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On the Sine–Gordon Equation with a Self-Consistent Source of the Integral Type

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It is shown that the solutions of the Sine–Gordon equation with a source of the integral type can be found by the method of the inverse scattering problem for the Dirac type operator on the real line.

Key words: Sine–Gordon equation, inverse scattering method, Jost solutions, scattering data.
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1. Introduction

In this paper we consider the problem of integration of the following system of equations

$$\begin{cases} u_{xt} = \sin u + \int\limits_{-\infty}^{\infty} \left(\phi_1^2 - \phi_2^2\right) d\eta \ , \\ L\phi = \eta\phi, \end{cases} \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in R, \tag{2}$$

where $L(t) = i\begin{pmatrix} \frac{d}{dx} & \frac{u_x}{2} \\ \frac{u_x}{2} & -\frac{d}{dx} \end{pmatrix}$, $u_x = \frac{\partial u(x,t)}{\partial x}$, $u_{xt} = \frac{\partial^2 u(x,t)}{\partial x \partial t}$, and $u_0(x)$ $(-\infty < x < \infty)$ is a function satisfying the conditions:

1)
$$\begin{gathered} u_0(x) \equiv 0 (\text{mod } 2\pi) \ \text{ as } \ |x| \to \infty, \\ \int\limits_{-\infty}^{\infty} \left((1+|x|)\left|u_0'(x)\right| + \left|u_0''(x)\right|\right) dx < \infty; \end{gathered} \tag{3}$$

2) the operator $L(0)$ does not have the points of spectral singularity (see [6]) and has only simple eigenvalues $\xi_1(0)$, $\xi_2(0)$, $\dots$, $\xi_N(0)$.

We assume that the vector function $\phi = (\phi_1(x, \eta, t),\ \phi_2(x, \eta, t))^T$ is a solution of the equation $L\phi = \eta\phi$ satisfying the condition

$$\phi \to A(\eta, t) \begin{pmatrix} \exp(-i\eta x) \\ \exp(i\eta x) \end{pmatrix} \quad \text{as } x \to \infty, \tag{4}$$

where $A(\eta, t)$ is a continuous function satisfying the condition

$$A(-\eta, t) = A(\eta, t), \quad \int\limits_{-\infty}^{\infty} |A(\eta, t)|^2 d\eta < \infty, \tag{5}$$

for all nonnegative values of t.

We assume that the solution $u(x, t)$ of the problem (1)–(5) exists, possesses the required smoothness, and tends to its limits sufficiently rapidly as $x \to \pm\infty$, i.e., for all $t \geq 0$ it satisfies the condition

$$\begin{gathered} u(x, t) \equiv 0 (\text{mod } 2\pi) \quad \text{as} \quad |x| \to \infty, \\ \int\limits_{-\infty}^{\infty} ((1 + |x|)\, |u_x(x, t)| + |u_{xx}(x, t)|)\, dx < \infty. \end{gathered} \tag{6}$$

The main objective of this paper is to derive representations for the solutions $u(x, t)$, $\phi(x, \eta, t)$ within the framework of the inverse scattering method for $L(t)$ operator.

The full description of the solutions of the Sine–Gordon equation without sources was given in [1–2].

The scattering problem for $L(t)$ operator was studied in the papers by V.E. Zakharov, A.B. Shabat [3], L.P. Nizhnik, Fam Loy Woo [4], I.S. Frolov [5], A.B. Khasanov [6] and in many others.

Note that the similar problem for the KdV equation was considered in the paper [7]. In the V.K. Mel'nikov's paper [8] there was obtained evolution of the scattering dates for the selfadjoint Dirac type operator with the potential which is a solution of the NLS equation with the integral type source. Notice however that in our case operator $L(t)$ is not self-adjoint. As it is well known, under the condition (6) the not self-adjoint operator $L(t)$ has a finite number of complex eigenvalues (in general multiple). Moreover, operator $L(t)$ may have a finite number of real points of spectral singularity. The continuous spectrum of the operator $L(t)$ fills up the real line, i.e., $\sigma_{ess}(L(t)) = (-\infty,\ \infty)$. For simplicity we suppose that operator $L(t)$ has a finite number of simple complex eigenvalues, and does not have points of singular spectrum.

