Journal of Mathematical Physics, Analysis, Geometry *Short Notes*
2006, vol. 2, No. 1, pp. 87–93

On the Generalized Solution of the Boundary-Value Problem for the Operator-Differential Equations of the Second Order with Variable Coefficients

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Received, April 23, 2005

Boundary-value problem for a class of operator-differential equations of the second order with variable coefficients on $[0;+\infty)$ is studied. The principal part of investigated operator-differential equation has discontinuities. Sufficient conditions for the existence and uniqueness of generalized solutions of the boundary-value problem for such equations are given. These conditions are expressed only in terms of coefficients of the operator-differential equation.

Key words: generalized solution, boundary-value problem, discontinuous coefficient, selfadjoint operators, Hilbert space.
Mathematics Subject Classification 2000: 47E05, 34B40.

Let H be a separable Hilbert space, A is a selfadjoint positive definite operator in H.

Define the following Hilbert spaces:

$$L_2(R_+;H) = \left\{ u(t): \ \|u\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|u(t)\|_H^2 \, dt \right)^{1/2} < +\infty \right\},$$

$$W_2^2(R_+;H)$$
$$= \left\{ u(t): \ \|u\|_{W_2^2(R_+;H)} = \left(\int_0^{+\infty} \left(\|u''(t)\|_H^2 + \|A^2u(t)\|_H^2 \right) dt \right)^{1/2} < +\infty \right\}$$

(see [1, ch. 1; 2]).

Derivatives will be understood in the sense of generalized functions theory.

Now consider the following boundary-value problem:

$$P(d/dt)u(t) \equiv -u''(t) + \rho(t)A^2u(t)$$
$$+A_1(t)u'(t) + A_2(t)u(t) = f(t), \quad t \in R_+ = [0; +\infty), \tag{1}$$
$$u(0) = 0, \tag{2}$$

where $f(t) \in L_2(R_+;H)$, $A_1(t)$ and $A_2(t)$ are linear, generally speaking, unbounded operators, defined for all $t \in R_+$, moreover, $A_1(t)$ has the strong derivative for each $t \in R_+$ on each element of $D(A)$, and $\rho(t)$ is a scalar positive piecewise constant function.

Assume for simplicity that $\rho(t)$ has discontinuity only at one point, i.e., $\rho(t) = \alpha$, if $0 \le t \le T$ and $\rho(t) = \beta$, if $T < t < +\infty$, where α and β are possitive, and generally speaking, distinct numbers.

Introduce the following notations.

Denote by $D^1(R_+;H)$ the linear set of infinitely differentiable functions with the values in $D(A^2)$, which have compact support in R_+. Introducing the norm

$$\|u\|_{W_2^1(R_+;H)} = \left(\|u'\|^2_{L_2(R_+;H)} + \|Au\|^2_{L_2(R_+;H)} \right)^{1/2},$$

we obtain a pre-Hilbert space, whose completion we denote by $W_2^1(R_+;H)$ (see [1, ch. 1, p. 23–24]).

Denote by $\overset{\circ}{W}{}_2^1(R_+;H)$ the Hilbert space

$$\overset{\circ}{W}{}_2^1(R_+;H) = \{u(t): \quad u(t) \in W_2^1(R_+;H), \quad u(0) = 0\},$$

$L(X;Y)$ — the set of linear bounded operators, acting from Hilbert space X into the other Hilbert space Y, and $L_\infty(R_+;B)$ — the set of B-valued essentially bounded operator-functions in R_+, where B is Banach space.

First of all let us formulate the following lemma, which has auxiliary character.

Lemma 1. *Let A be a selfadjoint positive definite operator in H, $A_1(t)A^{-1}$, $A^{-1}A_1'(t)A^{-1}$ and $A^{-1}A_2(t)A^{-1} \in L_\infty(R_+;L(H;H))$. Then the bilinear form*

$$\mathcal{P}_1(u,\psi) \equiv (P_1(d/dt)u,\psi)_{L_2(R_+;H)} \equiv \left(A_1(t)u' + A_2(t)u,\psi\right)_{L_2(R_+;H)},$$

defined for all vector-functions $u(t) \in D^1(R_+;H)$ and $\psi(t) \in \overset{\circ}{W}{}_2^1(R_+;H)$, can be extended on the space $W_2^1(R_+;H) \oplus \overset{\circ}{W}{}_2^1(R_+;H)$ by continuity. The extension $\widetilde{\mathcal{P}_1}(u,\psi)$ acts by the following way:

$$\widetilde{\mathcal{P}_1}(u,\psi) = -\left(A_1(t)u,\psi'\right)_{L_2(R_+;H)} - \left(A_1'(t)u,\psi\right)_{L_2(R_+;H)} + (A_2(t)u,\psi)_{L_2(R_+;H)}.$$

