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Inverse Scattering Problem on the Axis for the Schrödinger Operator with Triangular 2×2 Matrix Potential. I. Main Theorem

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The necessary and sufficient conditions for solvability of ISP under consideration are obtained.

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The paper deals with the scattering problem on the axis for the Schrödinger equation system with the upper triangular matrix potential $V(x) = (v_{rd}(x))_1^2$: $v_{21} \equiv 0$ whose principal diagonal is real: $\mathrm{Im} v_{rr}(x) \equiv 0$, $r = 1, 2$,

$$l[Y] \equiv -Y'' + V(x)Y = k^2 Y, \qquad -\infty < x < \infty. \tag{1}$$

The scalar selfadjoint case of ISP for (1) was considered in [1] under the condition

$$\int_{-\infty}^{\infty} (1 + |x|^m)|V(x)|\mathrm{dx} < \infty \tag{2}$$

with $m = 1$; in [2] and [3] essentially for $m = 2$. The problem on the semi-axis for selfadjoint systems with the condition (2) for $m = 1$ was considered in [4], for triangular $n \times n$ systems in [5]. The scalar not selfadjoint case on the semi-axis with an exponentially small potential at infinity was considered in [9], and a similar case of the axis in [6].

We also mention the monographs [13, 14].

Consider the case $m = 2$.

Besides (1), we also consider the tilde($\sim$)-equation

$$\widetilde{l}[\widetilde{Z}] \equiv -\widetilde{Z}'' + \widetilde{Z}V(x) = k^2\widetilde{Z}, \qquad -\infty < x < \infty. \tag{3}$$

The solutions $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ of equations (1), (3) with asymptotics

$$E_\pm(x,k) \sim e^{\pm ikx}I, \qquad \widetilde{E}_\pm(x,k) \sim e^{\pm ikx}I, \qquad x \to \pm\infty, \quad \operatorname{Im} k \geq 0, \tag{4}$$

where I is a unit matrix, are called the Jost solutions. For them, the representations by B.Ya. Levin are known [7] (see also [1, 3, 4])

$$\begin{aligned} E_\pm(x,k) &= Ie^{\pm ikx} \pm \int_x^{\pm\infty} K_\pm(x,t)e^{\pm ikt}\mathrm{dt}, \\ \widetilde{E}_\pm(x,k) &= Ie^{\pm ikx} \pm \int_x^{\pm\infty} \widetilde{K}_\pm(x,t)e^{\pm ikt}\mathrm{dt}, \qquad \operatorname{Im} k \geq 0, \end{aligned} \tag{5}$$

in terms of transformation operators, with

$$V(x) = \mp 2dK_\pm(x,x)/dx = \mp 2d\widetilde{K}_\pm(x,x)/dx. \tag{6}$$

In addition to the Jost solutions (4), we need the solutions

$$E^\wedge_\pm(x,k) \sim e^{\mp ikx}I, \quad \widetilde{E}^\wedge_\pm(x,k) \sim e^{\mp ikx}I, \quad x \to \pm\infty, \quad \operatorname{Im} k \geq 0,\ k \neq 0, \tag{7}$$

which form fundamental systems together with $E_\pm(x,k)$ (respectively, together with $\widetilde{E}_\pm(x,k)$). The matrix solutions $E^\wedge_\pm(x,k)$ were constructed and studied in [4]; we form the solutions $\widetilde{E}^\wedge_\pm(x,k)$ in a similar way. However, unlike the Jost solutions, the solutions (7) are not determined unambiguously by their asymptotics for $\operatorname{Im} k > 0$. Nevertheless, if one of the solutions (7), (e.g., $E^\wedge_+(x,k)$) is fixed, then the associated solution $\widetilde{E}^\wedge_+(x,k)$ is determined unambiguously by the additional condition

$$\begin{gathered} W\{\widetilde{E}^\wedge_+(x,k), E^\wedge_+(x,k)\} \equiv \widetilde{E}^\wedge_+(x,k)\tfrac{d}{dx}E^\wedge_+(x,k) - \tfrac{d}{dx}\widetilde{E}^\wedge_+(x,k)E^\wedge_+(x,k) = 0, \\ \operatorname{Im} k \geq 0, \quad k \neq 0. \end{gathered} \tag{8}$$

For every $\rho_0 > 0$ solutions (7) can be chosen as analytic in k for $|k| > \rho_0$, $\operatorname{Im} k > 0$, and we will assume this in what follows (for the scalar case see also [8, 9].)

Since with real $k \neq 0$ the pairs of functions $E_+(x,\pm k)$, $E_-(x,\pm k)$ (respectively, $\widetilde{E}_+(x,\pm k)$, $\widetilde{E}_-(x,\pm k)$) form the fundamental systems of solutions of (1) (respectively, of (3)), and their Wronski determinants do not depend on x, one has the following relations (see, for example, [10]):

$$\begin{aligned} E_+(x,k) &= E_-(x,-k)A(k) + E_-(x,k)B(k), \\ E_-(x,k) &= E_+(x,-k)C(k) + E_+(x,k)D(k), \\ \widetilde{E}_+(x,k) &= C(k)\widetilde{E}_-(x,-k) - D(-k)\widetilde{E}_-(x,k), \\ \widetilde{E}_-(x,k) &= A(k)\widetilde{E}_+(x,-k) - B(-k)\widetilde{E}_+(x,k), k \in \mathbf{R}\backslash\{0\}, \end{aligned} \tag{9}$$

with

$$A(k) = \tfrac{1}{2ik}W\{\widetilde{E}_-(x,k), E_+(x,k)\}; C(k) = -\tfrac{1}{2ik}W\{\widetilde{E}_+(x,k), E_-(x,k)\};$$

$$B(k) = -\tfrac{1}{2ik}W\{\widetilde{E}_-(x,-k), E_+(x,k)\}; D(k) = \tfrac{1}{2ik}W\{\widetilde{E}_+(x,-k), E_-(x,k)\}. \tag{10}$$

