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Homogenization of Electrostatic Problems in Nonlinear Medium with Thin Perfectly Conducting Grids

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The asymptotic behavior of solutions of the family of nonlinear elliptic equations in domains with thin grids concentrating near a hypersurface when measure of the wires tends to zero and the density tends to infinity is investigated. The homogenized equations and the homogenized boundary conditions are derived. The homogenization technique is based on applying of the abstract theorem on homogenization of the nonlinear variational functionals in the Sobolev-Orlicz spaces.

Key words: homogenization, domains with grids, electrostatic, Sobolev–Orlicz spaces.
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1. Introduction

In the present paper we study the asymptotic behavior of the electrostatic field in the domains $\Omega^{(s)} = \Omega \setminus F^{(s)} \subset \Omega \subset \mathbb{R}^n$, $s = 1, 2, \dots$, where $F^{(s)}$ is a connected set of the net type with a density that tends to infinity as $s \to \infty$. These structures having the form of metal wire nets with different cell shapes are widely applied in radio engineering, antenna technique and radio-relay links.

Usually a complex structure of the domain in which the initial problem is considered does not cause additional difficulties in the proof of the existence of solutions contrary to the way of finding solution either analytically or numerically. However, one can expect that when the grid is dense enough, then it acts as an effective continuous medium (or film), and its behavior can be approximately described by the homogenized differential equations (or boundary conditions). To derive these equations we are to analyze asymptotic behavior of solutions in the domains with grids.

The homogenized equations describing distribution of the electrostatic potential in the domains with dense thin grids in the case of linear dependence of the permeability of the medium on the intensity of electric field were derived in [1]. The case of nonlinear medium with a zero potential on the grid was studied in [2]. In the present work we study the case of nonlinear medium with nonzero potential on the grid.

The method used is based on the variational principles [3] and the Sobolev–Orlicz spaces technique [4], [5]. This technique helps us to show that sequence of solutions of the initial systems converges to the solution of the homogenized system.

The paper is composed as follows. In Section 2 we consider a number of basic definitions of the Sobolev–Orlicz spaces. In section 3 we formulate the problem statement and the main result which is proved in Sect. 4. In Section 5 we apply the main result for studying the asymptotic behavior of the electrostatic potential in weakly nonlinear medium with thin perfectly conducting grids.

2. Basic Definitions of the Sobolev–Orlicz Spaces Theory

In this section we present some basic definitions and properties of the Orlicz and Sobolev–Orlicz spaces. More information on the subject can be found, for example, in [4, 5].

Let $M(u)$ be a real-valued function of the real variable u satisfying

$$M(u) = \int_0^{|u|} M'(t)\, dt, \tag{2.1}$$

where $M'(t)$ is a positive for $t > 0$, right–continuous for $t \geq 0$, nondecreasing function such that $M'(0) = 0$, $M'(\infty) = \lim_{t\to\infty} M'(t) = \infty$. A function having the above-mentioned properties is called N function. We will further assume that for $u \geq 0$

$$uM'(u) \leq \alpha M(u) \ (\alpha > 1). \tag{2.2}$$

Let us suppose that $M(u)$ satisfies the following condition : there exist such $l > 0, u_0 > 0$, that

$$M(u) \leq \frac{1}{2l} M(lu), \quad u \geq u_0. \tag{2.3}$$

Inequality (2.2) guaranties that $M(u)$ satisfies Δ_2-condition: there exists such a function $k(l) \geq 0$ that, for all $l \geq 0$ and $u \geq 0$

$$M(lu) \leq k(l)M(u) \tag{2.4}$$

where $k(l)$ is a monotone increasing and differentiable for $l > 0$ function such that $k(0) = 0$ and for all $l > 0$

$$lk'(l) \leq Ck(\frac{l}{2}). \tag{2.5}$$

For any N function $M(u)$ we introduce the complementary function $N(v)$ given by

$$N(v) = \max_{u>0} \left[u|v| - M(u)\right]. \tag{2.6}$$

$N(v)$ also satisfies Δ_2- condition, because of (2.3).

The following inequalities hold

$$N\left[M(u)/u\right] < M(u), u > 0 \tag{2.7}$$

$$uv \leq M(u) + N(v) \quad \text{(the Young inequality)}. \tag{2.8}$$

Let $\Omega \subset \mathbb{R}^n$ be a domain with a piecewise smooth boundary. Then the Orlicz class $L_M(\Omega)$ consists of all the functions $u(x)$ such that

$$\rho(u, M, \Omega) \stackrel{def}{=} \int\limits_{\Omega} M\left(u(x)\right)\ dx < \infty. \tag{2.9}$$

Let us introduce the Orlicz norm

$$\|u\|_{M,\,\Omega} = \sup_{\rho(v,N,\Omega)\leq 1} \left| \int\limits_{\Omega} u(x)v(x)\ dx \right|, \tag{2.10}$$

where $N(v)$ is a complementary function to $M(u)$.

Taking into account the Young inequality, this norm makes sense for all $u(x) \in L_M(\Omega)$, and if $M(u), N(u)$ satisfy Δ_2-condition, then $L_M(\Omega)$ becomes a Banach reflexive space, which is called as Orlicz space [4] denoted by $L_M(\Omega)$).

Let $u \in L_M(\Omega)$ and $v \in L_N(\Omega)$. The following inequalities hold

$$\int\limits_{\Omega} uv\ dx \leq \|u\|_{M,\,\Omega} \|v\|_{N,\,\Omega} \quad \text{(the Holder inequality)} \tag{2.11}$$

$$\|u\|_{M,\,\Omega} \leq \rho(u, M, \Omega) + 1 \tag{2.12}$$

and if $\|u\|_{M,\,\Omega} \leq 1$, then

$$\rho(u, M, \Omega) \leq \|u\|_{M,\,\Omega}. \tag{2.13}$$

We say that the sequence of functions $\{u_k(x) \in L_M(\Omega),\ k = 1, 2, \dots\}$ converges on average to $u(x) \in L_M(\Omega)$, if

$$\rho(u_k - u, M, \Omega) \to 0, \quad k \to \infty. \tag{2.14}$$

If $M(u)$ satisfies Δ_2-condition then convergence on average is equivalent to strong convergence in space $L_M(\Omega)$.

Let us now consider the classes $W^1_M(\Omega)$ consisting of all the functions $u(x)$ from the Orlicz spaces $L_M(\Omega)$ such that distributional derivatives $D^\alpha u$ are contained in $L_M(\Omega)$ for all α with $|\alpha| \le 1$. Here we denote by α the multi-index of integers $[\alpha_1, \dots, \alpha_n]$ and by $|\alpha|$ the sum $\sum_{i=1}^{n} \alpha_i$. These classes of functions can be supplied with the norm

$$\|u\|^1_{M,\,\Omega} = \max_{|\alpha| \le 1} \left\{ \|D^\alpha u\|_{M,\,\Omega} \right\}, \tag{2.15}$$

where $\|\cdot\|_{M,\,\Omega}$ is a suitable norm in $L_M(\Omega)$ (as the norm defined above). These classes are the Banach spaces under this norm. We shall refer to the spaces of the form $W^1_M(\Omega)$ as to the Orlicz–Sobolev spaces. They form generalization of the Sobolev spaces in the same way as the Orlicz spaces form generalization of L^p spaces. Next we define another Orlicz–Sobolev space, $W^{1,0}_M(\Omega)$, as the closure of C_0^∞ in $W^1_M(\Omega)$.

The following imbedding theorems are valid [5]:

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary, then $W^1_M(\Omega) \hookrightarrow L_M(\Omega)$, where we use "$\hookrightarrow$" to indicate the compact imbedding. Furthermore, the following inequality holds*

$$\|u\|_{M,\,\Omega} \le C\,\|Du\|_{M,\,\Omega}\,, \tag{2.16}$$

for any $u(x) \in \overset{\circ}{W}{}^1_M(\Omega)$.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let Γ be a smooth hypersurface of dimension $n-1$ such that $\Gamma \subset \Omega$. If $\int_1^\infty \frac{M^{-1}(t)}{t^{1+\frac{1}{n}}}\,dt = \infty$, then $W^1_M(\Omega) \longrightarrow L_{[M^*]^{\frac{n-1}{n}}}(\Gamma)$, where*

$$(M^*)^{-1}(|x|) = \int_0^{|x|} \frac{M^{-1}(t)}{t^{1+\frac{1}{n}}}\,dt, \tag{2.17}$$

and $(M^*)^{-1}(u)$ and $M^{-1}(u)$ are the functions inverted to the N-functions of $M^*(u)$ and $M(u)$ respectively.

