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Homogenization of the Neumann–Fourier Problem in a Thick Two-Level Junction of Type 3:2:1

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We consider a mixed boundary-value problem for the Poisson equation in a two-level junction Ω_ε which is the union of a domain Ω_0 and a large number of thin cylinders with cross-section of order $\mathcal{O}(\varepsilon^2)$. The thin cylinders are divided into two levels depending on their lengths. In addition, the thin cylinders from each level are ε-periodically alternated. The nonuniform Neumann conditions are given on the lateral sides of the thin cylinders from the first level and the uniform Fourier conditions are given on the lateral sides of the thin cylinders from the second level. We study the asymptotic behavior of the solution as $\varepsilon \to 0$. The convergence theorem and the convergence of the energy integral are proved.

Key words: homogenization, multi-level junctions, asymptotic behavior of solutions.

Mathematics Subject Classification 2000: 35B27, 35J25, 35C20, 35B25.

1. Introduction and Statement of the Main Result

Asymptotic methods for the investigation of boundary-value problems in domains with complex dependence on a small parameter (perforated domains, partially perforated domains, skeleton structures, and thin domains) were considered in numerous papers (see, e.g., [1]–[14]) and the references therein). Boundary-value problems in thick singularly degenerating junctions (the number of components of such junctions increases infinitely if the perturbation parameter ε tends to zero) have specific difficulties and deserve special attention. As shown in [15], boundary-value problems in thick singularly degenerating junctions lose coercitivity as $\varepsilon \to 0$, that essentially complicates asymptotic researches.

It is necessary to note that boundary-value problems in domains with quickly oscillating boundaries, when ratio of the amplitude to the period of the oscillation is bounded or infinitesimal quantity as the period of the oscillation tends to zero, have no such asymptotic difficulties and properties (see, e.g., [13, 16]). For thick junctions this ratio tends to infinity.

The first works in this direction were papers [17]–[19] in which the asymptotic behavior of the Green function of the Neumann problem for the Helmholtz equation in an unbounded thick junction was studied. In [20]–[30] thick singularly degenerating junctions were classified, asymptotic methods for the investigation of the main boundary-value problems of mathematical physics in thick junctions of different types were developed, the convergence theorems were proved, the first terms of asymptotic expansions were constructed, the corresponding estimates were proved, and the influence of boundary conditions given at the boundaries of thick junctions and the geometric configuration of thick junctions on the asymptotic behavior of solutions was investigated.

A thick junction Ω_ε of type $k : p : d$ is a domain in $\mathbb{R}^n$ which consists of some domain Ω_0 and a large number of ε-periodically situated thin domains along some manifold on the boundary of Ω_0. This manifold is called the joint zone and the domain Ω_0 is called the junction's body. Here ε is a small parameter which characterizes the distance between the neighboring thin domains and their thicknesses. In general, the junction's body and the joint zone can depend on ε as well. The type $k : p : d$ of a thick junction refers to the limiting dimensions of the body, the joint zone, and each of the attached thin domains respectively.

These thick junctions are the prototypes of widely used engineering constructions, industrial installations, spaceship grids as well as of other physical and biological systems with very distinct characteristic scales.

The aim of researches is to develop rigorous asymptotic methods for boundary-value problems in thick junctions as the parameter ε goes to 0, i.e., when the number of the attached thin domains infinitely increases and their thicknesses tend to zero.

In the present paper we consider a new kind of thick junctions, namely, thick multi-level junctions. A thick multi-level junction is a thick junction in which the thin domains are divided into finitely many levels depending on their lengths. In addition the thin domains from each level are ε-periodically alternated along the joint zone.

For the fist time the problem in a plane two-level junction was considered in [31] where the asymptotic behavior of eigenvalues and eigenfunctions of the spectral problem was studied (the full proofs were published in [32]). In [33], with the help of special extension operators, a convergence theorem was proved for a solution to the Poisson equation in a plane two-level junction with homogeneous Fourier boundary conditions at the boundaries of thin rods. In [34] the authors proved the convergence theorem and the convergence of the energy integral for a solution to the Poisson equation in a plane two-level junction with ε-periodically alternated boundary Neumann and Dirichlet conditions at the boundaries of thin rods from the first and the second levels respectively. In [35], with the method of matched asymptotic expansions being used, the first terms of the asymptotic expansion of a solution to a boundary-value problem with minimum smoothness conditions imposed on the right-hand side were constructed and asymptotic estimates in the Sobolev space $H^1(\Omega_\varepsilon)$ as $\varepsilon \to 0$ were proved. It should be noted that these plain thick multi-level junctions have type $2:1:1$ according to the classification given in [20]–[30].

In the present paper we study the asymptotic behavior of a solution to a mixed boundary-value problem in the three-dimensional thick two-level junction of type $3:2:1$ and investigate the influence of boundary conditions on the asymptotic behavior. In particular, the inhomogeneous Neumann boundary conditions are given on the lateral sides of the thin cylinders from the first level and the homogeneous Fourier boundary conditions are given on the lateral sides of the thin cylinders from the second level. Besides, the thin cylinders from the first and from the second levels have both more dense packing on the cell of the joining. Thus, except special perturbation of the domain, the boundary conditions are ε-periodically changed in the problem.

1.1. Statement of the Problem

Let B be the finite union of smooth plane domains which are not crossed and touched. In addition, the set B is strongly situated in the square $\{(\xi_1, \xi_2) : 0 < \xi_1 < 1, 0 < \xi_2 < 1\}$. Let us divide B into two classes: $B^{(1)} = \bigcup_{k=1}^{K_1} B_k^{(1)}$ and $B^{(2)} = \bigcup_{k=1}^{K_2} B_k^{(2)}$ (see. Fig. 1).

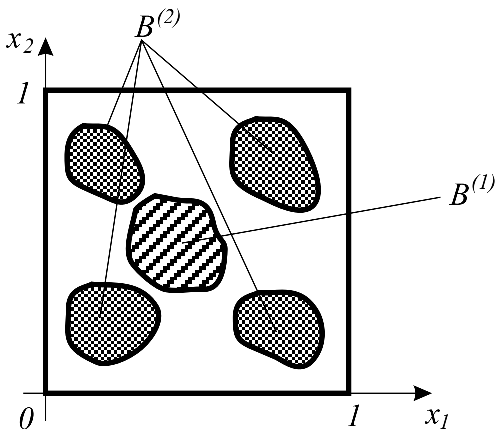

Figure 1.