The values

$$R^+(k) = D(k)C(k)^{-1}; \qquad R^-(k) = B(k)A(k)^{-1} \tag{11}$$

are called the right and the left reflection coefficients, and

$$T^+(k) = C(k)^{-1}; \qquad T^-(k) = A(k)^{-1}, \tag{12}$$

the right and the left transmission coefficients, respectively.*

Use the following well-known relations for coefficients

$$\begin{gathered} A(-k)C(k) = I - B(k)D(k); \qquad C(-k)A(k) = I - D(k)B(k); \\ B(-k)C(k) + A(k)D(k) = D(-k)A(k) + C(k)B(k) = 0, \end{gathered} \tag{13}$$

which follow from (9), (10), to deduce that (see [1, 10])

$$\begin{aligned} (I - R^-(-k)R^-(k))^{-1} &= A(k)C(-k); \\ (I - R^+(-k)R^+(k))^{-1} &= C(k)A(-k). \end{aligned} \tag{14}$$

The eigenvalues of problem (1) k_j^2, $j = \overline{1,p}$, coincide with the set of eigenvalues for the scalar scattering problems with the real diagonal elements of the matrix potential $V(x)$, since they are the roots of the equation

$$\det A(k) = a_{11}(k)a_{22}(k) = 0, \qquad \mathrm{Im} k > 0.$$

Therefore the number of eigenvalues is finite, and $k_j^2 < 0$. Besides that, we assume the absence of a virtual level, i.e., the absence of a bounded on the x-axis nontrivial solution of (1) at $k = 0$.

*These were derived from the first two relations in (9). In a similar way, the last two relations in (9) can be used to determine the tilde-coefficients $\widetilde{R}^{\pm}(k)$ и $\widetilde{T}^{\pm}(k)$ corresponding to (3) which coincide with $R^{\pm}(k)$ and $T^{\pm}(k)$ (see Lemma 2 below).

We call the polynomials

$$
\begin{aligned}
Z_j^+(t) &= -ie^{-ik_jt}\mathrm{Res}_{k_j}\{W^+(k)C(k)^{-1}e^{ikt}\},\\
Z_j^-(t) &= -ie^{ik_jt}\mathrm{Res}_{k_j}\{W^-(k)A(k)^{-1}e^{-ikt}\}, \qquad j=\overline{1,p}, \quad t\in\mathbf{R},
\end{aligned}
\tag{15}
$$

with

$$
W^{\pm}(k) = \pm\frac{1}{2ik}W\{\widetilde{E}_{\pm}^{\wedge}(x,k), E_{\mp}(x,k)\}, \tag{16}
$$

respectively, the right and the left normalizing polynomials (cf. [8, 9, 6] in the scalar case). Normalizing polynomials do not depend on a choice of $\widetilde{E}_{\pm}^{\wedge}$ in the expression for $W^{\pm}(k)$ (16) and possess the properties:

Lemma 1. *a) One has the following inequalities that involve degrees of normalizing polynomials:*

$$
\deg Z_j^{\pm}(t) \le \sum_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]\pm} - 1 \le 1, \qquad j=\overline{1,p}, \tag{17}
$$

with the diagonal elements $z_{ll}^{[j]\pm}$ being nonnegative and independent of t. (The degree of the identically zero polynomial is assumed to be negative.)
b) The ranks of normalizing polynomials satisfy the following relations:

$$
\operatorname{rg} Z_j^{\pm}(t) = \operatorname{rg}\operatorname{diag} Z_j^{\pm}(t) = \operatorname{rg}\operatorname{diag} Z_j^{\pm}(0), \qquad j=\overline{1,p}. \tag{18}
$$

c) The matrix elements of normalizing polynomials and those of the matrices $C(k)$, $A(k)$ are related as follows:

$$
\begin{aligned}
a_{11}(k_j)z_{12}^{[j]-}(0) + c_{12}(k_j)z_{22}^{[j]-} + i\dot a_{11}(k_j)(z_{12}^{[j]-})'(0) = 0;\\
z_{11}^{[j]-}a_{12}(k_j) + z_{12}^{[j]-}(0)a_{22}(k_j) + i(z_{12}^{[j]-})'(0)\dot a_{22}(k_j) = 0;
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
z_{11}^{[j]+}c_{12}(k_j) + z_{12}^{[j]+}(0)a_{22}(k_j) - i(z_{12}^{[j]+})'(0)\dot a_{22}(k_j) = 0;\\
a_{11}(k_j)z_{12}^{[j]+}(0) + a_{12}(k_j)z_{22}^{[j]+} - i\dot a_{11}(k_j)(z_{12}^{[j]+})'(0) = 0;
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
a_{11}(k_j)(z_{12}^{[j]\pm})'(0) = a_{22}(k_j)(z_{12}^{[j]\pm})'(0) = 0;\\
a_{ll}(k_j)z_{ll}^{[j]\pm} = 0, \qquad l=1,2; \quad j=\overline{1,p}.
\end{aligned}
\tag{21}
$$