3. Problem Statement and Formulation of the Main Result

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and $F^{(s)}$ be a closed set in Ω of the arbitrary shape depending on a parameter s ($s = 1, 2, \dots$) such that $\lim\limits_{s\to\infty} mesF^{(s)} = 0$. We will assume that $F^{(s)}$ belongs to the indefinitely small neighborhood of some $(n-1)$-dimensional smooth surface $\Gamma \subset \Omega$ and that the distance from each point $x \in \Gamma$ to $F^{(s)}$ tends to zero as $s \to \infty$.

For every fixed s in domain $\Omega^{(s)} = \Omega \backslash F^{(s)}$ we consider the nonlinear variational problem

$$J^{(s)}[u^{(s)}] = \int\limits_{\Omega^{(s)}} F(x, u^{(s)}, \nabla u^{(s)}) dx \to \inf, \tag{3.1}$$

$$u^{(s)}_{|_{\partial F^{(s)}}} = A^s, \quad x \in \partial F^{(s)}, \tag{3.2}$$

$$u^{(s)}_{|_{\partial\Omega}} = f(x), \quad x \in \partial\Omega, \tag{3.3}$$

where the infinum is taken over the class of functions $u^{(s)} \in W^1_M(\Omega^{(s)})$ such that $u^{(s)} = f(x)$ for $x \in \partial\Omega$, $u^{(s)} = A^s$ for $x \in \partial F^{(s)}$ and parameters A^s are some unknown constants. Without loss of generality, we may assume that $f(x) \in C^1(\overline{\Omega})$.

Let $F(x, u, p)$ be a function that is defined and continuous for $\{(x, u, p) : x \in \Omega \subset \mathbb{R}^n; u \in \mathbb{R}^1; p \in \mathbb{R}^n\}$. Let this function possess continuous partial derivatives F_u, F_{p_i}, $i = 1, \dots, n$ and satisfy the following conditions:

$$F(x, u, p) - F(x, u, q) - \sum_{i=1}^{n} F_{p_i}(x, u, q)(p_i - q_i) \geq 0, \tag{3.4}$$

$$A_1 M(|p|) - A_2 M(u) \leq F(x, u, p) \leq A_3[1 + M(u) + M(|p|)], \tag{3.5}$$

$$|F(x, u, p) - F(x, v, q)|$$
$$\leq A_4[1 + M'(|u|) + M'(|v|) + M'(|p|) + M'(|q|)](|u - v| + |p - q|), \tag{3.6}$$

$$\forall x \in \Omega : \ |u(x)| > k \quad F(x, u, 0) > F(x, k, 0), \tag{3.7}$$

where $A_i > 0$, $i = 1, 3, 4$, $A_2 \geq 0$, $u, v \in \mathbb{R}^1$, $p, q \in \mathbb{R}^n$.

The functional $J^{(s)}[u^{(s)}]$ is assumed to be bounded from below so that for any function $u^{(s)} \in W^1_M(\Omega^{(s)})$, $u^{(s)} = A^{(s)}$ for $x \in \partial F^{(s)}$ and $u^{(s)} = f(x)$ for $x \in \partial\Omega$ ($f(x) \in C^1(\bar{\Omega})$) the following inequality holds

$$\int\limits_{\Omega^{(s)}} F(x, u^{(s)}, \nabla u^{(s)}) dx \geq \Phi\left(\left\|u^{(s)}\right\|^{(1)}_{M,\, \Omega^{(s)}}\right), \tag{3.8}$$

where $\Phi(t)$ is a continuous, increasing for $t \to \infty$, function; $\|\cdot\|^{(1)}_{M,\,\Omega^{(s)}}$ is a norm in the Sobolev–Orlicz space $W^1_M(\Omega^{(s)})$.

It is well known [6], that there exists at least one solution $u^{(s)}(x) \in W^1_M(\Omega^{(s)})$ of problem (3.1–3.3). We continue $u^{(s)}$ to $F^{(s)}$ by setting $u^{(s)}(x) = A^s$, $x \in F^{(s)}$ and consider the sequence of the continued functions, still denoted by $\{u^{(s)}\}$, as a sequence in $W^1_M(\Omega)$. The problem is to describe the asymptotic behavior of sequence $u^{(s)}(x)$ as $s \to \infty$.

Let us introduce quantitative characteristics of the sets $F^{(s)}$ [3]. Consider an arbitrary piece S of the hypersurface Γ. Let us draw normals of the length $h > 0$ from every point of S to both sides of Γ. For h small enough these normals are not intersected and their ends form two smooth surfaces Γ^+_h and Γ^-_h. We denote by $T_h(S)$ the subdomain of Ω which is formed by the segments of the mentioned above normals (the layer of $2h$ thickness with the central surface S).

For every fixed $h > 0$ $F^{(s)}$ belongs to $T_h(S)$ for sufficiently large s ($s > s(h)$). Consider the functional

$$C(S, s, h; b) = \inf_{v^{(s)}} \int\limits_{T_h(S)} \{F(x, 0, \nabla v^{(s)}) + \Gamma(h) M(v^{(s)} - b)\} dx, \tag{3.9}$$

where the infinum is taken over the class of functions $v^{(s)} \in W^1_M(\Omega^{(s)})$ vanishing on $F^{(s)} \cap T_h(S)$; $\Gamma(h) = k(\frac{1}{h^{1+\gamma}})$, $\gamma > 0$, the function $k(l)$ is defined by Δ_2-condition (2.4) and $b \in \mathbb{R}^1$.

It is seen clearly that $C(S, s_1, h; b) \leq C(S, s_2, h; b)$ for $F^{(s_1)} \bigcap T_h(S) \subset F^{(s_2)} \bigcap T_h(S)$. Hence, function $C(S, s, h; b)$ is a local characteristic of massiveness of the set $F^{(s)}$ generated by $F(y, u, p)$.

Theorem 3.1. *Let the following condition hold: for any arbitrary piece S of the surface Γ, $\forall\, b \in \mathbb{R}^n$ and $\gamma > 0$ there exist the following limits:*

$$\lim_{h\to 0} \limsup_{s\to\infty} C(S, s, h, b) = \lim_{h\to 0} \liminf_{s\to\infty} C(S, s, h, b) = \int\limits_S c(x, b) d\Gamma, \tag{3.10}$$

where $c(x, b)$ is a nonnegative continuous on Γ function.

Then, for any sequence $\{u^{(s)}(x)\}$ of the solutions of problem (3.1–3.3) (continued on $F^{(s)}$ by setting $u^{(s)} = A^{(s)}$) there exists a subsequence $\{u^{s_j}(x)\}$ that weakly converges in the space $W^1_M(\Omega)$ to function $u(x)$ such that the pair $\{u(x), A\}$ is a solution of the following problem:

$$\int\limits_\Omega F(x, u, \nabla u) dx + \int\limits_\Gamma c(x, u - A) d\Gamma \to \inf, \tag{3.11}$$

$$u_{|\partial\Omega} = f(x), \tag{3.12}$$

where $A = \lim\limits_{s\to\infty} A^s$.

4. Proof of Theorem 3.1

Since $u^{(s)}$ is the minimum of variational problem (3.1)–(3.3), we have

$$\int\limits_{\Omega^{(s)}} F(x, u^{(s)}, \nabla u^{(s)})dx \le \int\limits_{\Omega^{(s)}} F(x, \tilde{f}_B(x), \nabla \tilde{f}_B(x)\ dx \le \int\limits_{\Omega} |F(x, \tilde{f}_B(x), \nabla \tilde{f}_B(x))|dx, \tag{4.1}$$

where

$$\tilde{f}_B(x) = \begin{cases} f(x),\ x \in \partial\Omega; \\ B,\ x \in F^{(s)}\ ; \end{cases} \tag{4.2}$$

it follows from (3.8) that

$$\Phi\left(\left\|u^{(s)}\right\|^{(1)}_{M,\ \Omega^{(s)}}\right) \le Const \text{ (does not depend on } s). \tag{4.3}$$

This means that the sequence of the continued functions $\{u^{(s)}(x)\}$ is weakly compact in $W^1_M(\Omega)$, hence, one can extract a subsequence $u^{(s)}, s = s_j \to \infty$ that weakly converges to function $u(x) \in W^1_M(\Omega)$. Let us prove that $\{u(x), A\}$ is a solution of problem (4.94–4.95).