2. Scattering Problem for Zakharov–Shabat Eigenvalue Problem

In this section we present some facts from the theory of the direct and inverse scattering problems for the operator $L(t)$ (for example, see [9]). For a while in this section we omit the dependence of functions on t.

We consider the eigenvalue problem

$$\begin{cases} v_{1x} + i\xi v_1 = u'(x)v_2 \\ v_{2x} - i\xi v_2 = -u'(x)v_1, \end{cases} \tag{7}$$

on the interval $-\infty < x < \infty$. The potential $u'(x)$ is assumed to satisfy the condition

$$u(x) \equiv 0 (\text{mod } 2\pi) \ \text{ as } \ |x| \to \infty, \quad \int_{-\infty}^{\infty} ((1+|x|)\,|u'(x)|)\,dx < \infty. \tag{8}$$

We define the Jost solution of the problem (7)–(8) with the following asymptotic values

$$\left.\begin{array}{l} \varphi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \\ \bar\varphi \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \end{array}\right\} \text{ as } x \to -\infty, \qquad \left.\begin{array}{l} \psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \\ \bar\psi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \end{array}\right\} \text{ as } x \to \infty.$$

For real ξ the pairs of functions $\{\varphi,\ \bar\varphi\}$ and $\{\psi,\ \bar\psi\}$ are the pairs of linearly independent solutions of (7), and therefore

$$\varphi = a(\xi)\bar\psi + b(\xi)\psi, \quad \bar\varphi = -\bar a(\xi)\psi + \bar b(\xi)\bar\psi\,, \tag{9}$$

where $a(\xi) = W\{\varphi,\psi\} \equiv \varphi_1\psi_2 - \varphi_2\psi_1$, $b(\xi) = W\{\bar\psi,\varphi\}$, $a(\xi)a(-\xi) + b(\xi)b(-\xi) = 1$.

For real ξ the coefficient $b(\xi)$ has the following asymptotic $b(\xi) = O\left(\frac{1}{|\xi|}\right)$ as $|\xi| \to \infty$, $Im\xi = 0$. The coefficient $a(\xi)$ $(\bar a(\xi))$ can be analytically extended into the upper (lower) half-plane $Im\ \xi > 0$ $(Im\xi < 0)$. The function $a(\xi)$ has the asymptotic $a(\xi) = 1 + O\left(\frac{1}{|\xi|}\right)$ as $|\xi| \to \infty$, $Im\xi \geq 0$. Besides, in the half-plane $Im\ \xi > 0$ $(Im\xi < 0)$ the function $a(\xi)$ $(\bar a(\xi))$ has a finite number of zeros at the points ξ_k $\left(\bar\xi_k\right)$, and these points are the eigenvalues of the operator

$$L = i\begin{pmatrix} \frac{d}{dx} & \frac{u'(x)}{2} \\ \frac{u'(x)}{2} & -\frac{d}{dx} \end{pmatrix},$$

so that $\varphi(x,\xi_k)=C_k\psi(x,\xi_k)$ $(\bar{\varphi}(x,\xi_k)=\bar{C}_k\bar{\psi}(x,\xi_k))$, $k=1,2,\dots,N$. It is clear that the function $\varphi_k\equiv\varphi(x,\xi_k)$ is an eigenfunction of the operator L corresponding to the eigenvalue ξ_k.

We assume that the operator L does not have multiple eigenvalues. The requirement of absence of the points of spectral singularity of the operator $L(t)$ means the absence of real zeros of function $a(\xi)$. The class of the potentials satisfying $a(\xi)\neq 0$ as $\xi\in R^1$ is not empty. For example, this class contains "unreflected" potentials, i.e., potential for which $b(\xi)=0$. In this case the equation $a(\xi)a(-\xi)=1$, $\xi\in R^1$ is valid.

