Journal of Mathematical Physics, Analysis, Geometry *Short Notes*
2006, vol. 2, No. 1, pp. 104–110

Minimal Surfaces in Standard Three-Dimensional Geometry Sol^3

L.A. Masaltsev

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine

E-mail:masaltsev@univer. kharkov.ua

Received, April 19, 2005

We study minimal and totally geodesic surfaces in the standart three-dimensional geometry Sol^3 with the left-invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$.

Key words: minimal surface, totally geodesic surface, Sol-manifold.
Mathematics Subject Classification 2000: 53A10, 53C42.

A three-dimensional geometry Sol^3 can be presented as matrix group

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix},$$

homeomorphic to R^3 with the left-invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ [1, p. 127]. Its group of isometries is of dimension 3, consists from 8 components, and the component of unit $e = (0, 0, 0)$ coincides with Sol^3, acting by left translations. The stabilizer of origin consists of 8 linear transformations of space R^3 taking the form $(x, y, z) \to (\pm x, \pm y, z)$ and $(x, y, z) \to (\pm y, \pm x, -z)$. These eight transformations are isomorphisms and isometries of group Sol^3. In this note we find some examples of ruled minimal surfaces and minimal surfaces, invariant under the action of some 1-parameter group of isometris of Sol^3. The techniques of finding ruled minimal surfaces is similar to those, wich we used in [2], but in contrast to geometry Nil^3, in geometry Sol^3 there is a family of totally geodesic surfaces.

Work is partialy supported with the grant DFFD of Ministery of education and science of Ukraine 01.07/00132.

1. Minimal ruled surfaces in geometry Sol^3

Acting as in [2], we at first write down the system of ordinary differential equations for geodesics of Sol^3:

$$x''(t) + 2x'(t)z'(t) - 0,$$

$$y''(t) - 2y'(t)z'(t) - 0,$$

$$z''(t) - e^{2z}(x')^2 + e^{-2z}(y')^2 = 0.$$

The obvious solutions to the system are 1) "vertical" geodesics ($x = x_0, y = y_0, z = t$), 2) "horizontal" geodesics ($x = \pm\frac{1}{\sqrt{2}}e^{-z_0}t + x_0, y = \pm\frac{1}{\sqrt{2}}e^{z_0}t + y_0, z = z_0$)). Find at first all ruled minimal surfaces, composed of "vertical" geodesics: $r(s,t) = (x(s), y(s), t)$. Compute the first and the second fundamental forms of this surface.

Proposition 1. *1) The first fundamental form of the surface $r(s,t) = (x(s), y(s), t)$ in the geometry Sol^3 is*

$$I = (e^{2t}(x'_s)^2 + e^{-2t}(y'_s)^2)ds^2 + dt^2;$$

2) the second fundamental form of the surface is

$$II = \frac{(x''y' - x'y'')ds^2 + 4x'y'dsdt}{(e^{2t}(x')^2 + e^{-2t}(y')^2)^{1/2}}.$$

P r o o f. The nonzero Cristoffel symbols of Sol^3 metric are $\Gamma^1_{13} = 1$, $\Gamma^2_{23} = -1$, $\Gamma^3_{11} = -e^{2z}$, $\Gamma^3_{22} = e^{-2z}$. The tangent vectors to the surface are $r_s = (x', y', 0)$, $r_t = (0,0,1)$, and from here we easily obtain the first fundamental form. The normal vector is $n = \frac{(e^{-2t}y', -e^{2t}x', 0)}{(e^{2t}x'^2 + e^{-2t}y'^2)^{1/2}}$. The coefficients of the second fundamental form can be computed using formulas (43.4), (43.5) from [3, p. 180], which for given surface in Sol^3 take the form (latin indices vary from 1 to 2, and greek indices from 1 to 3):

$$b_{ij} = e^{2t}n^1(r^1_{ij} + \Gamma^1_{\mu\nu}r^\mu_{,i}r^\nu_{,j}) + e^{-2t}n^2(r^2_{ij} + \Gamma^2_{\mu\nu}r^\mu_{,i}r^\nu_{,j}) + n^3(r^3_{ij} + \Gamma^3_{\mu\nu}r^\mu_{,i}r^\nu_{,j}).$$