P r o o f. The claims a) and b) can be proved similarly to [5]. Prove (20), (21) for $Z_j^+(t)$. Use (17) to deduce from the definition (15) that $Z_j^+(0) = -i\frac{d}{dk}(W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}$; $(Z_j^+)'(0) = (W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}$, that is
$(Z_j^+)'(0)C(k_j) = (W^+(k)C(k)^{-1}C(k)(k-k_j)^2)_{k_j} = (W^+(k)(k-k_j)^2)_{k_j} = 0;$

$Z_j^+(0)C(k_j) - i(Z_j^+)'(0)\dot{C}(k_j) = -i\frac{d}{dk}(W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}C(k_j) - i(W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}\dot{C}(k_j) = -i\frac{d}{dk}(W^+(k)C(k)^{-1}C(k)(k-k_j)^2)_{k_j} = -i\frac{d}{dk}(W^+(k)(k-k_j)^2)_{k_j} = 0.$

Now write down the latter relations separately for the matrix elements to get (20) and (21) for $Z_j^+(t)$, $j = \overline{1,p}$. The validity of (19), (21) for $Z_j^-(t)$ can be proved in a similar way. Lemma 1 is proved.

The set of values $\{R^+(k), k \in \mathbf{R}; k_j^2 < 0, Z_j^+(t), j = \overline{1,p}\}$, respectively, $\{R^-(k), k \in \mathbf{R};\ k_j^2 < 0, Z_j^-(t), j = \overline{1,p}\}$ are called the right and the left scattering data (SD) for equation (1). In this context, as in the scalar case [1–3], the right scattering data is determined unambiguously by the left data, and conversely (see (29) and Lemma 3 below). SD of the scalar problem that corresponds to some diagonal entry of the triangular matrix potential coincides with the system of corresponding diagonal entries of the matrix SD.

Lemma 2. *The associated right scattering data (SD) and the tilde-scattering data*

$$\{R^+(k),\ k \in \mathbf{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = \overline{1,p}\} \tag{22}$$

for problems (1) and (3) of the above form coincide. Analogously, the left data for problems (1) and (3) coincide.

P r o o f. We present a proof for the case of the right SD. Define the reflection coefficient for problem (3) by

$$\widetilde{R}^+(k) = -A^{-1}(k)B(-k), \qquad k \in \mathbf{R}, \tag{23}$$

with $A(k)$ and $B(k)$ determined by (10). The third equality in (13), together with (23), implies

$$R^+(k) = D(k)C^{-1}(k) = -A^{-1}(k)B(-k) = \widetilde{R}^+(k), \qquad k \in \mathbf{R}. \tag{24}$$

The coincidence of the eigenvalues k_j^2, $\operatorname{Im} k_j > 0$, for problems (1) and (3) follows from the upper-triangular form of the problems and the fact that the diagonal elements of the matrix potential $V(x)$ are real, that is $\det A(k) = \det C(k) = a_{11}(k)a_{22}(k)$, $\operatorname{Im} k \geq 0$. By the definition of a normalizing polynomial for problem (3) one has

$$\widetilde{Z}_j^+(t) = -ie^{-ik_jt}\mathrm{Res}_{k_j}\{A^{-1}(k)\widetilde{W}^+(k)e^{ikt}\}, \tag{25}$$

with

$$\widetilde{W}^+(k) = -\frac{1}{2ik}W\{\widetilde{E}_-(x,k); E_+^\wedge(x,k)\}. \tag{26}$$

In the upper half-plane, similarly to (9), one has the following representations:

$$\begin{gathered} E_+(x,k) = E_-(x,k)W^-(k) + E_-^\wedge(x,k)A(k), \\ E_+^\wedge(x,k) = E_-(x,k)W^\wedge(k) - E_-^\wedge(x,k)\widetilde{W}^+(k), \\ E_-(x,k) = E_+(x,k)W^+(k) + E_+^\wedge(x,k)C(k), \qquad \operatorname{Im} k > 0, \end{gathered} \tag{27}$$

with $W^{\pm}(k)$ defined by (16) and $\widetilde{W}^{+}(k)$ by (26), $W^{\wedge}(k) = -\frac{1}{2ik}W\{\widetilde{E}_{-}^{\wedge}(x,k), E_{+}^{\wedge}(x,k)\}$. Now substitute the initial two relations of (27) into the third one to obtain $E_{-}(x,k) = E_{-}(x,k)W^{-}(k)W^{+}(k) + E_{-}^{\wedge}(x,k)A(k)W^{+}(k) + E_{-}(x,k)W^{\wedge}(k)C(k) - E_{-}^{\wedge}(x,k)\widetilde{W}^{+}(k)C(k)$. Group the summands to get $E_{-}(x,k)(I - W^{-}(k)W^{+}(k) - W^{\wedge}(k)C(k)) = E_{-}^{\wedge}(x,k)(A(k)W^{+}(k) - \widetilde{W}^{+}(k)C(k))$, with $\mathrm{Im}\,\mathrm{k} > 0$.