A model thick two-level junction Ω_ε consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^3 : \ x' = (x_1, x_2) \in Q, \ 0 < x_3 < \gamma(x')\},$$

where $Q = (0,a) \times (0,a)$, $\gamma \in C^1(\overline{Q})$, $\min_{x' \in \overline{Q}} \gamma(x') = \gamma_0 > 0$, and a large number of the thin cylinders

$$G_\varepsilon^{(1)} = \bigcup_{i,j=0}^{N-1} \left(\bigcup_{k=1}^{K_1} \left\{ x : \ (\varepsilon^{-1}x_1 - i, \varepsilon^{-1}x_2 - j) \in B_k^{(1)}, \ x_3 \in (-d_1, 0] \right\} \right),$$

$$G_\varepsilon^{(2)} = \bigcup_{i,j=0}^{N-1} \left(\bigcup_{k=1}^{K_2} \left\{ x : \ (\varepsilon^{-1}x_1 - i, \varepsilon^{-1}x_2 - j) \in B_k^{(2)}, \ x_3 \in (-d_2, 0] \right\} \right).$$

Here N is a large natural number, $\varepsilon = a/N$ is a small discrete parameter that characterizes the distance between nearby thin cylinders and their thicknesses; $0 < d_2 \le d_1$. Thus, $\Omega_\varepsilon = \Omega_0 \bigcup G_\varepsilon^{(1)} \bigcup G_\varepsilon^{(2)}$. The thin cylinders are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ depending on their lengths, and they are ε-periodically alternated along the Ox_1-direction and Ox_2-direction and they are joined with Ω_0 over the ε-homothetic images $\varepsilon(i + j + B_k^{(1)})$, $i, j = 0, 1, \ldots, N-1$, $k = 1, \ldots, K_1$, and $\varepsilon(i + j + B_k^{(2)})$, $i, j = 0, 1, \ldots, N-1$, $k = 1, \ldots, K_2$, of the classes $B^{(1)}$ and $B^{(2)}$ respectively. The cell of alternation is shown on Fig. 2.

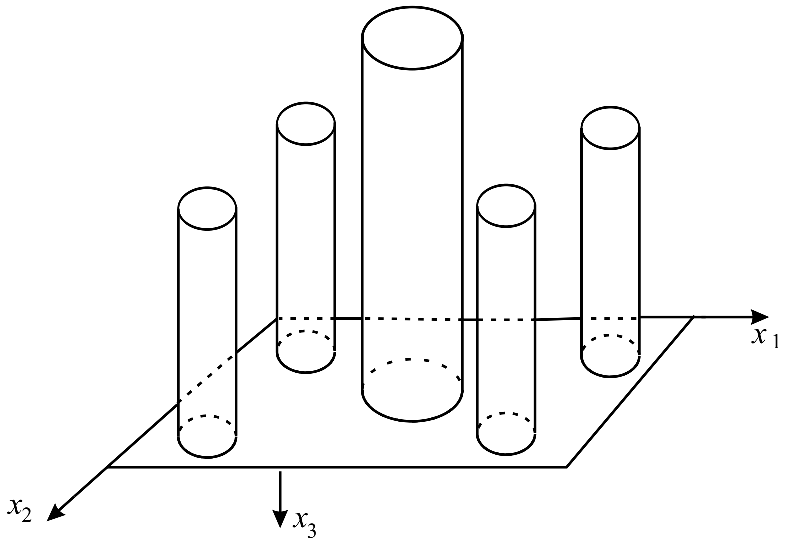

Figure 2.

In Ω_ε we consider the following problem

$$\begin{array}{rcll} -\Delta u_\varepsilon(x) & = & f_\varepsilon(x), & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(x) & = & \varepsilon g_\varepsilon(x), & x \in S_\varepsilon^{(1)}, \\ \partial_\nu u_\varepsilon(x) & = & -\varepsilon k_0 u_\varepsilon(x), & x \in S_\varepsilon^{(2)}, \\ \partial_\nu u_\varepsilon(x) & = & 0, & x \in \partial\Omega_\varepsilon \backslash (S_\varepsilon^{(1)} \cup S_\varepsilon^{(2)}), \end{array} \qquad (1)$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, $S_\varepsilon^{(i)},\ i = 1,2$ are the unions of the lateral surfaces of the thin cylinders from the level $G_\varepsilon^{(i)}$.

Without loss of generality, we can assume that $f_\varepsilon \in L^2(\Omega_1)$ where $\overline{\Omega}_1 = \overline{\Omega}_0 \cup \overline{D}_1,\ D_1 = Q \times (-d_1, 0)$. Analogously we define $D_2 = Q \times (-d_2, 0)$ and $\overline{\Omega}_2 = \overline{\Omega}_0 \cup \overline{D}_2$. Assume that

$$f_\varepsilon \longrightarrow f_0 \quad \text{in} \quad L^2(\Omega_1) \quad \text{as} \quad \varepsilon \to 0. \qquad (2)$$

We also suppose that the function g_ε and its generalized derivatives with respect to x_1 and x_2 belong to $L^2(D_1)$ and

$$\begin{array}{c} \exists\, C_0 > 0 \quad \forall\, \varepsilon > 0 \quad \|\partial_{x_m} g_\varepsilon\|_{L^2(D_1)} \le C_0, \qquad m = 1,2; \\ g_\varepsilon \longrightarrow g_0 \quad \text{in} \quad L^2(D_1) \quad \text{as} \quad \varepsilon \to 0. \end{array} \qquad (3)$$

The function $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is called a generalized solution to problem (1) if it satisfies the integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \varepsilon k_0 \int\limits_{S_\varepsilon^{(2)}} u_\varepsilon \varphi \, d\sigma_x = \int\limits_{\Omega_\varepsilon} f_\varepsilon \varphi \, dx + \varepsilon \int\limits_{S_\varepsilon^{(1)}} g_\varepsilon \varphi \, d\sigma_x \quad \forall \varphi \in H^1(\Omega_\varepsilon). \qquad (4)$$

It follows from the fundamental statements of the theory of boundary-value problems that for every fixed value $\varepsilon > 0$ there exists a unique generalized solution to problem (1).

The aim of the present paper is to study the asymptotic behavior of the solution to problem (1) as $\varepsilon \to 0$, i.e., as the number of thin cylinders increases infinitely and their thicknesses tend to zero, and to investigate the influence of the alternation of the boundary Neumann and the Fourier conditions on the asymptotic behavior of the solution.

1.2. Features of Investigation and Formulation of the Main Result

For Neumann boundary-value problems in perturbed domains E.Ya. Khruslov introduced the notion of strongly connected domains D_ϵ depending on a small parameter ϵ. This means that we suppose the existence of an extension operator from $H^1(D_\epsilon)$ into $H^1(\mathbb{R}^n)$ uniformly bounded with respect to ϵ. Later, D. Cioranescu, J. Saint Jean Paulin, O.A. Oleinik, G.A. Iosif'yan, and A.S. Shamaev (see, e.g., [4, 10]) proved the existence of the extension operators and proposed a procedure for their construction in perforated domains of an ϵ-periodic structure. Uniformly bounded extension operators play a very important role in the investigation of boundary-value problems in the domains with complex dependence on a small parameter.

