Journal of Mathematical Physics, Analysis, Geometry
2006, vol. 2, No. 2, pp. 138–148

On Pseudospherical Surfaces in E^4 with Grassmann Image of Prescribed Type

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Received October 5, 2005

It is demonstrated that there exist surfaces of constant negative Gauss curvature in E^4 whose Grassmann image consists of either hyperbolic or parabolic or elliptic points. As a consequence, there exist surfaces of constant negative Gauss curvature in E^4 which do not admit Backlund transformations with help of pseudospherical congruencies. A geometric representation for pseudospherical surfaces in E^4 with parabolic Grassmann image is proposed.

Key words: pseudospherical surface, Grassman image.
Mathematics Subject Classification 2000: 53A07, 53A25, 53B25.

1. Introduction

The theory of surfaces with constant negative Gauss curvature $K = -1$ in three-dimensional Euclidean space E^3 is one of the most attractive branches of the classical differential geometry. Initially the pseudospherical surfaces in E^3 were of a great interest for many geometers since each surface with constant negative Gauss curvature realizes locally the hyperbolic geometry. Another important reason for the studying of pseudospherical surfaces was the discovery of elegant geometric transformations constructed and studied by L. Bianchi, A. Backlund, G. Darboux and others. The corresponding geometric construction based on the notion of *pseudospherical congruencies* in E^3 has various nontrivial properties, we shall mention two of them. Firstly, if two surfaces in E^3 are connected by a pseudospherical congruence, then both surfaces are of the same constant negative Gauss curvature. Secondly, an arbitrary pseudospherical surface in E^3 admits a large family of Backlund transformations. So an iteration of Backlund transformations generates a family of pseudospherical surfaces from a given one [1–3].

The pseudospherical surfaces in E^3 may be interpreted via solutions of the well-known sine-Gordon equation $\partial_{xy}\varphi = \sin\varphi$, and the geometric Backlund transformations of pseudospherical surfaces correspond to analytic transformations of solutions of SGE. This interpretation resulted in the general fundamental idea of Backlund transformations for solutions of nonlinear partial differential equations (integrable systems) [1–3].

Large majority of results from the classical theory of pseudospherical surfaces in E^3 and their Backlund transformations were generalized for n-dimensional pseudospherical submanifolds in E^{2n-1} [4–7].

An attempt to generalize the mentioned constructions and ideas for two-dimensional pseudospherical surfaces in *four-dimensional* Euclidean space E^4 necessarily leads to the consideration of *Cartan* surfaces. By definition, a Cartan surface in E^4 is characterized by the existence of a well-defined net of conjugate curves. Besides, a Cartan surfaces in E^4 may be defined in terms of the Grassman image (generalized Gauss image): the Grassmann image of a Cartan surface consists of hyperbolic points (here we apply a classification of points on surfaces in E^4 proposed by Yu. Aminov [1, Ch. 8, § 6]). It was remarked in [8–10], that pseudospherical congruencies in E^4 and corresponding Backlund transformations may be constructed either for pseudospherical Cartan surfaces in E^4 or for pseudospherical hypersurfaces in $E^3 \subset E^4$ only.

On the other hand, there are many surfaces in E^4 which are neither Cartan surfaces nor hypersurfaces in $E^3 \subset E^4$. We speak about the surfaces whose Grassmann image consists of elliptic and/or parabolic points. By definition, a surface with elliptic Grassmann image, an E-surface, is characterized by the absence of conjugate directions in tangent planes. A surface with parabolic Grassmann image, a P-surface, has a well-defined asymptotic direction at each its point. So a P-surfaces in E^4 is foliated in a unique way by asymptotic curves; the ruled surfaces in E^4 which are not hypersurfaces in $E^3 \subset E^4$ present a particular class of P-surfaces.

The following question was of the major interest for us: *do there exist surfaces of constant negative Gauss curvature in E^4 which don't admit Backlund transformations with help of pseudo-spherical congruencies?* We reformulate this question as follows: *do there exist E-surfaces and/or P-surfaces with constant negative Gauss curvature in E^4?* It turns out that the answer is positive.