Let us cover Γ with a finite number of segments S'_i, $i = 1, \ldots, N$ with sufficiently small diameter $\delta = d_i(N)$. Every set S'_i is immersed into a more extensive open on Γ set $\tilde{S}_i$, moreover, we choose a subset $S_i = S'_i \setminus (\bigcup\limits_{i\ne j} \tilde{S}_j)$ to satisfy the following conditions:

1. $\overline{S}_i \subset S'_i \subset \overline{S'_i} \subset \tilde{S}_i,\ diam\tilde{S}_i < Cd_i(N) \to 0,\ N \to \infty;$ (4.4)
2. $\tilde{S}_i$ are bounded with the finite set of smooth $(n-2)$-dimension manifolds. The number of intersections between $\tilde{S}_i$ do not exceed a fixed number M, that doesn't depend on N;
3. $\sum\limits_{i=1}^{N} \mu\left(\tilde{S}_i \setminus S_i\right) \le \eta(N), \quad \eta(N) \to 0,\ N \to \infty.$ (4.5)

It is obvious that the unit of sets $\{\tilde{S}_i\}$ covers Γ. With this coverage we associate a partition of unity $\{\varphi_i(\overline{x}),\ \overline{x} = \{t_1, \ldots, t_{n-1}, 0\} \in \Gamma\}$ satisfying the conditions: $0 \le \varphi_i(\overline{x}) \le 1$, $\varphi_i(\overline{x}) = 0$, $\overline{x} \notin \tilde{S}_i$, $\varphi_i(\overline{x}) = 1$, $\overline{x} \in S_i$, $\sum\limits_i \varphi_i(\overline{x}) \equiv 1$, $|D^\beta \varphi_i(\overline{x})| \le C_\beta(N)$.

Consider an arbitrary function $w(x) \in C^2(\Omega)$ such that $w(x) = f(x)$, $x \in \partial\Omega$. Let U_θ be a subset of layers $T_h(\tilde{S}_i)$ covering Γ so that $|w(x) - \tilde{f}_B(x)| > \theta > 0$ for any point $x \in T_h(\tilde{S}_i)$. We denote by $b_i = w(x^i) - B$, $x^i \in \Gamma$ for $T_h(\tilde{S}_i) \in U_\theta$ and $b_i = 1$ for $T_h(\tilde{S}_i) \notin U_\theta$. Let $v_i^{(s)}(x)$ be a function that minimizes $C(\tilde{S}_i, s, h, b)$. For any $T_h(\tilde{S}_i)$ we define

$$B_i(\varepsilon, s, h) = \{x \in T_h(\tilde{S}_i) : v_i^{(s)}(x) \operatorname{sign} b_i \le |b_i| - \varepsilon\} \tag{4.6}$$

and

$$\hat{v}_i^s(x) = \begin{cases} v_i^s(x), & x \in B_i(\varepsilon, s, h), \\ b_i^\varepsilon = (|b_i| - \varepsilon) \operatorname{sign} b_i, & x \in T_h(S_i) \setminus \ B_i(\varepsilon, s, h), \end{cases} \tag{4.7}$$

where $0 < \varepsilon < \frac{\theta}{2}$.

In the domain $\Omega^{(s)}$ we define the function

$$w_h^{(s)}(x)$$

$$= \left(w(x) + \sum_{i=1}^{N_\delta} [w(x) - \tilde{f}_B(x)] \, (\hat{v}_i^s(x) - b_i^\varepsilon)(b_i^\varepsilon)^{-1} \, \varphi_i(\overline{x}) \right) \psi(x) + w(x) \, [1 - \psi(x)] \, , \tag{4.8}$$

where $\psi(x) \in C^\infty\left(\overline{\Omega}\right)$, such that $0 \le \psi(x) \le 1$, $\psi(x) = 1$, $x \in T_{h'}(\Gamma)$, $\psi(x) = 0$, $x \notin T_h(\Gamma)$ and $|\nabla \psi(x)| \le C r^{-1}$; $T_{h'}(\Gamma)$ is the layer of $2h' = 2(h - r)$ thickness, $r = h^{1+\gamma}$. It follows from the properties of functions $\hat{v}_i^s(x)$, $\varphi_i(\overline{x})$, $\psi(x)$ that w_h^s belongs to $W_M^1(\Omega^{(s)})$, w_h^s is equal to $f(x)$ on $\partial\Omega^{(s)}$ and to some constant B on $F^{(s)}$. Since $u^{(s)}(x)$ is the solution of problem (3.1–3.3) we get

$$J^{(s)}\left(u^{(s)}\right) \le J^{(s)}\left(w_h^{(s)}\right). \tag{4.9}$$

It is clear that

$$\int_{\Omega^{(s)}} F(x, w_h^s, \nabla w_h^s) dx \le \int_{\Omega \setminus T_h(\Gamma)} F(x, w_h^{(s)}, \nabla w_h^{(s)}) \, dx$$

$$+ \int_{T_h(\Gamma) \setminus T_{h'}(\Gamma) \bigcap \Omega^{(s)}} F(x, w_h^{(s)}, \nabla w_h^{(s)}) \, dx + \sum_i \int_{T_{h'}(S_i) \bigcap \Omega^{(s)}} F(x, w_h^{(s)}, \nabla w_h^{(s)}) \, dx$$

$$+ \sum_{\substack{i,j \\ i \ne j}} \int_{T_{h'}(\tilde{S}_i) \bigcap T_{h'}(\tilde{S}_j) \bigcap \Omega^{(s)}} \left| F(x, w_h^{(s)}, \nabla w_h^{(s)}) \right| \, dx. \tag{4.10}$$

The second term in the right-hand side of (4.10) can be estimated with the use of (3.5)

$$\int_{T_h(\Gamma) \setminus T_{h'}(\Gamma)} F(x, w_h^s, \nabla w_h^s) \, dx$$

$$\leq C \int\limits_{T_h(\Gamma)\setminus T_{h'}(\Gamma)} [M(|w_h^s|)] + [M(|\nabla w_h^s|)]\,dx + C\cdot mes(T_h(\Gamma)\setminus T_{h'}(\Gamma)). \quad (4.11)$$

$M(u)$ is a convex function that satisfies Δ_2–condition, so $M(u+v)\leq C(M(u)+M(v))$. Further we will use this inequality to estimate $\int\limits_{T_h(\Gamma)} M(w_h^{(s)})\,dx$ and $\int\limits_{T'_h(\Gamma)} M(\nabla w_h^{(s)})\,dx$.

Let us denote $\tilde{w}(x) = w(x) - \tilde{f}_B(x)$. It follows from (4.8) that

$$|w_h^{(s)}(x)| \leq |w(x)| + \sum_i^{N_\delta} \frac{|\tilde{w}(x)|}{|b_i^\varepsilon|}|\hat{v_i^s} - b_i^\varepsilon||\varphi_i(\overline{x})|. \quad (4.12)$$

Using the properties of $\varphi_i(\overline{x})$ and the fact that $M(u)$ satisfies Δ_2-condition, we get

$$\int\limits_{T_h(\Gamma)\setminus T_{h'}(\Gamma)} M(|w_h^{(s)}|)\,dx$$

$$\leq C\sum_i \int\limits_{T_h(S'_i)\setminus T_{h'}(S'_i)} M(|\hat{v_i^s} - b_\alpha^\varepsilon|)\,dx + const\cdot mes(T_h(\Gamma)\setminus T_{h'}(\Gamma)). \quad (4.13)$$

To estimate $\sum_i \int\limits_{T_h(S'_i)\setminus T_{h'}(S'_i)} M(|\hat{v_i^s} - b_\alpha^\varepsilon|)\,dx$ we divide $T_h(S'_i)\setminus T_{h'}(S'_i)$ into two subsets:

$$B_i \cap \big(T_h(S'_i)\setminus T_{h'}(S'_i)\big) := U_i^1, \quad (4.14)$$

$$\big(T_h(S'_i)\setminus T_{h'}(S'_i)\setminus B_i\big) := U_i^2. \quad (4.15)$$

Since $\hat{v}_i^s = b_i^\varepsilon$ when $x\notin B_i$, then

$$\int\limits_{T_h(S'_i)\setminus T_{h'}(S'_i)} M(|\hat{v_i^s} - b_i^\varepsilon|)\,dx = \int\limits_{U_i^2} + \int\limits_{U_i^1} = \int\limits_{U_i^1} M(|\hat{v_i^s} - b_i^\varepsilon|)\,dx. \quad (4.16)$$

From the last equality, the condition of Th. 3.1 and the fact that $|\hat{v_i^s} - b_i^\varepsilon| \leq |v_i^s - b_i|$ when $x\in B_i$, we have

$$\int\limits_{T_h(S'_i)\setminus T_{h'}(S'_i)} M(|\hat{v_i^s} - b_\alpha^\varepsilon|)\,dx \leq \int\limits_{U_i^1} M(|v_i^s - b_i|)\,dx \leq \underline{O}(\mu(S'_i))\Gamma^{-1}(h). \quad (4.17)$$

The estimation for $T_h(S_i') \notin U_\theta$ is obtained in the same way as for $T_h(S_i') \in U_\theta$. So,

$$\int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M(|\hat{v}_i^s - 1|) \le \int\limits_{U_i^1} M(|v_i^s - 1|)dx \le \underline{O}(\mu(S_i'))\Gamma^{-1}(h). \tag{4.18}$$

Then from (4.13), (4.17) and (4.18) we deduce

$$\lim_{h\to 0}\limsup_{s\to\infty} \int\limits_{T_h(\Gamma)\setminus T_{h'}(\Gamma)} M(w_h^{(s)})\, dx = 0. \tag{4.19}$$