However, as it was shown in [20]–[29], thick junctions do not belong to the class of strongly connected (as well as weakly connected) domains, i.e., for these domains there are no extension operators that would be bounded uniformly with respect to the parameter ε in the corresponding Sobolev spaces. This is one of the main specific features of investigation of boundary-value problems in thick junctions. In [20]–[29] the procedures were developed for the construction of special extension operators preserving the class of a space for solutions of boundary-value problems in thick junctions of different types and with the help of these operators the asymptotic behavior of solutions was studied and convergence theorems were proved.

Later, in [36] where the homogeneous Neumann boundary-value problem in a thick one-level junction was studied, it was shown that if the boundaries of thin cylinders are rectilinear along the Ox_3-axis, then the solution of the boundary-value problem can be extended by zero to prove the convergence theorem. This is explained by the fact that due to the rectilinearity of the boundaries of cylinders this extension preserves the generalized derivative with respect to x_3. We use this fact in the present paper. However, for thick two-level junctions it is necessary to construct two special operators of zero extension into two different domains. In the case, when the thin cylinders of a thick two-level junction are of variable thickness, it is necessary to construct special extension operators (for the thick plane two-level junctions it was made in [33]).

To formulate the main result we introduce the following operations of extension by zero for functions from the space $H^1(\Omega_\varepsilon)$:

$$\widetilde{y_\varepsilon}^{(1)}(x) = \begin{cases} y_\varepsilon, & x \in \Omega_0 \cup G_\varepsilon^{(1)}, \\ 0, & x \in D_1 \setminus G_\varepsilon^{(1)}, \end{cases} \qquad \widetilde{y_\varepsilon}^{(2)}(x) = \begin{cases} y_\varepsilon, & x \in \Omega_0 \cup G_\varepsilon^{(2)}, \\ 0, & x \in D_2 \setminus G_\varepsilon^{(2)}, \end{cases} \tag{5}$$

where $D_1 = Q \times (-d_1, 0)$ and $D_2 = Q \times (-d_2, 0)$ are parallelepipeds filled up with thin cylinders of the first and the second levels respectively in the limit passage as $\varepsilon \to 0$. It is obvious that $\widetilde{y_\varepsilon}^{(1)}$ and $\widetilde{y_\varepsilon}^{(2)}$ belong to the anisotropic Sobolev spaces $W^{0,1}(D_i) = \big\{v \in L^2(D_i):$ there exists a generalized derivative $\partial_{x_3} v \in L^2(D_i)\big\}$, $i = 1, 2$.

Theorem 1. *The solution u_ε to problem (1) satisfies the following relations*

$$\left.\begin{array}{lll} u_\varepsilon \xrightarrow{w} v_0^+ & in & H^1(\Omega_0), \\ \widetilde{u_\varepsilon}^{(1)} \xrightarrow{w} |B^{(1)}|\, v_0^{(1,-)} & in & W^{0,1}(D_1), \\ \widetilde{u_\varepsilon}^{(2)} \xrightarrow{w} |B^{(2)}|\, v_0^{(2,-)} & in & W^{0,1}(D_2), \end{array}\right\} \quad as \quad \varepsilon \to 0,$$

where

$$\mathbf{v}_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{(1,-)}(x), & x \in D_1, \\ v_0^{(2,-)}(x), & x \in D_2, \end{cases} \tag{6}$$

is a solution of the following problem

$$\begin{array}{rcll} -\Delta v_0^+(x) &=& f_0(x), & x \in \Omega_0, \\ \partial_\nu v_0^+(x) &=& 0, & x \in \partial\Omega_0 \setminus Q, \\ -|B^{(1)}|\, \partial_{x_3}^2 v_0^{(1,-)}(x) &=& |B^{(1)}|\, f_0(x) + l^{(1)} g_0(x), & x \in D_1, \\ \partial_{x_3} v_0^{(1,-)}(x', -d_1) &=& 0, & x' \in Q. \\ -|B^{(2)}|\, \partial_{x_3}^2 v_0^{(2,-)}(x) + k_0 l^{(2)} |B^{(2)}| v_0^{(2,-)}(x) &=& |B^{(2)}|\, f_0(x), & x \in D_2, \\ \partial_{x_3} v_0^{(2,-)}(x', -d_2) &=& 0, & x' \in Q. \\ v_0^{(1,-)}(x',0) = v_0^{(2,-)}(x',0) &=& v_0^+(x',0), & x' \in Q. \\ |B^{(1)}| \partial_{x_3} v_0^{(1,-)}(x',0) + |B^{(2)}| \partial_{x_3} v_0^{(2,-)}(x',0) &=& \partial_{x_3} v_0^+(x',0), & x' \in Q. \end{array} \tag{7}$$

Here $|B^{(i)}| = \sum_{k=1}^{K_i} |B_k^{(i)}|$, $l^{(i)} = \sum_{k=1}^{K_i} l_k^{(i)}$, *where* $|B_k^{(i)}|$, $l_k^{(i)}$ *are the area and the perimeter of the plane domain* $B_k^{(i)}$ *respectively,* $i = 1, 2$.

2. Auxiliary Asymptotic Estimates

Investigation of the boundary-value problems in thick junctions with inhomogeneous Neumann, Fourier, or Steklov boundary conditions on the boundaries of the attached thin domains encounters special difficulties. In [37, 26, 27, 28] for the homogenization of these boundary-value problems there was suggested a new approach with the special integral identities being used.

