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Variation of Subharmonic Function under Transformation of its Riesz Measure

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The paper combines two aspects. First, it contains a compressed comparative review of the well-known results on the change of growth of entire (subharmonic resp.) function under the shifts of its zeros (under T-shift of its Riesz measure resp.). There was B.Ya. Levin who stood at the sources of these results. Second, the first coauthor proves new results obtained in this direction: the estimates of change of subharmonic function under integral restrictions for T-shift of its Riesz measure, and also, in a certain sense, an optimal approximation of an entire function by the entire function with simple zeros.*

Key words: entire function, subharmonic function, shift of zeros, Riesz measure, T-shift of measure.

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Dedicated to the centennial of the birthday of B.Ya. Levin

1. Introduction: Initial Results

Denote by $D(z,t)$ *an open disk of the radius* r *centered at* $z \in \mathbb{C}$ in *the complex plane* $\mathbb{C}$, $D(r) := D(0,r)$. For $t \leqslant 0$, $D(z,t)$ is *an empty set* $\varnothing$ by definition. For $S \subset \mathbb{C}$, we denote its *boundary in* $\mathbb{C}$ by ∂S. In particular, $\partial D(z,t)$ is *a circumference of the radius* r *centered at* z.

By $\mathcal{M}^+$ denote the class of all *positive Borel measures* μ on $\mathbb{C}$, $\operatorname{supp}\mu$ is *a support* of $\mu \in \mathcal{M}^+$; $\mu(z,t) := \mu\bigl(D(z,t)\bigr)$, $\mu^{\mathrm{rad}}(r) := \mu(0,r) = \mu\bigl(D(r)\bigr)$. By $\mu\,\big|_B$ denote restriction of the measure μ to the Borel subset $B \subset \mathbb{C}$.

Let $\Lambda = \{\lambda_k\}$, $k = 1,2,\dots$, be a sequence of points in the complex plane without accumulating points in $\mathbb{C}$. With Λ we associate an *integer-valued measure* n_Λ on $\mathbb{C}$ by the rule

$$n_\Lambda(D) := \sum_{\lambda_k \in D} 1, \quad D \subset \mathbb{C}, \tag{1.1}$$

i.e., $n_\Lambda(D)$ is the number of points from Λ occurring in D. In this connection we set $\operatorname{supp}\Lambda := \operatorname{supp} \mathrm{n}_\Lambda$. By definition, the inclusion $\Lambda \subset D$ means that $\operatorname{supp}\Lambda \subset$ D; $z \in \Lambda$ ($z \notin \Lambda$ resp.) signifies $z \in \operatorname{supp}\Lambda$ ($z \notin \operatorname{supp}\Lambda$ resp.);

$$n_\Lambda^{\mathrm{rad}}(r) = n_\Lambda\bigl(D(r)\bigr) = \sum_{|\lambda_k|<r} 1, \quad r \geqslant 0,$$

is *a counting function of sequence* Λ, i.e., $n_\Lambda^{\mathrm{rad}}(r)$ is the number of all points of this sequence from the disk $D(r)$.

Our treatment of the sequence differs from that commonly used one considering the sequence to be a function of natural or integer argument. *Two sequences* $\Lambda \subset \mathbb{C}$ and $\Gamma = \{\gamma_{k'}\} \subset \mathbb{C}$ *are equal* (we write $\Lambda = \Gamma$) if we have the equality $n_\Lambda = n_\Gamma$ for the measures n_Λ and n_Γ from (1.1). In other words, we consider each *sequence of points* as a representative of the equivalence class which consists of the sequences with equal associate integer-valued measures (1.1). A sequence Λ includes a sequence $\Lambda \subset \mathbb{C}$ if $n_\Gamma \leqslant n_\Lambda$. In this case we write $\Gamma \subset \Lambda$, and Γ is *a subsequence* of the sequence Λ. *The union* $\Lambda \cup \Gamma$ (*the intersection* $\Lambda \cap \Gamma$ resp.) is defined by the equality $n_{\Lambda\cup\Gamma} = n_\Lambda + n_\Gamma$ ($n_{\Lambda\cap\Gamma} = \min\{n_\Lambda, n_\Gamma\}$ resp.). Given $\Gamma \subset \Lambda$, *the difference* $\Lambda \setminus \Gamma$ is defined by the measure $n_{\Lambda\setminus\Gamma} = n_\Lambda - n_\Gamma$. A sequence Λ *consists of single points* if $n_\Lambda\bigl(\{z\}\bigr) \leqslant 1$ for all $z \in \mathbb{C}$.

If *a numeration of the points of sequence* Λ *is important* in principle, then we represent it as $\Lambda = (\lambda_k)$, i.e., within round brackets.

For a nonzero entire function f, by $\mathrm{Zero_f}$ denote *the sequence of zeros* of this function counting multiplicities, i.e., $n_{\mathrm{Zero_f}}\bigl(\{z\}\bigr)$ is equal to multiplicity of zero of f for every point $z \in \mathbb{C}$.

In Levin's fundamental monograph the results about estimations of change of growth of entire function under shifts of its zeros [1, Ch. II, Lemmas 1, 4] played an important role in creating the theory of the functions of finite order ρ of completely regular growth.

We represent here only a more simple case of the noninteger order ρ in the form convenient for parallels with the subsequent results (in the original the formulation is given for the proximate order $\rho(\cdot)$).

Further images of the point $z \in \mathbb{C}$ and the set $D \subset \mathbb{C}$ under mapping (transformation) $T\colon \mathbb{C} \to \mathbb{C}$ are frequently written as Tz and TD.

Theorem L ([1, Ch. II, Lemma 1]). *Let f be an entire function with zero sequence $\mathrm{Zero_f} = \{\lambda_k\}$, $k = 1, 2, \dots$, having finite density with respect to a noninteger order $\rho > 0$, i. e., there exists the finite limit*

$$\lim_{r \to +\infty} \frac{n^{\mathrm{rad}}_{\mathrm{Zero_f}}(r)}{r^\rho}. \tag{1.2}$$

Then, for every $\varepsilon > 0$ and $\beta > 0$, we can select the number $d > 0$ such that, for each mapping $T\colon \mathrm{Zero_f} \to \mathbb{C}$ satisfying the conditions

$$\bigl|T\lambda_k\bigr| = |\lambda_k|, \quad \bigl|\arg T\lambda_k - \arg \lambda_k\bigr| \leqslant d, \quad k = 1, 2, \dots, \tag{1.3}$$

there is an entire function f_T with zero sequence

$$\mathrm{Zero}_{\mathrm{f_T}} = \mathrm{TZero_f} := \{\mathrm{T}\lambda_\mathrm{k}\} =: \{\gamma_\mathrm{k}\}, \quad \mathrm{k} = 1, 2, \dots, \tag{1.4}$$

for which the estimate

$$\bigl|\log|f_T(z)| - \log|f(z)|\bigr| \leqslant \varepsilon |z|^\rho$$

holds for all $z \in \mathbb{C} \setminus E$, where an exceptional set

$$E = \bigcup_{j=1}^{\infty} D(z_j, t_j) \tag{1.5}$$

has the upper density $\leqslant \beta$, i. e., $\limsup\limits_{r \to +\infty} \dfrac{1}{r} \sum\limits_{|z_j| < r} t_j \leqslant \beta$.

A.A. Gol'dberg showed in [2, § 6, Lemma] that this result holds if $\mathrm{Zero_f}$ is only a sequence finite *upper* density with respect to order ρ (or proximate order $\rho(\cdot)$), i. e., we can substitute the superior limit for the limit in (1.2).

Further more general quantitative results were obtained by I.F. Krasichkov-Ternovskiĭ [3] who applied them to the problem of spectral synthesis and the problems of completeness of exponential systems [4]. In paper [3] it was supposed, that zeros of entire function moved in arbitrary directions:

$$\left|1-\frac{\gamma_k}{\lambda_k}\right| < d \leqslant \frac{1}{2} \quad \text{for all sufficiently large } k \text{ where } T\lambda_k = \gamma_k. \tag{1.6}$$

Then denote various positive constants by const^+.

Theorem K-T ([3]). *Under the condition* (1.6), *for each entire function f with $\mathrm{Zero_f} = \{\lambda_\mathrm{k}\}$ of finite upper density with respect to a noninteger order $\rho > 0$, there exists an entire function/ f_T with $\mathrm{Zero_{f_T}} = \mathrm{TZero_f} = \{\gamma_\mathrm{k}\}$ such that for any $\alpha, \beta \in (0,1)$ and for a const^+ independent of α, $\beta \in (0,1)$, the estimate*

$$\bigl|\log|f_T z| - \log|f(z)|\bigr| \leqslant \mathrm{const}^+ \frac{d^{1-\alpha}}{\alpha\beta\sin\pi\alpha}\,|z|^\rho \tag{1.7}$$

holds for all $z \in \mathbb{C}\setminus E$, where the exceptional set E from (1.5) *has the upper density $\leqslant \beta d^{\alpha^2}$.*

It is easy to see that condition (1.3) follows from (1.6) under the restriction $|T\lambda_k| = |\lambda_k|$ with constant $1{,}033d$ in place of d. Therefore Th. K-T implies Levin's theorem L by a non-complicated choice of constants α, β, d. A certain development of Th. K-T was obtained later in [5, Th. A].

In [6] V.S. Azarin gave a general *subharmonic interpretation* for the conception *of shifts of zeros of entire function* and obtained the result which was used for asymptotic approximation of subharmonic functions by the logarithm of modulus of entire function, for construction of entire function of completely regular growth on arbitrary closed system of rays, and also, as in [4, Cor. 4.3], for a decomposition of entire function into product of entire functions of the prescribed growth at the infinity [7]. One more Azarin's result [8, § 5, Variational theorem] concerning the subjects studied in the paper is formulated in terms of convergence in the distribution space or, rather, in the language of the theory of limiting sets in the sense of V. S. Azarin.

A.F. Grishin [9] used the precise methods in studying asymptotic behavior of the difference of subharmonic functions under shift of argument, i. e., the asymptotic of $|u(z+hz)-u(z)|$ under $z\to\infty$ depending on h. The results can also be interpreted as an influence on change of subharmonic function of special variation of its Riesz measure generated by transformation of the complex plane of the form $T\colon z\mapsto z+hz$, $z\in\mathbb{C}$, for the fixed h (see the approach in [10], and with generalizations in [11], [12]). The edited version of A.F. Grishin's technique from his recent paper written together with T.I. Malyutina [13, Th. 6 etc.], might be helpful in a number of cases for studying the change of behavior of subharmonic (entire resp.) function under a sufficiently arbitrary variation of its Riesz measure (distribution of zeros resp.).