Since the solutions $E_{-}(x,k)$ and $E_{-}^{\wedge}(x,k)$ form a fundamental system with $\mathrm{Im}\,k > 0$, the following relations are valid:
$A(k)W^{+}(k) = \widetilde{W}^{+}(k)C(k)$,
$I = W^{-}(k)W^{+}(k) + W^{\wedge}(k)C(k), \ \mathrm{Im}\,k > 0$.
So, $A(k)^{-1}\widetilde{W}^{+}(k) = W^{+}(k)C(k)^{-1}$, that is $Z_j^{+}(t) = \widetilde{Z}_j^{+}(t)$, which was to be proved. Lemma 2 is proved.

Remark 1. *Similarly to (24), one has for the left reflection coefficient*

$$R^{-}(k) := B(k)A(k)^{-1} = -C(k)^{-1}D(-k) =: \widetilde{R}^{-}(k), \qquad k \in \mathbf{R}, \qquad (28)$$

and for the left normalizing polynomial
$Z_j^{-}(t) = -ie^{ik_jt}\mathrm{Res}_{k_j}\{W^{-}(k)A(k)^{-1}e^{-ikt}\} = -ie^{ik_jt}\mathrm{Res}_{k_j}\{C(k)^{-1}\widetilde{W}^{-}(k)e^{-ikt}\}$, $j = \overline{1,p}$, *with* $\widetilde{W}^{-}(k) = \frac{1}{2ik}W\{\widetilde{E}_{+}(x,k), E_{-}^{\wedge}(x,k)\}$.

(28) and (24) allow to write down a relationship between the right and the left reflection coefficients:

$$R^{-}(k) = -A(-k)R^{+}(-k)A(k)^{-1} = -C(k)^{-1}R^{+}(-k)C(-k), \qquad k \in \mathbf{R}, \quad (29)$$

A relationship between the right and the left normalizing polynomials is set up by the following

Lemma 3. *One has the relations as follows between the right and the left normalizing polynomials for any* $\tau \in \mathbf{R}$*:*

$$\begin{aligned} Z_j^{-}(t) &= -C_j(t-\tau)[Z_j^{+}(\tau) + Q_j]^{-1}A_{-1}^{<k_j>}, \\ Z_j^{+}(t) &= -A_j(t-\tau)[Z_j^{-}(\tau) + Q_j]^{-1}C_{-1}^{<k_j>}; \end{aligned} \qquad (30)$$

$$\begin{aligned} Z_j^{-}(t) &= -C_{-1}^{<k_j>}[Z_j^{+}(\tau) + Q_j]^{-1}A_j(\tau - t), \\ Z_j^{+}(t) &= -A_{-1}^{<k_j>}[Z_j^{-}(\tau) + Q_j]^{-1}C_j(\tau - t), \end{aligned} \qquad (31)$$

with $C_j(t) = e^{ik_jt}\mathrm{Res}_{k_j}\{C(k)^{-1}e^{-ikt}\} = C_{-1}^{<k_j>} + (-it)C_{-2}^{<k_j>}$, $A_j(t) = e^{-ik_jt}\times \mathrm{Res}_{k_j}\{A(k)^{-1}e^{ikt}\} = A_{-1}^{<k_j>} + itA_{-2}^{<k_j>}$, Q_j *being the arbitrary upper triangular matrices with the property* $q_{ll}^{[j]} = 0$ *if* $z_{ll}^{[j]\pm} \neq 0$ *and* $q_{ll}^{[j]} \neq 0$ *if* $z_{ll}^{[j]\pm} = 0$, $l = 1, 2$, $\mathrm{rg}\,Q_j = 2 - \mathrm{rg}\,Z_j^{\pm}(t)$.

P r o o f. It is easy to show that at k_j, the eigenvalues of problems (1) and (3), one has the following relations:

$$\left\{\begin{aligned} E_+(x,k_j)A_{-2}^{<k_j>} &= i^2E_-(x,k_j)(Z_j^-)'(0);\\ E_+(x,k_j)A_{-1}^{<k_j>} + \dot{E}_+(x,k_j)A_{-2}^{<k_j>} &= iE_-(x,k_j)Z_j^-(0) + i^2\dot{E}_-(x,k_j)(Z_j^-)'(0); \end{aligned}\right. \tag{32}$$

$$\left\{\begin{aligned} E_-(x,k_j)C_{-2}^{<k_j>} &= -i^2E_+(x,k_j)(Z_j^+)'(0);\\ E_-(x,k_j)C_{-1}^{<k_j>} + \dot{E}_-(x,k_j)C_{-2}^{<k_j>} &= iE_+(x,k_j)Z_j^+(0) - i^2\dot{E}_+(x,k_j)(Z_j^+)'(0); \end{aligned}\right. \tag{33}$$

$$\left\{\begin{aligned} C_{-2}^{<k_j>}\widetilde{E}_+(x,k_j) &= i^2(Z_j^-)'(0)\widetilde{E}_-(x,k_j);\\ C_{-1}^{<k_j>}\widetilde{E}_+(x,k_j) + C_{-2}^{<k_j>}\dot{\widetilde{E}}_+(x,k_j) &= iZ_j^-(0)\widetilde{E}_-(x,k_j) + i^2(Z_j^-)'(0)\dot{\widetilde{E}}_-(x,k_j); \end{aligned}\right. \tag{34}$$

$$\left\{\begin{aligned} A_{-2}^{<k_j>}\widetilde{E}_-(x,k_j) &= -i^2(Z_j^+)'(0)\widetilde{E}_+(x,k_j);\\ A_{-1}^{<k_j>}\widetilde{E}_-(x,k_j) + A_{-2}^{<k_j>}\dot{\widetilde{E}}_-(x,k_j) &= iZ_j^+(0)\widetilde{E}_+(x,k_j) - i^2(Z_j^+)'(0)\dot{\widetilde{E}}_+(x,k_j). \end{aligned}\right. \tag{35}$$