For problem (1) this will be integral identities (9). Analogously as in [26], for the 1-periodic extensions with respect to ξ_1 and ξ_2 of solutions $Y_k^{(i)}$, $k = 1, \ldots, K_i$

of the following problems

$$\begin{array}{rcll} \Delta_\xi Y_k^{(i)}(\xi) & = & l_k^{(i)}|B_k^{(i)}|^{-1}, & \xi=(\xi_1,\xi_2)\in B_k^{(i)}, \\ \partial_{\nu(\xi)}Y_k^{(i)}(\xi) & = & 1, & \xi\in\partial B_k^{(i)}, \\ \int_{B_k^{(i)}} Y_k^{(i)}(\xi)d\xi & = & 0, & \end{array} \tag{8}$$

we prove

$$\varepsilon\int\limits_{S_\varepsilon^{(i)}} v\,d\sigma_x=\sum_{k=1}^{K_i}\frac{l_k^{(i)}}{|B_k^{(i)}|}\int\limits_{-d_i}^{0}\int\limits_{\bigcup\limits_{i,j=0}^{N-1}\varepsilon(i+j+B_k^{(i)})} v\,dx'dx_3$$

$$+\,\varepsilon\sum_{k=1}^{K_i}\int\limits_{-d_i}^{0}\int\limits_{\bigcup\limits_{i,j=0}^{N-1}\varepsilon(i+j+B_k^{(i)})}\nabla_\xi Y_k^{(i)}|_{\xi=\frac{x'}{\varepsilon}}\cdot\nabla_{x'}\,v\,dx'dx_3\quad\forall\,v\in H^1\big(G_\varepsilon^{(i)}\big),\quad i=1,2. \tag{9}$$

From (9) it follows that for any function $v^2\in H^1\big(G_\varepsilon^{(i)}\big)$

$$c\int\limits_{G_\varepsilon^{(i)}} v^2\,dx\le\varepsilon\int\limits_{S_\varepsilon^{(i)}} v^2\,d\sigma_x+\varepsilon\sum_{k=1}^{K_i}\int\limits_{-d_i}^{0}\int\limits_{\bigcup\limits_{i,j=0}^{N-1}\varepsilon(i+j+B_k^{(i)})}|\nabla_\xi Y_k^{(i)}|\cdot|\nabla_{x'}\;(v^2)|\,dx'dx_3,$$

where $c=\min\{l_k^{(i)}/|B_k^{(i)}|\}$. Taking into account that $\sup_{\xi\in B_k^{(i)}}|\nabla_\xi Y_k^{(i)}|\le c_k$, we obtain the following identities

$$\int\limits_{G_\varepsilon^{(i)}} v^2\,dx\le C_0\varepsilon\left(\int\limits_{S_\varepsilon^{(i)}} v^2\,d\sigma_x+\int\limits_{G_\varepsilon^{(i)}}|\nabla_{x'}(v^2)|\,dx\right),\quad i=1,2. \tag{10}$$

In the Sobolev space H^1 along with the norm $\|u\|_{H^1(\Omega_\varepsilon)}=\big(\int_{\Omega_\varepsilon}(|\nabla u|^2+u^2)dx\big)^{\frac{1}{2}}$, we introduce a new norm $\|\cdot\|_\varepsilon$ generated by the scalar product

$$(u,v)_\varepsilon=\int\limits_{\Omega_\varepsilon}\nabla u\cdot\nabla v\,dx+\varepsilon k_0\int\limits_{S_\varepsilon^{(2)}} uv\,d\sigma_x\qquad\forall\,u,v\in H^1(\Omega_\varepsilon).$$

Lemma 1. *The norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ are uniformly equivalent, i.e., there exist constants $C_1>0$, $C_2>0$ and $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and $u\in H^1(\Omega_\varepsilon)$ the following relations hold*

$$C_1\|u\|_{H^1(\Omega_\varepsilon)}\le\|u\|_\varepsilon\le C_2\|u\|_{H^1(\Omega_\varepsilon)}. \tag{11}$$

P r o o f. The right inequality in (11) follows from the inequality

$$\varepsilon\int\limits_{S_\varepsilon^{(i)}}v^2\,d\sigma_x\le C_3\Big(\int\limits_{G_\varepsilon^{(i)}}v^2\,dx+\varepsilon^2\int\limits_{G_\varepsilon^{(i)}}|\nabla v|^2\,dx\Big)\quad\forall\,v\in H^1\big(G_\varepsilon^{(i)}\big),\quad i=1,2, \tag{12}$$

which was proved in [26]. Let us prove the left inequality in (11). Using (9) and (10) we get

$$\|u\|^2_{H^1(\Omega_\varepsilon)}=\int\limits_{\Omega_0}|\nabla u|^2\,dx+\int\limits_{\Omega_0\cup G_\varepsilon^{(1)}}u^2\,dx+\int\limits_{G_\varepsilon^{(2)}}u^2\,dx\le\int\limits_{\Omega_0}|\nabla u|^2\,dx+\int\limits_{\Omega_0\cup G_\varepsilon^{(1)}}u^2dx$$
$$+\,\varepsilon C_0\int\limits_{S_\varepsilon^{(2)}}u^2\,d\sigma_x+\varepsilon C_0\int\limits_{G_\varepsilon^{(i)}}|2u\nabla_{x'}\,u|\,dx'dx_3,$$

whence

$$\|u\|^2_{H^1(\Omega_\varepsilon)}\le c_1\|u\|^2_\varepsilon+\int\limits_{\Omega_0\cup G_\varepsilon^{(1)}}u^2\,dx. \tag{13}$$

Now let us show that there exists a positive constant c_2 such that for ε small enough

$$\int\limits_{\Omega_0\cup G_\varepsilon^{(1)}}u^2\,dx\le c_2\|u\|^2_\varepsilon\qquad\forall\,u\in H^1(\Omega_\varepsilon). \tag{14}$$

We argue by contradiction. If not, then there exist sequences $\{\varepsilon_n:n\in\mathbb{N}\}$ and $\{v_{\varepsilon_n}\}\in H^1(\Omega_{\varepsilon_n})$ such that $\lim\limits_{n\to\infty}\varepsilon_n=0$,

$$\int\limits_{\Omega_0\cup G_{\varepsilon_n}^{(1)}}v^2_{\varepsilon_n}\,dx=1, \tag{15}$$

$$\|v_{\varepsilon_n}\|^2_{\varepsilon_n}=\int\limits_{\Omega_{\varepsilon_n}}|\nabla v_{\varepsilon_n}|^2\,dx+\varepsilon k_0\int\limits_{S_{\varepsilon_n}^{(2)}}v^2_{\varepsilon_n}\,d\sigma_x<\frac{1}{n}\qquad\forall n\in\mathbb{N}. \tag{16}$$

