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On the Polynomial Asymptotics of Subharmonic Functions of Finite Order and their Mass Distributions

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We obtain the results analogous of those of [5] on the polynomial asymptotics with arbitrary $0 < \rho_n < \ldots < \rho_1 < \rho$, defining multipolynomial terms.

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1. Introduction and the Main Results

1.1. In papers [1–8], it is considered, in particular, the polynomial asymptotics of subharmonic functions of finite order ρ and their mass distributions in terms of the growth of reminder terms and topology of exceptional sets. Besides, the exponents $\rho_1, \ldots, \rho_n$ of terms had to satisfy the conditions $[\rho] < \rho_n < \ldots < \rho$ for a noninteger ρ. We are going to represent another point of view by studying the polynomial asymptotics in $\mathcal{D}'$-topology and a little bit stronger topology and relax restriction on the exponent to the natural $\rho > \rho_1 > \ldots > \rho_n > 0$. It occurs that this change of topology together with the consideration of more narrow class than in [5] allows to obtain a multiterm asymptotic analog of Levin–Pfluger's theory of completely regular growth and make simpler (in our opinion) formulating of the results and proofs.

By "$\mathcal{D}'$-topology" we call the topology of the space $\mathcal{D}'(\mathbb{C}\backslash 0)$ of distributions (i.e., linear bounded functionals) over the basic space $\mathcal{D}(\mathbb{C}\backslash 0)$ of finite infinitely differentiable functions. Recall that a sequence $u_j \to 0$, $j \to \infty$ in this space if the linear functionals

$$< u_j, g > \to 0 \tag{1.1.1}$$

for all $g \in \mathcal{D}$.

About connection between $\mathcal{D}'$-topology and the topology of exceptional sets for subharmonic functions see [9], [10, Ch. 3].

We also use $C^{\infty}_{q,p}{}^*$-topology, i.e., the topology of linear functionals over the basic space $C^{\infty}_{q,p}$ with the convergence defined like in (1.1.1). The space $C^{\infty}_{q,p}$ is one of the infinitely differentiable functions in $\mathbb{C}\setminus 0$ that tends to ∞ not faster then $O(|z|^{-q})$ as $z\to 0$ and tends to zero not slower then $O(|z|^{-p})$ as $z\to\infty$. Let us note that this topology is stronger that $\mathcal{D}'$-topology because $C^{\infty}_{q,p}\supset\mathcal{D}(\mathbb{C}\backslash 0)$.

Let $u(z)$ be a subharmonic function in $\mathbb{C}$ of normal type with respect to a finite order ρ, i.e.,

$$0<\sigma[u]:=\limsup_{r\to\infty} M(r,u)r^{-\rho}<\infty,$$

where $M(r,u):=\max_{|z|=r}u(z)$.We write $u\in SH(\rho)$.

Let μ be a mass distribution in $\mathbb{C}$ with no mass in zero. It has normal type with respect to the exponent ρ if

$$0<\overline{\Delta}[\mu]:=\limsup_{r\to\infty}\mu(K_r)r^{-\rho}<\infty,$$

where $K_r:=\{z:|z|<r\}$. We write $\mu\in\mathcal{M}(\rho)$. Define by μ_u the mass distribution associated with u. Recall

Borel's Theorem. *Let $[\rho]<\rho$. If $u\in SH(\rho)$, then $\mu\in\mathcal{M}(\rho)$ and vice versa.*

Let $\rho=[\rho]:=p$. Set

$$\delta_R(z,\mu,p):=\frac{1}{p}\int\limits_{|\zeta|<R}\Re\left(\frac{z}{\zeta}\right)^{\rho}\mu(d\xi d\eta).$$

This is a family of the homogeneous harmonic polynomial of degree p. Recall in an equivalent formulation

Lindelöf's Theorem. *If $u\in SH(\rho)$ then $\mu_u\in\mathcal{M}(\rho)$ and the family $\{\delta_R\}$ is precompact as $R\to\infty$, and vice versa.*

Denote $u_t(z):=u(tz)t^{-\rho}$. The function $u(z)\in SH(\rho)$ is called a function of the *completely regular growth* (CRG-function) if $u_t\to h_\rho(z)$ in $\mathcal{D}'$-topology, as $t\to\infty$. Here

$$h_\rho(z):=r^\rho h(e^{i\phi}) \tag{1.1.2}$$

and the function $h(e^{i\phi})$ is a ρ-trigonometrically convex function (ρ-t.c. function) (see, e.g., [10, Ch. 1, §§15, 16]), i.e., it is a 2π-periodic generalized solution of the equation

$$h''+\rho^2h=\Delta(d\phi), \tag{1.1.3}$$

where Δ is a 2π-periodic positive measure.