Prove the first relation in (30). The second one of (30) and (31) can be proved in a similar way. Transform the system (32) as follows:

$$\left\{\begin{aligned} -iE_+(x,k_j)A_j'(-t) &= i^2E_-(x,k_j)(Z_j^-)'(t);\\ E_+(x,k_j)A_j(-t) - i\dot{E}_+(x,k_j)A_j'(-t) &= iE_-(x,k_j)Z_j^-(t) + i^2\dot{E}_-(x,k_j)(Z_j^-)'(t); \end{aligned}\right.$$

to be rewritten in a block matrix form:

$$\begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j)\\ 0 & E_+(x,k_j)\end{pmatrix}\cdot\begin{pmatrix} A_j(-t)\\ -iA_j'(-t)\end{pmatrix}$$

$$=\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j)\\ 0 & E_-(x,k_j)\end{pmatrix}\cdot\begin{pmatrix} iZ_j^-(t)\\ i^2(Z_j^-)'(t).\end{pmatrix}.$$

Furthermore, it follows from (33) that

$$\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j)\\ 0 & E_-(x,k_j)\end{pmatrix}\cdot\begin{pmatrix} C_{-1}^{<k_j>}\\ C_{-2}^{<k_j>}\end{pmatrix}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}$$

$$=\begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j)\\ 0 & E_+(x,k_j)\end{pmatrix}\cdot\begin{pmatrix} iZ_j^+(0)\\ -i^2(Z_j^+)'(0)\end{pmatrix}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}.$$

Applying the relation

$$Z_j^+(\tau)[Z_j^+(t)+Q_j]^{-1}A_{-1}^{<k_j>} = A_j(\tau - t), \tag{36}$$

to be proved below, we deduce that

$$\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \cdot \begin{pmatrix} C_{-1}^{<k_j>} \\ C_{-2}^{<k_j>} \end{pmatrix} (Z_j^+(t)+Q_j)^{-1} A_{-1}^{<k_j>}$$

$$= \begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j) \\ 0 & E_+(x,k_j) \end{pmatrix} \cdot \begin{pmatrix} iA_j(-t) \\ -i^2(A_j)'(-t) \end{pmatrix}$$

$$= i \begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \cdot \begin{pmatrix} iZ_j^-(t) \\ i^2(Z_j^-)'(t) \end{pmatrix}.$$

Compare the left- and the right-hand sides to deduce

$$Z_j^-(t) = -C_{-1}^{<k_j>}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>};$$

$$(Z_j^-)'(t) = iC_{-2}^{<k_j>}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}.$$

Multiply the second relation by $(\tau - t)$ and add the result to the first one to get $Z_j^-(\tau) = -C_j(\tau - t)(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}$, or, equivalently, $Z_j^-(t) = -C_j(t - \tau)(Z_j^+(\tau)+Q_j)^{-1}A_{-1}^{<k_j>}$.

Now prove (36). Consider the cases:

1) $a_{11}(k_j) = a_{22}(k_j) = 0$, then $z_{ll}^{[j]\pm} \neq 0$, hence $Q_j = 0$, and so by a virtue of the second relation in (20):

$$Z_j^+(\tau)Z_j^+(t)^{-1}A_{-1}^{<k_j>}$$

$$= \begin{pmatrix} 1 & \frac{z_{12}^{[j]+}(\tau) - z_{12}^{[j]+}(t)}{z_{22}^{[j]+}} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\dot{a}_{11}(k_j)} & -\frac{d}{dk}\left(\frac{a_{12}(k)(k-k_j)^2}{a_{11}(k)a_{22}(k)}\right)_{k_j} \\ 0 & \frac{1}{\dot{a}_{22}(k_j)} \end{pmatrix}$$

$$= A_{-1}^{<k_j>} + i(\tau - t)A_{-2}^{<k_j>} = A_j(\tau - t).$$

2) $a_{11}(k_j) = 0$; $a_{22}(k_j) \neq 0$, hence $q_{11}^{[j]} = 0$; $q_{22}^{[j]} \neq 0$ and $Z_j^+(t) = Z_j^+(\tau) = Z_j^+$; $A_j(\tau - t) = A_{-1}$. Thus (36) is equivalent to $Q_j(Z_j^+ + Q_j)^{-1}A_{-1}^{<k_j>} = 0$. So, in our case

$$Q_j(Z_j^+ + Q_j)^{-1}A_{-1}^{<k_j>}$$

$$= \begin{pmatrix} 0 & q_{12}^{[j]} \\ 0 & q_{22}^{[j]} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{z_{11}^{[j]+}} & -\frac{z_{12}^{[j]+} + q_{12}^{[j]}}{z_{11}^{[j]+} q_{22}^{[j]}} \\ 0 & \frac{1}{q_{22}^{[j]}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\dot{a}_{11}(k_j)} & -\frac{a_{12}(k_j)}{\dot{a}_{11}(k_j)a_{22}(k_j)} \\ 0 & 0 \end{pmatrix} = 0.$$