Since the sequence $\{v_{\varepsilon_n}\}$ is bounded in $H^1(\Omega_0)$, there exists a subsequence that is fundamental in $L^2(\Omega_0)$. Denote this subsequence again by $\{v_{\varepsilon_n}\}$. Furthermore,

$$\begin{aligned}\|v_{\varepsilon_n} - v_{\varepsilon_m}\|^2_{H^1(\Omega_0)} &\le \|v_{\varepsilon_n} - v_{\varepsilon_m}\|^2_{L^2(\Omega_0)} + 2\|\nabla v_{\varepsilon_n}\|^2_{L^2(\Omega_0)} + 2\|\nabla v_{\varepsilon_m}\|^2_{L^2(\Omega_0)} \\ &\le \|v_{\varepsilon_n} - v_{\varepsilon_m}\|^2_{L^2(\Omega_0)} + \frac{2}{n} + \frac{2}{m} \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Hence $\{v_{\varepsilon_n}\}$ is fundamental in $H^1(\Omega_0)$ and therefore it converges to an element $v_0 \in H^1(\Omega_0)$. Using relation (16) we get $\int_{\Omega_0} |\nabla v_0|^2\, dx = 0$, which implies that $v_0 = \text{const}$ in $H^1(\Omega_0)$. Taking into account properties of the trace operator, we conclude that

$$v_{\varepsilon_n}\big|_{x_3=0} \xrightarrow{s} v_0 \equiv \text{const} \quad \text{in} \quad L^2(Q) \quad \text{as} \quad n \to \infty. \tag{17}$$

From (16) and (10) it follows that

$$\int\limits_{B^{(2)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x', 0)\, dx' \le c_6 \left(\int\limits_{G^{(2)}_{\varepsilon_n}} |\nabla v_{\varepsilon_n}|^2\, dx + \int\limits_{G^{(2)}_{\varepsilon_n}} v^2_{\varepsilon_n}\, dx \right) \longrightarrow 0, \qquad n \to \infty, \tag{18}$$

where $B^{(2)}_{\varepsilon_n} = \bigcup\limits_{i,j=0}^{N-1} \left(\bigcup\limits_{k=1}^{K_2} \varepsilon_n (i + j + B^{(2)}_k) \right)$.

Consider 1-periodic function $\chi_2(\xi)$, $\xi \in \mathbb{R}^2$ which is defined in the square $[0,1]^2$ as follows:

$$\chi_2(\xi) = \begin{cases} 1, & \xi \in B^{(2)}, \\ 0, & \xi \in [0,1]^2 \setminus B^{(2)}. \end{cases}$$

It is easy to verify that

$$\chi_2\left(\frac{x'}{\varepsilon}\right) \xrightarrow{w} |B^{(2)}| \quad \text{in} \quad L^2([0,1]^2) \quad \text{as} \quad \varepsilon \to 0, \tag{19}$$

where $|B^{(2)}|$ denotes the Lebesgue measure of $B^{(2)}$. Using relations (17) and (19) we obtain

$$\int\limits_{B^{(2)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x', 0)\, dx' = \int\limits_Q \chi_2\left(\frac{x'}{\varepsilon_n}\right) v^2_{\varepsilon_n}(x', 0)\, dx' \longrightarrow |B^{(2)}| \int\limits_Q v_0^2\, dx', \quad n \to \infty.$$

On the other hand, according to (18) we have

$$|B^{(2)}| \int\limits_Q v_0^2\, dx' = 0. \tag{20}$$

Since $v_0 \equiv \text{const}$ in Ω_0, it follows from (20) that $v_0 \equiv 0$ almost everywhere in Ω_0.

Let us find the limit of $\int_{G^{(1)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x)\,dx$ as $n \to \infty$. According to (17) and (20) we get

$$\int\limits_{B^{(1)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x',0)\,dx' = \int\limits_{Q} \chi_1\left(\frac{x'}{\varepsilon_n}\right) v^2_{\varepsilon_n}(x',0)\,dx' \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{21}$$

where $B^{(1)}_{\varepsilon_n} = \bigcup\limits_{i,j=0}^{N-1}\left(\bigcup\limits_{k=1}^{K_1} \varepsilon_n(i+j+B^{(1)}_k)\right)$ and $\chi_1(\xi), \xi \in \mathbb{R}^2$, is 1-periodic function which is defined in the square $[0,1]^2$ as follows

$$\chi_1(\xi) = \begin{cases} 1, & \xi \in B^{(1)}, \\ 0, & \xi \in [0,1]^2 \setminus B^{(1)}. \end{cases}$$

The inequality

$$\int\limits_{G^{(1)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x)\,dx \le 2d_1^2 \int\limits_{G^{(1)}_{\varepsilon_n}} |\nabla v_{\varepsilon_n}|^2\,dx + 2d_1 \int\limits_{B^{(1)}_{\varepsilon_n}} v^2_{\varepsilon_n}(x',0)\,dx'$$

and relations (16) , (21) yield that $\|v_{\varepsilon_n}\|^2_{L^2\left(G^{(1)}_{\varepsilon_n}\right)} \longrightarrow 0$ as $n \to \infty$.

Thus $\|v_{\varepsilon_n}\|^2_{L^2\left(\Omega_0 \cup G^{(1)}_{\varepsilon_n}\right)} \longrightarrow 0$ as $n \to \infty$. However, this is at variance with (15). This contradiction establishes estimate (14). Now, by virtue of (13) and (14), we obtain the left inequality in (11). The lemma is proved.

R e m a r k 1. Here and in what follows, all constants c_i and C_i in inequalities are independent of ε.

Let us prove uniform estimates for the solution of problem (1). Setting $\varphi = u_\varepsilon$ in the integral identity (4) and using (12) we obtain

$$\|u_\varepsilon\|^2_\varepsilon \le C_4\left(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|g_\varepsilon\|_{L^2\left(S^{(1)}_\varepsilon\right)}\right)\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

Taking Lemma 1 into account we get

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_5\left(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|g_\varepsilon\|_{L^2\left(S^{(1)}_\varepsilon\right)}\right). \tag{22}$$

Assuming (3) and using the integral identity (9) we deduce the following inequality

$$\sqrt{\varepsilon}\|g_\varepsilon\|_{L^2\left(S^{(1)}_\varepsilon\right)} \le C_6. \tag{23}$$

Thus, taking into account this inequality and relation (2) we conclude from (22) that there exist constant $C_7 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_7. \tag{24}$$

3. Proof of the Convergence Theorem

1. We extend the solution u_ε by zero (see (5)). Since the boundaries of the thin cylinders are rectilinear, we get $\widetilde{u_\varepsilon}^{(1)} \in W^{0,1}(D_1)$ and $\widetilde{u_\varepsilon}^{(2)} \in W^{0,1}(D_2)$. Furthermore,

$$\partial_{x_3}\big(\widetilde{u_\varepsilon}^{(1)}\big) = \widetilde{\partial_{x_3} u_\varepsilon}^{(1)}, \qquad \partial_{x_3}\big(\widetilde{u_\varepsilon}^{(2)}\big) = \widetilde{\partial_{x_3} u_\varepsilon}^{(2)}. \tag{25}$$