Corollary 1. *The ruled minimal surfaces, composed from 'vertical' geodesics in Sol^3, are the surfaces of the form $r(s,t) = (s, as+b, t)$ or $r(s,t) = (as+b, s, t)$, where a, b — arbitrary constants.*

P r o o f. For the surface, composed from 'vertical' geodesics, minimality condition $2H = b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11} = 0$ takes the form $x''y' - x'y'' = 0$, whence the statement follows.

Corollary 2. *The totally geodesic surfaces in Sol^3, composed from "vertical" geodesics are the surfaces of the form $r(s,t) = (s,b,t)$ and $r(s,t) = (a,s,t)$.*

P r o o f. It must be fulfilled the condition $b_{11} = b_{12} = 0$, whence the statement follows.

Remark, that isometries of Sol^3 of the form $(x,y,z) \to (y,x,-z)$ transform "vertical" totally geodesic surfaces, "parallel" to xOz to the "vertical" totally geodesic surfaces, "parallel" to yOz.

Proposition 2. *Arbitrary minimal surface, composed from "vertical" geodesics, is stable.*

P r o o f. For the surface in Sol^3 with parametrization $r(s,t) = (s, as+b, t)$ the coefficients of the first fundamental form are $g_{11} = e^{2t} + a^2e^{-2t}$, $g_{12} = 0$, $g_{22} = 1$. The coefficients of the second fundamental form are $b_{11} = b_{22} = 0$, $b_{12} = \frac{2a}{(e^{2t}+a^2e^{-2t})^{1/2}}$. Nonzero components of Rimann tensor of geometry Sol^3 are $R_{1212} = 1$, $R_{1313} = -e^{2z}$, $R_{2323} = -e^{-2z}$. The unique nonzero component of Ricci tensor of geometry Sol^3 is $R_{33} = -2$. Since the normal to studed surface is of the form $n = \frac{(ae^{-2t}, -e^{2t}, 0)}{(e^{2t}+a^2e^{-2t})^{1/2}}$, the Ricci curvature in the normal direction is $Ric(n,n) = R_{\alpha\beta}n^\alpha n^\beta = 0$. The norm of squared second fundamental $||b||^2$ (the sum of squared principal curvatures) is $||b||^2 = \frac{8a^2}{(e^{2t}+a^2e^{-2t})^2}$. For the Laplace–Beltrami Δ_M operator of the surface we obtain the following expression:

$$\Delta_M = \frac{1}{e^{2t}+a^2e^{-2t}}\left(\frac{\partial^2}{\partial s^2} + (e^{2t} - a^2e^{-2t})\frac{\partial}{\partial t}\right) + \frac{\partial^2}{\partial t^2}.$$

Hence, for the Jacobi operator $L = \Delta_M + Ric(n,n) + ||b||^2$ we find the expression

$$L = \frac{1}{e^{2t}+a^2e^{-2t}}\left(\frac{\partial^2}{\partial s^2} + (e^{2t} - a^2e^{-2t})\frac{\partial}{\partial t}\right) + \frac{\partial^2}{\partial t^2} + \frac{8a^2}{(e^{2t}+a^2e^{-2t})^2}.$$

It is directly checked that the following positive function $f(t) = (e^{2t} + a^2e^{-2t})^{-1/2}$ solves the equation $Lf = 0$. Then according to theorem of Fisher–Colbrie–Schoen [4, Th. 1] the studed minimal surface is stable.