Theorem 1.

1. There exist P-surfaces with constant negative Gauss curvature in E^4.
2. There exist E-surfaces with constant negative Gauss curvature in E^4.
3. There exist Cartan surfaces with constant negative Gauss curvature in E^4.

The E-surfaces as well as Cartan surfaces form two general classes of surfaces in E^4. Hence the existence of pseudospherical E-surfaces in E^4 seems to be rather expected. We demonstrate that each pseudospherical E-surface in E^4 may be represented by one function of two variables and two functions of one variable. The same situation is valid for the Cartan surfaces of constant negative curvature in E^4. As for the P-surfaces, they form a very particular class of surfaces in E^4. So the existence of pseudospherical P-surfaces is very surprising. We demonstrate, that each pseudospherical P-surface in E^4 may be represented by four functions of one variable. From the geometric point of view, the corresponding initial data for constructing of a pseudospherical P-surface F^2 in E^4 are a generic curve γ in some $E^3_0 \subset E^4$ and a generic field of two-planes π along γ which are tangent to γ and transversal to E^3_0; the surface F^2 passes through γ and at each point x of γ the tangent plane T_xF^2 coincides with $\pi(x)$, (see Th. 2 in Sect. 4).

The proven theorem leads us directly to the positive answer for the principal question stated above.

Corollary. *There exist pseudospherical surfaces in E^4 which don't admit Backlund transformations with help of pseudospherical congruencies.*

It is an open question whether a similar statement holds for pseudo-spherical Cartan surfaces in E^4.

In order to deduce Corollary from Theorem 1, one can apply, for instance, Theorem 4 from [8] which asserts that a pseudospherical E-surface in E^4 doesn't admit linear congruencies. The same is true for pseudospherical P-surfaces in E^4.

Thus the direct generalization of Backlund transformations with help of pseudospherical congruencies can not be applied to *all* pseudospherical surfaces in E^4. It would be very interesting to find an analogue of Backlund transformations for pseudospherical E- and P-surfaces in E^4, such an analogue has to be constructed without any use of pseudospherical congruencies.

2. Classification of Points on Surfaces in E^4

Let F^2 be a regular two-dimensional surface in the four-dimensional Euclidean space E^4. Choose two normal fields, $\vec{n}_1$, $\vec{n}_2$, on F^2 which form a frame in the normal planes N_xF^2. Let $II^\sigma = L^\sigma_{ij}du^i du^j$, $\sigma = 1, 2$, stand for the second fundamental forms of F^2 corresponding to the above choice of normals $\vec{n}_1$, $\vec{n}_2$.

Similarly to the classical case, various properties of the second fundamental forms II^σ may be applied to classify points of $F^2 \subset E^4$. We briefly describe one such classification, which is based on the consideration of conjugate and asymptotic directions in tangent planes to F^2.

Let us recall some fundamental notions. The point codimension of F^2 at a point $x \in F^2$ (the dimension of the first normal space to F^2 at x) is defined by the following formula:

$$codim_x = Rank \begin{pmatrix} L^1_{11} & L^1_{12} & L^1_{22} \\ L^2_{11} & L^2_{12} & L^2_{22} \end{pmatrix}.$$

Two directions $X = (X^1 : X^2)$, $Y = (Y^1 : Y^2)$ in the tangent plane T_xF^2 are called *conjugate* if

$$L^1_{11}X^1Y^1 + L^1_{12}(X^1Y^2 + X^2Y^1) + L^1_{22}X^2Y^2 = 0, \tag{1}$$

$$L^2_{11}X^1Y^1 + L^2_{12}(X^1Y^2 + X^2Y^1) + L^2_{22}X^2Y^2 = 0. \tag{2}$$

A self-conjugate direction $X = (X^1 : X^2)$ in T_xF^2 is called an *asymptotic* direction, its coordinates solve the following equations:

$$L^1_{11}(X^1)^2 + 2L^1_{12}X^1X^2 + L^1_{22}(X^2)^2 = 0,$$

$$L^2_{11}(X^1)^2 + 2L^2_{12}X^1X^2 + L^2_{22}(X^2)^2 = 0.$$

The existence of conjugate directions in T_xF^2 depends on the solvability of the system of algebraic equations (1)–(2). Write (1)–(2) as a system of two linear equations with respect to Y^1, Y^2:

$$\begin{pmatrix} L^1_{11}X^1 + L^1_{12}X^2 & L^1_{12}X^1 + L^1_{22}X^2 \\ L^2_{11}X^1 + L^2_{12}X^2 & L^2_{12}X^1 + L^2_{22}X^2 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