Now we derive the estimation for

$$\int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M(|\nabla w_h^s|)\, dx. \tag{4.20}$$

It follows from definition (4.8) of the function $w_h^{(s)}$ that

$$|\nabla w_h^s| \le |\nabla w| + \sum_i \left[|\nabla \tilde{w}| \frac{|\hat{v}_i^s - b_i^\varepsilon|}{|b_i^\varepsilon|} + \frac{|\tilde{w}|}{|b_i^\varepsilon|}|\nabla \hat{v}_i^s| + \frac{|\tilde{w}||\hat{v}_i^s - b_i^\varepsilon|}{|b_i^\varepsilon|} C(N_\delta) \right]$$

$$+|\nabla\psi(x)| \sum_i \frac{|\tilde{w}||\hat{v}_i^s - b_i^\varepsilon|}{|b_i^\varepsilon|}. \tag{4.21}$$

The terms containing $\dfrac{|\tilde{w}|}{|b_i^\varepsilon|}|\hat{v}_i^s - b_i^\varepsilon|$ and $\dfrac{|\nabla\tilde{w}|}{|b_i^\varepsilon|}|\hat{v}_i^s - b_i^\varepsilon|$ are estimated similarly to the previous ones. To get the estimation for the terms containing $|\nabla \hat{v}_i^s|$ we need the following lemma.

Lemma 4.1. *Under the assumptions of Th. 3.1 we have*

$$\int\limits_{T_h(\tilde{S}_i)\setminus T_{h'}(\tilde{S}_i)} \left[F(x, 0, \nabla v_i^{(s)}) + \Gamma(h) M(v_i^{(s)} - b_i^\varepsilon) \right] dx = \overline{o}(1),\ h \to 0. \tag{4.22}$$

From Lemma 4.1 it immediately follows that

$$\lim_{h\to 0}\limsup_{s\to\infty} \int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M(|\nabla \hat{v}_i^s|)\, dx = 0. \tag{4.23}$$

Now let us get the estimation for

$$\int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M\left(\left|\tilde{w}\,(\hat{v}_i^s - b_i^\varepsilon)(b_i^\varepsilon)^{-1}\right| |\nabla\psi(x)| \right)\, dx. \tag{4.24}$$

Since Lemma 4.1 implies that

$$\Gamma(h) \int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M(|\hat{v}_i^s - b_i^\varepsilon|)\, dx = \overline{o}(1), h \to 0, \tag{4.25}$$

then, taking into account the properties of the function $\psi(x)$ and that $M(u)$ satisfies Δ_2-condition, we have

$$\int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M\left(|\nabla\psi|\frac{|\tilde{w}||\hat{v}_i^s - b_i^\varepsilon|}{|b_i^\varepsilon|}\right) dx \leq k(\frac{1}{h^{1+\gamma}})\Gamma^{-1}(h)\overline{o}(1),\ h \to 0, \tag{4.26}$$

and so

$$\lim_{h\to 0}\limsup_{s\to\infty} \int\limits_{T_h(S_i')\setminus T_{h'}(S_i')} M\left(|\nabla\psi|\frac{|\tilde{w}||\hat{v}_i^s - b_i^\varepsilon|}{|b_i^\varepsilon|}\right) dx = 0. \tag{4.27}$$

Correspondingly

$$\lim_{h\to 0}\limsup_{s\to\infty} \int\limits_{T_h(\Gamma)\setminus T_{h'}(\Gamma)} M\left(|\nabla w_h^{(s)}|\right)\, dx = 0. \tag{4.28}$$

It follows from (4.19), (4.11) and (4.28) that

$$\lim_{h\to 0}\limsup_{s\to\infty} \int\limits_{T_h(\Gamma)\setminus T_{h'}(\Gamma)} F(x, w_h^{(s)}, \nabla w_h^{(s)})\, dx = 0. \tag{4.29}$$

Moreover, it can be shown that

$$\lim_{\delta\to 0}\lim_{h\to 0}\limsup_{s\to\infty}\sum_{\substack{i,j\\ i\neq j}} \int\limits_{T_{h'}(\tilde{S}_i)\bigcap T_{h'}(\tilde{S}_j)} |F(x, w_h^{(s)}, \nabla w_h^{(s)})|\, dx = 0. \tag{4.30}$$

Further a useful estimation for $mesB_i(\varepsilon, s, h)$ is to be obtained as

$$\int\limits_{T_h(\tilde{S}_i)\bigcap\Omega^{(s)}} M(v_i^{(s)} - b_i)\, dx = \underline{O}(\mu(\tilde{S}_i))\Gamma^{-1}(h), \tag{4.31}$$

then

$$\int\limits_{T_h(\tilde{S}_i)\bigcap\Omega^{(s)}} M(v_i^{(s)} - b_i)\, dx = \int\limits_{B_i} M(v_i^{(s)} - b_i)\, dx + \int\limits_{T_h(\tilde{S}_i)\setminus B_i} M(v_i^{(s)} - b_i)\, dx$$

$$\geq \int\limits_{B_i} M(v_i^{(s)} - b_i)\, dx. \tag{4.32}$$

Since $|b_i - v_i^{(s)}| \geq \varepsilon$ when $x \in B_i$ then, with the use of (4.32), we get

$$mesB_i \leq \underline{O}(\mu(\tilde{S}_i))\Gamma^{-1}(h)\frac{1}{M(\varepsilon)}. \tag{4.33}$$

From now on we assume that $\varepsilon = M^{-1}(\Gamma^{-1+\delta}(h))$, where $M^{-1}(v)$ is the function inverse to $M(u)$, $0 < \delta < 1$. It is clear that if h is small enough and $s \geq \hat{s}(h)$, then

$$mesB_i(\varepsilon, s, h) \leq \underline{o}(\mu(\tilde{S}_i))\Gamma^{-\delta}(h). \tag{4.34}$$

Denote $B_1^i(s,h) = T_{h'}(S_i) \cap B^i(\varepsilon, s, h) \cap \Omega^{(s)}$ and $B_2^i(s,h) = \Omega^{(s)} \cap T_{h'}(S_i) \setminus B_1^i(s,h)$. So long as $w(x)$ is smooth in Ω and $w_h^{(s)}(x) = w(x)$ for $x \in B_2^i(s,h)$, it follows from (4.34) that

$$\int\limits_{B_1^i(s,h)} F(x, w_h^s, \nabla w_h^s)dx = \int\limits_{B_2^i(s,h)} F(x, w, \nabla w)dx$$

$$= \int\limits_{T_{h'}(S_i)} F(x, w, \nabla w)\ dx + \underline{O}(\mu(S_i)) + \frac{1}{M(b)}\Gamma^{-1}(h)\underline{O}(\mu(\tilde{S}_i)). \tag{4.35}$$

Let us consider the case $T_h(\tilde{S}_i) \subset U_\theta$.

If $T_h(\tilde{S}_i) \subset U_\theta$, we can rewrite the integral over the set $B_1^i(s,h)$ in the following way

$$\int\limits_{B_1^i(s,h)} F(x, w_h^s, \nabla w_h^s)dx = \int\limits_{B_1^i(s,h)} F(x, 0, \nabla v_i^{(s)})dx$$

$$+ \int\limits_{B_1^i(s,h)} [F(x, w_h^s, \nabla w_h^s) - F(x, 0, \nabla v_i^{(s)})]dx. \tag{4.36}$$

Since $w(x)$, $\nabla w(x)$, $v_i^{(s)}(x)$ are bounded in $B_1^i(s,h)$ and $\psi(x) \equiv 1$, $\varphi_i(\overline{x}) \equiv 1$ for all $x \in T_{h'}(S_i)$, then with the use of (4.8), (3.6) we get

$$\int\limits_{B_1^i(s,h)} \Big[F(x, w_h^s, \nabla w_h^s) - F(x, 0, \nabla v_i^{(s)})\Big]\ dx \leq C \int\limits_{B_1^\alpha(s,h)} M'(C_1 + C_2|\nabla v_i^{(s)}|)\ dx$$

$$+\Theta(\delta, h) \int\limits_{B_1^i(s,h)} M(|\nabla v_i^{(s)}|)\ dx, \tag{4.37}$$

where constants C_1, C_2, C do not depend on s, h, ε.