We do not concern here numerous works in which interrelations are between shifts of zeros of entire functions and changes of behavior of these functions for the cases when very special shifts of zero or classes are examined as well as the works on the problems of approximation of subharmonic functions, stability of completeness, minimality, basis properties, etc. for the systems of functions in functional spaces. In part these works are marked in [14, 15].

2. Subharmonic Interpretation According to V.S. Azarin

Let us consider in details subharmonic interpretation of shift of zeros of entire function given by V.S. Azarin. Everywhere below in Sect. 2, for the mapping $T\colon \mathbb{C} \to \mathbb{C}$ we assume the following two conditions:

- *this mapping T is Borel measurable*;
- *the preimage $T^{-1}B$ of every bounded subset $B \subset bC$ is also bounded in $\mathbb{C}$.*

For $\nu \in \mathcal{M}^+(\mathbb{C})$ its *T-shift* $\nu_T \in \mathcal{M}^+(\mathbb{C})$ or, in other words, *the image $T\nu$ of ν under T* (see [16, Ch. IV, § 6]) is defined by the rule

$$\nu_T(B) := \nu\bigl(T^{-1}B\bigr), \quad B \subset \mathbb{C} \text{ is a Borel subset.} \tag{2.1}$$

Thus we have the equality

$$\int\limits_B f \,\mathrm{d}\nu_T = \int\limits_{T^{-1}B} f(Tz)\,\mathrm{d}\nu(z) \tag{2.2}$$

for every Borel function f on the Borel subset $B \subset \mathbb{C}$.

Without loss of generality, we everywhere assume that *supports of the measures do not contain* 0.

Following L. Schwarz [16, Ch. I, § 4], everywhere we understand the *positivity* of number, function, measure, etc. as $\geqslant 0$, and > 0 is *a strict positivity*; we accept similar agreements also for the *negativity* and *strict negativity.*

If, for a mapping (a function) f on the set X we have $f(x) \equiv a$ for all $x \in X$, then we write "$f \equiv a$ *on* X", and if it is not so, then "$f \not\equiv a$ *on* X". A function f on a subset X of the real axis $\mathbb{R}$ is called *increasing* (*strictly increasing* resp.) if the inequality $x_1 < x_2$ for $x_1, x_2 \in X$ implies $f(x_1) \leqslant f(x_2)$ ($f(x_1) < f(x_2)$ resp.). Similarly, we distinguish decrease and strict decrease.

Theorem A ([6, Main Lemma]). *Let* $d\colon [0,+\infty)\to(0,+\infty)$ *be a decreasing function satisfying* $\limsup\limits_{t\to+\infty}\dfrac{d(t)}{d(2t)}<+\infty$, *and, for* $T\colon \mathbb{C}\to\mathbb{C}$,

$$\left|1-\frac{Tz}{z}\right|\leqslant d(|z|)\quad \textit{for all}\quad z\in\mathbb{C}. \tag{2.3}$$

If $\nu\in\mathcal{M}^+$ *is a measure of finite type with respect to a noninteger order* $\rho>0$, *i. e.,*

$$\limsup_{r\to+\infty}\frac{\nu^{\mathrm{rad}}(r)}{r^\rho}<+\infty,$$

and, by definition, $p:=[\rho]$ *is an integer part of* ρ, *then for each subharmonic function* u *with the Riesz measure* ν *there exists a subharmonic function* u_T *with the Riesz measure* ν_T *such that for any number* $\beta\in(0,1/\sqrt{2})$, *the estimate*

$$\begin{aligned}&\bigl|u_T(z)-u(z)\bigr|\\ &\leqslant \mathrm{const}^+\frac{|z|^\rho}{\beta^2}\left(\int_0^1\frac{d(|z|t)\,\mathrm{d}t}{t^{1+p-\rho}}+\int_1^{+\infty}\frac{d(|z|t)\,\mathrm{d}t}{t^{2+p-\rho}}\right) &&(2.4\mathrm{l})\\ &\qquad\qquad + \mathrm{const}^+|z|^\rho\beta\log\frac{1}{\beta} &&(2.4\mathrm{r})\end{aligned}$$

holds for all $z\in\mathbb{C}\backslash E$, *where the exceptional set* E *from* (1.5) *has the upper density* $\leqslant\mathrm{const}^+\cdot\beta$. *Here all three arising constants* const^+ *do not depend on* β, d.

In particular, it follows from Th. A that for an entire function f with sequence *of simple* * zeros $\mathrm{Zero_f}=\{\lambda_\mathrm{k}\}$ of finite upper density with respect to an order $\rho>0$, under condition (2.3), there is an entire function f_T with zero sequence (1.4), for which the estimate (2.4) with $u:=\log|f|$ and $u_T:=\log|f_T|$ holds outside the exceptional set (1.5) of the upper density $\leqslant\mathrm{const}^+\cdot\beta$.

Under the conditions of Th. K-T for $d(t)\equiv d(0)=:d\in(0,1/2]$, the estimate (2.4) allows to replace the right-hand side of the estimate (1.7) by a somewhat more convenient quantity $\mathrm{const}^+\left(\dfrac{d}{\beta^2}+\beta\log\dfrac{1}{\beta}\right)|z|^\rho$ performing this estimate outside some exceptional set (1.5) of the upper density $\leqslant\mathrm{const}^+\beta$. Moreover, if we choose $\beta=\sqrt[3]{d}$ and again replace $\sqrt[3]{d}$ by d, it is possible to get rid of the parameters α,β without loss of pithiness: *for any number* $d\in(0,1)$ *the left-hand side of* (1.7) *in Th. K-T can be estimated from above by* $\mathrm{const}^+d|z|^\rho\log(1/d)$ *outside the exceptional set of a kind of* (1.5) *of the upper density* $\leqslant\mathrm{const}^+d$ *where the constant* const^+ *does not depend on* d.

*It is easy to get rid of the restrictions on multiplicity of zeros by the general scheme from Sect. 3.

The kind of the last summand in the right part of the estimate (2.4) shows that the order of the estimation $|u_T - u|$ by Th. A cannot be less than $|z|^\rho$, even if function d from (2.3) decreases very quickly. It means that the conclusion of Azarin's theorem A "feels" the degree of closeness of measures ν and ν_T insufficiently. The incompleteness mentioned above is compensated in B. N. Khabibullin's paper [17]. Moreover, the estimations set up in this paper are in a certain degree the best possible ones on terms of behavior of function $|Tz - z|$ and cover a multivariate case as well. To formulate one of the versions of these results for the complex plane we will need additional designations. We put

$$\delta(z) := \bigl|Tz - z\bigr|, \ z \in \mathbb{C}; \quad \delta_T(z) := \begin{cases} \sup\limits_{T\zeta = z} |z - \zeta|, & z \in T\mathbb{C}, \\ \delta_T(z) := 0, & z \notin T\mathbb{C}. \end{cases} \tag{2.5}$$

For a measure $\sigma \in \mathcal{M}^+$ with $\operatorname{supp}\sigma \cap \{0\} = \varnothing$, we define the characteristic (cf. (**??**I))

$$K_q(r, \mathrm{d}\sigma) : = r^q \int\limits_0^{r-0} \frac{\mathrm{d}\sigma^{\mathrm{rad}}(t)}{t^q} + r^{1+q} \int\limits_{r-0}^{+\infty} \frac{\mathrm{d}\sigma^{\mathrm{rad}}(t)}{t^{1+q}} \tag{2.6d}$$

$$= qr^q \int\limits_0^r \sigma^{\mathrm{rad}}(t)\, \frac{\mathrm{d}t}{t^{1+q}} + (q+1) r^{q+1} \int\limits_r^{+\infty} \sigma^{\mathrm{rad}}(t)\, \frac{\mathrm{d}t}{t^{2+q}} \tag{2.6p}$$

$$= q \int\limits_0^1 \sigma^{\mathrm{rad}}(rt)\, \frac{\mathrm{d}t}{t^{1+q}} + (q+1) \int\limits_1^{+\infty} \sigma^{\mathrm{rad}}(rt)\, \frac{\mathrm{d}t}{t^{2+q}}\,, \quad r \geqslant 0, \tag{2.6c}$$

where q should be a positive integer, not smaller than a genus of the measure σ. Let us remind, that the genus of the measure σ is the least integer $q \geqslant 0$ for which the second integral in (**??**d) or in (**??**p), (**??**c) is finite.

Denote by $[-\infty, +\infty]$ *an extended real axis* equipped by natural order relation. In particular, $-\infty \leqslant x \leqslant +\infty$, $x \in [-\infty, +\infty]$. Given a number $\varepsilon > 0$, a function $f \colon \mathbb{C} \to [-\infty, +\infty]$, and a subset $B \subset \mathbb{C}$, define

$$f^{(\varepsilon)}(z) := \sup\bigl\{f(\zeta) \colon \zeta \in D(z, \varepsilon|z|)\bigr\}, \ z \in \mathbb{C}; \quad B^\varepsilon := \bigcup_{z \in B} D\bigl(z, \varepsilon|z|\bigr). \tag{2.7}$$

For $0 < \varepsilon < 1$, the ratios

$$\bigl(f^{(\varepsilon)}\bigr)^{(\varepsilon)}(z) \leqslant f^{(3\varepsilon)}(z), \ z \in \mathbb{C}, \quad \bigl(B^\varepsilon\bigr)^\varepsilon \subset B^{3\varepsilon} \tag{2.8}$$

are valid.

