||\mathrm{w}(\cdot,T)|||^s \le \sqrt{2T^2+3}\left[\frac{a(Ta)^{2K+3}}{(2K+3)!} - \frac{Ta^{2(s+1)}}{2(1+s)}\right]. \tag{2.26}$$

Applying the Stirling formula, we have

$$\frac{(Ta)^{2K+3}}{(2K+3)!} \le \left(\frac{Tae}{2K+3}\right)^{2K+3} \frac{1}{\sqrt{2\pi(2K+3)}}.$$

Setting $a=(2K+3)/(2Te)$, we obtain from (2.26) that

$$|||\mathrm{w}(\cdot,T)|||^s \le \sqrt{2T^2+3}\left[\frac{\sqrt{2K+3}}{Te4^{K+2}} - \frac{T}{2s+2}\left(\frac{2K+3}{2Te}\right)^{2s+2}\right] \to 0 \text{ as } K\to\infty. \tag{2.27}$$

The theorem is proved.

Denote

$$\begin{aligned} \mathcal{B}^N(0,T) \;=\;& \{u\in\mathcal{B}(0,T) \mid \exists T_*\in(0,T)(|u(t)|=1 \text{ a.e. on } (0,T_*)) \\ \wedge\;& (u(t)=0 \text{ a.e. on } (T_*,T)) \\ \wedge\;& (u \text{ has no more than } N \text{ discontinuity points on } (0,T_*))\}. \end{aligned}$$

It is well known [11, 12] that if Markov power moment problem (2.24) is solvable then there exists its solution $u \in \mathcal{B}^N(0,T)$. Taking into account Theorem 2.3, we conclude that under the conditions ot this theorem we can find a solution $u_K \in \mathcal{B}^{2K+1}(0,T)$ of Markov power moment problem (2.24) for $N = 2K+1$ and such solutions $\{u_K\}_{K=1}^{\infty}$ give us bang-bang controls solving the approximate null-controllability problem (see also (2.27)).

Thus the following theorem is true.

Theorem 2.4. *Let $T > 0$, $\mathrm{w}^0 \in \widetilde{H}^s$, $s < -1$. Let also conditions (1.11)–(1.14) be fulfilled and $\{\varpi_n\}_{n=0}^{\infty}$ be defined by (2.14), (2.22), (2.23). Then $\forall K \in \mathbb{N}$ there exists a solution $u_K \in \mathcal{B}^{2K+1}(0,T)$ of moment problem (2.24) with $N = 2K+1$. Moreover, for this u_K the corresponding solution* w *of control system (0.1), (1.3) satisfies the estimate*

$$\|\mathrm{w}(\cdot,T)\|^s \le \sqrt{2T^2+3}\left[\frac{\sqrt{2K+3}}{Te4^{K+2}} - \frac{T}{2s+2}\left(\frac{2K+3}{2Te}\right)^{2s+2}\right]. \tag{2.28}$$

Let us show that the condition $s < -1$ of Theorems 2.3, 2.3 is essential. Precisely if $-1/2 \le s \le 0$ then $\exists \mathrm{w}^0 \in \widetilde{H}^s$ $\forall T > 0$ $\forall u \in \cup_{N\in\mathbb{N}}\mathcal{B}^N(0,T)$ $\exists \varepsilon_0 > 0$ such that for a solution w of (0.1), (1.3), corresponding to the control u we have $\|\mathrm{w}(\cdot,T)\|^s \ge \varepsilon_0$. Thus the state w^0 is not approximate null-controllable at the time T by bang-bang controls in space $\widetilde{H}^s$, if $-1/2 \le s \le 0$.