We have the following integral representation for the function φ [9]

$$\psi=\begin{pmatrix}0\\1\end{pmatrix}e^{i\xi x}+\int\limits_x^\infty K\ (x,s)\,e^{i\xi\, s}ds, \tag{10}$$

where the kernel $K\ (x,s)=\begin{pmatrix}K_1\,(x,s)\\K_2\,(x,s)\end{pmatrix}$ does not depend on ξ and is related to the potential $u(x)$ by the formulae

$$u'\,(x)=4K_1\,(x,x)\,,\qquad (u'\,(x))^2=8\frac{dK_2\,(x,x)}{dx}\,. \tag{11}$$

Components $K_1(x,y)$, $K_2(x,y)$ of the kernel $K\ (x,y)$ in the representation (10), for $y>x$ are solutions of the integral Gelfand–Levitan–Marchenko equations

$$K_1(x,y)-F(x+y)+\int\limits_x^\infty\int\limits_x^\infty K_1(x,z)F(z+s)F(s+y)dsdz=0,$$
$$K_2(x,y)+\int\limits_x^\infty F(x+s)F(s+y)ds+\int\limits_x^\infty\int\limits_x^\infty K_2(x,z)F(z+s)F(s+y)dsdz=0,$$

where $F(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{b(\xi)}{a(\xi)}e^{i\xi x}d\xi-i\sum\limits_{j=1}^{N}C_je^{i\xi_j x}$.

Now the potential can be expressed via $K_1\,(x,y)$ by the formula (11).

The set of the quantities $\left\{r^+\,(\xi)=\frac{b(\xi)}{a(\xi)},\ \zeta_k,\ C_k,\ k=1,2,\dots,N\right\}$ is called the scattering data for equations (7).

It is worthy to remark that the vector functions

$$h_n\,(x)=\frac{\left.\frac{d}{d\xi}\,(\varphi-C_n\psi)\right|_{\xi=\xi_n}}{\dot{a}\,(\xi_n)},\quad n=1,2,\dots,N, \tag{12}$$

are solutions of the equations $Lh_n = \xi_n h_n$ and have the following asymptotics

$$\begin{aligned} h_n &\sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x} \qquad \text{as } x \to -\infty, \\ h_n &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x} \qquad \text{as } x \to \infty. \end{aligned} \tag{13}$$

According to (13) we obtain

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1}h_{n2} - \varphi_{n2}h_{n1} = -C_n, \quad n = 1, 2, \ldots, N. \tag{14}$$

It is easy to see that the following statement is true.

Lemma 1. *If $Y(x,\zeta)$ and $Z(x,\eta)$ are solutions of the equations $LY = \zeta Y$ and $LZ = \eta Z$, then*

$$\frac{d}{dx}(y_1z_2 - y_2z_1) = -i(\zeta - \eta)(y_1z_2 + y_2z_1),$$

$$\frac{d}{dx}(y_1z_1 + y_2z_2) = -i(\zeta + \eta)(y_1z_1 - y_2z_2).$$

3. Evolution of the Scattering Data

Let the potential $u(x,t)$ of the problem (7) be a solution of the system of equations

$$\begin{cases} u_{xt} = \sin u + \int\limits_{-\infty}^{\infty} (\phi_1^2 - \phi_2^2)\, d\eta, \\ L\phi = \eta\phi. \end{cases} \tag{15}$$

We put $G(x,t) = \int\limits_{-\infty}^{\infty} (\phi_1^2 - \phi_2^2)\, d\eta$. According to (4)

$$\phi(x,\eta,t) = A(\eta,t)\left(\bar{\psi}(x,\eta,t) + \psi(x,\eta,t)\right),$$

and therefore, by using (9), as well as the asymptotic for the Jost solution and $a(\xi), b(\xi)$ and Riemann–Lebesgue lemma in each nonnegative t, we have $G(x,t) = o(1)$ as $x \to \pm\infty$. The first equation of (15) can be rewritten in the form

$$u_{xt} = \sin u + G. \tag{16}$$

Lemma 2. *If potential $u(x,t)$ of the problem (7) is a solution of equation (16), then the scattering data depend on t as*

$$\frac{dr^+}{dt} = -\frac{i}{2\xi}r^+ + \frac{1}{2a^2}\int\limits_{-\infty}^{\infty}\left(G\varphi_2^2 + G\varphi_1^2\right)dx, \quad (Im\xi = 0),$$

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + \int\limits_{-\infty}^{\infty}\frac{G}{2}\left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right)dx\right)C_n,$$

$$\frac{d\xi_n}{dt} = \frac{i\int\limits_{-\infty}^{\infty}\left(G\varphi_{n2}^2 + G\varphi_{n1}^2\right)dx}{4\int\limits_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx}, \quad n = 1, 2, \ldots, N.$$