Recall also that ρ-t.c. function as a distribution is equivalent to a continuous function and can be represented for noninteger ρ in the form

$$h(\phi) = \frac{1}{2\rho \sin \pi\rho} \int_0^{2\pi} * \cos \rho(\phi - \psi - \pi)\ \Delta(d\psi), \tag{1.1.4}$$

where the function $* \cos \rho(\phi)$ is a 2π-periodic extension of the function $\cos \rho\phi$ from the interval $(-\pi, \pi)$ on $(-\infty, \infty)$. If $\rho(> 0)$ is integer, then Δ must satisfy the condition

$$\int_0^{2\pi} e^{i\rho\phi} \Delta(d\phi) = 0, \tag{1.1.5}$$

and the representation has the form

$$h(\phi) = \Re\{Ce^{i\phi)}\} + \frac{1}{2\rho} \int_0^{2\pi} *(\phi - \psi) \sin \rho(\phi - \psi)\Delta(d\psi), \tag{1.1.6}$$

where C is a complex constant, the function $*\psi$ means the 2π-periodic continuation of the function $f(\psi) := \psi$ from the interval $[0, 2\pi)$ on $(-\infty, \infty)$.

Recall (see [9], [10, Ch.3, § 1]) that μ_t (do not confuse with μ_u) is the mass distribution defined by the equality

$$< \mu_t, g >:= t^{-\rho} \int g(z/t)\mu(dxdy)$$

for all $g \in \mathcal{D}$. It can also be defined by the equality

$$\mu_t(E) := \mu(tE)t^{-\rho},$$

where E is every Borel set and tE is the homothety of E.

Let $\rho > [\rho]$. Recall that the mass distribution μ is called *regular* if

$$\mu_t \to \Delta(d\phi) \otimes \rho r^{\rho-1} dr \tag{1.1.7}$$

in $\mathcal{D}'$-topology as $t \to \infty$. $\Delta(d\phi)$ is a measure on the unit circle which is necessarily positive.

Let ρ be an integer number $p = [\rho]$. Then the mass distribution is called regular if, in addition to (1.1.7), $\delta_R(z, \mu_t, p)$ converges in $\mathcal{D}'$-topology as $t \to \infty$ for some R.

Since $\delta_R(z, \mu_t, p)$ is a homogeneous harmonic polynomial, the convergence in $\mathcal{D}'$-topology is equivalent to uniform convergence in every bounded domain.

In such terms Levin–Pfluger's theorem (see [9, Chs. 2, 3], [10, Ch. 3, Th. 3]) may be formulated as follows.

Levin–Pfluger's Theorem. *If u is a CRG-function, then its mass distribution is regular and vice versa.*

1.2. Let $\hat{\rho} = \{\rho > \rho_1 > \ldots > \rho_n > 0\}$ be a finite monotonic system of numbers. We call a function $u \in SH(\rho)$ *completely $\hat{\rho}$-regular* if

$$u_t = h_\rho + t^{\rho_1 - \rho} h_{\rho_1} + \ldots + t^{\rho_n - \rho} h_{\rho_n} + t^{\rho_n - \rho} o(1), \tag{1.2.1}$$

where h_ρ is a ρ-t.c. function and $h_{\rho_j}(z)$, $j = 1, 2, \ldots, n$, are of the form of (1.1.2) with the corresponding h's being the differences of ρ_j-t.c. functions. Therefore h_{ρ_j} can be represented in the form of (1.1.4) or (1.1.6) with Δ's being the functions of bounded variation. Besides, $o(1) \to 0$ in $\mathcal{D}'$ topology.

Let $\rho > [\rho]$ and $\rho_j \in ([\rho], \rho)$, $j = 1, 2, \ldots, n$. We call $\mu \in \mathcal{M}(\rho)$ *$\hat{\rho}$-regular* if

$$\mu_t = \mu_{(\rho)} + \sum_{j=1}^{j=n} t^{\rho_j - \rho} \mu_{(\rho_j)} + t^{\rho_n - \rho} o(1)) \tag{1.2.2}$$

as $t \to \infty$, where

$$\mu_{(\rho)} = \Delta_\rho(d\psi) \otimes \rho r^{\rho - 1} dr, \tag{1.2.3}$$

with Δ_ρ positive and summable, and $\mu_{(\rho_j)}$, $j = 0, 1, \ldots, n$, are of the same form as $\rho = \rho_j$, $j = 0, 1, \ldots, n$, and arbitrary $\Delta_{(\rho_j)}$'s that are the functions of bounded variation on the circle.

If $o(1) \to 0$ in $\mathcal{D}'$-topology, then μ is *$\hat{\rho}$-regular* in $\mathcal{D}'$-topology. However it is possible to say that μ is *$\hat{\rho}$-regular* in other topology if $o(1) \to 0$ in this topology.