E x a m p l e 2.1. Let $-1/2 \le s \le 0$, $T > 0$,

$$\begin{aligned}
\mathrm{w}_0^0(x) &= \frac{x_2 T}{2\pi|x|^2\sqrt{T^2-|x|^2}}\left[H(|x|) - H(|x|-T)\right],\\
\mathrm{w}_1^0(x) &= \frac{x_2}{2\pi\sqrt{(T^2-|x|^2)^3}}\left[H(|x|) - H(|x|-T)\right].
\end{aligned}$$

Obviously, $\mathrm{w}^0(x) = \dfrac{1}{\sqrt{2\pi}}\Phi\begin{pmatrix}\widetilde{\mathcal{U}}\\ \widetilde{\mathcal{U}}'\end{pmatrix}(|x|)$, where $\widetilde{\mathcal{U}}(t) = \dfrac{1}{2}\left[H(t) - H(t-T)\right]$. Therefore $\mathrm{w}^0 \in \widetilde{H}^s$ satisfies (1.11)–(1.14). Let $u \in \mathcal{B}^N(0,T)$, $n \in \mathbb{N}$. Hence

$$u(t) = \alpha\sum_{k=0}^{N}(-1)^k\left[H(t-t_k) - H(t-t_{k+1})\right],$$

where $\alpha = \pm 1$, $0 = t_0 < t_1 < t_2 \cdots < t_{N+1} = T_* \le T$, $\mathcal{U}(t) = [H(t) - H(t-T)]$. Let w be a solution of (0.1), (1.3) corresponding to the control u. According to (1.10), we have

$$\sqrt{2\pi}E(|x|,-T) * \mathrm{w}(x,T) = \frac{x_2}{|x|}\Phi\begin{pmatrix}\widetilde{\mathcal{U}} - \mathcal{U}\\ \widetilde{\mathcal{U}}' - \mathcal{U}'\end{pmatrix}(|x|).$$

Put $a = \pi/(12T)$. With regard to Lemma 4.4 we get

$$\|\!|\mathrm{w}(x,T)|\!\|^s \geq \frac{1}{\sqrt{2\pi}\sqrt{4T^2+6}} \left\|\!\left| \frac{x_2}{|x|}\Phi \begin{pmatrix} \widetilde{\mathcal{U}} - \mathcal{U} \\ \widetilde{\mathcal{U}}' - \mathcal{U}' \end{pmatrix} (|x|) \right|\!\right\|^s$$

$$\geq \frac{1}{\sqrt{\pi}\sqrt{4T^2+6}} \left(\int_0^\infty \left(1+\rho^2\right)^{-1/2} \left| \int_0^T \sin(\rho t) \left(\widetilde{\mathcal{U}}(t) - \mathcal{U}(t)\right) dt \right|^2 \rho\, d\rho \right)^{1/2}$$

$$\geq \frac{\sqrt{a}}{\sqrt{\pi}\sqrt[4]{1+a^2}\sqrt{4T^2+6}} \left(\int_a^\infty \left| \int_0^T \sin(\rho t) \left(\widetilde{\mathcal{U}}(t) - \mathcal{U}(t)\right) dt \right|^2 d\rho \right)^{1/2}$$

$$\geq \frac{\sqrt{a}}{\sqrt{2}\sqrt[4]{1+a^2}\sqrt{4T^2+6}} \left[\left(\int_{-\infty}^\infty \left| \mathcal{F}\Omega\left(\widetilde{\mathcal{U}} - \mathcal{U}\right)(\rho) \right|^2 d\rho \right)^{1/2} - \left(\int_{-a}^a \left| \mathcal{F}\Omega\left(\widetilde{\mathcal{U}} - \mathcal{U}\right)(\rho) \right|^2 d\rho \right)^{1/2} \right]. \quad (2.29)$$

We have

$$\int_{-\infty}^\infty \left| \mathcal{F}\Omega\left(\widetilde{\mathcal{U}} - \mathcal{U}\right)(\rho) \right|^2 d\rho \geq \frac{1}{4} \int_{-\infty}^\infty |(H(t+T) - H(t-T))|^2\, dt = \frac{T}{2}. \quad (2.30)$$