P r o o f. Here we use the method of [10] (see also [11]).
We set

$$A = \begin{pmatrix} \frac{i\cos u}{4\xi} & \frac{i\sin u}{4\xi} \\ \frac{i\sin u}{4\xi} & -\frac{i\cos u}{4\xi} \end{pmatrix}.$$

It is easy to see that

$$[L, A] \equiv LA - AL = -i\begin{pmatrix} 0 & \frac{\sin u}{2} \\ \frac{\sin u}{2} & 0 \end{pmatrix}. \tag{17}$$

The operator $L(t)$ depends on time t as a parameter and therefore

$$\frac{\partial L}{\partial t} = i\begin{pmatrix} 0 & \frac{u_{xt}}{2} \\ \frac{u_{xt}}{2} & 0 \end{pmatrix}. \tag{18}$$

Comparing formulas (17) and (18) with the equation (16), we can see that the equation (16) is identical to the operator relation

$$\frac{\partial L}{\partial t} + [L, A] = iR, \tag{19}$$

where $R = \begin{pmatrix} 0 & \frac{G}{2} \\ \frac{G}{2} & 0 \end{pmatrix}$.

Let $\varphi(x, \xi, t)$ be the Jost solution of the equation

$$L\varphi = \xi\varphi.$$

We differentiate this relation with respect to time

$$L_t\varphi + L\varphi_t = \xi\varphi_t, \tag{20}$$

and substitute L_t from (19) into (20). This results to

$$(L-\xi)(\varphi_t - A\varphi) = -iR\varphi. \tag{21}$$

We seek the solutions of (21) in the form

$$\varphi_t - A\varphi = \alpha(x)\psi + \beta(x)\varphi. \tag{22}$$

To find $\alpha(x)$ and $\beta(x)$ we use the equation

$$M\alpha_x\psi + M\beta_x\varphi = -R\varphi, \tag{23}$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to (9)

$$\hat{\psi}^T M\varphi = -\hat{\varphi}^T M\psi = a, \quad \hat{\psi}^T M\psi = \hat{\varphi}^T M\varphi = 0,$$

where $\hat{\varphi} = \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix}$.

Multiplying (23) by $\hat{\varphi}^T$ and $\hat{\psi}^T$ we yield

$$\alpha_x = \frac{\hat{\varphi}^T R\varphi}{a}, \qquad \beta_x = -\frac{\hat{\psi}^T R\varphi}{a}. \tag{24}$$

On the basis of (6) and the asymptotic of the Jost solution we have

$$\varphi_t - A\varphi \to -\frac{i}{4\xi}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \quad \text{as } x \to -\infty.$$

Therefore from (22) one gets

$$\beta(x) \to -\frac{i}{4\xi}, \qquad \alpha(x) \to 0 \qquad \text{as } x \to -\infty.$$

By solving (24) we obtain

$$\alpha(x) = \frac{1}{a}\int_{-\infty}^{x} \hat{\varphi}^T R\varphi dx, \quad \beta(x) = -\frac{1}{a}\int_{-\infty}^{x} \hat{\psi}^T R\varphi dx - \frac{i}{4\xi}.$$

Therefore the relation (22) can be rewritten in the form

$$\varphi_t - A\varphi = \frac{1}{a}\int\limits_{-\infty}^{x} \hat{\varphi}^T R\varphi dx \cdot \psi + \left(-\frac{1}{a}\int\limits_{-\infty}^{x} \hat{\psi}^T R\varphi dx - \frac{i}{4\xi} \right) \varphi. \tag{25}$$

Using (9) we take the limit in (25) as $x \to \infty$ and obtain

$$a_t = -\int\limits_{-\infty}^{\infty} \hat{\psi}^T R\varphi \, dx,$$

$$b_t = -\frac{i}{2\xi} b + \frac{1}{a}\int\limits_{-\infty}^{\infty} \hat{\varphi}^T R\varphi \, dx - \frac{b}{a}\int\limits_{-\infty}^{\infty} \hat{\psi}^T R\varphi \, dx.$$