To get the estimation for the first term of the right-hand side of (4.37) we divide set $B_1^i(s,h)$ into two subsets $B_{11}^i(s,h)$ and $B_{12}^i(s,h)$

$$B_{11}^i(s,h) = \left\{ x \in B_1^i(s,h) \ : \ a_i^{(s)}(x) \le m(h) \right\},$$
$$B_{12}^i(s,h) = \left\{ x \in B_1^i(s,h) \ : \ a_i^{(s)}(x) > m(h) \right\}, \tag{4.38}$$

where $a_i^{(s)}(x) = C_8 + C_9|\nabla v_i^{(s)}(x)|$ and $m(h) = \sup\left\{ s \ : \ M'(s) < \Gamma^{\delta/2}(h) \right\}$. It follows from the properties of functions $\Gamma(h)$ and $M'(s)$ that $m(h) \to \infty$ as $h \to 0$. The convexity and monotony of the N-function imply that $M'(a_i^{(s)}(x)) \le M'(m(h))$ when $x \in B_{11}^i(s,h)$. Therefore

$$\int\limits_{B_{11}^i(s,h)} M'(C_8 + C_9|\nabla v_i^{(s)}(x)|)\, dx \le \Gamma^{\delta/2} mes B_{11}^i(s,h). \tag{4.39}$$

From (2.2) for $x \in B_{12}^i(s,h)$ we have

$$m(h)M'(a_i^{(s)}(x)) < a_i^{(s)}(x)M'(a_i^{(s)}(x)) \le CM(a_i^{(s)}(x)). \tag{4.40}$$

This means that $M'(a_i^{(s)}(x)) < CM(a_i^{(s)}(x))m^{-1}(h)$. And using the convexity of $M(u)$ and (4.34), we get

$$\int\limits_{B_{12}^i(s,h)} M'(C_8 + C_9|\nabla v_i^{(s)}(x)|)\, dx \le Cmes[B_{12}^i(s,h)]m^{-1}(h) + m^{-1}(h)\underline{O}(\mu(\tilde{S}_i)). \tag{4.41}$$

It follows from (4.39), (4.41) and (3.9) that for all $T_h(\tilde{S}_i) \subset U_\theta$ the following inequality holds

$$\limsup_{s\to\infty} \int\limits_{T_{h'}(S_i)\bigcap\Omega^{(s)}} F(x, w_h^{(s)}, \nabla w_h^{(s)})\, dx \le \int\limits_{T_{h'}(S_i)} F(x, w, \nabla w)\, dx$$
$$+ \limsup_{s\to\infty} C(S_i', s, h, b_i) + \Theta(\delta, h)\underline{O}(\mu(\tilde{S}_i)) + \overline{o}(1), \ h \to 0, \tag{4.42}$$

where $b_i = w(x^i) - \tilde{f}_B(x^i)$.

Now we shall study the case when $T_{h'}(\tilde{S}_i) \not\subset U_\theta$.

Using the same technique we have

$$\int\limits_{B_2^i(s,h)} F(x, w_h^{(s)}, \nabla w_h^{(s)})\, dx \le \int\limits_{T_{h'}(S_i)} F(x, w, \nabla w) + \underline{O}(\mu(\tilde{S}_i))\Gamma^{-\delta}(h) + \overline{o}(1), s \to \infty. \tag{4.43}$$

The integral over the set $B_1^i(s,h)$ can be written in the following form:

$$\int\limits_{B_1^i(s,h)} F(x, w_h^{(s)}, \nabla w_h^{(s)})\ dx = \int\limits_{B_1^i(s,h)} F(x, w, \nabla w)$$

$$+ \int\limits_{B_1^i(s,h)} \left[F(x, w_h^{(s)}, \nabla w_h^{(s)}) - F(x, w, \nabla w)\right]\ dx. \qquad (4.44)$$

Since $w(x)$, $\nabla w(x)$, $v_\alpha^s(x)$ are bounded on the set $B_1^\alpha(s,h)$ and $|w(x) - \tilde{f}_B(x)| < \theta$ it follows from (3.6) that

$$\int\limits_{B_1^i(s,h)} \left[F(x, w_h^{(s)}, \nabla w_h^{(s)}) - F(x, w, \nabla w)\right]\ dx$$

$$\leq A \cdot C_1 \int\limits_{B_1^i(s,h)} M'(C_2 + C_3|\nabla v_i^{(s)}|)\ dx + k(\theta) \int\limits_{B_1^i(s,h)} M(|\nabla v_i^{(s)}|)\ dx, \qquad (4.45)$$

where $k(\theta)$ is defined in (2.4). Then

$$\limsup_{s\to\infty} \int\limits_{T_{h'} \bigcap \Omega^{(s)}} F(x, w_h^{(s)}, \nabla w_h^{(s)}) \leq \int\limits_{T_{h'}} F(x, w, \nabla w)$$

$$+k(\theta)\underline{O}(\mu(\tilde{S}_i)) + \overline{o}(1), h \to 0. \qquad (4.46)$$

Now let us summarize (4.42) and (4.46) over corresponding elements $T_h(\tilde{S}_i)$ covering layer $T(\Gamma, h)$ and pass to the limit first where $h \to 0$, then $\delta \to 0$ and finally $\theta \to 0$. Then from (4.10), (4.29), (4.30) and the condition of Th. 3.1, we get

$$\lim_{\delta} \lim_{h\to 0} \int \Omega^{(s)} \limsup_{s\to\infty} F(x, w_h^{(s)}, \nabla w_h^{(s)})$$

$$\leq \int\limits_{\Omega} F(x, w, \nabla w)\ dx + \int\limits_{\Gamma} c\left(x, w(x) - B\right)\ d\sigma, \qquad (4.47)$$

and from (4.10)

$$\lim_{h\to 0} \limsup_{s\to\infty} J^{(s)}\left[w_h^{(s)}\right] \leq \int\limits_{\Omega} F(x, w, \nabla w)\ dx + \int\limits_{\Gamma} c\left(x, w(x) - B\right)\ d\sigma = J_c\left[w, B\right], \qquad (4.48)$$

as well as using (4.9), we get

$$\lim_{s\to\infty} J^{(s)}\left[u^{(s)}\right] \leq J_c\left[w, B\right]. \qquad (4.49)$$

This inequality was obtained under the assumption that $w(x) \in C^1(\overline{\Omega})$. Hence this holds true for any $w(x) \in W^1_M(\Omega)$ such that $w(x) = f(x)$ when $x \in \partial\Omega$. This statement goes out from the density of $C^1(\overline{\Omega})$ in the space $W^1_M(\Omega)$ and the following lemma.

Lemma 4.2. *The functional $J_c(w)$ is continuous on the Sobolev–Orlicz space $W^1_M(\Omega)$.*

Let the sequence $\{u^{(s)}(x),\ s = 1, 2, 3, \dots\}$ of the solutions of problem (3.1)–(3.3) (continued on $F^{(s)}$ by setting $u^{(s)} = A^{(s)}$) weakly converge in the space $W^1_M(\Omega)$ to some function $u(x)$. Let us show that

$$\lim_{s=s_k\to\infty} J^{(s)}\left[u^{(s)}\right] \geq J_c\left[u, A\right], \tag{4.50}$$

where A is the limit of $A^{(s)}$ as $s \to \infty$. It is clear that $|A| < \infty$ because of the principle of the maximum [6] and (3.7).

We denote by $\overset{\circ}{W}{}^1_M(\Omega, F^{(s)})$ a class of functions $u(x) \in W^1_M(\Omega)$ that are vanishing on $F^{(s)}$. The following lemma is essential.

Lemma 4.3. *Let $w(x)$ be an arbitrary function from $W^1_M(\Omega)$ such that $\|w\|^1_{M,\,\Omega} < 1$ and let conditions of Th. 3.1 be satisfied. Then there exists a sequence $\{\hat{w}(x) \in \overset{\circ}{W}{}^1_M(\Omega, F^{(s)}),\ s = 1, 2, \dots\}$ that weakly converges in $W^1_M(\Omega)$ to $w(x)$ and*

$$\|\hat{w}(x)\|^1_{M,\,\Omega} \leq \Phi\left(\|w\|^1_{M,\,\Omega}\right), \tag{4.51}$$

for sufficiently large s ($s \geq s(w)$), where $\Phi(t)$ is a nonnegative function defined on $[0, \infty)$ and $\Phi(t) \to 0$ as $t \to 0$.