On the other hand

$$\int_{-a}^a \left| \mathcal{F}\Omega\left(\widetilde{\mathcal{U}} - \mathcal{U}\right)(\rho) \right|^2 d\rho \leq \frac{3}{\pi} \int_{-a}^a \left(\frac{1}{\rho} \sum_{k=0}^N |\cos(t_k\rho) - \cos(t_{k+1}\rho)| \right)^2 d\rho$$

$$= \frac{6}{\pi} \int_0^a \left(\frac{2}{\rho} \sum_{k=0}^N \left| \sin\left(\rho \frac{t_{k+1} - t_k}{2}\right) \sin\left(\rho \frac{t_{k+1} + t_k}{2}\right) \right| \right)^2 d\rho$$

$$\leq \frac{6}{\pi} \int_0^a \left(\sum_{k=0}^N \frac{t_{k+1}^2 - t_k^2}{2} \right)^2 d\rho \leq \frac{3T^2 a}{2\pi} = \frac{T}{8}. \quad (2.31)$$

Comparing (2.29), (2.31), we obtain

$$\|\!|\mathrm{w}(\cdot,T)|\!\|^s \geq \frac{\sqrt{a}}{\sqrt{2}\sqrt[4]{1+a^2}\sqrt{4T^2+6}} \left[\sqrt{\frac{T}{2}} - \sqrt{\frac{T}{8}} \right] \geq \frac{T}{4(4T^2+6)^{3/4}} = \varepsilon_0. \quad (2.32)$$

That was to be proved.

3. Operators Φ and Φ^*

In this section we introduce and study operators Φ and Φ^*.

Let the operator $\Phi^* : \mathcal{S} \longrightarrow \mathcal{S}$ be defined by the rule

$$(\Phi^*\varphi)(t) = -\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{H(r(t-r))}{\sqrt{t^2-r^2}} \varphi'(r)\, dr, \qquad \varphi \in \mathcal{S}. \tag{3.1}$$

Obviously, $(\Phi^*\varphi)(t) = -\sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \varphi'(t\sin\alpha)\, d\alpha$, $\varphi \in \mathcal{S}$. Hence $\Phi^*\varphi \in \mathcal{S}$, if $\varphi \in \mathcal{S}$. It is easy to see that $\Phi^{*-1} : \mathcal{S} \longrightarrow \mathcal{S}$ can be defined by the rule

$$\left(\Phi^{*-1}\psi\right)(t) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{H(t(r-t))}{\sqrt{r^2-t^2}} t\psi(t)\, dt, \qquad \psi \in \mathcal{S}. \tag{3.2}$$

It is clear that $\left(\Phi^{*-1}\psi\right)(t) = \sqrt{\frac{2}{\pi}}\, t \int_0^{\pi/2} \psi(t\sin\alpha)\sin\alpha\, d\alpha$, $\varphi \in \mathcal{S}$, and $\Phi^{*-1}\psi \in \mathcal{S}$, if $\psi \in \mathcal{S}$. Thus

$$\Phi^*(\mathcal{S}) = \mathcal{S} = \Phi^{*-1}(\mathcal{S}).$$

Let the operator $\Phi : \mathcal{S}' \longrightarrow \mathcal{S}'$ be defined by the rule

$$(\Phi f, \varphi) = (f, \Phi^*\varphi), \qquad \varphi \in \mathcal{S},\ f \in \mathcal{S}'.$$

Obviously, Φ^{-1} is defined by

$$\left(\Phi^{-1} f, \varphi\right) = \left(f, \Phi^{*-1}\varphi\right), \qquad \varphi \in \mathcal{S},\ f \in \mathcal{S}'.$$

Thus

$$\Phi(\mathcal{S}') = \mathcal{S}' = \Phi^{-1}(\mathcal{S}').$$

One can easily show that the following three lemmas are true.

Lemma 3.1. *If $f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $\mathcal{S}'$ then $\Phi f_n \longrightarrow \Phi f$ and $\Phi^{-1} f_n \longrightarrow \Phi^{-1} f$ as $n \longrightarrow \infty$ in $\mathcal{S}'$.*

Lemma 3.2. *Let $0 < A \le +\infty$, $f \in \mathcal{S}'$, $\operatorname{supp} f \subset [0, A]$ and $\forall a \in (0, A)$ $f \in L^1(a, A)$. Then $\operatorname{supp} \Phi f \subset [0, A]$ and*

$$(\Phi f)(r) = -\sqrt{\frac{2}{\pi}} \frac{d}{dr} \int_r^A \frac{f(t)\, dt}{\sqrt{t^2-r^2}}, \qquad r \in (0, A).$$