P r o o f. Since $u(t) \in D^1(R_+;H)$, $\psi(t) \in \overset{\circ}{W}{}_2^1(R_+;H)$, then integrating by parts the corresponding item, we obtain that

$$\mathcal{P}_1(u,\psi) = \left(A_1(t)u' + A_2(t)u,\psi\right)_{L_2(R_+;H)}$$
$$= \left(A_1(t)u',\psi\right)_{L_2(R_+;H)} + (A_2(t)u,\psi)_{L_2(R_+;H)}$$
$$= -\left(A_1(t)u,\psi'\right)_{L_2(R_+;H)} - \left(A_1'(t)u,\psi\right)_{L_2(R_+;H)} + (A_2(t)u,\psi)_{L_2(R_+;H)}.$$

On the other hand, from theorem on intermediate derivatives we have

$$\left|\left(A_1(t)u,\psi'\right)_{L_2(R_+;H)}\right| = \left|\left(A_1(t)A^{-1}Au,\psi'\right)_{L_2(R_+;H)}\right|$$
$$\le \sup_t \left\|A_1(t)A^{-1}\right\|_{H\to H} \|Au\|_{L_2(R_+;H)} \left\|\psi'\right\|_{L_2(R_+;H)}$$
$$\le \sup_t \left\|A_1(t)A^{-1}\right\|_{H\to H} \|u\|_{W_2^1(R_+;H)} \|\psi\|_{W_2^1(R_+;H)}.$$

Analogously we obtain

$$\left|\left(A_1'(t)u,\psi\right)_{L_2(R_+;H)}\right| = \left|\left(A^{-1}A_1'(t)A^{-1}Au,A\psi\right)_{L_2(R_+;H)}\right|$$
$$\le \sup_t \left\|A^{-1}A_1'(t)A^{-1}\right\|_{H\to H} \|Au\|_{L_2(R_+;H)} \|A\psi\|_{L_2(R_+;H)}$$
$$\le \sup_t \left\|A^{-1}A_1'(t)A^{-1}\right\|_{H\to H} \|u\|_{W_2^1(R_+;H)} \|\psi\|_{W_2^1(R_+;H)},$$

$$\left|(A_2(t)u,\psi)_{L_2(R_+;H)}\right| = \left|\left(A^{-1}A_2(t)A^{-1}Au,A\psi\right)_{L_2(R_+;H)}\right|$$
$$\le \sup_t \left\|A^{-1}A_2(t)A^{-1}\right\|_{H\to H} \|Au\|_{L_2(R_+;H)} \|A\psi\|_{L_2(R_+;H)}$$
$$\le \sup_t \left\|A^{-1}A_2(t)A^{-1}\right\|_{H\to H} \|u\|_{W_2^1(R_+;H)} \|\psi\|_{W_2^1(R_+;H)}.$$

As the set $D^1(R_+;H)$ is dense in the space $W_2^1(R_+;H)$ (see [1, ch. 1]), then $\mathcal{P}_1(u,\psi)$ is extended on the space $W_2^1(R_+;H)\oplus \overset{\circ}{W}{}_2^1 (R_+;H)$ by continuity. Lemma is proved.

Definition 1. *If vector-function $u(t) \in W_2^1(R_+;H)$ satisfies condition (2) and for any $\psi(t) \in \overset{\circ}{W}{}_2^1 (R_+;H)$ the identity*

$$\left(u',\psi'\right)_{L_2(R_+;H)} + \left(\rho^{1/2}(t)Au, \rho^{1/2}(t)A\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u,\psi) = (f,\psi)_{L_2(R_+;H)}$$

is fulfilled, then $u(t)$ is called the generalized solution of the boundary-value problem (1), (2).

Note that the conditions, providing correct and unique solvablity of the boundary-value problem (1), (2) in the space $W_2^2(R_+;H)$, are given in work [3] in terms of the operator coefficients of the equation (1). For $\rho(t) \equiv 1$, $t \in R_+$ the boundary-value problems for equation (1) with constant operator coefficients are extensively studied in [4]. This case is also considered in work [5], investigating the existence of generalized solutions for the conditions, which are different from the conditions of [4], moreover, A^{-1} is assumed to be a compact operator in H.