3) $a_{11}(k_j) \neq 0$; $a_{22}(k_j) = 0$, hence $q_{11}^{[j]} \neq 0$; $q_{22}^{[j]} = 0$ and $Z_j^+(t) = Z_j^+(\tau) = Z_j^+$; $A_j(\tau - t) = A_{-1}^{<k_j>}$. Thus (36) is equivalent to $Q_j(Z_j^+ + Q_j)^{-1}A_{-1}^{<k_j>} = 0$. By a virtue of the second relation in (20) one has

$$Q_j(Z_j^+ + Q_j)^{-1}A_{-1}^{<k_j>}$$
$$= \begin{pmatrix} q_{11}^{[j]} & q_{12}^{[j]} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{q_{11}^{[j]}} & -\frac{z_{12}^{[j]+} + q_{12}^{[j]}}{q_{11}^{[j]} z_{22}^{[j]+}} \\ 0 & \frac{1}{z_{22}^{[j]+}} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{a_{12}(k_j)}{a_{11}(k_j)\dot{a}_{22}(k_j)} \\ 0 & \frac{1}{\dot{a}_{22}(k_j)} \end{pmatrix} = 0.$$

Lemma 3 is proved.

Lemma 4. *For the problem (1), (2) under consideration with $m \geq 1$ one has the following decomposition of the δ-function:*

$$\begin{aligned} \delta(x-t)I = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} E_+(x,k)A(k)^{-1}\widetilde{E}_-(t,k)\,dk \\ + \sum_{j=1}^{p}\sum_{l=0}^{1} \tfrac{d^l}{i^l dk^l}\{E_+(x,k)(Z_j^+)^{(l)}(0)\widetilde{E}_+(t,k)\}_{k_j}, \end{aligned} \tag{37}$$

or, equivalently,

$$\begin{aligned} \delta(x-t)I = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \{E_+(x,k)\widetilde{E}_+(t,-k) + E_+(x,k)R^+(k)\widetilde{E}_+(t,k)\}\,dk \\ + \sum_{j=1}^{p}\sum_{l=0}^{1} \tfrac{d^l}{i^l dk^l}\{E_+(x,k)(Z_j^+)^{(l)}(0)\widetilde{E}_+(t,k)\}_{k_j}, \end{aligned} \tag{38}$$

which is known to be equivalent to the Parceval equation.

P r o o f (cf. [11]) elaborates the method of contour integration combined with the passage to weak limit for the resolvent $zR_z(L) \to E$ as $z \to \infty$, with E being the identity operator generated by the kernel $\delta(x-t)$ as an integral operator, along with (9), (11).

Lemma 3 and (29) indicate that for determining the right scattering data by a given left SD or conversely, it is sufficient to retrieve simultaneously the matrices $A(k)$ and $C(k)$.

Lemma 5. *Suppose that an upper triangular 2×2 matrix $R^+(k)$ is continuous in $k \in \mathbf{R}$; $R^+(0) = -I$, $R^+(k) = O(k^{-1})$ as $k \to \pm\infty$, and $I - \overline{R^+(-k)R^+(k)} = O(k^2)$ as $k \to 0$. Assume that its diagonal elements are such that $\overline{r_{ll}^+(k)} = r_{ll}^+(-k)$, $|r_{ll}^+(k)| \leq 1 - \frac{C_l k^2}{1+k^2}$, $l = 1,2$, and the associated functions*

$$za_{ll}(z) \equiv zc_{ll}(z) = z\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln(1-|r_{ll}^+(k)|^2)}{k-z}dk\right\}, \quad l = 1,2,\ \operatorname{Im} z > 0, \tag{39}$$

are continuous in the closed upper half-plane (see [1], [2]) such that the function $ka_{11}(-k)a_{22}(k)\left\{r_{11}^{+}(-k)r_{12}^{+}(k)+r_{12}^{+}(-k)r_{22}^{+}(k)\right\}$ *satisfies the Hölder condition on the real axis, with the infinite point included* [*]. *Then the following Riemann problem is solvable uniquely with respect to* $c_{12}(k)$ *and* $a_{12}(-k)$ *(cf. (14)), which are regular in the upper and the lower half-planes:*

$$\begin{gathered} kc_{12}(k)=\tfrac{a_{11}(k)}{a_{22}(-k)}\cdot[-ka_{12}(-k)]+ka_{22}(k)|a_{11}(k)|^2 \\ \cdot\left\{r_{11}^{+}(-k)r_{12}^{+}(k)+r_{12}^{+}(-k)r_{22}^{+}(k)\right\}, \\ k\in\mathbf{R}, \end{gathered} \tag{40}$$

and it turns out that this solution satisfies the assumption

$$R^{-}(0)=-I, \tag{41}$$

with $R^{-}(k)$ *produced as in (29) via the matrices* $A(k)$ *and* $C(k)$*, determined from (39), (40),* I *is an identity matrix. The above solution admits a representation in the form*