Theorem Kh1 ([17, Theorem 2]). *Let u be a subharmonic function with the Riesz measure ν, $0 \notin \operatorname{supp} \nu$, and the measure $\sigma \in \mathcal{M}^+$ is defined by the equality*

$$\mathrm{d}\sigma := \delta \, \mathrm{d}\nu + \delta_T \, \mathrm{d}\nu_T \tag{2.9}$$

in designations from (2.5). *Let $q \geqslant 0$ be an integer such that $q_\sigma \leqslant q$ where q_σ is a genus of the measure σ. Then there exists a subharmonic function u_T with the Riesz measure ν_T such that, for any Borel function $N \colon \mathbb{C} \to (1, +\infty)$ and for any $\varepsilon \in (0, 1)$, the estimate*

$$\begin{aligned} \bigl|u(z) - u_T(z)\bigr| \leqslant \mathrm{const}^+ N^{(\varepsilon)}(z) \, &\frac{K_q\bigl(|z|, \mathrm{d}\sigma\bigr)}{\varepsilon^2 |z|} \\ &\times \log\left(2 + \frac{\varepsilon |z|}{K_q\bigl(|z|, \mathrm{d}\sigma\bigr)} \bigl(\nu(z, \varepsilon|z|) + \nu_T(z, \varepsilon|z|)\bigr)\right), \end{aligned} \tag{2.10}$$

with a constant const^+ *depending only on q, is fulfilled everywhere outside the exceptional set of a kind of* (1.5) *satisfying the inequalities*

$$\sum_{z_j \in B} t_j \leqslant \int\limits_{B^\varepsilon} \frac{\mathrm{d}m(z)}{N(z)|z|}, \quad t_j \leqslant \frac{\varepsilon}{3} |z_j| \text{ for all } j = 1, 2, \ldots, \tag{2.11}$$

and for every Borel subset B of $\mathbb{C}$ where m is a Lebesgue measure on $\mathbb{C}$.

If the function N depends only on $|z|$ and its restriction on $[0, +\infty)$ is increasing, then, by definitions and properties (2.7)–(2.8), it is easy to see that $N^{(\varepsilon)}(z) \equiv N\bigl((1 + \varepsilon)|z|\bigr)$, $z \in \mathbb{C}$. In that case, in view of the restrictions* $t_j \leqslant \dfrac{\varepsilon}{3} |z_j|$, $j = 1, 2, \ldots$, the estimate of sum from (2.11) could be replaced by a more convenient and useful estimate

$$\sum_{D(z_j, t_j) \bigcap \bigl(D(R) \setminus D(r)\bigr) \neq \varnothing} t_j \leqslant 2\pi \int\limits_{\max\{(1-3\varepsilon)r, 0\}}^{(1+3\varepsilon)R} \frac{\mathrm{d}t}{N(t)} \,. \tag{2.12}$$

Let us apply Th. Kh1 to some development of Th. A.

Note, that Azarin's theorem A is substantial only for small decreasing function d. Therefore, without loss of generality, the restriction $d(0) \leqslant 1/2$ can be added to its conditions. Hence the condition (2.3) in notations (2.5) implies

*There are no such restrictions in the original formulation [17, Th. 2], but they are obviously present in the proof (see also dissertation [14] or its abstract [15]).

$\delta(z) \leqslant d(|z|)|z|$ and, as a consequence, $\delta_T(z) \leqslant \mathrm{const}^+ d(|z|)|z|$. The representation and inequality (see definition (2.9) of the measure σ from Th. Kh1)

$$\sigma^{\mathrm{rad}}(t) = \int\limits_0^t \delta(s)\,\mathrm{d}\nu^{\mathrm{rad}}(s) + \int\limits_0^t \delta_T(s)\,\mathrm{d}\nu_T^{\mathrm{rad}}(s) \leqslant \mathrm{const}^+ t^{\rho+1}, \tag{2.13}$$

show that its genus q_σ is not larger than $q = p + 1 = [\rho] + 1$. Hence, for the characteristic (2.6) in version (**??**d), a series of somewhat tiring integrations by parts and the change of variable allow to obtain the estimate

$$\begin{aligned} \frac{K_q(r,\,\mathrm{d}\sigma)}{r} &\leqslant \mathrm{const}^+ r^\rho \left(\int\limits_0^1 \frac{d(rt)\,\mathrm{d}t}{t^{1+p-\rho}} + \int\limits_1^{+\infty} \frac{d(rt)\,\mathrm{d}t}{t^{2+p-\rho}} \right) \\ &\leqslant \mathrm{const}^+ r^\rho \left(\int\limits_0^1 \frac{d(rt)\,\mathrm{d}t}{t^{1+p-\rho}} + \frac{1}{1+p-\rho}\,d(r) \right). \end{aligned} \tag{2.14}$$

The function $x \log(2 + a/x)$ is increasing on $[0, +\infty)$ when $a > 0$. Thus, if we choose $\varepsilon = 1/3$ and put $N(t) \equiv 4\pi/\beta$, $t \geqslant 0$, then, by Th. Kh1, the estimates (2.10)–(2.12) together with (2.14) entail the following:

$$\begin{aligned} \bigl|u(z) - u_T(z)\bigr| &\leqslant \mathrm{const}^+ \frac{|z|^\rho}{\beta} \left(\int\limits_0^1 \frac{d(|z|t)\,\mathrm{d}t}{t^{1+p-\rho}} + \frac{1}{1+p-\rho}\,d(|z|) \right) \\ &\times \log \left(2 + \mathrm{const}^+ \frac{\nu(z,|z|/3) + \nu_T(z,|z|/3)}{|z|^\rho \left(\int\limits_0^1 \frac{d(|z|t)\,\mathrm{d}t}{t^{1+p-\rho}} + \frac{1}{1+p-\rho}\,d(|z|) \right)} \right). \end{aligned}$$

Some weakening of the last estimation gives

Corollary Kh1. *Under the conditions of Th. A the estimate* (2.4) *can be replaced by*

$$\bigl|u_T(z) - u(z)\bigr| \leqslant \mathrm{const}^+ \frac{|z|^\rho}{\beta} \left(\int\limits_0^1 \frac{d(|z|t)\,\mathrm{d}t}{t^{1+p-\rho}} + d(|z|) \right) \log\left(2 + \frac{1}{d(|z|)}\right) \tag{2.15}$$

outside the exceptional set (1.5) *of the upper density* $\leqslant \beta$.

The estimate (2.15) is frequently more refined than (2.4) from Th. A. For example, if $d(t) \equiv (1+t)^{-\alpha}$, $t \geqslant 0$, where $0 < \alpha < \rho - [\rho]$, then the right-hand side of (2.15) can be easily estimated from above by the quantity $O\bigl(|z|^{\rho-\alpha}\log|z|\bigr)$, $z\to\infty$, whose order of growth at infinity is strictly less than for $|z|^{\rho}$ (compare with the comment directly ahead of (2.5)).

Using of the genus q_σ of measure σ in Th. Kh1 (below $q_\sigma \leqslant [\rho-\gamma]+1$) allows to give considerably more general and improved version of Cor. Kh1.

Corollary Kh2 ([14, Cor. 3.1]). *Let u be a subharmonic function of finite type with respect to the order $\rho \geqslant 0$ with the Riesz measure ν. If, for a number $\gamma \in [0,\rho+1]$ and a function $\varphi\colon [0,\infty)\to[0,1/2]$ $[a,+\infty)$ that is decreasing on a ray $[a,+\infty)$ where $a>0$, and $\varphi\equiv 0$ on $[0,a)$, the mapping $T\colon \mathbb{C}\to\mathbb{C}$ satisfies*

$$\Bigl|1-\frac{Tz}{z}\Bigr| \leqslant \varphi(|z|)|z|^{1-\gamma}, \quad z\neq 0, \tag{2.16}$$

then there exists a subharmonic function u_T of the order $\leqslant \rho$ with the Riesz measure ν_T such that for any increasing function $N\geqslant 2$ on $[0,+\infty)$ the following estimate

$$\bigl|u_T(z)-u(z)\bigr| \leqslant \mathrm{const}^+\,|z|^{\rho-\gamma}\left(\int\limits_0^1 \frac{\varphi(|z|t)\,\mathrm{d}t}{t^{1+[\rho-\gamma]-(\rho-\gamma)}}\right.$$
$$\left. +\varphi\bigl(|z|/2\bigr)\,N\bigl(2|z|\bigr)\log\Bigl(2+\frac{|z|^{\gamma}}{\varphi\bigl(|z|/2\bigr)}\Bigr)\right) \tag{2.17}$$

is fulfilled outside the exceptional set of a kind of (1.5) satisfying

$$\sum_{D(z_j,t_j)\bigcap\bigl(D(R)\setminus D(r)\bigr)\neq\varnothing} t_j \leqslant \int\limits_{r/2}^{2R}\frac{\mathrm{d}t}{N(t)} \text{ for all sufficiently large } r<R. \tag{2.18}$$

Unfortunately, in Cor. Kh2 the function u_T can have the infinite type with respect to the order ρ. Finiteness of the type of this function can be provided due to any of the following three conditions (see [14, Remark 2, p. 57]): 1) ρ is *a noninteger number*; 2) $\gamma>0$; 3) $\int\limits_1^{+\infty}\varphi(t)\dfrac{\mathrm{d}t}{t}<+\infty$.

Here is one more addition to Azarin's Theorem A.

Corollary Kh3 ([14, Remark 1, p. 56]). *Suppose that under conditions of the previous Cor.* Kh2 *the function φ from (2.16) is increasing on $[a,+\infty)$, $\lim\limits_{t\to+\infty}\dfrac{\log\varphi(t)}{\log t}=0$, and $0<\gamma\leqslant\rho+1$. Then we can draw the conclusions* *similar to those in Consequence 2, with the same increasing function $N \geqslant 2$, but with the estimate*

$$\big|u_T(z) - u(z)\big| \leqslant \mathrm{const}^+ |z|^{\rho-\gamma} \left(\int\limits_1^{+\infty} \frac{\varphi(|z|t)\,\mathrm{d}t}{t^{2+[\rho-\gamma]-(\rho-\gamma)} } + \varphi\big(2|z|\big)\, N\big(2|z|\big) \log\Big(2 + \frac{|z|^\gamma}{\varphi\big(2|z|\big)}\Big)\right) \tag{2.19}$$

outside the exceptional set of a kind of (1.5) *satisfying* (2.18).