Lemma 3.3. *Let* $0 < A \le +\infty$, $g \in \mathcal{S}'$, $\operatorname{supp} g \subset [0, A]$ *and* $\forall a \in (0, A)$ $g \in L^1(a, A)$. *Then* $\operatorname{supp} \Phi^{-1} g \subset [0, A]$ *and*

$$\left(\Phi^{-1} g\right)(t) = \sqrt{\frac{2}{\pi}} t \int_t^A \frac{g(r)\, dr}{\sqrt{r^2 - t^2}}, \qquad t \in (0, A).$$

4. Auxiliary statements

In this section we denote by $\mathcal{S}_n$ the space of functions $\varphi \in \mathcal{S}$ defined on $\mathbb{R}^n$, if we want to indicate the dimension. For each functional $f \in \mathcal{S}'_1$, $\operatorname{supp} f \subset [0, +\infty)$, we can define $f(|x|) \in \mathcal{S}'_2$ by the rule

$$(f(|x|), \psi(x)) = (f(r), r S_r[\psi]), \tag{4.1}$$

where $S_r[\psi] = \int_0^{2\pi} \psi(r\cos\alpha, r\sin\alpha)\, d\alpha$, $r \in \mathbb{R}$. Obviously, if $\psi \in \mathcal{S}_2$ then $S_r[\psi] \in \mathcal{S}_1$.

To prove conditions for (approximate) null-controllability we need the following four lemmas.

Lemma 4.1. *Let* $T > 0$, $u \in \mathcal{B}(0, T)$, $\mathcal{U}(t) = u(t)\,[H(t) - H(t - T)]$, $\sigma \in \mathbb{R}^2$, $x \in \mathbb{R}^2$. *Then*

$$\mathcal{F}^{-1}\left[i\sigma_2 \int_0^T \begin{pmatrix} -\dfrac{\sin(|\sigma| t)}{|\sigma|} \\ \cos(|\sigma| t) \end{pmatrix} u(t)\, dt \right] = \sqrt{\frac{\pi}{2}} \frac{x_2}{|x|} \Phi \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} (|x|). \tag{4.2}$$

P r o o f. Denote $h(\rho, t) = \begin{pmatrix} -\dfrac{\sin(\rho t)}{\rho} \\ \cos(\rho t) \end{pmatrix} H(\rho)$. We have

$$\mathcal{F}^{-1}\left[i\sigma_2 \int_0^T h(|\sigma|, t) u(t)\, dt \right] = \frac{\partial}{\partial x_2} \mathcal{F}^{-1}\left[\int_0^T h(|\sigma|, t) u(t)\, dt \right]. \tag{4.3}$$

For each $\varphi \in \mathcal{S}_2$ we get

$$\begin{aligned} \left(\mathcal{F}^{-1}\left[\int_0^T h(|\sigma|, t) u(t)\, dt \right], \varphi \right) &= \left(\int_0^T h(\rho, t) u(t)\, dt, \rho S_\rho[\mathcal{F}\varphi] \right) \\ &= \int_0^\infty \left(\int_0^T h(\rho, t) u(t)\, dt \right) \overline{\rho S_\rho[\mathcal{F}\varphi]}\, d\rho, \end{aligned}$$

where $\overline{z}$ means the complex conjugation of z. Since $\rho S_\rho\left[\mathcal{F}\varphi\right] \in \mathcal{S}_1$ we obtain