To solve the problem of classification of all totally geodesic surfaces in the geometry Sol^3 we need the expressions for the coefficients of the first and second fundamental forms of the surface $r(x,y) = (x,y,z(x,y))$, which has nondegenerate projection on the plane xOy. In this case the tangent vectors and normal to the surface are $r_x = (1,0,z_x)$, $r_y = (0,1,z_y)$, $n = \frac{(-z_xe^{-2z}, -z_ye^{2z}, 1)}{(z_x^2e^{-2z}+z_y^2e^{2z}+1)1/2}$. The coefficients of the first and second fundamental forms are

$$g_{11} = e^{2z} + z_x^2,\ g_{12} = z_xz_y,\ g_{22} = e^{-2z} + z_y^2,$$

$$b_{11} = \frac{z_{xx} - 2z_x^2 - e^{2z}}{(z_x^2 e^{-2z} + z_y^2 e^{2z} + 1)^{1/2}}\,,\ b_{12} = \frac{z_{xy}}{(z_x^2 e^{-2z} + z_y^2 e^{2z} + 1)^{1/2}}\,,$$

$$b_{22} = \frac{z_{yy} + 2z_y^2 + e^{-2z}}{(z_x^2 e^{-2z} + z_y^2 e^{2z} + 1)^{1/2}}\,.$$

Proposition 3. *There is no totally geodesic surface in the geometry* Sol^3 *with nondegenerate projection to the* xOy.

P r o o f. Suppose that there exists the totally geodesic surface in the form $(x, y, z(x, y))$. Then the condition $b_{12} = 0$ implies $z(x, y) = \phi(x) + \psi(y)$. The conditions $b_{11} = b_{22} = 0$ yield the following system for $\phi(x)$ and $\psi(y)$:

$$\phi_{xx} - 2\phi_x^2 - e^{2(\phi+\psi)} = 0,\ \psi_{yy} + 2\psi_y^2 + e^{-2(\phi+\psi)} = 0.$$

It can be rewritten in the form

$$(e^{-2\phi(x)})''_{xx} = -2e^{\psi(y)},\ (e^{2\psi(y)})''_{yy} = -2e^{-2\phi(x)}.$$

Since the left hand side of the second equation does not depend of x, we can differentiate it two times by x, getting $(e^{-2\phi})''_{xx} = 0$, but then the first equation takes the form $-2e^{2\psi(y)} = 0$, that is impossible.

We will find now all ruled minimal surfaces composed of "horizontal" geodesics. This surface admits the parametrization

$$x(s, t) = \frac{1}{\sqrt{2}} e^{-z(s)} t + a(s),\ y(s, t) = \frac{1}{\sqrt{2}} e^{z(s)} t + b(s),\ z(s, t) = z(s).$$

The problem consists in finding of triple of unknown functions $(a(s), b(s), z(s))$, which yield minimal surface in the geometry Sol^3. Note, that by virtue of mentioned dihedral isometries it is sufficient to restrict search to the case of the pointed out surfaces. The tangent vectors and normal to the surface are

$$r'_s = (-\frac{1}{\sqrt{2}} e^{-z} z' t + a', \frac{1}{\sqrt{2}} e^{z} z' t + b', z'),\ r'_t = (\frac{1}{\sqrt{2}} e^{-z}, \frac{1}{\sqrt{2}} e^{z}, 0),$$

$$n = \frac{(z', -e^{2z} z',\ \sqrt{2} e^{z} z' t - a' e^{2z} + b')}{A},$$

where $A = \left(2e^{2z} z'^2 + (\sqrt{2} e^{z} z' t - a' e^{2z} + b')^2\right)^{1/2}$.

The calculations yield the following values for the coefficients of the first and second fundamental forms:

$$g_{12} = \frac{1}{\sqrt{2}} (e^{z} a' + e^{-z} b'),\ g_{22} = 1,$$

$$b_{11} = \frac{e^z}{A}(z''(b'e^{-z} - a'e^z) + 2z'^2(a'e^z + b'e^{-z})$$

$$+(a''e^z - b''e^{-z})z' + (e^z a' + e^{-z}b')(\sqrt{2}z't - a'e^z + b'e^{-z})^2),$$

$$b_{12} = \frac{\sqrt{2}e^z}{A}(z't - \frac{e^z a'}{\sqrt{2}} + \frac{e^{-z}b'}{\sqrt{2}})^2, \ b_{22} = 0.$$