R e m a r k 1. The right-hand sides of the estimates (2.10) from Th. Kh1, (2.17) from Cor. Kh2, and (2.19) from Cor. Kh3 demonstrate that these results cannot give the order of closeness of the functions u and u_T less than $O(1/|z|)$ as $z \to \infty$. Generally speaking, it is impossible to lower this order of closeness without additional conditions. For example, if $u(z) \equiv \log|z-\lambda|$, $z \in \mathbb{C}$, and T takes λ to $\gamma \neq \lambda$, then it is not difficult to understand that the function $u_T(z) \equiv \log|z-\gamma|$ is asymptotically most close to u. At the same time, for $|z| \geqslant 2\max\{|\lambda|, |\gamma|\}$ we have the estimate

$$\big|u_T(z) - u(z)\big| \equiv \big|\log|z-\gamma| - \log|z-\lambda|\big| \geqslant \frac{2|\lambda-\gamma|}{|z|}\,.$$

R e m a r k 2. As well as in the comment following Th. A, all previous subharmonic results can be considered as a statement on the change of growth of entire function f under transformation of the sequence of its zeros $\mathrm{Zero_f} = \{\lambda_\mathrm{k}\}$ in the sequence of zeros (1.4) of some entire function f_T with the corresponding reformulations for $u := \log|f|$ and $u_T := \log|f_T|$. Thus, for example, conditions (2.3) and (2.16) will be written as

$$\left|1 - \frac{\gamma_k}{\lambda_k}\right| \leqslant d(|\lambda_k|), \quad \left|1 - \frac{\gamma_k}{\lambda_k}\right| \leqslant \varphi(|\lambda_k|)|\lambda_k|^{1-\gamma}, \quad k = 1, 2, \ldots; \quad T\lambda_k := \gamma_k.$$

As the subharmonic results were formulated *for mappings T*, application of these results to the entire functions f is possible, generally speaking, only for the case when *the sequences* $\mathrm{Zero_f}$ *have no multiple* (repeating) *points.* Indeed, if $\lambda_k = \lambda_{k'}$ are two points of $\Lambda = \mathrm{Zero_f}$, $k \neq k'$, but $\gamma_k = T\lambda_k \neq \gamma_{k'} = T\lambda_{k'}$, then such transformation of T *is not mapping any more,* whereas all the results of V. S. Azarin and B. N. Khabibullin were proved just for mappings T. This difficulty can be overcome in some ways. For example, one of them is to consider the multiple-valued mappings T, i. e., to do all reasonings and calculations once again with the probable complications at least of technical character. An alternative way is offered in Sect. 3.

3. Approximation by Entire Functions with Simple Zeros

First, we define the joint result of V.V. Napalkov and M.I. Solomeshch [18] which directly relates to our subject. The proof is in the dissertation by M.I. Solomeshch [19].

Let f be an entire function with $\mathrm{Zero_f} = (\mathrm{\lambda_k})$, $0 \notin \mathrm{Zero_f}$ represented by the Weierstrass canonical product

$$f(z) = R(z) \prod_{k=1}^{\infty} \Big(1 - \frac{z}{\lambda_k}\Big) \exp p_k(z/\lambda_k), \quad z \in \mathbb{C}, \tag{3.1}$$

where R is an entire function without zeros, and p_k, $k = 1, 2, \dots$, are polynomials.

As well as in [18], considered is a sequence of points $d = (d_k) \subset \mathbb{C}$ such that $\lambda_k + d_k \neq 0$ for all $k = 1, 2, \dots$. Suppose that $d_k = d_{k'}$ *in all cases when* $\lambda_k = \lambda_{k'}$.

By (3.1) let us construct a formal product

$$f_d(z) = R(z) \prod_{k=1}^{\infty} \Big(1 - \frac{z}{\lambda_k + d_k}\Big) \exp p_k(z/\lambda_k), \quad z \in \mathbb{C}. \tag{3.2}$$

Let a family of disks $D(\lambda_k, t_k)$, $t_k > 0$, $k = 1, 2, \dots$, such that $t_k = t_{k'}$ *if* $\lambda_k = \lambda_{k'}$.

Theorem N–S ([18, Prop. 1], [19, Props. 7–9])**.** *In the assumed notations and agreements, let* $|d_k| < t_k$ *for all* $k = 1, 2, \dots$, *and*

$$\sum_{k=1}^{\infty} \frac{|d_k|}{t_k} < +\infty. \tag{3.3}$$

Then product (3.2) *converges if* $z \notin E = \bigcup_{k=1}^{\infty} D(\lambda_k, t_k)$ *and determines an analytic function outside* E, *and for* const^+ *we have*

$$\big|\log |f_d(z)| - \log |f(z)|\big| \leqslant \mathrm{const}^+, \quad z \in \mathbb{C} \setminus E.$$

If each connected component of E *is bounded, then product* (3.2) *converges to the entire function* f_d *with* $\mathrm{Zero_{f_d}} = (\mathrm{\lambda_k + d_k})$, $k = 1, 2, \dots$.

If connected components of the set E from Napalkov–Solomeshch's Theorem N–S are unbounded, then product (3.2), generally speaking, can diverge at points $z \in E$. Thus, a condition on the connected components of set E in the last paragraph of Th. N–S is essential.

Taking into account properties of the sequences (d_k) and (t_k) in relation to the sequence (λ_k), we define an auxiliary notion.

We say that *a sequence* (a_k), $k = 1, 2, \dots$, *is linked with a sequence* (b_k), $k = 1, 2, \dots$, if $b_k = b_{k'}$ implies $a_k = a_{k'}$. In particular, according to the condition above the sequences (d_k), (t_k), and $(\lambda_k + d_k)$ are linked with the sequence (λ_k). By virtue of the last, Theorem of Napalkov–Solomeshch cannot be used to solve the main problem of Sect. 3 on the approximation of entire function by the entire function with simple zeros. More specifically, we cannot "split" multiple zeros of function f to simple zeros of function f_d with the help of Th. N–S, because, by the construction of sequence $\mathrm{Zero_{f_d}} = (\mathrm{\lambda_k + d_k})$, the coincident points $\lambda_k = \lambda_{k'}$ are transformed to the same points $\lambda_k + d_k = \lambda_{k'} + d_{k'}$ from $\mathrm{Zero_{f_d}}$.

The main result of this paragraph was announced rather long ago in the paper [20], but its proof is given here for the first time.

Theorem 1. *Let f be an entire function with $\mathrm{Zero_f} = (\mathrm{\lambda_k})$, $k = 1, 2, \ldots$. For every given decreasing function $\beta\colon [0, +\infty) \to (0, +\infty)$ and number $\varepsilon > 0$ we can find an entire function g with the sequence of simple zeros $\mathrm{Zero_g} = (\mathrm{\gamma_k})$ and $(t_k) \subset (0, +\infty)$ that is linked with the sequence (λ_k) such that:*

1) *for $\lambda_k \neq \lambda_{k'}$, the disks $D(\lambda_k, t_k)$ and $D(\lambda_{k'}, t_{k'})$ are not intersected; for $r \geqslant 0$, $\sum\limits_{|\lambda_k| \geqslant r} t_k \leqslant \beta(r)$; finally, $|\gamma_k - \lambda_k| < t_k$ for all $k = 1, 2, \ldots$;*

2) *the inequality*
$$\big|\log|g(z)| - \log|f(z)|\big| \leqslant \frac{\varepsilon}{|z|^2} \tag{3.4}$$
takes place for all $z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} D(\lambda_k, t_k)$.

P r o o f. First, we consider the case when

(!) *multiplicity of zeros of the function f at any point is an even number.*

In this case the sequence $\mathrm{Zero_f} = \{\mathrm{\lambda_k}\} =: \Lambda$ can be represented as the union $\Lambda = \Lambda' \cup \Lambda''$, where the sequences $\Lambda' = (\lambda'_k)$ and $\Lambda'' = (\lambda''_k)$, $k = 1, 2, \ldots$, such that $\lambda'_k = \lambda''_k$ for each $k = 1, 2, \ldots$.

Now we choose a sequence of strictly positive numbers (t_k) linked with (λ'_k) such that the disks $D(\lambda'_k, t_k)$ are mutually disjoint and for all $r \geqslant 0$, $\sum_{|\lambda'_k| \geqslant r} t_k \leqslant \beta(r)$, i.e., 1) is fulfilled. One can always do it as the imposed conditions are not mutually exclusive in the sense that both restrictions take only a sufficient rapid decrease of the sequence (t_k). Given $\varepsilon > 0$, we select strictly positive numbers $d_k \leqslant t_k/2$ so small that (cf. (3.3))
$$\sum_{k=1}^{\infty} d_k |\lambda'_k| \leqslant \frac{\varepsilon}{2^6}, \quad \sum_{|\lambda'_k| \geqslant r} \frac{d_k}{t_k^2} |\lambda'_k| \leqslant \frac{\varepsilon}{2^6 r^2} \quad \text{for all } r > 0. \tag{3.5}$$

To each pair of the coincident points $\lambda'_k = \lambda_k$" we assign *two diametrically opposite points γ'_k and γ''_k on the circumference $\partial D(\lambda'_k, d_k)$*, i. e.,

$$\gamma'_k + \gamma''_k = 2\lambda'_k = 2\lambda''_k = \lambda'_k + \lambda''_k, \quad |\gamma'_k - \lambda'_k| = |\gamma''_k - \lambda''_k| = d_k. \tag{3.6}$$

Since every point of $\mathbb{C}$ coincides only with the finite number of points λ'_k, it follows that we can construct the distinct diametrically opposite points (γ'_k, γ''_k) so that $|\gamma'_k| \leqslant |\lambda'_k|$ and

(∗) *the union $\Gamma_0 := \Gamma' \cup \Gamma''$ of the sequences $\Gamma' = (\gamma'_k)$ and $\Gamma'' = (\gamma''_k)$ consists of simple points and at the same time, by construction, $\Lambda \cap \Gamma_0 = \varnothing$.*

Now we estimate the sum of differences

$$\Sigma(z) := \sum_k \Bigl(\log\bigl|(z-\gamma'_k)(z-\gamma''_k)\bigr| - \log\bigl|(z-\lambda'_k)(z-\lambda''_k)\bigr|\Bigr) \text{ for } z \notin \bigcup_k D(\lambda'_k, t_k). \tag{3.7}$$

Using (3.6), the identity

$$L_k(z) := \log\bigl|(z-\gamma'_k)(z-\gamma''_k)\bigr| - \log\bigl|(z-\lambda'_k)(z-\lambda''_k)\bigr| = \log\Bigl|1 - \frac{\lambda'_k\lambda''_k - \gamma'_k\gamma''_k}{(z-\lambda'_k)^2}\Bigr|$$

implies an upper bound

$$L_k(z) \leqslant \log\Bigl(1 + \frac{|\lambda'_k\lambda''_k - \gamma'_k\gamma''_k|}{|z-\lambda'_k|^2}\Bigr) \leqslant \frac{|\lambda''_k||\lambda'_k - \gamma'_k| + |\gamma'_k||\lambda''_k - \gamma''_k|}{|z-\lambda'_k|^2} \leqslant \frac{2d_k|\lambda'_k|}{|z-\lambda'_k|^2}\,. \tag{3.8}$$