There exists a nonzero solution $Y = (Y^1 : Y^2)$ of (3) if and only if

$$\det \begin{pmatrix} L^1_{11}X^1 + L^1_{12}X^2 & L^1_{12}X^1 + L^1_{22}X^2 \\ L^2_{11}X^1 + L^2_{12}X^2 & L^2_{12}X^1 + L^2_{22}X^2 \end{pmatrix} = 0,$$

i.e., if $X = (X^1 : X^2)$ solves the following second-order homogeneous equation:

$$(X^1)^2(L^1_{11}L^2_{12} - L^1_{12}L^2_{11}) + X^1X^2(L^1_{11}L^2_{22} - L^1_{22}L^2_{11}) + (X^2)^2(L^1_{12}L^2_{22} - L^1_{22}L^2_{12}) = 0. \tag{4}$$

By symmetry, the same equation is satisfied by $Y = (Y^1 : Y^2)$.

Thus, the conjugate directions in T_xF^2 are determined by solutions of (4). The solvability of (4) depends on the coefficients L^σ_{ij}. Firstly, it is easy to see that (4) is nondegenerate if and only if $codim_x$ is equal to 2. Secondly, in the nondegenerate case the number of solutions of (4) depends on the sign of the following discriminant:

$$D = (L^1_{11}L^2_{22} - L^1_{22}L^2_{11})^2 - 4(L^1_{11}L^2_{12} - L^1_{12}L^2_{11})(L^1_{12}L^2_{22} - L^1_{22}L^2_{12}).$$

If $D > 0$, then there exist two independent solutions $X = (X^1 : X^2)$ and $Y = (Y^1 : Y^2)$, they determine a well-defined pair of independent conjugate directions in T_xF^2; in this case the point $x \in F^2$ is said to be *hyperbolic*.

If $D = 0$, then there exist a unique solution $X = (X^1 : X^2)$, it determines a well-defined asymptotic direction in T_xF^2; in this case the point $x \in F^2$ is said to be *parabolic*.

If $D < 0$, then there no exist nonzero solutions of (4), so the tangent plane T_xF^2 doesn't contain neither conjugate directions nor asymptotic ones; in this case the point $x \in F^2$ is said to be *elliptic*.

If $codim_x$ is equal to 1, the left hand side of (4) vanishes. It means that for an arbitrary direction $X = (X^1, X^2)$ in T_xF^2 there exist a well-defined conjugate direction $Y = (Y^1, Y^2)$, which may be determined from (3).

If $codim_x$ is equal to 0, then the equations (1)–(2) degenerate, so arbitrary directions in T_xF^2 are conjugate.

Thus, we described five classes of points in $F^2 \subset E^4$ with very different extrinsic-geometric properties depending on the point codimension and on the sign of the discriminant D. The proposed classification is well known, it corresponds to the so-called affine and Grassmannian classifications introduced by A.A. Borisenko and by Yu.A. Aminov respectively (see [1, Ch. 8, § 6], [11, Ch. 3, § 1]).

If a surface in E^4 consists of hyperbolic points, it is called a Cartan surface; such a surface carries a well-defined net of conjugate curves. A surface in E^4 is called a P-surface, if it consists of parabolic points; such a surface is foliated in a unique way by asymptotic curves. A surface in E^4 is called an E-surface, if it consists of elliptic points; such a surface does not admit neither conjugate tangent directions nor asymptotic ones.

If the point codimension is less than 2 at all points of a surface in E^4, then either such a surface belongs to some hyperplane $E^3 \subset E^4$ or it is ruled and has a degenerate Grassmann image.