Sufficient conditions on the coefficients of operator-differential equation (1), providing the existence and uniqueness of generalized solutions of the boundary-value problem (1), (2) are obtained in the present work.

Before we formulate a theorem on the existence and uniqueness of generalized solution of the problem (1), (2), let us consider the equation, presenting the principal part of (1):

$$P_0(d/dt)u(t) \equiv -u''(t) + \rho(t)A^2u(t) = f(t), \quad t \in R_+. \tag{3}$$

Theorem 1. *Equation (3) with boundary condition (2) has unique generalized solution.*

We will outline briefly the proof of this theorem.

The validity of the statement follows from the fact that in work [3] there is theorem on the existence of unique solution $u_0(t)$ from the space $W_2^2(R_+;H)$ of the boundary-value problem (3), (2). Since $W_2^2(R_+;H) \subset W_2^1(R_+;H)$ (see [1, ch. 1]), then $u_0(t) \in W_2^1(R_+;H)$ and it is not difficult to verify that

$$\left(u_0',\psi'\right)_{L_2(R_+;H)} + \left(\rho^{1/2}(t)Au_0, \rho^{1/2}(t)A\psi\right)_{L_2(R_+;H)} = (f,\psi)_{L_2(R_+;H)},$$

hence the vector-function $u_0(t)$ is also the generalized solution of problem (3), (2).

Now let us give the main result of this paper.

Theorem 2. *Let A be a selfadjoint positive-definite operator in H, $A_1(t)A^{-1}$, $A^{-1}A_1'(t)A^{-1}$, $A^{-1}A_2(t)A^{-1} \in L(H,H)$ and the inequality*

$$\omega = \frac{1}{2}\sup_t \left\| A_1(t)A^{-1} \right\| + \sup_t \left\| A^{-1}A_1'(t)A^{-1} \right\|$$

$$+ \sup_t \left\| A^{-1}A_2(t)A^{-1} \right\| < \min(1;\alpha;\beta)$$

is satisfied. Then problem (1), (2) has unique generalized solution.

P r o o f. First of all we show that for $\omega < \min(1;\alpha;\beta)$ for any $\psi(t) \in \overset{\circ}{W}{}_2^1 (R_+;H)$ the following inequality is valid:

$$\left| (P(d/dt)\psi,\psi)_{L_2(R_+;H)} \right| \geq (\min(1;\alpha;\beta) - \omega) \left\| \psi \right\|^2_{W_2^1(R_+;H)}. \tag{4}$$

As

$$\left| (P(d/dt)\psi,\psi)_{L_2(R_+;H)} \right| \geq \left| (P_0(d/dt)\psi,\psi)_{L_2(R_+;H)} \right| - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$= \left| \left(-\psi'',\psi\right)_{L_2(R_+;H)} + \left(\rho(t)A^2\psi,\psi\right)_{L_2(R_+;H)} \right| - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$= \left| \left(\psi',\psi'\right)_{L_2(R_+;H)} + \left(\rho^{1/2}A\psi,\rho^{1/2}A\psi\right)_{L_2(R_+;H)} \right| - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$= \left\| \psi' \right\|^2_{L_2(R_+;H)} + \left\| \rho^{1/2}(t)A\psi \right\|^2_{L_2(R_+;H)} - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$\geq \left\| \psi' \right\|^2_{L_2(R_+;H)} + \min(\alpha;\beta) \left\| A\psi \right\|^2_{L_2(R_+;H)} - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$\geq \min(1;\alpha;\beta) - \left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right|, \tag{5}$$

then taking into consideration the form of $(P_1(d/dt)\psi,\psi)_{L_2(R_+;H)}$, we obtain

$$\left| (P_1(d/dt)\psi,\psi)_{L_2(R_+;H)} \right| = \left| \left(A_1(t)\psi' + A_2(t)\psi,\psi\right)_{L_2(R_+;H)} \right|$$

$$= \left| \left(A_1(t)\psi',\psi\right)_{L_2(R_+;H)} + (A_2(t)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$= \left| -\left(A_1(t)\psi,\psi'\right)_{L_2(R_+;H)} - \left(A_1'(t)\psi,\psi\right)_{L_2(R_+;H)} + (A_2(t)\psi,\psi)_{L_2(R_+;H)} \right|$$

$$\leq \left| \left(A_1(t)\psi,\psi'\right)_{L_2(R_+;H)} \right| + \left| \left(A_1'(t)\psi,\psi\right)_{L_2(R_+;H)} \right| + \left| (A_2(t)\psi,\psi)_{L_2(R_+;H)} \right|.$$