Similarly, it follows from

$$-L_k(z) = \log\Bigl|1 + \frac{\lambda'_k\lambda''_k - \gamma'_k\gamma''_k}{(z-\gamma'_k)(z-\gamma''_k)}\Bigr| \leqslant \log\Bigl(1 + \frac{|\lambda'_k\lambda''_k - \gamma'_k\gamma''_k|}{|z-\gamma'_k||z-\gamma''_k|}\Bigr)$$

that, in view of (3.6), and for $|z-\lambda'_k| > d_k$,

$$-L_k(z) \leqslant \frac{2d_k|\lambda'_k|}{\bigl(|z-\lambda'_k| - |\lambda'_k - \gamma'_k|\bigr)\bigl(|z-\lambda''_k| - |\lambda''_k - \gamma''_k|\bigr)} \leqslant \frac{2d_k|\lambda'_k|}{\bigl(|z-\lambda'_k| - d_k\bigr)^2}\,.$$

But for $|z-\lambda'_k| \geqslant t_k \geqslant 2d_k$ we have $|z-\lambda'_k| - d_k \geqslant |z-\lambda'_k|/2$. Hence

$$-L_k(z) \leqslant \frac{2^3 d_k|\lambda'_k|}{|z-\lambda'_k|^2} \quad \text{for } |z-\lambda'_k| \geqslant t_k.$$

The last estimate together with (3.8), (3.7) gives

$$\bigl|\Sigma(z)\bigr| \leqslant \sum_k |L_k(z)| \leqslant \sum_k \frac{2^3 d_k|\lambda'_k|}{|z-\lambda'_k|^2} \quad \text{for } z \notin \bigcup_k D(\lambda'_k, t_k) =: E.$$

If we fix the point $z \notin E$, then

$$\begin{aligned}\bigl|\Sigma(z)\bigr| &\leqslant \left(\sum_{|\lambda_k'|<|z|/2} + \sum_{|\lambda_k'|\geqslant|z|/2}\right)\frac{2^3 d_k|\lambda_k'|}{|z-\lambda_k'|^2}\\ &\leqslant \sum_{|z-\lambda_k'|\geqslant|z|/2}\frac{2^3 d_k|\lambda_k'|}{|z-\lambda_k'|^2} + \sum_{|\lambda_k'|\geqslant|z|/2}\frac{2^3 d_k|\lambda_k'|}{t_k^2}\,.\end{aligned}$$

Here, using (3.5), we can estimate the first sum in the right-hand side as

$$\sum_{|z-\lambda_k'|\geqslant|z|/2}\frac{2^3 d_k|\lambda_k'|}{|z-\lambda_k'|^2} \leqslant \sum_k \frac{2^5 d_k|\lambda_k'|}{|z|^2} \leqslant \frac{\varepsilon}{2|z|^2}\,,$$

and the second sum as

$$\sum_{|\lambda_k'|\geqslant|z|/2}\frac{2^3 d_k|\lambda_k'|}{t_k^2} \leqslant \frac{\varepsilon}{2^3\bigl(|z|/2\bigr)^2} \leqslant \frac{\varepsilon}{2|z|^2}\,.$$

Thus, the last three estimates imply

$$\bigl|\Sigma(z)\bigr| \leqslant \frac{\varepsilon}{|z|^2} \quad \text{for } z\notin E=\bigcup_k D(\lambda_k',t_k). \tag{3.9}$$

For the case (!), our construction is finished.

Now, if the function f has zeros of odd multiplicity, then we represent the $\mathrm{Zero_f}$ in the form $\mathrm{Zero_f}=\Lambda_0\cup\Lambda$, where $\Lambda_0=\{\lambda_k^0\}$ is a sequence of simple points and Λ *is a sequence of points of even multiplicity*, i. e., $n_\Lambda\bigl(\{z\}\bigr)$ is an even number for each point $z\in\mathbb{C}$. In this case we choose $\Gamma:=\Lambda_0\cup\Gamma_0=(\gamma_k)=:\mathrm{Zero_g}$, where the sequence Γ_0 is constructed by the sequence Λ similarly to that one above. In view of $(*)$, the sequence Γ consists only of *simple points* whereas the exceptional set $E=\bigcup_k D(\lambda_k',t_k)$ is identical to Λ. Besides, considering (3.9), for appropriate renumbering and denotation (if necessary) of the points in $\mathrm{Zero_f}=\Lambda_0\cup\Lambda:=(\lambda_\mathrm{k})$, $\Gamma:=(\gamma_k)$, and (t_k), we get

$$\bigl|\Sigma(z;\Lambda,\Gamma)\bigr| := \left|\sum_k\bigl(\log|z-\gamma_k|-\log|z-\lambda_k|\bigr)\right| \leqslant \frac{\varepsilon}{|z|^2} \tag{3.10}$$

for $z\notin\bigcup_k D(\lambda_k,t_k)=E$ where $t_k=0$ if the point λ_k is simple, i.e., $n_\Lambda\bigl(\{\lambda_k\}\bigr)=1$.

To conclude the proof, we use the Weierstrass representation (3.1) of f and define a function g in the form of product (cf. (3.2))

$$g(z)=R(z)\prod_{k=1}^{\infty}\frac{\gamma_k-z}{\lambda_k}\,\exp p_k(z/\lambda_k), \quad z\in\mathbb{C}, \tag{3.11}$$

for which, according to (3.10),

$$\big|\log|g(z)| - \log|f(z)|\big| \equiv \big|\Sigma(z;\Lambda,\Gamma)\big| \leqslant \frac{\varepsilon}{|z|^2} \quad \text{for } z \notin E.$$

Hence, using maximum-modulus principle for increasing sequence of bounded domains with the boundaries disjointed from E, we see that the product (3.11) is uniformly bounded on compacta. Therefore, by the Montel theorem the product (3.11) determines a desired entire function g with simple zeros satisfying (3.4) outside E.

This completes the proof of Th. 1.

R e m a r k 1. The polynomial $f\colon z \mapsto z^2$ shows that *the estimate* (3.4) *is unimprovable*. Indeed, for any pair of different points $\{\gamma_1, \gamma_2\}$ there exists a constant $\text{const}^+ > 0$ such that

$$\Big|\log\big|(z-\gamma_1)(z-\gamma_2)\big| - \log|z^2|\Big| \geqslant \frac{\text{const}^+}{|z|^2} \quad \text{for all } |z| \geqslant 2\max\big\{|\gamma_1|, |\gamma_2|\big\}.$$

R e m a r k 2. Theorem 1 completely solves the problem set in Remark 2 from Sect. 2 and even more, since under Remark 1 from Sect. 2 the highest possible closeness of functions has the order $O(1/|z|)$.

To conclude Sect. 3 we note without the proof the result similar to Th. N–S. It is obtained analogously to Th. 1.

Theorem 2. *Let f be an entire function with $\mathrm{Zero_f} = (\lambda_k) =: \Lambda$, $k = 1, 2, \dots$, and a sequence of strictly positive numbers (t_k) is linked with Λ. Suppose that all connected components of $E := \bigcup_k D(\lambda_k, t_k)$ are bounded. If, for a sequence (d_k), $0 \leqslant d_k < t_k$, $k = 1, 2, \dots$ (cf. (3.3))*

$$\sum_{k=1}^{\infty} d_k < +\infty, \qquad \sum_{|\lambda_k| \geqslant r} \frac{d_k}{t_k} = O(1/r),\ r \to +\infty, \tag{3.12}$$

then, for any sequence of points $(\gamma_k) \subset \mathbb{C}$ satisfying the inequalities $|\lambda_k - \gamma_k| \leqslant d_k$, $k = 1, 2, \dots$, there exists an entire function g with $\mathrm{Zero_g} = (\gamma_k)$ and a constant const^+ such that

$$\big|\log|g(z)| - \log|f(z)|\big| \leqslant \frac{\text{const}^+}{|z|} \quad \textit{for all } z \in \mathbb{C} \setminus E.$$

Evidently, if the sequence (t_k) is bounded, then the convergence of the first sum from (3.12) follows from the convergence of the second sum from (3.12).

4. Integral Condition on T-shift. Main Result

As usual, denote by $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}$ the sets of all natural numbers and all integers resp.; $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For $q < s$, by definition, $\prod_{m=s}^{q} \cdots := 1$, $\sum_{m=s}^{q} \cdots := 0$.

Given $q \in \mathbb{Z}_+$, the function

$$E_q(z, \zeta) := \Bigl(1 - \frac{z}{\zeta}\Bigr) \prod_{m=1}^{q} \exp \frac{z^m}{m\zeta^m}, \quad z \in \mathbb{C},\ \zeta \in \mathbb{C} \setminus \{0\},$$

is called the *Weierstrass primary factors of genus* $q \in \mathbb{Z}_+$.

The following special case of the classical Lindelöf theorem on an interconnection between the growth of entire function and the distribution of its zeros [1, Ch. I, § 11, Th. 15] relates to the sources of the main theorem of Sect. 4. This result was announced in [21].

Proposition 1. *If, for a sequence* $\Lambda = \{\lambda_k\} \subset \mathbb{C}$, $k \in \mathbb{N}$, *the sum* $\sum\limits_{k \in \mathbb{N}} \dfrac{1}{|\lambda_k|^\rho}$ *is finite for a number* $\rho > 0$, *then, for*

$$q := \begin{cases} [\rho] := \text{integer part of } \rho & \text{if } \rho \text{ is noninteger}, \\ \rho - 1 = [\rho] - 1 & \text{if } \rho \text{ is integer}, \end{cases} \tag{4.1}$$

the Weierstrass–Hadamard product $W_\Lambda(z) := \prod_{k=1}^{\infty} E_q(z, \lambda_k)$ *of the genus* q, $z \in \mathbb{C}$, *is an entire function of zero type with respect to the order* ρ *with* $\mathrm{Zero}_{\mathrm{W}} = \Lambda$.