3. Pseudospherical Surfaces in E^4

The surface $F^2 \subset E^4$ is said to be *pseudospherical* if its Gauss curvature K is equal to -1. In view of the classification discussed in the previous section, it is naturally to analyze what kind of points may belong to F^2.

Let us suppose that $F^2 \subset E^4$ is represented explicitly:

$$x^1 = u, \quad x^2 = v, \quad x^3 = U(u,v), \quad x^4 = V(u,v), \tag{5}$$

where $U(u,v)$, $V(u,v)$ are some C^k, $k \geq 2$, functions defined on a neighborhood of the origin $(0,0) \in R^2$. An easy calculation provides us with the following simple formula for the Gauss curvature K [1, Ch. 6, § 7, p. 176]:

$$K = \big((1 + (U_u)^2 + (U_v)^2)(V_{uu}V_{vv} - (V_{uv})^2)$$
$$+(1 + (V_u)^2 + (V_v)^2)(U_{uu}U_{vv} - (U_{uv})^2)$$

$$-(U_u V_u + U_v V_v)(U_{uu} V_{vv} - 2U_{uv} V_{uv} + U_{vv} V_{uu}))$$
$$/ \left(1 + (U_u)^2 + (U_v)^2 + (V_u)^2 + (V_v)^2 + (U_u V_v - U_v V_u)^2\right)^2.$$

Therefore the explicit surface $F^2 \subset E^4$ is pseudospherical, $K \equiv -1$, if and only if the functions $U(u,v)$ and $V(u,v)$ satisfy the following second order partial differential equation:

$$(1 + (U_u)^2 + (U_v)^2)(V_{uu} V_{vv} - (V_{uv})^2) + (1 + (V_u)^2 + (V_v)^2)(U_{uu} U_{vv} - (U_{uv})^2)$$
$$-(U_u V_u + U_v V_v)(U_{uu} V_{vv} - 2U_{uv} V_{uv} + U_{vv} V_{uu})$$
$$+ \left(1 + (U_u)^2 + (U_v)^2 + (V_u)^2 + (V_v)^2 + (U_u V_v - U_v V_u)^2\right)^2 = 0. \tag{6}$$

It is easy to see, that (6) may be written in the following form:

$$U_{uu} = \frac{A(U_u, U_v, U_{uv}, U_{vv}, V_u, V_v, V_{uu}, V_{uv}, V_{vv})}{(1 + (V_u)^2 + (V_v)^2) U_{vv} - (U_u V_u + U_v V_v) V_{vv}}, \tag{7}$$

where A denotes some polynomial. By a corresponding existence and uniqueness theorem from PDE theory [12, Ch. I, § 2, p. 24], for an arbitrary choice of analytical functions $V(u,v)$, $U(0,v) = P(v)$, $U_u(0,v) = Q(v)$ which obeys

$$(1 + (V_u)^2 + (V_v)^2) U_{vv} - (U_u V_u + U_v V_v) V_{vv}|_{(0,0)}$$
$$= (1 + (V_u)^2 + (V_v)^2) P_{vv} - (Q V_u + P_v V_v) V_{vv}|_{(0,0)} \neq 0,$$

there exists a unique analytical solution $U(u,v)$ of (7) defined in a neighborhood of $(0,0) \in R^2$. Such a solution describes some pseudospherical surface in E^4 via the explicit representation (5).

In order to control the kind of points on F^2, we have to write the discriminant D in terms of functions $U(u,v)$ and $V(u,v)$:

$$D = (U_{uu} V_{vv} - V_{uu} U_{vv})^2 - 4(U_{uu} V_{uv} - V_{uu} U_{uv})(U_{uv} V_{vv} - V_{uv} U_{vv}). \tag{8}$$

As for the point codimension *codim*, it is determined by the following formula:

$$codim_{(u,v)} = Rank \begin{pmatrix} U_{uu} & U_{uv} & U_{vv} \\ V_{uu} & V_{uv} & V_{vv} \end{pmatrix}. \tag{9}$$