Let us find the limits of the extensions for the solution u_ε. Using relation (24) we conclude that the quantities $\|u_\varepsilon\|_{H^1(\Omega_0)}$, $\|\widetilde{u_\varepsilon}^{(1)}\|_{W^{0,1}(D_1)}$, $\|\widetilde{u_\varepsilon}^{(2)}\|_{W^{0,1}(D_2)}$ are uniformly bounded with respect to ε. Hence, there exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$, again denoted by ε such that

$$\left.\begin{array}{llll} u_\varepsilon & \xrightarrow{w} & v_0^+ & \text{in } H^1(\Omega_0), \\ \widetilde{u_\varepsilon}^{(1)} & \xrightarrow{w} & |B^{(1)}| v_0^{(1,-)} & \text{in } W^{0,1}(D_1), \\ \widetilde{u_\varepsilon}^{(2)} & \xrightarrow{w} & |B^{(2)}| v_0^{(2,-)} & \text{in } W^{0,1}(D_2), \\ \widetilde{\partial_{x_m} u_\varepsilon}^{(1)} & \xrightarrow{w} & \gamma_m^{(1)} & \text{in } L^2(D_1),\ m=1,2,3, \\ \widetilde{\partial_{x_m} u_\varepsilon}^{(2)} & \xrightarrow{w} & \gamma_m^{(2)} & \text{in } L^2(D_2),\ m=1,2,3, \end{array}\right\} \text{ as } \varepsilon \to 0, \tag{26}$$

where v_0^+, $v_0^{(i,-)}$, $\gamma_m^{(i)}$, $i=1,2$, $m=1,2,3$, are certain functions which will be determined in what follows.

Let us determine $\gamma_3^{(i)}$, $i=1,2$. Consider an arbitrary function $\psi \in C_0^\infty(D_i)$. Using (25) we get

$$\int_{D_i} \widetilde{\partial_{x_3} u_\varepsilon}^{(i)}\, \psi\, dx = \int_{D_i} \partial_{x_3} \widetilde{u_\varepsilon}^{(i)}\, \psi\, dx = -\int_{D_i} \widetilde{u_\varepsilon}^{(i)}\, \partial_{x_3}\psi\, dx \quad \forall \psi \in C_0^\infty(D_i),\ \ i=1,2.$$

Passing to the limit as $\varepsilon \to 0$ in this equality we obtain

$$\int_{D_i} \gamma_3^{(i)} \psi\, dx = -|B^{(i)}| \int_{D_i} v_0^{(i,-)} \partial_{x_3}\psi\, dx \qquad \forall \psi \in C_0^\infty(D_i), \tag{27}$$

which implies that $\gamma_3^{(i)} = |B^{(i)}| \partial_{x_3} v_0^{(i,-)}$ almost everywhere in D_i, $i=1,2$.

Let us determine $\gamma_m^{(i)}$, $i=1,2$, $m=1,2$. Let $(b_1^{(i)}(k), b_2^{(i)}(k))$ be the geometric center of gravity of the domain $B_k^{(i)}$ $(k=1,\dots K_i)$. Consider the functions

$$Z_{m,k}^{(i)}(\xi_m) = -\xi_m + b_m^{(i)}(k) + [\xi_m], \quad k=1,\dots K_i, \ \ i=1,2, \quad m=1,2,$$

where $[t]$ is the integer part of t. With the help of this functions we determine the following test function

$$\Phi_m^{(1)}(x) = \begin{cases} 0, & x \in \Omega_0 \bigcup G_\varepsilon^{(2)}, \\ \varepsilon Z_{m,k}^{(1)}\left(\frac{x_m}{\varepsilon}\right)\psi(x), & x \in G_\varepsilon^{(1)}(k), \ \ k=1,\dots,K_1, \ \ m=1,2, \end{cases}$$

$$\forall \psi \in C_0^\infty(D_1),$$

$$\Phi_m^{(2)}(x) = \begin{cases} 0, & x \in \Omega_0 \bigcup G_\varepsilon^{(1)}, \\ \varepsilon Z_{m,k}^{(2)}\left(\frac{x_m}{\varepsilon}\right)\psi(x), & x \in G_\varepsilon^{(2)}(k), \quad k = 1, \ldots, K_2, \quad m = 1, 2, \end{cases}$$

$$\forall \psi \in C_0^\infty(D_2),$$

where $G_\varepsilon^{(i)}(k) = \bigcup\limits_{i,j=0}^{N-1} \Big(\{x : \ (\varepsilon^{-1}x_1 - i, \varepsilon^{-1}x_1 - j) \in B_k^{(i)}, \ x_3 \in (-d_i, 0]\} \Big)$. It is easy to see that $\Phi_m^{(i)}(x) \in H^1(\Omega_\varepsilon)$ and

$$\nabla \Phi_1^{(i)} = \left(-\psi + \varepsilon Z_{1,k}^{(i)}\left(\tfrac{x_1}{\varepsilon}\right)\partial_{x_1}\psi, \ \varepsilon Z_{1,k}^{(i)}\left(\tfrac{x_1}{\varepsilon}\right)\partial_{x_2}\psi, \ \varepsilon Z_{1,k}^{(i)}\left(\tfrac{x_1}{\varepsilon}\right)\partial_{x_3}\psi \right),$$

$$\nabla \Phi_2^{(i)} = \left(\varepsilon Z_{2,k}^{(i)}\left(\tfrac{x_2}{\varepsilon}\right)\partial_{x_1}\psi, \ -\psi + \varepsilon Z_{2,k}^{(i)}\left(\tfrac{x_2}{\varepsilon}\right)\partial_{x_2}\psi, \ \varepsilon Z_{2,k}^{(i)}\left(\tfrac{x_2}{\varepsilon}\right)\partial_{x_3}\psi \right),$$

$$x \in G_\varepsilon^{(i)}(k), \quad k = 1, \ldots K_i, \quad i = 1, 2.$$

Substituting the functions $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$ into the integral identity (4) we get

$$\sum_{k=1}^{K_1} \int\limits_{G_\varepsilon^{(1)}(k)} \left(-\frac{\partial u_\varepsilon}{\partial x_m}\psi + \varepsilon Z_{m,k}^{(1)} \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \varepsilon Z_{m,k}^{(1)} \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \varepsilon Z_{m,k}^{(1)} \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \psi}{\partial x_3} \right) dx$$

$$= \sum_{k=1}^{K_1} \left(\int\limits_{G_\varepsilon^{(1)}(k)} \varepsilon f_\varepsilon Z_{m,k}^{(1)} \psi \, dx + \varepsilon^2 \int\limits_{S_\varepsilon^{(1)}(k)} Z_{m,k}^{(1)} g_\varepsilon \psi \, d\sigma_x \right), \quad m = 1, 2.$$