Theorem 1.2.1. *Let $\rho > [\rho]$ and $[\rho_n, \rho] \cap \mathbb{N} = \emptyset$. If u is completely $\hat{\rho}$-regular in $\mathcal{D}'$-topology then its mass distribution μ is $\hat{\rho}$-regular in $\mathcal{D}'$ topology. If μ is $\hat{\rho}$-regular in $C^\infty_{p,p+1}$*-topology, then u is completely $\hat{\rho}$-regular in $\mathcal{D}'$-topology.*

Let us notice that the classical Levin–Pfluger theorem of completely regular growth function for noninteger ρ can be obtained from here by using the following

Proposition 1.2.2. *Let $\mu \in \mathcal{M}(\rho)$ and $\mu_t \to \mu_{(\rho)}$ in $\mathcal{D}'$ as $t \to \infty$. Then the same holds in $C^\infty_{p,p+1}$*.*

We suppose further that ρ is an exponent of the convergence of μ.

Let us consider the situation, when $\hat{\rho}$ consists of noninteger numbers, but the interval $(0, \rho)$ contains integer numbers.

Theorem 1.2.3. *Let u_t have the representation*

$$u_t = h_\rho + t^{\rho_1 - \rho} h_{\rho_1} + \ldots + t^{\rho_n - \rho} h_{\rho_n} + \sum_1^{[\rho]} \Re\{a_k z^k\} t^{k - \rho} + t^{\rho_n - \rho} o(1), \tag{1.2.4}$$

where $o(1) \to 0$ *in* $\mathcal{D}'$.

Then

$$\mu_t = \mu_{(\rho)} + \sum_{j=1}^{j=n} t^{\rho_j - \rho} \mu_{(\rho_j)} + t^{\rho_n - \rho} o(1) \tag{1.2.5}$$

with $o(1) \to 0$ *in* $\mathcal{D}'$.

The inverse theorem is the following

Theorem 1.2.4. *Let* $u \in SH(\rho)$ *and its mass distribution have the representation (1.2.5) with* $o(1) \to 0$ *in* ${C^\infty_{p,p+1}}^*$ *and*

$$\int_0^{2\pi} e^{ik\phi} \Delta_{\rho_j}(d\phi) = 0 \tag{1.2.6}$$

for all k, $\rho > k > \rho_j$.

Then (1.2.4) holds for u_t *with* $o(1) \to 0$ *in* $\mathcal{D}'$.

Let us notice that the conditions (1.2.6) are not necessary for the validness of (1.2.4).

The similar theorems can be formulated for the case when ρ or some of ρ_j are integers.

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2. Proofs

2.1. Consider the case when $\rho > [\rho]$ and $[\rho_n, \rho] \cap \mathbb{N} = \emptyset$. Let u_t have the representation (1.2.1) and the remainder term be $o(1)$ in $\mathcal{D}'$-topology. Applying to (1.2.1), the operator $(1/2\pi)\Delta$ (here Δ is the Laplace operator) we obtain (1.2.2), as $(1/2\pi)\Delta u_t = \mu_t$, $(1/2\pi)\Delta h_{\rho_j} = \Delta_{\rho_j}(d\phi)$, $j = 0, \ldots, n$, and $(1/2\pi)\Delta o(1) = o(1)$ since the Laplace operator is continuous in $\mathcal{D}'$-topology. The first assertion of Th. 1.2.1 is proved.

Let (1.2.2) hold with $o(1)$ in ${C^\infty_{p,p+1}}^*$. Apply to it the operator Ad^*_ρ which is conjugated to

$$Ad_\rho[\bullet] := \int_{\mathbb{C}\setminus 0} H(z/\zeta, [\rho]) \bullet (dxdy)$$

that acts from $\mathcal{D}$ to $C_\infty p, p+1$. By definition, for $g \in \mathcal{D}$ we have

$$< Ad^*_\rho \mu_t, g >=< \mu_t, Ad_\rho[g] > .$$

Now substitute (1.2.2) for μ_t. The integral of the first n terms of (1.2.2) are, in fact, the first n terms of (1.2.1). Let us verify it.