$$\left(\mathcal{F}^{-1}\left[\int_0^T h(|\sigma|,t)u(t)\,dt\right],\varphi\right) = \int_0^\infty \mathcal{U}(t)\int_0^\infty h(\rho,t)u(t)\overline{\rho S_\rho\left[\mathcal{F}\varphi\right]}\,d\rho\,dt$$
$$= \left(\mathcal{U}(t),\int_0^\infty h(\rho,t)u(t)\rho S_\rho\left[\mathcal{F}\varphi\right]\,d\rho\right)$$
$$= -\left(\begin{pmatrix}\mathcal{U}(t)\\ \mathcal{U}'(t)\end{pmatrix},\int_0^\infty \sin(\rho t)S_\rho\left[\mathcal{F}\varphi\right]\,d\rho\right)$$
$$= -\left(\begin{pmatrix}\mathcal{U}(t)\\ \mathcal{U}'(t)\end{pmatrix},\int_{\mathbb{R}^2} \frac{\sin(|\sigma|t)}{|\sigma|}\left(\mathcal{F}\varphi\right)(\sigma)\,d\sigma\right)$$
$$= -\left(\begin{pmatrix}\mathcal{U}(t)\\ \mathcal{U}'(t)\end{pmatrix},\int_{\mathbb{R}^2} \mathcal{F}^{-1}\left[\frac{\sin(|\sigma|t)}{|\sigma|}\right](x)\varphi(x)\,dx\right). \quad (4.4)$$

With regard to (4.4) and (1.8) that gives

$$\left(\mathcal{F}^{-1}\left[\int_0^T h(|\sigma|,t)u(t)\,dt\right],\varphi\right) = -\left(\begin{pmatrix}\mathcal{U}(t)\\ \mathcal{U}'(t)\end{pmatrix},\int_{-\infty}^\infty \frac{H\left(t(t-r)\right)}{\sqrt{t^2-r^2}}rS_r[\varphi]\,dr\right). \quad (4.5)$$

Consider the operator $\Psi^* : \mathcal{S} \longrightarrow \mathcal{S}$ such that

$$(\Psi^*\mu) = \int_{-\infty}^\infty \frac{H\left(t(t-r)\right)}{\sqrt{t^2-r^2}}\mu(r)\,dr = \int_0^{\pi/2} \mu(t\sin\alpha)\,d\alpha, \qquad \mu \in \mathcal{S}.$$

It is clear that if $\mu \in \mathcal{S}$ then $\Psi^*\mu \in \mathcal{S}$. Denote by Ψ the operator $\Psi : \mathcal{S}' \longrightarrow \mathcal{S}'$ such that

$$\left(\Psi f,\mu\right) = \left(f,\Psi^*\mu\right), \qquad \mu \in \mathcal{S},\ f \in \mathcal{S}'.$$

Evidently, if $\operatorname{supp} f \subset [0,+\infty)$ $(f \in \mathcal{S}')$ then $\operatorname{supp}\Psi f \subset [0,+\infty)$. One can see that $\Phi = -\sqrt{\frac{\pi}{2}}\frac{d}{dr}\Psi$. All this implies that

$$\mathcal{F}^{-1}\left[\int_0^T h(|\sigma|,t)u(t)\,dt\right] = -\frac{\partial}{\partial x_2}\Psi\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix}(|x|) = \sqrt{\frac{\pi}{2}}\frac{x_2}{|x|}\Phi\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix}(|x|).$$

That was to be proved.

Lemma 4.2. *Let* $f \in \mathcal{S}'$, $\operatorname{supp} f \subset [0,+\infty)$ *and* $\forall a > 0$ $f \in L^1(a,+\infty)$. *Then*

$$\left(\Phi \frac{d}{dt} \Phi^{-1} f\right)(r) = \frac{d}{dr}\left[f(r) + \int\limits_{-\infty}^{\infty} f(\xi) k(\xi,r)\, d\xi\right], \tag{4.6}$$

where $k(\xi,r) = \dfrac{2}{\pi} H\left(\xi(\xi - r)\right) \displaystyle\int_0^{\pi/2} \frac{\sin^2\alpha\, d\alpha}{\sqrt{\xi^2 \sin^2\alpha + r^2\cos^2\alpha}}$.