Then, using relations (2), (3),(23) and (24) we have

$$\left| \int\limits_{G_\varepsilon^{(1)}} \frac{\partial u_\varepsilon}{\partial x_m}\psi \, dx \right|$$

$$\le \varepsilon \sum_{k=1}^{K_1} \left(\int\limits_{G_\varepsilon^{(1)}(k)} \left| Z_{m,k}^{(1)} \left(\nabla u_\varepsilon \cdot \nabla \psi - f_\varepsilon \psi \right) \right| dx + \varepsilon \int\limits_{S_\varepsilon^{(1)}(k)} \left| Z_{m,k}^{(1)} \right| |g_\varepsilon \psi| \, d\sigma_x \right)$$

$$\le \varepsilon c_1 \Big(\|\nabla u_\varepsilon\|_{L^2\left(G_\varepsilon^{(1)}\right)} \|\nabla \psi\|_{L^2\left(G_\varepsilon^{(1)}\right)} + \|f_\varepsilon\|_{L^2\left(G_\varepsilon^{(1)}\right)} \|\psi\|_{L^2\left(G_\varepsilon^{(1)}\right)}$$

$$+ \sqrt{\varepsilon}\|g_\varepsilon\|_{L^2\left(S_\varepsilon^{(1)}\right)} \sqrt{\varepsilon}\|\psi\|_{L^2\left(S_\varepsilon^{(1)}\right)} \Big) \le \varepsilon c_1 \Big(c_2 \|\psi\|_{H^1\left(G_\varepsilon^{(1)}\right)}$$

$$+ c_3 \|\psi\|_{H^1\left(G_\varepsilon^{(1)}\right)} + c_4 \|\psi\|_{H^1\left(G_\varepsilon^{(1)}\right)} \Big) \le \varepsilon c_5 \|\psi\|_{H^1(D_1)}, \quad m = 1, 2.$$

Passing to the limit as $\varepsilon \to 0$ in these inequalities we obtain

$$\int_{D_1} \gamma_m^{(1)} \psi \, dx = 0 \qquad \forall \psi \in C_0^\infty(D_1), \quad m = 1, 2, \tag{28}$$

i.e., $\gamma_1^{(1)} = \gamma_2^{(1)} = 0$ almost everywhere in D_1.

Substituting the functions $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$ into the integral identity (4) we get

$$\sum_{k=1}^{K_2} \int_{G_\varepsilon^{(2)}(k)} \left(-\frac{\partial u_\varepsilon}{\partial x_m} \psi + \varepsilon Z_{m,k}^{(2)} \frac{\partial u_\varepsilon}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \varepsilon Z_{m,k}^{(2)} \frac{\partial u_\varepsilon}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \varepsilon Z_{m,k}^{(2)} \frac{\partial u_\varepsilon}{\partial x_3} \frac{\partial \psi}{\partial x_3} \right) dx$$

$$+ \varepsilon^2 k_0 \sum_{k=1}^{K_2} \int_{S_\varepsilon^{(2)}(k)} Z_{m,k}^{(2)} u_\varepsilon \psi \, d\sigma_x = \sum_{k=1}^{K_2} \int_{G_\varepsilon^{(2)}(k)} \varepsilon f_\varepsilon Z_{m,k}^{(2)} \psi \, dx, \quad m = 1, 2.$$

Analogously we obtain that $\gamma_1^{(2)} = \gamma_2^{(2)} = 0$ almost everywhere in D_2.

2. It remains to determine the functions v_0^+, $v_0^{(1,-)}$ and $v_0^{(2,-)}$. First, we find the traces of these functions on Q. By virtue of the compactness of the trace operator in the anisotropic spaces $W^{0,1}$ and the first three relations in (26) we have

$$\begin{array}{lcllll} v_\varepsilon(x', 0) & \xrightarrow{s} & v_0^+(x', 0) & \text{in} & L^2(Q) & \text{as} \quad \varepsilon \to 0, \\ \widetilde{v_\varepsilon}^{(1)}(x', 0) & \xrightarrow{s} & |B^{(1)}| v_0^{(1,-)}(x', 0) & \text{in} & L^2(Q) & \text{as} \quad \varepsilon \to 0, \\ \widetilde{v_\varepsilon}^{(2)}(x', 0) & \xrightarrow{s} & |B^{(2)}| v_0^{(2,-)}(x', 0) & \text{in} & L^2(Q) & \text{as} \quad \varepsilon \to 0. \end{array} \tag{29}$$

Since

$$\widetilde{v_\varepsilon}^{(1)}(x', 0) = \chi_1\left(\frac{x'}{\varepsilon}\right) v_\varepsilon(x', 0), \qquad \widetilde{v_\varepsilon}^{(2)}(x', 0) = \chi_2\left(\frac{x'}{\varepsilon}\right) v_\varepsilon(x', 0), \quad x' \in Q, \tag{30}$$

then, passing to the limit as $\varepsilon \to 0$ in(30) and using relation (29), we obtain

$$v_0^+(x', 0) = v_0^{(1,-)}(x', 0) = v_0^{(2,-)}(x', 0), \quad x' \in Q.$$

Using the extension operators (5) and the integral identities (9) we rewrite the integral identity (4) in the following way

$$\int\limits_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int\limits_{D_1} \left(\widetilde{\partial_{x_1} u_\varepsilon}^{(1)} \partial_{x_1}\varphi + \widetilde{\partial_{x_2} u_\varepsilon}^{(1)} \partial_{x_2}\varphi + \widetilde{\partial_{x_3} u_\varepsilon}^{(1)} \partial_{x_3}\varphi \right) dx$$

$$+ \int\limits_{D_2} \left(\widetilde{\partial_{x_1} u_\varepsilon}^{(2)} \partial_{x_1}\varphi + \widetilde{\partial_{x_2} u_\varepsilon}^{(2)} \partial_{x_2}\varphi + \widetilde{\partial_{x_3} u_\varepsilon}^{(2)} \partial_{x_3}\varphi \right) dx$$

$$+ k_0 \sum_{k=1}^{K_2} \frac{l_k^{(2)}}{|B_k^{(2)}|} \int\limits_{-d_2}^{0} \int\limits_{\bigcup_{i,j=0}^{N-1} \varepsilon(i+j+B_k^{(2)})} u_\varepsilon \varphi\, dx' dx_3$$

$$+ \varepsilon k_0 \sum_{k=1}^{K_2} \int\limits_{-d_2}^{0} \int\limits_{\bigcup_{i,j=0}^{N-1} \varepsilon(i+j+B_k^{(2)})} \nabla_\xi Y_k^{(2)} \cdot \nabla_{x'} \ (u_\varepsilon \varphi)\, dx' dx_3$$

$$= \int\limits_{\Omega_0} f_\varepsilon \varphi\, dx + \int\limits_{D_1} \chi_1 \left(\frac{x'}{\varepsilon}\right) f_\varepsilon \varphi\, dx + \int\limits_{D_2} \chi_2 \left(\frac{x'}{\varepsilon}\right) f_\varepsilon \varphi\, dx$$

$$+ \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int\limits_{-d_1}^{0} \int_{\bigcup_{i,j=0}^{N-1} \varepsilon(i+j+B_k^{(1)})} g_\varepsilon \varphi\, dx' dx_3$$

$$+ \varepsilon \sum_{k=1}^{K_1} \int\limits_{-d_1}^{0} \int\limits_{\bigcup_{i,j=0}^{N-1} \varepsilon(i+j+B_k^{(1)})} \nabla_\xi Y_k^{(1)} \cdot \nabla_{x'} \ (g_\varepsilon \varphi)\, dx' dx_3 \qquad \forall \varphi \in H^1(\Omega_1). \quad (31)$$