We have

$$< \mu_{(\rho_j)}, Ad_\rho[g] >_z = \int g(z)dxdy \int H(z/re^{i\psi}, p)\Delta_j(d\psi)\rho_j r^{\rho_j-1}dr.$$

Counting the inner integral on dr (see, [11, Ch. 1, § 17, footnote 21]), we obtain

$$\int_0^\infty H(z/re^{i\psi}, p)\rho_j r^{\rho_j-1}dr = \frac{1}{2\rho_j \sin \pi\rho_j} * \cos \rho(\arg z - \psi - \pi)|z|^{\rho_j}. \qquad (2.1.1)$$

Hence, using (1.1.4), we obtain

$$< \mu_{\rho_j}, Ad_\rho[g] >_z = < h_{\rho_j}.g > . \qquad (2.1.2)$$

The last term is $t^{\rho_n-\rho}o(1)$ where $o(1)$ is understood in ${C^\infty_{p,p+1}}^*$. The function $Ad_\rho[g]$ is a canonical potential of the function $g \in \mathcal{D}$. Thus $Ad_\rho[g] \in C^\infty_{p,p+1}$. Therefore $< o(1), Ad_\rho[g] >_z \to 0$ as $t \to \infty$. This proves the second assertion of Th. 1.2.1.

2.2. Let us prove Proposition 1.2.2.

P r o o f. Let $g \in C^\infty_{p,p+1}$. Let τ_1, τ_2, τ_3 be a partition of unity by infinitely differentiable functions, such that *supp* $\tau_1 \subset (0, \epsilon)$, *supp* $\tau_2 \subset (\epsilon/2, 2R)$, *supp* $\tau_3 \subset (R, \infty)$. Then

$$\int_{\mathbb{C}} g(z)\mu_t(dxdy) = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_j(t) = \int_{\mathbb{C}} g(z)\tau_j(|z|)\mu_t(dxdy), \ j = 1, 2, 3.$$

The first integral has the estimate

$$|I_1(t)| \leq \lim_{\delta\to 0} \int_\delta^\epsilon Cr^{-p}\mu_t(dr),$$

because g is $O(|z|^{-p})$ as $z \to 0$. Integrating by parts, we obtain

$$I_1(t) \leq C \left[\mu_t(\epsilon)\epsilon^{\rho-p} + \lim_{\delta\to 0} \int_\delta^\epsilon r^{-p-1}\mu_t(r)(dr) \right].$$

Since $\mu(r) \le Cr^{\rho}$, also $\mu_t(r) \le Cr^{\rho}$. Thus

$$I_1(t) \le C\epsilon^{\rho-p} \tag{2.2.1}$$

uniformly with respect to t.

In the same way we obtain

$$I_3(t) \le CR^{\rho-p-1} \tag{2.2.2}$$

uniformly with respect to t.

Since $\mu_t \to \mu_\rho$ in $\mathcal{D}'$ and $g\tau_2 \in \mathcal{D}$, we have

$$I_2(t) \to \int\limits_{\mathbb{C}} g(z)\tau_2(|z|)\mu_\rho(dxdy),\ t \to \infty. \tag{2.2.3}$$

Moreover, (2.2.1),(2.2.2), and (2.2.3) imply that

$$< g, \mu_t > \to < g, \mu_(\rho) >$$

for every $g \in C^{\infty}_{p,p+1}$ because ϵ can be chosen to be arbitrarily small and R can be selected to be arbitrarily large. ■

For proving Th. 1.2.3 we should only repeat the first part of the proof of Th. 1.2.1.

2.3.

P r o o f o f T h e o r e m 1.2.4. As in the proof of Th. 1.2.1 we apply the operator Ad^*_ρ to μ_t and evaluate $< \mu_{\rho_j}, Ad_\rho[g] >_z$. Because of (1.2.3),

$$< \mu_{(\rho_j)}, Ad_\rho[g] >_z = < \rho_j r^{\rho_j - 1}, < \Delta_{\rho_j}, Ad_\rho[g] >_\phi >_r,$$

where

$$< \Delta_{\rho_j}, Ad_\rho[g] >_\phi := \int\limits_0^{2\pi} Ad_\rho[g](re^{i\phi})\Delta_{\rho_j}(d\phi).$$

Changing the order of integration and using (1.2.6) and (2.1.1), we obtain

$$< \mu_{(\rho_j)}, Ad_\rho[g] >_z = < \mu_{(\rho_j)}, Ad_{\rho_j}[g] >_z = < h_{\rho_j}, g > .$$

As it was explained in the proof of Th. 1.2.1, $< o(1), Ad_\rho[g] > \to 0$. Thus

$$Ad^*_\rho \mu_t = h_\rho + t^{\rho_1 - \rho} h_{\rho_1} + ... + t^{\rho_n - \rho} h_{\rho_n} + o(1)t^{\rho_n - \rho}. \tag{2.3.1}$$

By Adamar's theorem (see, e.g., [12, Ch. 4.2])

$$u(z) - Ad^*_\rho \mu(z) = \sum_{k=0}^{[\rho]} \Re\{a_k z^k\}. \tag{2.3.2}$$

Thus (2.3.1) and (2.3.2) imply (1.2.4). ■

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