P r o o f. For each $\varphi \in \mathcal{S}$ we have

$$\left(\Phi \frac{d}{dt}\Phi^{-1} f, \varphi\right) = -\left(f, \Phi^{*-1}\frac{d}{dt}\Phi^*\varphi\right). \tag{4.7}$$

With regard to (3.1), (3.2) for $\xi > 0$ we get

$$\begin{aligned}
\left(\Phi^{*-1}\frac{d}{dt}\Phi^*\varphi\right)(\xi) &= \frac{2}{\pi}\int\limits_0^{\xi} \frac{1}{\sqrt{\xi^2 - t^2}} \frac{d}{dt}\left[t\int\limits_0^{t} \frac{\varphi'(r)\, dr}{\sqrt{t^2 - r^2}}\right] dt \\
&= \frac{2}{\pi}\frac{1}{\xi}\frac{d}{d\xi}\int\limits_0^{\xi} \sqrt{\xi^2 - t^2}\frac{d}{dt}\left[t\int\limits_0^{t} \frac{\varphi'(r)\, dr}{\sqrt{t^2 - r^2}}\right] dt \\
&= \frac{2}{\pi}\frac{1}{\xi}\frac{d}{d\xi}\int_0^{\xi} \varphi'(r) \int\limits_r^{\xi} \frac{t^2\, dt}{\sqrt{\xi^2 - t^2}\sqrt{t^2 - r^2}}\, dr \\
&= \frac{2}{\pi}\frac{1}{\xi}\frac{d}{d\xi}\int\limits_0^{\xi} \varphi'(r) \int\limits_0^{\pi/2} \sqrt{\xi^2\sin^2\alpha + r^2\cos^2\alpha}\, d\alpha\, dr \\
&= \varphi'(\xi) + \frac{2}{\pi}\int_0^{\xi} \varphi'(r)\int\limits_0^{\pi/2} \frac{\sin^2\alpha\, d\alpha}{\sqrt{\xi^2\sin^2\alpha + r^2\cos^2\alpha}}\, d\alpha\, dr.
\end{aligned}$$

Taking into account (4.7), we obtain

$$\begin{aligned}
\left(\Phi\frac{d}{dt}\Phi^{-1} f, \varphi\right) &= -\left(f, \varphi'(\xi) + \int\limits_{-\infty}^{\infty} \varphi'(r) k(\xi,r)\, dr\right) \\
&= \left(\frac{d}{dr}\left[f(r) + \int\limits_{-\infty}^{\infty} f(\xi)k(\xi,r)\, d\xi\right], \varphi\right).
\end{aligned} \tag{4.8}$$

Hence (4.6) holds, and the lemma is proved.

Lemma 4.3. *Let* $u \in \mathcal{B}(0,T)$, $\mathcal{U}(t) = u(t)\,[H(t) - H(t-T)]$. *Then* $\operatorname{supp}\Phi\mathcal{U} \subset [0,T]$ *and*

$$|(\Phi\mathcal{U})(r)| \le \frac{\sqrt{2}T}{\sqrt{\pi}r\sqrt{T^2 - r^2}}, \qquad r \in (0,T). \tag{4.9}$$

P r o o f. According to the Lemma 3.2, we obtain $\operatorname{supp}\Phi\mathcal{U} \subset [0,T]$. We also have that

$$(\Phi\mathcal{U})(r) = \sqrt{\frac{2}{\pi}}\frac{d}{dr}\int_r^T \frac{u(t)\,dt}{\sqrt{t^2 - r^2}}, \qquad r \in (0,T).$$

Denote $f_n(r) = \int_r^T \frac{u(t)\,dt}{\sqrt{t^2 - (r-1/n)^2}}$, $f(r) = \int_r^T \frac{u(t)\,dt}{\sqrt{t^2 - r^2}}$ $(r \in (0,T])$.

One can see that

$$f_n(r) \to f(r) \qquad \text{as } n \to \infty, \quad r \in (0,T]. \tag{4.10}$$

First let us prove that $\forall r_0 \in (0,T)$ $\forall \varepsilon \in (0, T - r_0)$ we have

$$f_n'(r) \rightrightarrows f'(r) \qquad \text{as } n \longrightarrow \infty, \text{ on } [r_0, T-\varepsilon]. \tag{4.11}$$