The minimality condition $2H = b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11} = 0$ leads to the equation

$$z''(b'e^{-z} - a'e^z) + (a''e^z - b''e^{-z})z' + (a'e^z + b'e^{-z})(2z'^2 + (\sqrt{2}z't - a'e^z + b'e^{-z})^2)$$

$$= 2(a'e^z + b'e^{-z})(z't + b'e^{-z} - a'e^z)^2$$

After conversion we get linear by variable t equation:

$$z''(b'e^{-z} - a'e^z) + (a''e^z - b''e^{-z})z'$$

$$+(e^z a' + e^{-z}b')(2z'^2 + 2(\sqrt{2} - 2)(b'e^{-z} - a'e^z)z't - (b'e^{-z} - a'e^z)^2) = 0.$$

From here follows, that it must be fulfilled the system of two equations, getting by setting equal to zero the coefficient by t and constant term:

$$z'(a'e^z + b'e^{-z})(a'e^z - b'e^{-z}) = 0,$$

$$z''(b'e^{-z} - a'e^z) + (a''e^z - b''e^{-z})z' + (a'e^z + b'e^{-z})(2z'^2 - (b'e^{-z} - a'e^z)^2) = 0.$$

The analysis of the system gives that 1) if $z' = 0$, the solution is complete minimal surface $z = z_0$ (analog of the plane), 2) if $a'e^z - b'e^{-z} = 0$, then differentiate this relation we get $a''e^z - b''e^{-z} = -z'(a'e^z + b'e^{-z})$.

Then the second equation of the system takes the form $z'^2(a'e^z + b'e^{-z}) = 0$, if $z' \neq 0$, we may assume that $z(s) = s$, and then we find that $a = a_0, b = b_0$ — arbitrary constants. The solution obtained we can write in the form

$$x(s,t) = \frac{1}{\sqrt{2}}e^{-s}t + a_0, \ y(s,t) = \frac{1}{\sqrt{2}}e^{s}t + b_0, \ z(s,t) = s. \tag{1}$$

Finally, let us consider the last possibility 3) $a'e^z + b'e^{-z} = 0$. Differentiating this relation, we get $a''e^z = -b''e^{-z} + 2b'z'e^{-z}$. Substituting it to the second equation of the system, we get the following equation $z''b' - z'b'' + z'^2b' = 0$. Integrating it we find $z' = \frac{b'}{b+c}$. Whence, integrating it once more, we get $b = c_1e^z + c_2$. Then from the relation $a' = -e^{-2z}b'$ we find $a = c_1e^{-z} + c_3$. Hence the solution we get in the form $x(s,t) = \frac{1}{\sqrt{2}}e^{-z(s)}t + c_1e^{-z} + c_3$, $y(s,t) = \frac{1}{\sqrt{2}}e^{z(s)}t + c_1e^z + c_2$, $z(s,t) = z(s)$.

It is evident that if we introduce new variables $\bar{s} = z(s)$, $\bar{t} = t + \sqrt{2}c_1$, then we get the parametrization (1), that we find earlier in the case 2). Hence, we have proved the following statement.

Proposition 4. *Arbitrary complete minimal ruled surface composed from "horizontal" geodesics is either analog of the "plane" $z = z_0$, or the analog of the "helicoid" with the parametrization (1) (as well the surfaces obtained from they by dihedral isometries in Sol^3) .*

2. Minimal surfaces in Sol^3, invariant under action of 1-parameter subgroup

It is known that if the metric on the Lie group is biinvariant then every 1-parameter subgroup is geodesic with respect to the Levi–Civita connection [5, p. 184]. In the case of Sol^3 considered left-invariant metric is not biinvariant, so in general not every 1-parameter subgroup is geodesic. The law of multiplication in the Sol^3 can be written in the form

$$(x, y, z)(x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z').$$

The basis of the Lie algebra sol^3 consists of the vectors $e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with brackets $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$.