$$\begin{aligned} c_{12}(z)&=\tfrac{\psi^{+}(z)-\psi^{+}(0)}{z}a_{11}(z); \\ a_{12}(z)&=\tfrac{\psi^{-}(-z)-\psi^{+}(0)}{z}a_{22}(z), \qquad \operatorname{Im}z>0, \end{aligned} \tag{42}$$

with

$$\begin{gathered} \psi^{\pm}(z)=\tfrac{1}{2\pi i}\int\limits_{-\infty}^{\infty}ka_{11}(-k)a_{22}(k)\left\{r_{11}^{+}(-k)r_{12}^{+}(k)+r_{12}^{+}(-k)r_{22}^{+}(k)\right\}\tfrac{dk}{k-z}, \\ \pm\mathrm{Im}z>0, \\ \psi^{+}(0)=\tfrac{1}{2\pi i}\int\limits_{-\infty}^{\infty}a_{11}(-k)a_{22}(k)\left\{r_{11}^{+}(-k)r_{12}^{+}(k)+r_{12}^{+}(-k)r_{22}^{+}(k)\right\}dk. \end{gathered} \tag{43}$$

P r o o f. The Riemann problem (40) under the assumptions of the lemma has index $\nu=-1$, hence its solution exists and is unique (see [12, Sect. 14.7]). The related formulas get simpler since the coefficient of equation (40) is already written in a factorized form $\frac{a_{11}(k)}{a_{22}(-k)}$, so we obtain

$$\begin{aligned} zc_{12}(z)&=\{a_0+\psi^{+}(z)\}a_{11}(z), \qquad &\operatorname{Im}z>0, \\ -za_{12}(-z)&=\{a_0+\psi^{-}(z)\}a_{22}(-z), \qquad &\operatorname{Im}z<0, \end{aligned} \tag{44}$$

where a_0 is a constant determined by the requirement for the right hand sides to be continuous at $z=0$, i.e.,

$$a_0=-\psi^{+}(0)=-\psi^{-}(0), \tag{45}$$

[*] The latter assumption for a function $f(k)$ (see [12]) means the existence of constants A and λ, $0<\lambda\le 1$, such that for any x_1 and x_2 whose modulus exceeds one, $|f(x_1)-f(x_2)|\le A|\frac{1}{x_1}-\frac{1}{x_2}|^{\lambda}$.

and the functions $\psi^{\pm}(z)$ determined by

$$\psi^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ka_{22}(k)|a_{11}(k)|^2\{r_{11}^{+}(-k)r_{12}^{+}(k)+r_{12}^{+}(-k)r_{22}^{+}(k)\}}{a_{11}(k)}\frac{dk}{k-z}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} ka_{22}(k)a_{11}(-k)\{r_{11}^{+}(-k)r_{12}^{+}(k) + r_{12}^{+}(-k)r_{22}^{+}(k)\}\frac{dk}{k-z}, \quad \pm\mathrm{Im}\, z > 0, \tag{46}$$

where it is implicit that $a_{ll}(-k) = \overline{a_{ll}(k)}$, $l = 1, 2$, by a virtue of (39). The second equality in (45) follows from (46) and the assumptions of the lemma. The solution (44) is derived from (40), (46), and the Sokhotski–Plemelj formulas, which all together result in

$$\frac{kc_{12}(k)}{a_{11}(k)} - \psi^{+}(k) = -\frac{ka_{12}(-k)}{a_{22}(-k)} - \psi^{-}(k) = a_0, \tag{47}$$

where the second equality (with a constant as a right-hand side) follows from the Liouville theorem, since the left and the central parts of (47) appear to be analytic continuations of each other to the entire complex plane. Applying (45), one can deduce (42) from (47).

Now use (29) to construct a function

$$r_{12}^{-}(k) = -\frac{r_{11}^{+}(-k)c_{12}(-k)}{a_{11}(k)} - \frac{a_{22}(-k)}{a_{11}(k)}r_{12}^{+}(-k) - \frac{r_{22}^{-}(-k)c_{12}(k)}{a_{11}(k)}.$$

The application of the Sokhotskiy–Plemelj formulas for $\psi^{\pm}(z)$ (46) allows one to deduce from (44) with $k \neq 0$ on the real axis

$$r_{12}^{-}(k) = -\frac{a_{22}(-k)}{a_{11}(k)}r_{12}^{+}(-k) - \frac{r_{11}^{-}(k)}{k}\{a_0 + \psi^{+}(-k)\} - \frac{r_{22}^{-}(k)}{k}\{a_0 + \psi^{+}(k)\}.$$

It follows that as $k \to 0$ by (47), (45), one has $r_{12}^{-}(0) = 0$ since $r_{ll}^{-}(0) = -1$ due to the properties of a scalar inverse scattering problem, which yields (41). Lemma 5 is proved.