Let $u(x) \in W^1_M(\Omega)$ be a weak limit in $W^1_M(\Omega)$ of the subsequence of solutions $\{u^{(s)}(x),\ s = s_k \to \infty\}$ of problem (3.1)–(3.3), continued to $F^{(s)}$ by setting $u^{(s)}(x) = A^{(s)}$, $A^{(s)} \to A < \infty$. Since space $C^1(\overline{\Omega})$ is dense in $W^1_M(\Omega)$, it follows that for any $\varepsilon > 0$ there exists a function $u_\varepsilon(x) \in C^1(\overline{\Omega})$ such that

$$\|u_\varepsilon - u\|^1_{M,\,\Omega} \leq \varepsilon. \tag{4.52}$$

Furthermore, according to Lem. 4.3, there exists a sequence of functions $\{w^{(s)}_\varepsilon(x) \in \overset{\circ}{W}{}^1_M(\Omega,\ F^{(s)})\}$ that weakly converges in $W^1_M(\Omega)$ to $u_\varepsilon - u$. We set $u^{(s)}_\varepsilon = u^{(s)} - w^{(s)}_\varepsilon$. Then $u^{(s)}_\varepsilon = A^{(s)}$ in $F^{(s)}$, and $u^{(s)}_\varepsilon$ weakly converges to u_ε as $s = s_k \to \infty$. Therefore, by Lem. 4.3 we have

$$\lim_{s\to\infty} \left\|w^{(s)}_\varepsilon\right\|^1_{M,\,\Omega} \leq \Phi\left(\|u_\varepsilon - u\|^1_{M,\,\Omega}\right). \tag{4.53}$$

By this equality and using (4.52) we get

$$\lim_{\varepsilon\to 0}\limsup_{s=s_k\to\infty}\left\|u_\varepsilon^{(s)}-u^{(s)}\right\|^1_{M,\,\Omega}=0, \tag{4.54}$$

from which one can easily obtain

$$\lim_{\varepsilon\to 0}\limsup_{s=s_k\to\infty}|J^{(s)}[u_\varepsilon^{(s)}]-J^{(s)}[u^{(s)}]|=0. \tag{4.55}$$

Lemma 4.2 and (4.52) imply that

$$\lim_{\varepsilon\to 0}J_c[u_\varepsilon]=J_c[u]. \tag{4.56}$$

One can see now that to obtain (4.50) it is sufficient to prove the following inequality:

$$\lim_{s=s_k\to\infty}J^{(s)}[u_\varepsilon^{(s)}]\geq J_c[u_\varepsilon]. \tag{4.57}$$

Now let us divide the layer $T(\Gamma,\ h)$ into the elements $T(S'_i,h)$, that were defined above, and introduce the notation:

$$T_\theta^{\pm}=\{x\in T(\Gamma,\ h),\ \pm(u_\varepsilon-A)>\theta\}\,;\ \ \tilde T_\theta^{\pm}=\left\{\bigcup_i T(S'_i,h),\ T(S'_i,h)\in T_\theta^{\pm}\right\},$$

$$T_\theta=T_\theta^{+}\bigcup T_\theta^{-};\ \ \tilde T_\theta=\tilde T_\theta^{+}\bigcup\tilde T_\theta^{-};\ \ G_\theta=T(\Gamma,\ h)\setminus T_\theta,$$

$$T_\theta^{(s)}=T_\theta\bigcap\Omega^{(s)};\ \ \tilde T_\theta^{(s)}=\tilde T_\theta\bigcap\Omega^{(s)};\ \ G_\theta^{(s)}=G_\theta\bigcap\Omega^{(s)}. \tag{4.58}$$

Since $u_\varepsilon(x)$ and $f(x)$ are smooth in Ω, we have

$$\lim_{\delta\to 0}\lim_{h\to 0}\left[T_\theta^{(s)}\setminus\tilde T_\theta^{(s)}\right]=0, \tag{4.59}$$

uniformly with respect to s. It is clear that

$$\int\limits_{T(\Gamma,\,h)\bigcap F^{(s)}}F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx=\int\limits_{\tilde T_\theta^{(s)}}F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx$$

$$+\int\limits_{T_\theta\setminus\tilde T_\theta^{(s)}}F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx+\int\limits_{G_\theta^{(s)}}F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx. \tag{4.60}$$

Now we get the following estimation

$$F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\geq F(x,u_\varepsilon,\nabla u_\varepsilon)+\sum_{i=1}^{n}F_p(x,u_\varepsilon,\nabla_\varepsilon)\left(\frac{\partial u_\varepsilon^{(s)}}{\partial x_i}-\frac{\partial u_\varepsilon}{\partial x_i}\right)$$

$$-\left(1+2M'(|\nabla u_\varepsilon^{(s)}|)+M'(|u_\varepsilon^{(s)}|)+M'(|u_\varepsilon|)\right)(|u_\varepsilon^{(s)}-u_\varepsilon|), \tag{4.61}$$

which follows from (3.4) and (3.6). Applying the Holder inequality, we obtain

$$\int\limits_{T_\theta\setminus\tilde{T}_\theta^{(s)}} F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx$$

$$\geq \int\limits_{T_\theta\setminus\tilde{T}_\theta^{(s)}} F(x,u_\varepsilon,\nabla u_\varepsilon)\,dx+\sum_{i=1}^{n}\int\limits_{T_\theta\setminus\tilde{T}_\theta^{(s)}} F_p(x,u_\varepsilon,\nabla_\varepsilon)\left(\frac{\partial u_\varepsilon^{(s)}}{\partial x_i}-\frac{\partial u_\varepsilon}{\partial x_i}\right)dx$$

$$-\left\|\left(1+2M'(|\nabla u_\varepsilon^{(s)}|)+M'(|u_\varepsilon^{(s)}|)+M'(|u_\varepsilon|)\right)\right\|_{N,\,\Omega}\left\|u_\varepsilon^{(s)}-u_\varepsilon\right\|_{M,\,\Omega}. \tag{4.62}$$

Since $\nabla u_\varepsilon^{(s)}$ weakly converges to ∇u_ε in the space $L_M(\Omega)$, $u_\varepsilon^{(s)}$ converges to u_ε in $L_M(\Omega)$, $u_\varepsilon^{(s)}$ is bounded in $W_M^1(\Omega)$ and $F_p(x,u,p)$ belongs to the space $L_N(\Omega)$, with the help of (4.61), (4.59)and (2.7) we get

$$\lim_{\delta\to 0}\lim_{h\to 0}\liminf_{s=s_k\to\infty}\int\limits_{T_\theta\setminus\tilde{T}_\theta^{(s)}} F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx\geq 0. \tag{4.63}$$

The fact that $u_\varepsilon^{(s)}=A^s$ when $x\in F^{(s)}$ implies that

$$\int\limits_{G_\theta^{(s)}} F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx=\int\limits_{G_\theta} F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx-\int\limits_{G_\theta\bigcap F^{(s)}} F(x,A^{(s)},0)\,dx. \tag{4.64}$$

In the same way we obtain that

$$\limsup_{s=s_k\to\infty}\int\limits_{G_\theta^{(s)}} F(x,u_\varepsilon^{(s)},\nabla u_\varepsilon^{(s)})\,dx\geq\int\limits_{G_\theta} F(x,u_\varepsilon,\nabla u_\varepsilon)\,dx. \tag{4.65}$$

Consider the first term in the right -hand side of (4.60). Let $T(S_i',h)$ be an arbitrary element from $\tilde{\Omega}_\theta^+$. We set

$$b_i^\wedge=A+\delta,\quad b_i^\vee=\min_{T_h(S_i')}u_\varepsilon(x)-\delta,\quad b_i=b_i^\vee-b_i^\wedge,\ \delta>0, \tag{4.66}$$

where δ is a sufficiently small parameter that will be defined later.

Note that $b_i^{\vee} - b_i^{\wedge} = \min u_\varepsilon(x) - A - 2\delta > \frac{\theta}{2}$ for $\delta < \frac{\theta}{4}$. Let us decompose the set $T(S_i', h) \bigcap \Omega^{(s)}$ into three nonintersecting subsets:

$$\Omega_{1i}^{(s)} = \left\{ x \in T(S_i', h) \bigcap \Omega^{(s)} : \ u_\varepsilon^{(s)} < b_i^{\wedge} \right\},$$

$$\Omega_{2i}^{(s)} = \left\{ x \in T(S_i', h) \bigcap \Omega^{(s)} : \ b_i^{\wedge} \le u_\varepsilon^{(s)} \le b_i^{\vee} \right\}, \tag{4.67}$$

$$\Omega_{3i}^{(s)} = \left\{ x \in T(S_i', h) \bigcap \Omega^{(s)} : \ u_\varepsilon^{(s)} > b_i^{\vee} \right\}.$$

From the convergence of $u_\varepsilon^{(s)}$ to u_ε in space $L_M(\Omega)$ one can show that for sufficiently large $s = s(h)$ the following equality holds

$$\int\limits_{T(S_i', h) \bigcap \Omega^{(s)}} M(u_\varepsilon^{(s)} - u_\varepsilon)\, dx = \underline{O}(\mu(S_i'))\Gamma^{-2-\gamma}(h), \ 0 < \gamma < 1. \tag{4.68}$$

Therefore,

$$\int\limits_{\Omega_{1i}^{(s)} \bigcup \Omega_{2i}^{(s)}} M(|u_\varepsilon^{(s)} - u_\varepsilon|)\, dx u_\varepsilon|)\, dx$$

$$\ge \int\limits_{\Omega_{1i}^{(s)} \bigcup \Omega_{2i}^{(s)}} M(|\min_{T_h S_i'} u_\varepsilon^{(s)} - u_\varepsilon|)\, dx \ge M(\delta) mes \left[\Omega_{1i}^{(s)} \bigcup \Omega_{2i}^{(s)} \right], \tag{4.69}$$

this yields

$$M(\delta) mes \left[\Omega_{1i}^{(s)} \bigcup \Omega_{2i}^{(s)} \right] \le \underline{O}(\mu(S_i'))\Gamma^{-2-\gamma}(h). \tag{4.70}$$