Given $q \in \mathbb{Z}_+$, the function

$$e_q(z, \zeta) := \log |E_q(z, \zeta)| = \log\Bigl|1 - \frac{z}{\zeta}\Bigr| + \sum_{m=1}^{q} \frac{1}{m} \operatorname{Re} \frac{z^m}{\zeta^m}, \quad z \in \mathbb{C},\ \zeta \in \mathbb{C} \setminus \{0\}, \tag{4.2}$$

is said to be *the subharmonic Weierstrass kernel of genus* q.

If the function

$$w_\nu(z) := \int\limits_{\mathbb{C}} e_q(z, \zeta)\, \mathrm{d}\nu(\zeta), \quad z \in \mathbb{C}, \tag{4.3}$$

with values in $[-\infty, +\infty)$, is locally bounded above, then we may say that the function w_ν is *a Weierstrass–Hadamard potential of genus* q *of measure* ν.

A subharmonic version of Prop. 1 (particular case of [22, 4.2]) is

Proposition 2. *If* $0 \notin \operatorname{supp} \nu$ *for* $\nu \in \mathcal{M}^+$ *and* $\int\limits_{\mathbb{C}} \dfrac{1}{|\zeta|^\rho}\, \mathrm{d}\nu(\zeta) < +\infty$, *then* w_ν *is a subharmonic function of zero type with respect to the order* ρ *with the Riesz measure* ν.

As the next step for further developing these facts, when $\rho = 1$, it is possible to consider the following theorem formulated here with some losses in a substantial and constructive parts in comparison with the original treatment.

Theorem Kh2 ([23, Theorem 1]). *Let $f \not\equiv 0$ be an entire function of exponential type with* $\mathrm{Zero}_\mathrm{f} = (\lambda_\mathrm{k})$, *and* (γ_k) *be a sequence of points of* $\mathbb{C}$, $k = 1, 2, \dots$. *If the series*

$$\sum_{\gamma_k \cdot \lambda_k \neq 0} \left| \frac{1}{\lambda_k} - \frac{1}{\gamma_k} \right| \tag{4.4}$$

converges, then there exists an entire function of exponential type $g \not\equiv 0$ with $\mathrm{Zero}_\mathrm{g} = \Gamma$, *and with the same indicator function, as f.*

Natural expansion of the last result on entire functions of the finite order ρ was announced in [24]. For the role of condition (4.4), the convergence of series

$$\sum_{\lambda_k \neq 0} \frac{1}{|\lambda_k|^\rho} \left| 1 - \frac{\gamma_k}{\lambda_k} \right| \tag{4.5}$$

was offered [24, Cor. 1]. There was also formulated a subharmonic version, but without the proof and in a weaker and less precise form than the main one submitted in our paper.

Theorem 3 (partial formulations in [24, Theorem], [21, Theorem]). *Let $\nu \in \mathcal{M}^+$ be a measure of finite type with respect to the order $\rho > 0$, and $T \colon \mathbb{C} \to \mathbb{C}$ be a Borel mapping such that the preimage of each bounded set is bounded. If*

$$\liminf_{z \to \infty} \frac{|Tz|}{|z|} > 0, \qquad \int\limits_{\mathbb{C} \setminus D(1)} \frac{1}{|\zeta|^\rho} \left| 1 - \frac{T\zeta}{\zeta} \right| \, \mathrm{d}\nu(\zeta) < +\infty, \tag{4.6}$$

then, for every subharmonic function u with the Riesz measure ν, we can find a subharmonic function u_T with the Riesz measure ν_T such that, for any number $\varepsilon > 0$, there is an exceptional set $E_\varepsilon \subset \mathbb{C}$ of upper density $\leqslant \varepsilon$ for which

$$\bigl| u_T(z) - u(z) \bigr| \leqslant \varepsilon |z|^\rho \quad \textit{for all} \quad z \in \mathbb{C} \setminus E_\varepsilon. \tag{4.7}$$

In particular, for any $\varepsilon > 0$,

$$\begin{cases} u_T(z) \leqslant \sup\limits_{|\zeta - z| \leqslant \varepsilon |z|} u(\zeta) + \varepsilon |z|^\rho + \mathrm{const}^+ , \\ u(z) \leqslant \sup\limits_{|\zeta - z| \leqslant \varepsilon |z|} u_T(\zeta) + \varepsilon |z|^\rho + \mathrm{const}^+ , \end{cases} \quad |z| \geqslant 1. \tag{4.8}$$

In addition, if u is a function of finite type with respect to the order $\rho > 0$, then the function u_T is the same, and the indicator function of u coincides with the indicator function of u_T.

P r o o f. Fix $\varepsilon > 0$. We will prove the theorem in a few steps.

1. *Isolating of measure from origin.* For any given $R \geqslant 1$, we may suppose that $\operatorname{supp} \nu \cap \mathrm{D(R)} = \varnothing$. Indeed, if u is a subharmonic function with the Riesz measure ν, then we can represent the function u in a form of sum

$$u(z) = \int\limits_{\mathbb{C}} \log|\zeta - z| \, \mathrm{d}\left(\nu \mid_{D(R)}\right)(\zeta) + u^R(z) =: p_R(z) + u^R(z), \quad z \in \mathbb{C},$$

where $\nu \mid_{D(R)}$ is a restriction of the measure ν to the disk $D(R)$ and u^R is a subharmonic function with the Riesz measure $\nu \mid_{\mathbb{C}\setminus D(R)} = \nu - \nu \mid_{D(R)}$ for which $D(R) \cap \operatorname{supp}\left(\nu \mid_{\mathbb{C}\setminus \mathrm{D(R)}}\right) = \varnothing$. The logarithmic potential p_R satisfies condition [25, Th. 3.1.2]

$$p_R(z) = \nu \mid_{D(R)} (\mathbb{C}) \log|z| + O(1/|z|) \quad \text{as } z \to \infty. \tag{4.9}$$

By (2.1)–(2.2) and by boundness of $T^{-1}D(R)$, under the conditions of theorem, the support of T-shift $\left(\nu \mid_{D(R)}\right)_T$ of the measure $\nu \mid_{D(R)}$ is a compact set and $\left(\nu \mid_{D(R)}\right)_T (\mathbb{C}) = \nu \mid_{D(R)} (\mathbb{C})$. For

$$(p_R)_T z := \int\limits_{\mathbb{C}} \log|\zeta - z| \, \mathrm{d}\left(\nu \mid_{D(R)}\right)_T (\zeta) = \left(\nu \mid_{D(R)}\right)_T (\mathbb{C}) \log|z| + O(1/|z|), \; z \to \infty,$$

in view of (4.9), we have $p_R(z) - (p_R)_T z = O(1/|z|)$ as $z \to \infty$. The latter means that if functions u^R and u_T^R satisfy (4.7)–(4.8), then the addition of logarithmic potentials p_R and $(p_R)_T$ to them will give exactly (4.7)–(4.8) under possible increasing of the constant const^+, if necessary.

It follows from the first condition of (4.6) that for the number $b > 0$ we can choose a number $R \geqslant 1$ so large that

$$b|z| \leqslant |Tz| \quad \text{for all } z \notin D(R). \tag{4.10}$$

The second condition from (4.6) implies

$$\int\limits_{\mathbb{C}\setminus D(R)} \frac{1}{|\zeta|^\rho} \left|1 - \frac{T\zeta}{\zeta}\right| \mathrm{d}\nu(\zeta) =: \alpha(R) \to 0 \quad \text{as } R \to \infty. \tag{4.11}$$

Now we define more exactly a choice of R depending upon ε and other parameters. We consider only that the number $R \geqslant 1$ is chosen so that inequality (4.10) takes place and, besides,

$$\operatorname{supp} \nu \bigcap \mathrm{D(R)} = \varnothing. \tag{4.12}$$

Thus, obviously, for some number $B > 0$ the inequality

$$\nu^{\mathrm{rad}}(t) \leqslant Bt^{\rho} \tag{4.13}$$

holds for all $t > 0$. It is important to note that here the procedure of rejection of restriction of the measure ν on the disk $D(R)$ *does not increase the constant* B *under increasing of* R. Note also, that inclusion $T^{-1}D(t) \subset D(t/b)$ follows from (4.10) for all $t \geqslant R$. Hence, according to (2.1) we obtain

$$\nu_T^{\mathrm{rad}}(t) = \nu_T\bigl(D(t)\bigr) = \nu\bigl(T^{-1}D(t)\bigr) \leqslant \nu^{\mathrm{rad}}(t/b).$$

In particular, it means that under agreements (4.12)–(4.13) we have

$$\operatorname{supp}\nu_{\mathrm{T}} \cap \mathrm{D(bR)} = \varnothing, \quad \nu_{\mathrm{T}}^{\mathrm{rad}}(\mathrm{t}) \leqslant \frac{\mathrm{B}}{\mathrm{b}^{\rho}}\,\mathrm{t}^{\rho} \quad \text{for all } t > 0, \tag{4.14}$$

i. e., the measure ν_T has a finite type with respect to the order ρ, and $0 \notin \operatorname{supp}\nu_{\mathrm{T}}$.

2. *The main estimated integral*. For q from (4.1), consider the integral*

$$I(z) := \int\limits_{\mathbb{C}} e_q(z,\zeta)\,\mathrm{d}(\nu_T - \nu)(\zeta). \tag{4.15}$$

Our goal is to get the estimate $|I(z)| \leqslant \varepsilon|z|^{\rho}$ for all z laying outside some exceptional set of the upper density $\leqslant \varepsilon$.

We set

$$D_{1/9} := \left\{\zeta \in \mathbb{C}\colon \left|1 - \frac{T\zeta}{\zeta}\right| < \frac{1}{9}\right\}, \quad \nu^{1/9} := \nu\,\bigl|_{D_{1/9}} \tag{4.16}$$

is a restriction of the measure ν to the set $D_{1/9}$, and

$$\nu_T^{1/9} := (\nu^{1/9})_T, \quad \tilde\nu^{1/9} := \nu - \nu^{1/9}, \quad \tilde\nu_T^{1/9} := (\tilde\nu^{1/9})_T = \nu_T - (\nu^{1/9})_T. \tag{4.17}$$

Taking into account (2.2), (4.12) and (4.14), we represent $I(z)$ in the form of algebraic sum

$$\begin{aligned} I(z) = &\int\limits_{\mathbb{C}} \bigl(e_q(z,T\zeta) - e_q(z,\zeta)\bigr)\,\mathrm{d}\nu^{1/9}(\zeta) \\ &+ \int\limits_{\mathbb{C}} e_q(z,\zeta)\,\mathrm{d}\tilde\nu_T^{1/9}\zeta - \int\limits_{\mathbb{C}} e_q(z,\zeta)\,\mathrm{d}\tilde\nu^{1/9}(\zeta) =: I_{1/9}(z) + w_{\tilde\nu_T^{1/9}}(z) - w_{\tilde\nu^{1/9}}(z), \end{aligned} \tag{4.18}$$

*Convergence (finiteness) of the integrals arising further for the points z outside some exceptional set will follow from the estimates.

where we use the notation (4.3) for the Weierstrass–Hadamard potentials $w_{\tilde\nu_T^{1/9}}$ and $w_{\tilde\nu^{1/9}}$ of the measures $\tilde\nu_T^{1/9}$ and $\tilde\nu^{1/9}$ of genus q.