Remark 2. *Suppose that in Lemma 5, besides the matrix $R^{+}(k)$ that satisfies the assumptions of the lemma, we are given the numbers $k_j^2 < 0$ and polynomials $Z_j^{+}(t)$, $j = \overline{1,p}$, $t \in \mathbf{R}$, satisfying the assumptions a), b), and (21) of Lemma 1. Assume also that $za_{ll}(z)$ are given not by (39), but as follows:*

$$za_{ll}(z) \equiv zc_{ll}(z) := ze^{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-|r_{ll}^{+}(k)|^2)}{k-z}dk}\prod_{j=1}^{p}\left(\frac{z-k_j}{z+k_j}\right)^{s_j^l}, \tag{48}$$

with $s_j^l = \operatorname{sign} z_{ll}^{[j]+} \geq 0$, $l = 1, 2$. *In this case the Riemann problem (40) is also solvable uniquely under the assumptions (20) so that*

$$zc_{12}(z) = \left\{ \frac{-\psi^+(0) \prod\limits_{j=1}^{p} k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p} (z+k_j)^{\kappa_j}} + \psi^+(z) \right\} a_{11}(z) \prod_{j=1}^{p} \left(\frac{z+k_j}{z-k_j} \right)^{s_j^1},$$

$$\operatorname{Im} z > 0$$

$$-za_{12}(-z) = \left\{ \frac{-\psi^+(0) \prod\limits_{j=1}^{p} k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p} (z+k_j)^{\kappa_j}} + \psi^-(z) \right\} a_{22}(-z) \prod_{j=1}^{p} \left(\frac{z+k_j}{z-k_j} \right)^{s_j^1},$$

$$\operatorname{Im} z < 0, \tag{49}$$

with $a_1, \ldots, a_\kappa$ *being retrievable unambiguously due to the systems (20), (21), and the given normalizing polynomials,* $\kappa = \sum\limits_{j=1}^{p} \kappa_j = \sum\limits_{j=1}^{p} \left(\operatorname{sign} z_{11}^{[j]+} + \operatorname{sign} z_{22}^{[j]+} \right)$, *and the functions* $\psi^\pm(z)$ *and* $\psi^+(0)$ *determined by (43). The solution we get this way appears to be such that (43) is valid if* $R^-(k)$ *is constructed as in (29), which involves matrices* $A(k)$ *and* $C(k)$ *determined by (48), (49), (40). Note that problem (40) in our case has index* $\nu = \kappa - 1$.

The determinant of the system that defines $a_1, \ldots, a_\kappa$ is nonzero in the cases under consideration, as one can see from (21).

Theorem 1. *If a set of values* $\{R^+(k),\ k \in \mathrm{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = \overline{1,p}\}$ *forms the right SD for the scattering problem (1), (2) with* $m = 2$ *and an upper triangular* 2×2 *matrix potential (with a real diagonal and without virtual level), the conditions 1)–6) should be satisfied:*

1) $R^+(k)$ *is continuous in* $k \in \mathrm{R}$: $\overline{r_{ll}^+(k)} = r_{ll}^+(-k)$, $|r_{ll}^+(k)| \leq 1 - \frac{C_l k^2}{1+k^2}$, $l = 1, 2$, $R^+(0) = -I$; $I - R^+(-k)R^+(k) = O(k^2)$ *as* $k \to 0$ *and* $R^+(k) = O(k^{-1})$ *with* $k \to \pm\infty$ *(note, that replacing the last condition by* $R^+(k) = o(k^{-1})$ *we obtain a necessary condition too).*

2) The function

$$F_R^+(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} R^+(k) e^{ikx} dk \tag{50}$$

is absolutely continuous, and with $a > -\infty$ *one has* $\int\limits_a^{+\infty} \left(1 + x^2\right) \left| \frac{d}{dx} F_R^+(x) \right| dx < \infty$.

3) The functions $za_{ll}(z)$, $l = 1, 2$, *given by (48), are continuously differentiable in the closed upper half-plane.*

4) The function

$$F_R^-(x) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} C(k)^{-1}R^+(-k)C(-k)e^{-ikx}dk \tag{51}$$

is absolutely continuous, and with $a < +\infty$ *one has* $\int\limits_{-\infty}^{a} \left(1+x^2\right)\left|\frac{d}{dx}F_R^-(x)\right| dx < \infty$*. Here* $c_{12}(z)$ *is given by (49),* $c_{ll}(z) \equiv a_{ll}(z)$ *is determined by (48) (one can show that condition 4 is also necessary in the version 4a, namely, if* $c_{12}(z)$ *is constructed as in (42), (43), and* $c_{ll}(z) \equiv a_{ll}(z)$ *is constructed as in (39)).*

5) $\deg Z_j^+(t) \le \sum\limits_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]+} - 1, \quad j = \overline{1,p};$ $z_{ll}^{[j]+} \ge 0$ *and* $z_{ll}^{[j]+}$ *are constant.*

6) $\operatorname{rg} Z_j^+(t) = \operatorname{rg}\operatorname{diag} Z_j^+(t) = \operatorname{rg}\operatorname{diag} Z_j^+(0), \qquad j = \overline{1,p}.$

The necessary conditions 1)–6) listed above (with condition 4 being replaced by its version 4a) become sufficient together with the following assumption:

H) The function $ka_{11}(-k)a_{22}(k)\{r_{11}^+(-k)r_{12}^+(k) + r_{12}^+(-k)r_{22}^+(k)\}$ *satisfies the Hölder condition in the finite points as well as at infinity.*

(*The claims of the Theorem related solely to the diagonal matrix elements are direct consequences of [1, 2].*)

Remark 3. *In the case when the discrete spectrum is absent, the conditions 5) and 6) of the Theorem 1 become inapplicable, and the conditions 4 and 4a become the same.*

The proof of the theorem will be published in Part II.

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