Consequently, for $Im\xi = 0$ we get

$$\frac{dr^+}{dt} = -\frac{i}{2\xi} r^+ + \frac{1}{2a^2}\int\limits_{-\infty}^{\infty} \left(G\varphi_2^2 + G\varphi_1^2 \right) dx.$$

We differentiate the relation $\varphi_n = C_n \psi_n$ with respect to t

$$\left.\frac{\partial\varphi}{\partial t}\right|_{\xi = \xi_n} + \left.\frac{\partial\varphi}{\partial\xi}\right|_{\xi = \xi_n} \frac{d\xi_n}{dt}$$

$$= \frac{dC_n}{dt}\psi_n + C_n \left.\frac{\partial\psi}{\partial t}\right|_{\xi = \xi_n} + C_n \left.\frac{\partial\psi}{\partial\xi}\right|_{\xi = \xi_n} \frac{d\xi_n}{dt}, \tag{26}$$

and substitute $\left.\frac{d}{d\xi}(\varphi - C_n\psi)\right|_{\xi = \xi_n}$ from (12) into (26). This results in the following formula:

$$\frac{\partial\varphi_n}{\partial t} = \frac{dC_n}{dt}\psi_n + C_n \frac{\partial\psi_n}{\partial t} - \dot{a}(\xi_n) h_n \frac{d\xi_n}{dt}, \tag{27}$$

where $\frac{\partial\varphi_n}{\partial t} \equiv \left.\frac{\partial\varphi}{\partial t}\right|_{\xi = \xi_n}$.

Similarly to the continuous spectrum case, by using (14) for the discrete spectrum, we have

$$\frac{\partial\varphi_n}{\partial t} - A\varphi_n = \left(-\frac{1}{C_n}\int\limits_{-\infty}^{x} \hat{\varphi}_n^T R\varphi_n dx \right) h_n + \left(\frac{1}{C_n}\int\limits_{-\infty}^{x} \hat{h}_n^T R\varphi_n \, dx - \frac{i}{4\xi_n} \right) \varphi_n.$$

Hence, according to (27), we have

$$\frac{dC_n}{dt}\psi_n + C_n\frac{\partial\psi_n}{\partial t} - \dot{a}\left(\xi_n\right)\frac{d\xi_n}{dt}h_n - C_nA\psi_n = \left(-\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{\varphi}_n^T R\varphi_n dx\right)h_n + \left(\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{h}_n^T R\varphi_n\, dx - \frac{i}{4\xi_n}\right)C_n\psi_n\,. \tag{28}$$

Using (13) we pass to the limit in (28), as $x\to\infty$, and obtain

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + \int\limits_{-\infty}^{\infty}\hat{h}_n^T R\psi_n dx\right)C_n,$$

$$\frac{d\xi_n}{dt} = \frac{\int\limits_{-\infty}^{\infty}\hat{\varphi}_n^T R\varphi_n\, dx}{C_n\dot{a}\left(\xi_n\right)}\,.$$

Therefore

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + \int\limits_{-\infty}^{\infty}\frac{G}{2}\left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right)dx\right)C_n\,,$$

$$\frac{d\xi_n}{dt} = \frac{\int\limits_{-\infty}^{\infty}\left(G\varphi_{n2}^2 + G\varphi_{n1}^2\right)\,dx}{2C_n\dot{a}\left(\xi_n\right)}\,.$$

Hence, according to the relation

$$\dot{a}\left(\xi_n\right) = -\frac{2i}{C_n}\int\limits_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx,$$

we have

$$\frac{d\xi_n}{dt} = \frac{i\int\limits_{-\infty}^{\infty}\left(G\varphi_{n2}^2 + G\varphi_{n1}^2\right)\,dx}{4\int\limits_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx}.$$

Lemma 2 is proved.