Let us choose $\delta = M^{-1}(\Gamma^{-1}(h))$, where M^{-1} is a inverse function to $M(u)$. Then for $s > s(h)$ we have

$$mes \left[\Omega_{1i}^{(s)} \bigcup \Omega_{2i}^{(s)} \right] = \underline{O}(\mu(S_i'))\Gamma^{-1-\gamma}(h). \tag{4.71}$$

It is obvious that

$$\int\limits_{T(S_i', h) \bigcap \Omega^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)}) = \int\limits_{\Omega_{2i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx$$

$$+ \int\limits_{\Omega_{1i}^{(s)} \bigcup \Omega_{3i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx. \tag{4.72}$$

To estimate the integral over the set $\Omega_{2i}^{(s)}$ we introduce the following function

$$\tilde{v}_\alpha^s = \begin{cases} 0, & x \in \Omega_{1i}^{(s)} \bigcup \left(F^{(s)} \bigcap T(S_i', h)\right), \\ u_\varepsilon^{(s)} - b_i^\wedge, & x \in \Omega_{2i}^{(s)}, \\ b_i = b_i^\vee - b_i^\wedge, & x \in \Omega_{3i}^{(s)}. \end{cases} \quad (4.73)$$

Keeping in mind the definition of $\tilde{v}_\alpha^s(x)$, we find that $\nabla u_\varepsilon^{(s)}(x) = \nabla \tilde{v}_\alpha^s$ when $x \in \Omega_{2i}^{(s)}$. So,

$$\int\limits_{\Omega_{2i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx$$

$$= \int\limits_{\Omega_{2i}^{(s)}} \left[F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)}) - F(x, 0, \nabla u_\varepsilon^{(s)})\right] dx + \int\limits_{\Omega_{2i}^{(s)}} F(x, 0, \nabla v_\alpha^s)\, dx. \quad (4.74)$$

Using (3.6) and (4.71), and according to boundness of $u_\varepsilon^{(s)}$ in $W_M^1(\Omega)$ we obtain

$$\int\limits_{\Omega_{2i}^{(s)}} \left[F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)}) - F(x, 0, \nabla u_\varepsilon^{(s)})\right] dx$$

$$\leq A \int\limits_{\Omega_{2i}^{(s)}} \left[1 + M'(u_\varepsilon^{(s)}) + 2M'(|\nabla u_\varepsilon^{(s)}|)\right] |u_\varepsilon^{(s)}|\, dx$$

$$\leq C mes \left[\Omega_{1i}^{(s)} \bigcap \Omega_{1i}^{(s)}\right] + C \int\limits_{\Omega_{2i}^{(s)}} M'(|\nabla u_\varepsilon^{(s)}|)\, dx. \quad (4.75)$$

Analogously to (4.39) and (4.41) we find that for sufficiently large s, $s > s(h)$, the following equality holds

$$\int\limits_{\Omega_{2i}^{(s)}} \left[F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)}) - F(x, 0, \nabla u_\varepsilon^{(s)})\right] dx = \underline{O}(\mu(S_i'))\overline{o}(1),\ h \to 0. \quad (4.76)$$

Then from (4.74) and the definition of $\tilde{v}_\alpha^s$ we deduce that

$$\int\limits_{\Omega_{2i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx = \int\limits_{T_h(S_i')} F(x, 0, \nabla v_\alpha^s)\, dx + \Gamma(h) \int\limits_{T_h(S_i')} M(v_\alpha^s - b_i)\, dx$$

$$+\underline{O}(\mu(S_i'))\overline{o}(1),\ h\to 0, \tag{4.77}$$

therefore, from the definition of $C(S_i', s, h, b)$

$$\int\limits_{\Omega_{2i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\,dx \ge C(S_i', s, h, b_i) + \underline{O}(\mu(S_i'))\overline{o}(1),\ h\to 0. \tag{4.78}$$

It is clear that

$$\int\limits_{\Omega_{3i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\,dx - \int\limits_{T_h(S_i')} F(x, u_\varepsilon, \nabla u_\varepsilon)\,dx$$

$$= \int\limits_{T_h(S_i')} F(x, \overline{u_\varepsilon}, \nabla\overline{u_\varepsilon})\,dx - \int\limits_{T_h(S_i')} F(x, u_\varepsilon, \nabla u_\varepsilon)\,dx - \int\limits_{T_h(S_i')\setminus\Omega_{3i}^{(s)}} F(x, a_i, 0)\,dx, \tag{4.79}$$

where

$$\overline{u}_\varepsilon = \begin{cases} u_\varepsilon^{(s)}, & x\in\Omega_{3i}^{(s)} \\ a_i, & x\in T_h(S_i')\setminus\Omega_{3i}^{(s)}. \end{cases} \tag{4.80}$$

Then from (3.4) and (4.61) we obtain

$$\int\limits_{T_h(S_i')} F(x, \overline{u}_\varepsilon, \nabla\overline{u}_\varepsilon)\,dx - \int\limits_{T_h(S_i')} F(x, u_\varepsilon, \nabla u_\varepsilon)\,dx - \int\limits_{T_h(S_i')\setminus\Omega_{3i}^{(s)}} F(x, a_i, 0)\,dx$$

$$\ge \sum_{j=1}^{n}\int\limits_{T_h(S_i')} F_{u_{\varepsilon x_j}}(x, u_\varepsilon, \nabla u_\varepsilon)(\overline{u}_\varepsilon - u_\varepsilon)_{x_j}\,dx$$

$$+const\cdot\|\overline{u}_\varepsilon - u_\varepsilon\|_{M,T_h(S_i')} - Cmes\left(T_h(S_i')\setminus\Omega_{3i}^{(s)}\right), \tag{4.81}$$

thus weak convergence of $u_\varepsilon^{(s)}$ to u_ε in $W_M^1(\Omega)$ and definition of $\overline{u}_\varepsilon$ give us

$$\lim_{s=s_k\to\infty}\int\limits_{\Omega_{3i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\,dx \ge \int\limits_{T_h(S_i')} F(x, u_\varepsilon, \nabla u_\varepsilon)\,dx + \underline{O}(\mu(S_i'))\Gamma^{-1-\gamma}(h). \tag{4.82}$$

Now we obtain the estimation for $\int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx$. It follows from (3.5) that $F(x,u,p) - F(x,u,0) \geq A_1 M(|p|) \geq 0$, hence

$$\int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx \geq \int\limits_{\Omega_{1i}^{(s)}} \Big[F(x, u_\varepsilon^{(s)}, 0) - F(x, u_\varepsilon, 0)\Big]\, dx + \int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon, 0)\, dx. \tag{4.83}$$

With the help of (4.61) the last inequality leads to

$$\int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx$$
$$\geq \int\limits_{\Omega_{1i}^{(s)}} \Big(1 + 2M'(|\nabla u_\varepsilon^{(s)}|) + M'(|u_\varepsilon^{(s)}|) + M'(|u_\varepsilon|)\Big)(|u_\varepsilon^{(s)} - u_\varepsilon|)\, dx$$
$$+ \int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon, 0)\, dx. \tag{4.84}$$

Taking into account the weak convergence of $u_\varepsilon^{(s)}$ to u_ε in space $W_M^1(\Omega)$ and smothness of u_ε, we deduce from (4.71) and (4.84) that

$$\varliminf_{s=s_k\to\infty} \int\limits_{\Omega_{1i}^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx \geq \underline{O}(\mu(S_i'))\Gamma^{-1-\gamma}(h). \tag{4.85}$$

Combining of (4.76), (4.82), (4.85), (4.78) in the case when $T_h(S_i') \subset \Omega_\theta^+$ yields

$$\varliminf_{s=s_k\to\infty} \int\limits_{T_h(S_i')\bigcap\Omega^{(s)}} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})$$
$$\geq \int\limits_{T_h(S_i')} F(x, u_\varepsilon, \nabla u_\varepsilon)\, dx + \varliminf_{s=s_k\to\infty} C(S_i', s, h, b) + \overline{o}(1),\ h \to 0. \tag{4.86}$$

Besides, it follows from (4.61), (2.7) and from weak convergence of $u_\varepsilon^{(s)}$ to u_ε in $W_M^1(\Omega)$ that

$$\varliminf_{s=s_k\to\infty} \int\limits_{\Omega\backslash T(\Gamma,\, h)} F(x, u_\varepsilon^{(s)}, \nabla u_\varepsilon^{(s)})\, dx \geq \int\limits_{\Omega\backslash T(\Gamma,\, h)} F(x, u_\varepsilon, \nabla u_\varepsilon)\, dx. \tag{4.87}$$

Similarly to (4.86) inequalities hold for the elements $T_h(S'_i) \subset T^-_\theta$. We summarize them all over the elements $T_h(S'_i) \subset \tilde{T}^+_\theta \bigcup \tilde{T}^-_\theta$ and pass to the limit first as $h \to 0$ and then as $\delta \to 0$.