Then, passing to the limit as $\varepsilon \to 0$ in (31) and taking into account relations (2), (3), (19), (26)–(28) and the fact that the last terms both in the left-hand side and in the right-hand side tend to zero, we obtain

$$\int\limits_{\Omega_0} \nabla v_0^+ \cdot \nabla\varphi\, dx + |B^{(1)}| \int\limits_{D_1} \frac{\partial v_0^{(1,-)}}{\partial x_3} \frac{\partial\varphi}{\partial x_3}\, dx + |B^{(2)}| \int\limits_{D_2} \frac{\partial v_0^{(2,-)}}{\partial x_3} \frac{\partial\varphi}{\partial x_3}\, dx$$

$$+ k_0 |B^{(2)}|\, l^{(2)} \int\limits_{D_2} v_0^{(2,-)}\, \varphi\, dx = \int\limits_{\Omega_0} f_0 \varphi\, dx + |B^{(1)}| \int\limits_{D_1} f_0 \varphi\, dx \quad (32)$$

$$+ |B^{(2)}| \int_{D_2} f_0 \varphi\, dx + l^{(1)} \int\limits_{D_1} g_0 \varphi\, dx \quad \forall \varphi \in H^1(\Omega_1).$$

Identity (32) is the corresponding integral identity for problem (7) in the following anisotropic Sobolev vector-space

$$\mathcal{H} = \{\mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V} := L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2) \;\big|\; u_0 \in H^1(\Omega_0);$$

$$\exists\, \partial_{x_3} u_1 \in L^2(D_1);\ \exists\, \partial_{x_3} u_2 \in L^2(D_2);\ u_0(x', 0) = u_1(x', 0) = u_2(x', 0),\ x' \in Q\}$$

with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{H}} = \int\limits_{\Omega_0} \nabla u_0 \cdot \nabla v_0\, dx + \sum_{i=1}^{2} |B^{(i)}| \int\limits_{D_i} \partial_{x_3} u_i\, \partial_{x_3} v_i\, dx + k_0 |B^{(2)}|\, l^{(2)} \int\limits_{D_2} u\, v\, dx.$$

Obviously, the space $\mathcal{H}$ continuously embeds in $\mathcal{V}$. By using standard Hilbert space methods, we can state that there exists a unique weak solution $\mathbf{v}_0 \in \mathcal{H}$ to problem (7), which is called the limit problem for problem (1).

Due to the uniqueness of the solution to problem (7), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Therefore, the theorem is proved.

The fact that extensions of solutions to boundary-value problems in perforated domains converge weakly in the spaces H^1 enables to prove the convergence of the energy integrals (see, i.g., [5, 10]). Theorem 1 gives this possibility as well. We introduce the following notation

$$E_\varepsilon(u_\varepsilon) := \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \varepsilon \int\limits_{S_\varepsilon^{(2)}} u_\varepsilon^2\, d\sigma_x = (u_\varepsilon, u_\varepsilon)_\varepsilon,$$

$$E_0(\mathbf{v_0}) = (\mathbf{v_0}, \mathbf{v_0})_{\mathcal{H}} := \int\limits_{\Omega_0} |\nabla v_0^{+}|^2\, dx + |B^{(1)}| \int\limits_{D_1} |\partial_{x_3} v_0^{(1,-)}|^2\, dx$$
$$+ |B^{(2)}| \int\limits_{D_2} |\partial_{x_3} v_0^{(2,-)}|^2\, dx + k_0 |B^{(2)}|\, l^{(2)} \int\limits_{D_2} (v_0^{(2,-)})^2\, dx.$$

The quantities $E_\varepsilon(u_\varepsilon)$ and $E_0(\mathbf{v_0})$ determine the energy of the systems simulated by problems (1) and (7) respectively. It is easy to see that

$$E_\varepsilon(u_\varepsilon) = \int\limits_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\, dx + \varepsilon \int\limits_{S_\varepsilon^{(1)}} g_\varepsilon u_\varepsilon\, d\sigma_x.$$

Passing to the limit as $\varepsilon \to 0$ in this equality and taking into account relations (2), (3) and (26), as in Theorem 1 we obtain

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \int\limits_{\Omega_0} f_0 v_0^+ \, dx + |B^{(1)}| \int\limits_{D_1} f_0 v_0^{(1,-)} \, dx + |B^{(2)}| \int\limits_{D_2} f_0 v_0^{(2,-)} \, dx + l^{(1)} \int_{D_1} g_0 v_0^{(1,-)} \, dx = (\mathbf{v_0}, \mathbf{v_0})_{\mathcal{H}}.$$

Corollary 1. $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = E_0(\mathbf{v_0})$.

Conclusions

In the present paper we have studied the influence of boundary conditions on the asymptotic behavior of the solution to problem (1). We have shown that the limit boundary-value problem (7) consists of three boundary value problems joined together into one limit problem by certain conjugation conditions in the joint zone. The inhomogeneity in the Neumann boundary conditions on the lateral sides of the cylinders from the first level results in the appearance of a new term in the right-hand side of the homogenized boundary-value problem in the parallelepiped D_1. This fact was noted in [37] where the homogenization of elliptic equations that describe processes in strongly inhomogeneous thin perforated domains with rapidly varying thickness was made. Furthermore,in the differential equations of problem (7) there appear the coefficients $|B^{(i)}|/l^{(i)}$, $i = 1, 2$ which characterize "density of the packing" of the thin cylinders from the first and second levels.

It was noted in [5] that for functionals, that are defined on reflexive spaces and grow faster than the norm, there is, in fact, only one natural definition of homogenization of such functionals, namely, the definition in terms of the convergence of energies. For this reason, Corollary 1 is a very important result that enables to investigate variational problems in thick multi-level junctions.

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