On the other hand, applying the Bunyakovsky-Schwarz inequality and Hilbert inequality, we have

$$\begin{aligned}\left|\left(A_1(t)\psi,\psi'\right)_{L_2(R_+;H)}\right| &= \left|\left(A_1(t)A^{-1}A\psi,\psi'\right)_{L_2(R_+;H)}\right| \\ &\le \sup_t \left\|A_1(t)A^{-1}\right\|_{H\to H} \|A\psi\|_{L_2(R_+;H)} \left\|\psi'\right\|_{L_2(R_+;H)} \\ &\le \frac{1}{2}\sup_t \left\|A_1(t)A^{-1}\right\|_{H\to H} \left[\|A\psi\|^2_{L_2(R_+;H)} + \left\|\psi'\right\|^2_{L_2(R_+;H)}\right] \\ &= \frac{1}{2}\sup_t \left\|A_1(t)A^{-1}\right\|_{H\to H} \|\psi\|^2_{W_2^1(R_+;H)},\end{aligned}$$

$$\begin{aligned}\left|\left(A_1'(t)\psi,\psi\right)_{L_2(R_+;H)}\right| &= \left|\left(A^{-1}A_1'(t)A^{-1}A\psi,A\psi\right)_{L_2(R_+;H)}\right| \\ &\le \sup_t \left\|A^{-1}A_1'(t)A^{-1}\right\|_{H\to H} \|A\psi\|^2_{L_2(R_+;H)} \\ &\le \sup_t \left\|A^{-1}A_1'(t)A^{-1}\right\|_{H\to H} \|\psi\|^2_{W_2^1(R_+;H)},\end{aligned}$$

$$\begin{aligned}\left|\left(A_2(t)\psi,\psi\right)_{L_2(R_+;H)}\right| &= \left|\left(A^{-1}A_2(t)A^{-1}A\psi,A\psi\right)_{L_2(R_+;H)}\right| \\ &\le \sup_t \left\|A^{-1}A_2(t)A^{-1}\right\|_{H\to H} \|\psi\|^2_{W_2^1(R_+;H)}.\end{aligned}$$

Now taking into consideration last inequalities in (5), we obtain inequality (4).

Then by Theorem 1 problem (3), (2) has unique generalized solution $u_0(t)$. Writing the generalized solution of problem (1), (2) in the form $u(t) = u_0(t) + u_1(t)$, we have for $u_1(t)$

$$\left(-u_0'' + \rho(t)A^2u_0,\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u_0,\psi) + \left(-u_1'' + \rho(t)A^2u_1,\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u_1,\psi) = (f,\psi)_{L_2(R_+;H)}.$$

This implies

$$\left(u_0',\psi'\right)_{L_2(R_+;H)} + \left(\rho^{1/2}(t)Au_0,\rho^{1/2}(t)A\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u_0,\psi) + (u_1',\psi')_{L_2(R_+;H)} + \left(\rho^{1/2}(t)Au_1,\rho^{1/2}(t)A\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u_1,\psi) = (f,\psi)_{L_2(R_+;H)},$$

and finally we obtain

$$\left(u_1',\psi'\right)_{L_2(R_+;H)} + \left(\rho^{1/2}(t)Au_1,\rho^{1/2}(t)A\psi\right)_{L_2(R_+;H)} + \mathcal{P}_1(u_1,\psi) = -\mathcal{P}_1(u_0,\psi). \tag{6}$$

As we can see, the right hand side of (6) determines the continuous form in $W_2^1(R_+;H) \oplus \overset{\circ}{W}{}_2^1 (R_+;H)$, and the left hand side, satisfies the conditions of Lax–Milgram theorem (see [6, Part II]) in view of (4). That is why there exists a unique vector-function $u_1(t) \in \overset{\circ}{W}{}_2^1 (R_+;H)$, satisfying the equality (6), i.e., $u(t) = u_0(t) + u_1(t)$ is the generalized solution of the problem (1), (2). Theorem is proved.

R e m a r k 1. Note that the analogous analysis can be done for the boundary-value problem (1), (2) in the case, if $\rho(t)$ is any positive function, having the finite number of discontinuity points of the first order.

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