Since

$$\int\limits_{\Omega^{(s)}} F(x, u^{(s)}_\varepsilon, \nabla u^{(s)}_\varepsilon)\, dx$$

$$= \int\limits_{\Omega\setminus\, T(\Gamma,\, h)} F(x, u^{(s)}_\varepsilon, \nabla u^{(s)}_\varepsilon)\, dx + \int\limits_{T(\Gamma,\, h)\bigcap F^{(s)}} F(x, u^{(s)}_\varepsilon, \nabla u^{(s)}_\varepsilon)\, dx, \qquad (4.88)$$

it follows from (4.63), (4.60), (4.65), (4.87), (4.86), (4.88) and the condition of Th. 3.1 that

$$\lim_{s=s_k\to\infty} \int\limits_{\Omega^{(s)}} F(x, u^{(s)}_\varepsilon, \nabla u^{(s)}_\varepsilon)\, dx \geq \int\limits_{\Omega} F(x, u_\varepsilon, \nabla u_\varepsilon)\, dx + \int\limits_{\Gamma_\theta} c(x, u_\varepsilon - A)\, d\sigma, \qquad (4.89)$$

where $\Gamma_\theta = \{x \in \Gamma:\ \pm(u_\varepsilon(x) - A) > \theta\}$.

It is clear that $\bigcup\limits_{\theta>0} T_\theta = \{x \in T(\Gamma,\ h):\ |u_\varepsilon(x) - A| > 0\}$. Now we pass to the limit in (4.89) as $\theta \to 0$ and obtain (4.57) and, therefore, (4.50).

It follows from (4.50) and (4.49) that

$$J_c[u] \leq J_c[w]$$

holds for any $w(x) \in W^1_M(\Omega)$. This means that any weak limit in $W^1_M(\Omega)$ of the solutions of problem (3.1)–(3.3) (continued to set $F^{(s)}$ by setting $u^{(s)} = A^{(s)}$) is a solution of problem (4.94)–(4.95). Theorem 3.1 is proved.

Remark 1. *The theorem proved above corresponds to the generalization of the case of distribution of the electrostatic field in weakly nonlinear medium with a nonzero potential and a zero charge on the net. With the same methods being used one can show that Th. 3.1 can be modified in the following way.*

Theorem 4.1. *Let the following condition hold: for any arbitrary piece S of surface Γ, $\forall\, b \in \mathbb{R}^n$ and $\gamma > 0$ there exist the following limits:*

$$\lim_{h\to 0} \limsup_{s\to\infty} C(S, s, h, b) = \lim_{h\to 0} \liminf_{s\to\infty} C(S, s, h, b) = \int\limits_S c(x, b) d\Gamma, \qquad (4.90)$$

where $c(x, b)$ is a continuous on Γ nonnegative function such as if b is large enough, then $c(x, b) = \underline{O}(b^2)$.

Then for any sequence $\{u^{(s)}(x)\}$ of the problem solutions

$$J^{(s)}[u^{(s)}] = \int\limits_{\Omega^{(s)}} F(x, \nabla u^{(s)})dx - 2A^s q \to \inf, \tag{4.91}$$

$$u^{(s)}_{|_{\partial F^{(s)}}} = A^s, \quad x \in \partial F^{(s)}, \tag{4.92}$$

$$u^{(s)}_{|\partial\Omega} = f(x), \quad x \in \partial\Omega \tag{4.93}$$

(continued on $F^{(s)}$ by setting $u^{(s)} = A^{(s)}$) there exists a subsequence $\{u^{s_j}(x)\}$ that weakly converges in space $W^1_M(\Omega)$ to the function $u(x)$ such that the pair $\{u(x), A\}$ is a solution of the following problem:

$$\int\limits_{\Omega} F(x, \nabla u)dx + \int\limits_{\Gamma} c(x, u - A)d\Gamma - 2Aq \to \inf \tag{4.94}$$

$$u_{|\partial\Omega} = f(x), \quad x \in \partial\Omega, \tag{4.95}$$

where $A = \lim\limits_{s\to\infty} A^s$.

5. Asymptotic Behavior of the Electrostatic Potential in a Weakly Nonlinear Medium with Thin Perfectly Conducting Grids

Let $\Omega \subset \mathbb{R}^3$ be a dielectric. We suppose that a certain part of this dielectric is penetrated by thin perfectly conducting wires forming a periodic grid $F^{(s)}$ that concentrates in the neighborhood of plane $\Gamma \Subset \Omega$. We also suppose that the dielectric permeability $\varepsilon(E)$ depends on the electric field strength E as follows

$$\varepsilon(E) = \varepsilon_0 + \alpha \ln^{\beta}(1 + |E|^2) \ (\varepsilon_0 > 0, \alpha > 0, 0 \le \beta \le 1). \tag{5.1}$$

Thus the dielectric is a weakly nonlinear medium.

Assume that $F^{(s)}$ depends on a parameter s and has the following structure: $Q^{(s)}(\Gamma) \bigcap F^{(s)}_{\Pi}$, where $Q^{(s)}(\Gamma)$ is the neighborhood of Γ such as $\forall x \in Q^{(s)}(\Gamma)$: $x \to \Gamma$, $s \to \infty$, and $F^{(s)}_{\Pi}$ is a periodic set in $\mathbb{R}^3$. We suppose that $F^{(s)}_{\Pi}$ consists of the circular cylinders with radius $r^{(s)} = \frac{C}{s}$. The axes of the cylinders form a periodic net in $\mathbb{R}^2$ of the period $\delta^{(s)}$ and

$$\delta^{(s)} \sim \begin{cases} \delta\left(\ln \frac{1}{r^{(s)}}\right)^{\beta-1}, & 0 \le \beta < 1, \\ \delta\left(\ln\ln \frac{1}{r^{(s)}}\right)^{-1}, & \beta = 1, \end{cases} \tag{5.2}$$

as $s \to \infty$, where $\delta > 0$.

Let $f(x)$ be a potential defined on $\partial\Omega$ and let total charge on the grid be equal to q. Then potential $u^{(s)}(x)$ in domain $\Omega^{(s)} = \Omega \setminus F^{(s)}$ is described by the following boundary value problem

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\varepsilon(\nabla u^{(s)})\frac{\partial u^{(s)}}{\partial x_i}\right) = 0,\ x \in \Omega^{(s)}, \tag{5.3}$$

$$u^{(s)} = A^{(s)},\ x \in \partial F^{(s)}, \tag{5.4}$$

$$u^{(s)} = f(x),\ x \in \partial\Omega, \tag{5.5}$$

$$\int_{\partial F^{(s)}} \varepsilon(\nabla u^{(s)})\frac{\partial u^{(s)}}{\partial \nu}\, d\sigma = q, \tag{5.6}$$

where $\varepsilon : \mathbb{R}^3 \to \mathbb{R}^1$ is defined by (5.1).

It is clear that equations (5.4) and (5.1) are the Euler equation for functional (3.1) with the following integrant

$$F(x,u,\rho) = \varepsilon_0|\rho|^2 + \alpha \int_0^{|\rho|^2} \ln^{\beta}(1+t)\, dt = F(\rho). \tag{5.7}$$

It is easy to prove that $F(\rho)$ satisfies (3.4)–(3.6) with the function

$$M(u) = u^2 \ln^{\beta}(1+u^2). \tag{5.8}$$

We continue $u^{(s)}(x)$ to $F^{(s)}$ by setting $u^{(s)}(x) = A^{(s)}$, still denoting them as $u^{(s)}$.

From Th. 4.1 it follows that $u^{(s)}(x)$ strongly converges, as $s \to \infty$, in $L^2(\Omega)$ to the solution $u(x)$ of the following problem

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\varepsilon(\nabla u)\frac{\partial u}{\partial x_i}\right) = 0,\ x \in \Omega \setminus \Gamma, \tag{5.9}$$

$$\left[\varepsilon(\nabla u)\frac{\partial u}{\partial \nu}\right]^{\pm} = u - C_{\beta}\frac{1}{|\Gamma|}\left\{\int_{\Gamma} u\, d\sigma\ + q\right\}, x \in \Gamma, \tag{5.10}$$

$$[u]^{\pm} = 0,\ x \in \Gamma,$$

$$u = f(x),\ x \in \partial\Omega, \tag{5.11}$$

where

$$C_\beta = \begin{cases} 4(\alpha+\varepsilon_0)\pi\delta, & \beta = 0, \\ \frac{2^{\beta+2}\pi\alpha\delta}{(1-\beta)}, & 0<\beta<1, \\ 8\pi\alpha\delta, & \beta = 1. \end{cases} \tag{5.12}$$

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