Therefore, the surface $F^2 \subset E^4$ consists of elliptic points if and only if the functions $U(u,v)$ and $V(u,v)$ satisfy two conditions:

$$(U_{uu} V_{vv} - V_{uu} U_{vv})^2 - 4(U_{uu} V_{uv} - V_{uu} U_{uv})(U_{uv} V_{vv} - V_{uv} U_{vv}) < 0, \tag{10.1}$$

$$Rank \begin{pmatrix} U_{uu} & U_{uv} & U_{vv} \\ V_{uu} & V_{uv} & V_{vv} \end{pmatrix} = 2. \tag{10.2}$$

Moreover, in order to construct a pseudospherical E-surface in E^4 the function $V(u,v)$ and the initial data $U(0,v) = P(v)$, $U_u(0,v) = Q(v)$ for (7) has to be chosen in such a way that (10.1)–(10.2) hold at $(0,0)$. (For example, one can choose $V(u,v)$, $P(v)$, $Q(v)$ with $V_{uu}(0,0) = V_{vv}(0,0) = 0$, $V_{uv}(0,0) = 1/2$, $P_{vv}(0) \neq 0$, $|Q_v(0)| < \sqrt{3}/2$.) For such a choice of initial data there exists a solution $U(u,v)$ of (7), which satisfies (10.1) and (10.2) in a neighborhood of $(0,0) \in R^2$. Then (5) will describe a pseudospherical E-surface in E^4.

Similarly, the surface $F^2 \subset E^4$ consists of hyperbolic points if and only if the functions $U(u,v)$ and $V(u,v)$ satisfy two following conditions:

$$(U_{uu}V_{vv} - V_{uu}U_{vv})^2 - 4(U_{uu}V_{uv} - V_{uu}U_{uv})(U_{uv}V_{vv} - V_{uv}U_{vv}) > 0, \qquad (11.1)$$

$$Rank \begin{pmatrix} U_{uu} & U_{uv} & U_{vv} \\ V_{uu} & V_{uv} & V_{vv} \end{pmatrix} = 2. \qquad (11.2)$$

Besides, in order to construct a pseudospherical Cartan surface in E^4 the function $V(u,v)$ and the initial data $U(0,v) = P(v)$, $U_u(0,v) = Q(v)$ for (7) has to be chosen in such a way that (11.1)–(11.2) hold at $(0,0)$. (For example, one can choose $V(u,v)$, $P(v)$, $Q(v)$ with $V_{uu}(0,0) = V_{vv}(0,0) = 0$, $V_{uv}(0,0) = 1/2$, $P_{vv}(0) \neq 0$, $|Q_v(0)| > \sqrt{3}/2$.) For such a choice of initial data there exists a solution $U(u,v)$ of (7), which satisfies (11.1) and (11.2) in a neighborhood of $(0,0) \in R^2$. Then (5) will describe a pseudospherical Cartan surface in E^4.

Finally, the surface $F^2 \subset E^4$ consists of parabolic points if and only if the functions $U(u,v)$ and $V(u,v)$ satisfy two conditions

$$(U_{uu}V_{vv} - V_{uu}U_{vv})^2 - 4(U_{uu}V_{uv} - V_{uu}U_{uv})(U_{uv}V_{vv} - V_{uv}U_{vv}) = 0, \qquad (12.1)$$

$$Rank \begin{pmatrix} U_{uu} & U_{uv} & U_{vv} \\ V_{uu} & V_{uv} & V_{vv} \end{pmatrix} = 2. \qquad (12.2)$$

Therefore, in order to construct a pseudospherical P-surface in E^4, we have to solve two second order nonlinear partial differential equations, (6) and (12.1), completed by the partial differential relation (12.2).