By the second condition from (4.6), for $\zeta \in \mathbb{C} \setminus D_{1/9}$, i.e., $|1-(T\zeta)/\zeta| \geqslant 1/9$, in view of agreement (4.10), for $|\zeta| \geqslant R$ we obtain

$$b|\zeta| \leqslant |T\zeta|, \quad \frac{1}{|\zeta|^\rho} \leqslant \frac{9}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|, \quad \frac{1}{|T\zeta|^\rho} \leqslant \frac{1}{b^\rho|\zeta|^\rho} \leqslant \frac{9}{b^\rho|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|.$$

Hence, for the restriction $\tilde\nu^{1/9}$ of measure ν from (4.17), in view of (4.12) and (4.6), we have

$$\int\limits_{\mathbb{C}} \frac{1}{|\zeta|^\rho}\, \mathrm{d}\tilde\nu^{1/9}(\zeta) \leqslant 9 \int\limits_{\mathbb{C}\setminus D(R)} \frac{1}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\, \mathrm{d}\nu(\zeta) < +\infty.$$

Similarly, for T-shift $\tilde\nu_T^{1/9}$ from (4.17), using (4.14), (4.12), and (4.6), we get

$$\int\limits_{\mathbb{C}} \frac{1}{|\zeta|^\rho}\, \mathrm{d}\tilde\nu_T^{1/9}(\zeta) = \int\limits_{\mathbb{C}} \frac{1}{|T\zeta|^\rho}\, \mathrm{d}\tilde\nu^{1/9}(\zeta) \leqslant \frac{1}{b^\rho}\int\limits_{\mathbb{C}} \frac{1}{|\zeta|^\rho}\, \mathrm{d}\tilde\nu^{1/9}(\zeta) < +\infty.$$

By Proposition 2, the finiteness of these two integrals implies that the Weierstrass–Hadamard potentials $w_{\tilde\nu^{1/9}}$ and $w_{\tilde\nu_T^{1/9}}$ are subharmonic functions of *zero type with respect to the order* ρ.

Proposition 3 (partial case of [12, Th. 2])**.** *Let u be a subharmonic function on $\mathbb{C}$, and $N\colon [0,+\infty) \to [1,+\infty)$ be an increasing function. Then, for some absolute constants a_1, a_2, the inequality*

$$u(z) \geqslant -a_1\Bigl(\max_{|\zeta|=2|z|} u(\zeta)\Bigr)\cdot \log\bigl(a_2 N(|z|)\bigr)$$

holds for all $z \in \mathbb{C} \setminus E_0$, where E_0 is an exceptional set of the form (1.5) *such that*

$$\sum_{|z_j|<r} t_j \leqslant \int\limits_0^r \frac{\mathrm{d}t}{N(t)}\,.$$

If the function N increases to ∞ sufficiently slowly, then the application of Prop. 3 to each of functions $w_{\tilde\nu^{1/9}}$ and $w_{\tilde\nu_T^{1/9}}$ gives the relationships

$$\bigl|w_{\tilde\nu^{1/9}}(z)\bigr| + \bigl|w_{\tilde\nu_T^{1/9}}(z)\bigr| = o\bigl(|z|^\rho\bigr) \text{ as } z \in \mathbb{C}\setminus E_0,\ z \to \infty, \tag{4.19}$$

where E_0 is some set of zero upper density. In other words, the set E_0 is *a C_0-set* [1]. Hence, going back to (4.18) for the integral $I(z)$ from (4.15), the problem becomes simpler: we are to prove only the estimate $|I_{1/9}(z)| \leqslant \frac{\varepsilon}{2}|z|^\rho$ outside the set of upper density $\leqslant \varepsilon$. Further, for short, we designate $I_{1/9}(z)$ and $\nu^{1/9}$ as $I(z)$ and ν. By definitions (4.16) and (4.17), for the proof of estimate $|I(z)| \leqslant \frac{\varepsilon}{2}|z|^\rho$, we may suppose that *the mapping T also satisfies (together with* (4.10)–(4.11)*)*

$$\Bigl|1 - \frac{T\zeta}{\zeta}\Bigr| < \frac{1}{9}\,, \textit{ and, hence, also } \frac{8}{9}\,|\zeta| \leqslant |T\zeta| \leqslant \frac{10}{9}\,|\zeta| \quad \textit{for all } \zeta \in S_\nu, \tag{4.20}$$

where $S_\nu \subset \mathbb{C}$ is a supporting set of the measure ν.

3. *The integral $I(z)$.* Let us rewrite the integral $I(z)$ from (4.15) by the rule (2.2), taking into account the definition of subharmonic Weierstrass kernel of genus q from (4.2), in the following form:

$$\begin{aligned} I(z) &= \int\limits_{\mathbb{C}} \bigl(e_q(z,T\zeta) - e_q(z,\zeta)\bigr)\,\mathrm{d}\nu(\zeta) \\ &= \int\limits_{|\zeta|\geqslant 4|z|} \bigl(e_q(z,T\zeta) - e_q(z,\zeta)\bigr)\,\mathrm{d}\nu(\zeta) \\ &+ \int\limits_{|\zeta|<4|z|} \sum_{m=1}^{q} \frac{1}{m}\,\mathrm{Re}\Bigl(\frac{z^m}{(T\zeta)^m} - \frac{z^m}{\zeta^m}\Bigr)\,\mathrm{d}\nu(\zeta) \\ &+ \int\limits_{D(4|z|)\setminus D(z,|z|/2)} \Bigl(\log\Bigl|1 - \frac{z}{T\zeta}\Bigr| - \log\Bigl|1 - \frac{z}{\zeta}\Bigr|\Bigr)\,\mathrm{d}\nu(\zeta) \\ &+ \int\limits_{D(z,|z|/2)} \Bigl(\log\Bigl|1 - \frac{z}{T\zeta}\Bigr| - \log\Bigl|1 - \frac{z}{\zeta}\Bigr|\Bigr)\,\mathrm{d}\nu(\zeta) \\ &=: J_\infty(z) + J_0(z) + L(z) + L_0(z). \end{aligned} \tag{4.21}$$

3.1. *An estimate of the integral $J_\infty(z)$.* Under the condition $|T\zeta| \geqslant 8|\zeta|/9$ from (4.20), we use the expansion in series at $|\zeta| \geqslant 4|z|$ for

$$e_q(z,T\zeta) - e_q(z,\zeta) = -\sum_{m=q+1}^{\infty} \frac{1}{m}\,\mathrm{Re}\Bigl(\frac{z^m}{(T\zeta)^m} - \frac{z^m}{\zeta^m}\Bigr).$$

It implies

$$\bigl|e_q(z,T\zeta) - e_q(z,\zeta)\bigr| \leqslant \sum_{m=q+1}^{\infty} \frac{|z|^m}{m}\,\Bigl|\frac{1}{(T\zeta)^m} - \frac{1}{\zeta^m}\Bigr|. \tag{4.22}$$

For any $\zeta \neq 0$ and $T\zeta \neq 0$, we have

$$\left|\frac{1}{(T\zeta)^m} - \frac{1}{\zeta^m}\right| = |T\zeta - \zeta| \cdot \frac{\left|\sum\limits_{k=0}^{m-1} (T\zeta)^{m-1-k}\zeta^k\right|}{|T\zeta|^m|\zeta|^m},$$

and also

$$\left|\frac{1}{(T\zeta)^m} - \frac{1}{\zeta^m}\right| \leqslant |T\zeta - \zeta| \cdot \frac{m\Big(\max\{|T\zeta|, |\zeta|\}\Big)^{m-1}}{|T\zeta|^m|\zeta|^m}.$$

Hence, under the conditions (4.20), *for each $\zeta \neq 0$, $T\zeta \neq 0$, $m \geqslant 1$*, we have

$$\left|\frac{1}{(T\zeta)^m} - \frac{1}{\zeta^m}\right| \leqslant |T\zeta - \zeta| \cdot \frac{m\Big(\max\{|T\zeta|, |\zeta|\}\Big)^{m}}{|T\zeta|^m|\zeta|^{m+1}} \leqslant \left|1 - \frac{T\zeta}{\zeta}\right| \cdot \frac{m2^m}{|\zeta|^m}. \tag{4.23}$$

Using (4.22), for $|\zeta| \geqslant 4|z|$ we obtain

$$\big|e_q(z, T\zeta) - e_q(z,\zeta)\big| \leqslant \left|1 - \frac{T\zeta}{\zeta}\right| \sum_{m=q+1}^{\infty} \frac{2^m|z|^m}{|\zeta|^m} \leqslant \left|1 - \frac{T\zeta}{\zeta}\right| \frac{2^{q+2}|z|^{q+1}}{|\zeta|^{q+1}}$$

and

$$\begin{aligned}\big|J_\infty(z)\big| &\leqslant \int\limits_{|\zeta|\geqslant 4|z|} \big|e_q(z, T\zeta) - e_q(z,\zeta)\big|\, \mathrm{d}\nu(\zeta) \leqslant \int\limits_{|\zeta|\geqslant 4|z|} \left|1 - \frac{T\zeta}{\zeta}\right| \frac{2^{q+2}|z|^{q+1}}{|\zeta|^{q+1}}\, \mathrm{d}\nu(\zeta) \\ &= |z|^\rho \int\limits_{|\zeta|\geqslant 4|z|} \frac{2^{q+2}|z|^{q+1-\rho}}{|\zeta|^{q+1-\rho}} \left(\frac{1}{|\zeta|^\rho}\left|1 - \frac{T\zeta}{\zeta}\right|\right) \mathrm{d}\nu(\zeta) \quad \text{for all } z \in \mathbb{C}.\end{aligned}$$