Let $\forall r_0 \in (0,T)$ $\forall \varepsilon \in (0, T - r_0)$ be fixed. We have

$$f_n'(r) = -\frac{u(r)}{\sqrt{r^2 - (r-1/n)^2}} + (r - 1/n)\int_r^T \frac{u(t)\,dt}{\left(t^2 - (r-1/n)^2\right)^{3/2}}. \tag{4.12}$$

Let $n > m > 0$ be large enough. Denote

$$\begin{aligned} g_r(\xi) &= -\frac{u(r)}{\sqrt{r^2 - (r-1/n)^2}} \\ &+ (r - 1/n)\int_r^T \frac{u(t)\,dt}{\left(t^2 - (r-1/n)^2\right)^{3/2}}, \ \xi \in [r - 1/n, r - 1/m]. \end{aligned}$$

Applying the mean value theorem to $g_r(\xi)$ (with respect to ξ), we get

$$\begin{aligned} |f_n(r) - f_m(r)| &= |g_r(r - 1/n) - g_r(r - 1/m)| \\ &\le \sup_{\xi \in [r - \frac{1}{n}, r - \frac{1}{m}]}\left[\frac{2\xi}{(r^2 - \xi^2)^{3/2}} + \int_r^T \frac{t^2 + 2\xi^2}{(r^2 - \xi^2)^{5/2}}\right]\left(\frac{1}{m} - \frac{1}{n}\right) \\ &\le \sup_{\xi \in [r - \frac{1}{n}, r - \frac{1}{m}]}\left[\frac{2\xi\left(r^2 - \xi^2\right) + (T - r)\left(T^2 + 2\xi^2\right)}{(r^2 - \xi^2)^{5/2}}\right]\frac{2}{m} \\ &\le \frac{14T^3}{m^{7/2}r_0} \to 0 \quad \text{as } m \to \infty, \ r \in [r_0, T-\varepsilon]. \end{aligned}$$

With regard to (4.10) we conclude that the consequence $\{f_n'\}_{n=1}^{\infty}$ uniformly converges on $[r_0, T-\varepsilon]$ and (4.11) is true.

Finally let us prove (4.9). Due to (4.12) we have $\forall r \in (0,T)$

$$\begin{aligned} \left|f_n'(r)\right| &\le -\frac{1}{\sqrt{n}(r-1/n)\sqrt{2r-1/n}} \\ &+ \frac{T}{(r-1/n)\sqrt{T^2-(r-1/n)^2}} \to \frac{T}{r\sqrt{T^2-r^2}} \qquad \text{as } n\to\infty. \end{aligned}$$

Taking into account (4.11), we conclude that (4.9) holds that was to be proved.

Lemma 4.4. *If* $f \in H_0^s \times H_0^{s-1}$ *and* $g = \mathcal{F} f$ *then*

$$|\!|\!| E(|x|,t) * f |\!|\!|^s = [\![\Sigma(|\sigma|,t) g]\!]_s \le \sqrt{4t^2+6}\, [\![g]\!]_s = \sqrt{4t^2+6}\, |\!|\!| f |\!|\!|^s, \quad t \in \mathbb{R}. \tag{4.13}$$

P r o o f. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} |\!|\!| E(|x|,t) * f |\!|\!|^s &= [\![\Sigma(|\sigma|,t) g]\!]_s \\ &\le \left[\!\!\left[\begin{pmatrix} \cos(|\sigma|t) \\ -|\sigma|\sin(|\sigma|t) \end{pmatrix} g_0 \right]\!\!\right]_s + \left[\!\!\left[\begin{pmatrix} \frac{\sin(|\sigma|t)}{|\sigma|} \\ \cos(|\sigma|t) \end{pmatrix} g_1 \right]\!\!\right]_s \\ &\le \sqrt{2}\, \|g_0\|_s^0 + \left(\left(\left\| \frac{\sin(|\sigma|t)}{|\sigma|} g_1 \right\|_s^0 \right)^2 + \left(\|g_1\|_{s-1}^0 \right)^2 \right)^{1/2}. \end{aligned}$$

Since $\left(1+|\sigma|^2\right) \left| \frac{\sin(|\sigma|t)}{|\sigma|} \right|^2 \le 2\left(t^2+1\right)$ we obtain (4.13). The lemma is proved.

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