Denote by $G_{a,b}(t)$ the 1-parameter subgroup $exp(ae_1 + be_2)t = E + (ae_1 + be_2)t$. Consider the surface in Sol^3 , generated with the aid of curve $r(s) = (s, 0, z(s))$ in the following way:

$$R(s, t) = G_{a,b}(t)r(s) = (at, bt, 0)(s, 0, z(s)) = (at + s, bt, z(s)).$$

It is evident, that the surface $R(s, t)$ is invariant under the action of the group $G_{a,b}$, that is $G_{a,b}(\bar{t})R(s, t) = G_{a,b}(\bar{t} + t)R(s, 0)$.

Note that 1-parameter subgroup $G_{a,b}(t)$, in general, is not a geodesic of Sol^3, excepting the case $|a| = |b| = \frac{1}{\sqrt{2}}$, when we get "horizontal" geodesic. We will find minimal surfaces $R(s, t)$ in Sol^3, invariant under the action of subgroup $G_{a,b}(t)$. The tangent vectors and normal to the surface $R(s, t)$ are

$$R_s = (1, 0, z'),\ R_t = (a, b, 0),\ n = \frac{(bz'e^{-2z}, -az'e^{2z}, -b)}{B},$$

where $B = (a^2z'^2e^{2z} + b^2z'^2e^{-2z} + b^2)^{1/2}$.

Calculation of the first and second fundamental forms yields

$$g_{11} = e^{2z} + z'^2, \ g_{12} = ae^{2z}, \ g_{22} = a^2e^{2z} + b^2e^{-2z},$$

$$b_{11} = \frac{b}{B}(2z'^2 - z'' + e^2z), \ b_{12} = \frac{ab}{B}(2z'^2 + e^{2z}), \ b_{22} = -\frac{b}{B}(-a^2e^{2z} + b^2e^{-2z}).$$

Minimality condition $2H = g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11} = 0$ leads to the equation

$$z''(a^2e^{2z} + b^2e^{-2z}) + z'^2(a^2e^{2z} - b^2e^{-2z}) = 0.$$

Integrating it, we get $z'^2(a^2e^{2z} + b^2e^{-2z}) = c$, and further on $\int \sqrt{a^2e^{2z} + b^2e^{-2z}}dz = cs$.

So the following statement is valid.

Proposition 5. *Minimal surfaces in Sol^3, invariant under the action of the subgroup $exp(ae_1 + be_2)t$, admit the parametrization*

$$R(s,t) = (at + s, bt, z(s)),$$

where the function $z(s)$ can be found from the equation $\int \sqrt{a^2e^{2z} + b^2e^{-2z}}dz = cs$.

R e m a r k. The minimal surface equation in Sol^3, which admits nondegenerate projection on xOy, takes the form

$$(e^{-2z} + z_y^2)z_{xx} - 2z_xz_yz_{xy} + (e^{2z} + z_x^2)z_{yy} - e^{-2z}z_x^2 + e^{2z}z_y^2 = 0.$$

The analog of helicoid, founded in Sect. 1, admits in the domain $(x > 0, y > 0)$ the parametrization $(x, y, \frac{1}{2}ln(\frac{y}{x}))$, and among the surfaces discussed in 2, there are $(x, y, -ln(c - x))$, which obtained, taking $a = 0$, $b = 1$.

References

[1] *P. Scott*, The geometries of 3-manifolds. — *Bull. London Math. Soc.* **15** (1983), No. 56, 401–487.

[2] *L.A. Masaltsev and E.V. Petrov*, Minimal surfaces in Heisenberg group. — *Visnyk Kharkivs'kogo national'nogo universytetu. Ser. Mat., prikl. mat., mekh.* **602** (2003), 35–45. (Russian)

[3] *L.P. Eisenhart*, Riemannian geometry. Princeton univ., Princeton, 1926.

[4] *D. Fisher-Colbrie and R. Schoen*, The structure of complete stable minimal surface in 3-manifold of nonnegative scalar curvature. — *Comm. Pure Appl. Math.* **33** (1980), 199–211.

[5] *V.V. Trofimov*, Introduction to the geometry of manifolds with symmetries. Publ. MSU, Moscow, 1989. (Russian)