Consider (6) and (12.1) as a system of algebraic equations with respect to the partial derivatives U_{uu} and V_{uu}. The equation (6) is linear, whereas (12.1) is a second-order equation of parabolic type. So it's easy to see, that if U_u, U_v, U_{uv}, U_{vv}, V_u, V_v, V_{uv}, V_{vv} satisfy some polynomial inequality

$$\Phi(U_u, U_v, U_{uv}, U_{vv}, V_u, V_v, V_{uv}, V_{vv}) > 0, \qquad (13)$$

then (6) and (12.1) may be solved with respect to U_{uu} and V_{uu} as follows:

$$U_{uu} = A_1(U_u, U_v, U_{uv}, U_{vv}, V_u, V_v, V_{uv}, V_{vv}), \qquad (14.1)$$

$$V_{uu} = A_2(U_u, U_v, U_{uv}, U_{vv}, V_u, V_v, V_{uv}, V_{vv}), \tag{14.2}$$

where A_1 and A_2 are some analytical functions defined on an open domain

$$D = \left\{ (y^1, \ldots, y^8) \in R^8 \,|\, \Phi(y^1, \ldots, y^8) > 0 \right\}.$$

Choose analytical initial data

$$U(0,v) = P(v),\ U_u(0,v) = Q(v),\ V(0,v) = R(v),\ V_u(0,v) = S(v) \tag{15}$$

in such a way that

$$(Q(0), P_v(0), Q_v(0), P_{vv}(0), S(0), R_v(0), S_v(0), R_{vv}(0)) \in D. \tag{16}$$

Then by Cauchy–Kowalewskaya theorem [12, Ch. I, § 2, p. 24] there exists a unique analytical solution $U(u,v)$, $V(u,v)$ of (14.1)–(14.2), which is defined in a neighborhood of $(0,0) \in R^2$.

The constructed solution $U(u,v)$, $V(u,v)$ obeys the additional constraint (12.2) in a neighborhood of $(0,0) \in R^2$ if $P(v)$, $Q(v)$, $R(v)$ and $S(v)$ satisfy the following inequality:

$$Q_v(0)R_{vv}(0) - P_{vv}(0)S_v(0) \neq 0. \tag{17}$$

Consider the open domain

$$\tilde{D} = \left\{ (y^1, \ldots, y^8) \in R^8 | y^3 y^8 - y^4 y^7 \neq 0 \right\} \subset R^8.$$

It's easy to verify that the intersection $D \cap \tilde{D} \subset R^8$ is nonempty, for instance it contains the point $(0,0,1,1,0,0,0,1)$. Hence $D^\star = D \cap \tilde{D}$ is a nonempty open domain in R^8. Therefore, it is really possible to choose the initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ in such a way, that (16) and (17) hold. As consequence, the corresponding solution $U(u,v)$, $V(u,v)$ of (14.1)–(14.2) will satisfy (12.2) in a neighborhood of $(0,0) \in R^2$. So, the explicit representation (5) will describe a pseudospherical P-surface in E^4, q.e.d.

R e m a r k 1. If a point $x \in F^2$ is fixed, one can always specify the Cartesian coordinates in E^4 in such a way that P is the origin and $T_x F^2$ is the (x^1, x^2)-plane. For the explicit representation (5) such a specification means that

$$U(0,0) = 0,\ V(0,0) = 0,\ U_u(0,0) = 0,\ V_u(0,0) = 0,\ U_v(0,0) = 0,\ V_v(0,0) = 0.$$

So, without loss of generality one can consider the initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ which satisfy

$$P(0) = 0,\ R(0) = 0,\ Q(0) = 0,\ S(0) = 0,\ P_v(0) = 0,\ R_v(0) = 0. \tag{18}$$

If (18) hold then the initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ are said to be *reduced.*

It is easy to verify that

$$\Phi(0,0,y^3,y^4,0,0,y^7,y^8) = (y^4)^2 + (y^8)^2 - (y^3y^8 - y^4y^7)^2.$$

Therefore if $P(v)$, $Q(v)$, $R(v)$, $S(v)$ are reduced then (16) reads

$$P_{vv}^2 + R_{vv}^2 - (P_{vv}S_v - R_{vv}Q_v)^2 > 0. \tag{19}$$

The initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ are referred to as *appropriate* if they obey (17)–(19). For any choice of appropriate initial data corresponds a well-defined pseudospherical P-surface in E^4.