Let in Lemma 2

$$G = \int\limits_{-\infty}^{\infty}\left(\phi_1^2 - \phi_2^2\right)d\eta.$$

According to Lemma 1

$$\int_{-\infty}^{\infty} \left(\phi_1^2(x,\eta) - \phi_2^2(x,\eta)\right)\left(\varphi_1^2(x,\xi) + \varphi_2^2(x,\xi)\right) dx$$

$$= \frac{i}{2}\lim_{R\to\infty}\left(\frac{(\phi_1(x,\eta)\varphi_1(x,\xi)+\phi_2(x,\eta)\varphi_2(x,\xi))^2}{\eta+\xi} + \frac{(\phi_1(x,\eta)\varphi_2(x,\xi)-\phi_2(x,\eta)\varphi_1(x,\xi))^2}{\eta-\xi}\right)\Bigg|_{-R}^{R}.$$

By using (4), (5), (9) and the Riemann–Lebesgue lemma, we obtain

$$\int_{-\infty}^{\infty} \left(G\varphi_2^2 + G\varphi_1^2\right) dx = 2ab\left(\pi A^2(\xi,t) + iV.p.\int_{-\infty}^{\infty} \frac{A^2(\eta,t)}{\xi+\eta} d\eta\right).$$

Similarly,

$$\int_{-\infty}^{\infty} \left(G\varphi_{n2}^2 + G\varphi_{n1}^2\right)\, dx = 0,$$

$$\int_{-\infty}^{\infty} \left(Gh_{n2}\psi_{n2} + Gh_{n1}\psi_{n1}\right) dx = 2i\int_{-\infty}^{\infty} \frac{A^2(\eta,t)\bar{a}(\eta,t)a(\eta,t)}{\eta+\xi_n} d\eta.$$

By using Lemma 2 and the relation $\bar{a}(\xi)a(\xi) = \frac{1}{1+r^+(\xi)r^+(-\xi)}$, we have the following theorem

Theorem. *If the functions $u\,(x,t)$, $\phi_1(\eta,x,t)$, $\phi_2(\eta,x,t$ are solutions of the problem (1)–(6), then the scattering data of the operator $L\,(t)$ depend on t as*

$$\frac{dr^+}{dt} = \left(-\frac{i}{2\xi} + \pi A^2(\xi,t) + iV.p.\int_{-\infty}^{\infty} \frac{A^2(\eta,t)}{\xi+\eta} d\eta\right) r^+, \quad (Im\xi = 0)\,,$$

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + i\int_{-\infty}^{\infty} \frac{A^2(\eta,t)}{(1+r^+(\eta,t)r^+(-\eta,t))(\eta+\xi_n)}\right) C_n,$$

$$\frac{d\xi_n}{dt} = 0, \qquad n = 1,2,\ldots,N.$$

The above relations determine completely the evolution of the scattering data for the operator $L\,(t)$, which allows us to find the solutions of problem for (1)–(6) by using the inverse scattering problem method.

In conclusion we consider the following example. Let

$$u|_{t=0} = 4arctg\left(e^{2x}\right), \quad A(\eta,t) = \left(1+\eta^2\right)^{-\frac{1}{2}}.$$

In this case $r^+(\xi,0)=0$, $\xi_1(0)=i$, $C_1(0)=-2i$.
Therefore, by using the theorem

$$r^+(\xi,t)=0,\ \ \xi_1(t)=i,\ \ C_1(t)=-2i\exp\left(\frac{\pi-1}{2}t\right).$$

According to the inverse scattering problem method

$$u(x,t)=4arctg\left(\exp\left(2x-\frac{\pi-1}{2}t\right)\right),$$

$$\phi_1(x,\eta)=\frac{1}{\sqrt{\eta^2+1}}\left(\cos\eta x+\frac{\left(1-e^{-2x+g}\right)(\cos\eta x-\eta\sin\eta x}{(1+\eta^2)ch(2x-g)}\right)$$
$$+\frac{i}{\sqrt{\eta^2+1}}\left(-\sin\eta x+\frac{\left(1+e^{-2x+g}\right)(\sin\eta x+\eta\cos\eta x}{(1+\eta^2)ch(2x-g)}\right),$$
$$\phi_2(x,\eta)=\frac{1}{\sqrt{\eta^2+1}}\left(\cos\eta x-\frac{\left(1+e^{-2x+g}\right)(\cos\eta x-\eta\sin\eta x}{(1+\eta^2)ch(2x-g)}\right)$$
$$+\frac{i}{\sqrt{\eta^2+1}}\left(\sin\eta x+\frac{\left(1-e^{-2x+g}\right)(\sin\eta x+\eta\cos\eta x}{(1+\eta^2)ch(2x-g)}\right),$$

where $g(t)=\frac{(\pi-1)\,t}{2}$.

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