4. Geometric Representation of Pseudospherical P-Surfaces in E^4

The initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ may be interpreted geometrically. Due to (5), the functions $P(v)$, $R(v)$ represent a curve $\gamma \in F^2$ explicitly given by

$$x^1 = 0, \quad x^2 = v, \quad x^3 = P(v), \quad x^4 = R(v). \tag{20}$$

The curve γ is the intersection of F^2 with the hyperplane $E_0^3 \subset E^4$ given by $x^1 = 0$. At each point x of γ the tangent plane T_xF^2 is spanned by two vectors:

$$x_u(0,v) = (1,0,U_u(0,v),V_u(0,v)) = (1,0,Q(v),S(v))\,, \tag{20.1}$$

$$x_v(0,v) = (0,1,U_v(0,v),V_v(0,v)) = (0,1,P_u(v),R_u(v))\,. \tag{20.2}$$

Obviously, $x_v(0,v)$ is the tangent vector to γ. Besides, $T_xF^2 \not\subset E_0^3$. The curve γ being given, the tangent planes to F^2 along γ are one-to-one determined by $Q(v)$ and $S(v)$.

The initial data $P(v)$, $Q(v)$, $R(v)$, $S(v)$ are reduced, i.e., (18) holds, if and only if the origin O belongs to γ and the tangent plane T_OF^2 is the (x^1,x^2)-plane. The analytic constraints (17) and (19) impose some restrictions on the dynamical properties of γ and T_xF^2 at O.

Conversely, consider an arbitrary regular analytical curve γ in some $E_0^3 \subset E^4$ and an analytical field of two-planes π^2 along γ. Suppose that γ and π^2 satisfy the following conditions:

A1) the planes π^2 are tangent to γ, i.e., at each point x of γ the tangent vector to γ belongs to $\pi^2(x)$;

A2) there is a point $O \in \gamma$ such that $\pi^2(O)$ doesn't belong to E_0^3.

Introduce Cartesian coordinates $x^1,\dots,x^4$ in E^4 in such a way that O is the origin, the tangent line to γ at O is the x^2-axe, and $E_0^3 \subset E^4$ is the hyperplane $x^1 = 0$. Since the tangent line to γ at O is the x^2-axe, the curve γ may be represented explicitly, $x^1 = 0$, $x^2 = v$, $x^3 = P(v)$, $x^4 = R(v)$, where $P(v)$ and

$Q(v)$ are some functions. Moreover, since $\pi^2(O)$ contains the vector tangent to γ at O, i.e., $(0, 1, P_v(0), R_v(0)) \in \pi^2(O)$, and it doesn't belong to the hyperplane $x^1 = 0$, one may conclude that the orthogonal projection from $\pi^2(O)$ to the (x^1, x^2)-plane in E^4 is bijective. The same is valid for all $\pi^2(x)$ at points $x \in \gamma$ sufficiently close to O. Therefore, at each such point $x(v) \in \gamma$ the plane $\pi^2(x)$ is spanned by the vectors $(0, 1, P_v(v), R_v(v))$ and $(1, 0, Q(v), S(v))$. The functions $P(v)$, $Q(v)$, $R(v)$ and $S(v)$ constructed from γ and π^2 satisfy $P(0) = 0$, $P_v(0) = 0$, $R(0) = 0$, $R_v(0) = 0$. It is easy to see that these functions form reduced initial data if and only if γ and π^2 satisfy the additional condition:

A3) the plane $\pi^2(O)$ is orthogonal to E_0^3, i.e., it contains the normal straight line to $E_0^3 \subset E^4$ at O.

Definition. *γ and π^2 are referred to as appropriate geometric initial data if they satisfy A1)–A3) and if $P(v)$, $Q(v)$, $R(v)$ and $S(v)$ corresponding to γ and π^2 obey (17) and (19).*

Thus, the following representation statement holds.

Theorem 2. *Let γ and π^2 be appropriate geometric initial data. Then there exists a unique pseudospherical P-surface $F^2 \subset E^4$ which contains γ and whose tangent planes at points of γ coincide with two-planes π^2.*

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