By the definition from (4.1), for q we have $q + 1 - \rho \geqslant 0$. Therefore, for all $|\zeta| \geqslant 4|z|$,

$$\frac{2^{q+2}|z|^{q+1-\rho}}{|\zeta|^{q+1-\rho}} \leqslant \frac{2^{q+2}}{4^{q+1-\rho}} = 2^{2\rho-q} \leqslant 4^\rho.$$

So, we obtain the final estimate

$$\big|J_\infty(z)\big| \leqslant 4^\rho|z|^\rho \int\limits_{|\zeta|\geqslant 4|z|} \frac{1}{|\zeta|^\rho}\left|1 - \frac{T\zeta}{\zeta}\right| \mathrm{d}\nu(\zeta) \quad \textit{for all } z \in \mathbb{C}. \tag{4.24}$$

3.2. *An estimate of the integral* $J_0(z)$ *from* (4.21). Using (4.23) to the integration element of $J_0(z)$, *for any* $0<|\zeta|<4|z|$ *and* $T\zeta\neq 0$, we have

$$\left|\sum_{m=1}^{q}\frac{1}{m}\operatorname{Re}\Bigl(\frac{z^m}{(T\zeta)^m}-\frac{z^m}{\zeta^m}\Bigr)\,\mathrm{d}\nu(\zeta)\right|\leqslant\sum_{m=1}^{q}\frac{|z|^m}{m}\Bigl|\frac{1}{(T\zeta)^m}-\frac{1}{\zeta^m}\Bigr|\,\mathrm{d}\nu(\zeta)$$
$$\leqslant\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\sum_{m=1}^{q}\frac{2^m|z|^m}{|\zeta|^m}=|z|^\rho\frac{1}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\sum_{m=1}^{q}2^m\bigl(|\zeta|/|z|\bigr)^{\rho-m}.$$

Besides, by definition of q from (4.1), for all $m=1,2,\dots,q$, we obtain $\rho-m\geqslant 0$, and $|\zeta|/|z|<4$. Hence

$$\left|\sum_{m=1}^{q}\frac{1}{m}\operatorname{Re}\Bigl(\frac{z^m}{(T\zeta)^m}-\frac{z^m}{\zeta^m}\Bigr)\,\mathrm{d}\nu(\zeta)\right|\leqslant 2^{2\rho}\left(1-\frac{1}{2^q}\right)|z|^\rho\frac{1}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|.$$

So, we obtain the final estimate

$$|J_0(z)|\leqslant 4^\rho|z|^\rho\int\limits_{|\zeta|<4|z|}\frac{1}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\,\mathrm{d}\nu(\zeta)\quad \textit{for all } z\in\mathbb{C}. \tag{4.25}$$

3.3. *An estimate of the integral* $L(z)$ *from* (4.21). For the integration element of the integral $L(z)$, we have the following identity:

$$l_T(z,\zeta):=\log\Bigl|1-\frac{z}{T\zeta}\Bigr|-\log\Bigl|1-\frac{z}{\zeta}\Bigr|\equiv\log\left|1+\frac{z\zeta}{\zeta-z}\cdot\Bigl(\frac{1}{\zeta}-\frac{1}{T\zeta}\Bigr)\right|, \tag{4.26}$$

$\zeta T\zeta\neq 0$. Let us estimate *above* the right-hand side for $\zeta\in D(4|z|)\setminus D(z,|z|/2)$, i.e., for

$$0<|\zeta|<4|z|,\quad |\zeta-z|\geqslant\frac{1}{2}|z|,\quad T\zeta\neq 0. \tag{4.27}$$

Under these conditions, considering the inequality $|T\zeta|\geqslant 8|\zeta|/9$ from (4.20), in view of (4.26), we get

$$l_T(z,\zeta)\leqslant\log\left(1+\frac{|z||\zeta|}{|\zeta-z|}\cdot\Bigl|\frac{1}{\zeta}-\frac{1}{T\zeta}\Bigr|\right)\leqslant\frac{|z||\zeta|}{|\zeta-z||T\zeta|}\cdot\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|$$
$$\leqslant\frac{2|\zeta|}{|T\zeta|}\cdot\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\leqslant 3\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|\leqslant 3\cdot 4^\rho|z|^\rho\frac{1}{|\zeta|^\rho}\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|. \tag{4.28}$$

Under the same conditions, using the identity (4.26), we estimate above

$$-l_T(z,\zeta)\equiv\log\left|1+\frac{z\cdot T\zeta}{T\zeta-z}\cdot\Bigl(\frac{1}{T\zeta}-\frac{1}{\zeta}\Bigr)\right|\leqslant\frac{|z|}{|T\zeta-z|}\cdot\Bigl|1-\frac{T\zeta}{\zeta}\Bigr|. \tag{4.29}$$

In view of (4.20), we obtain $|\zeta - T\zeta| < |\zeta|/9$. Hence, under the conditions (4.27),

$$|T\zeta - z| \geqslant |\zeta - z| - |\zeta - T\zeta| \geqslant \frac{1}{2}\,|z| - \frac{1}{9}\,|\zeta| \geqslant \frac{1}{2}\,|z| - \frac{4}{9}\,|z| = \frac{1}{18}\,|z|.$$

Thus, we can extend (4.29) just as (4.28):

$$-l_T(z,\zeta) \leqslant 18 \cdot \Bigl|1 - \frac{T\zeta}{\zeta}\Bigr| \leqslant 18 \cdot 4^\rho |z|^\rho \frac{1}{|\zeta|^\rho}\Bigl|1 - \frac{T\zeta}{\zeta}\Bigr|.$$

The last one together with (4.28) gives a final estimate for the module of the integral

$$|L(z)| \leqslant \int\limits_{D(4|z|)\setminus D(z,|z|/2)} \bigl|l_T(z,\zeta)\bigr| \,\mathrm{d}\nu(\zeta) \leqslant 18 \cdot 4^\rho |z|^\rho \int\limits_{|\zeta|<4|z|} \frac{1}{|\zeta|^\rho}\Bigl|1 - \frac{T\zeta}{\zeta}\Bigr| \,\mathrm{d}\nu(\zeta)$$

for all $z \in \mathbb{C}$. Hence, using (4.25) and (4.24), under condition (4.11), we get an intermediate estimate

$$|I(z)| \leqslant 19{\cdot}4^\rho |z|^\rho \int\limits_{\mathbb{C}} \frac{1}{|\zeta|^\rho}\Bigl|1 - \frac{T\zeta}{\zeta}\Bigr| \,\mathrm{d}\nu(\zeta) + |L_0(z)| = 19{\cdot}4^\rho |z|^\rho \alpha(R) + |L_0(z)| \quad (4.30)$$

for all $z \in bC$ after simplifications of items **1** and **2**. The required estimate for the module of

$$L_0(z) = \int\limits_{D(z,|z|/2)} l_T(z,\zeta)\,\mathrm{d}\nu(z), \qquad (4.31)$$

with l_T from (4.26), is possible only outside some exceptional set constructed below.

4. *Normal points***.** Let us give a variant of the definition of normal points.

Definition ([17, § 2]). *Let $f \geqslant 0$ be a Borel function on $\mathbb{C}$, $f > 0$ on $\nu \in \mathcal{M}^+$. Let $d\colon \mathbb{C} \to (0, 1/2]$ be a Borel function on $\mathbb{C}$. We shall say that $z \in \mathbb{C}$ is (f,d)-normal with respect to ν if*

$$\nu(z,t) \leqslant f^{(d)}(z)\, t \quad \text{for all } \ t \leqslant d(z)|z| \quad \text{where } f^{(d)}(z) := \sup_{|\zeta - z| \leqslant d(z)|z|} f(\zeta). \qquad (4.32)$$

A partial case of [17, Normal Points Lemma] is

Lemma. *A set of points $z \in \mathbb{C}$ that are not (f,d)-normal with respect to the measure $\nu \in \mathcal{M}^+$ is contained in a union of the countable set of the disks $D(z_j, t_j)$, $j = 1, 2, \dots$, such that for any ν-measurable set $D \subset \mathbb{C}$*

$$\sum_{z_j \in D} t_j \leqslant a \int\limits_{D^d} \frac{\mathrm{d}\nu}{f} \quad \text{and} \quad t_j \leqslant d(z_j)|z_j| \ \text{for all } \ j \in \mathbb{N}, \qquad (4.33)$$

where a is an absolute constant (we can choose $a = 18$), and by definition $D^d := \bigcup\limits_{z\in D} D(z, d(z)|z|)$.

In our proof we choose $d \equiv 1/2$, and $f(z) \equiv M|z|^{\rho-1}$ for all $z \neq 0$ where $M > 0$ is *a constant* and then we consider the subset $E \subset \mathbb{C}$ of points z that are not (f, d)-normal with respect to ν *or* ν_T. In other words, $z \in \mathbb{C} \setminus E$ if

$$\max\{\nu(z,t), \nu_T(z,t)\} \leqslant f^{(1/2)}(z) \cdot t = Mc_\rho |z|^{\rho-1} t \textit{ for all } \quad 0 < t \leqslant |z|/2, \tag{4.34}$$

where $c_\rho := \frac{\max\{3^{\rho-1},1\}}{2^{\rho-1}}$. By Lemma, the set E can be covered with the disks $D(z_j, t_j)$, $j \in \mathbb{N}$, such that, according to (4.33), we have $t_j \leqslant |z_j|/2$, $j \in \mathbb{N}$, and

$$\sum_{|z_j|\leqslant r} t_j \leqslant a \int\limits_{(D(r))^{1/2}} \frac{\mathrm{d}(\nu+\nu_T)(z)}{M|z|^{\rho-1}} = \frac{a}{M} \int\limits_0^{3r/2} \frac{\mathrm{d}\big(\nu^{\mathrm{rad}}(t) + \nu_T^{\mathrm{rad}}(t)\big)}{t^{\rho-1}}\,.$$

Hence, in view of (4.12)–(4.13), (4.14), using integration by parts, we obtain

$$\begin{aligned}\sum_{|z_j|\leqslant r} t_j &\leqslant \frac{a}{M}\left(\left(\frac{2}{3}\right)^{\rho-1}\frac{\nu^{\mathrm{rad}}(3r/2)}{r^{\rho-1}} + (\rho-1)\int\limits_0^{3r/2}\frac{\nu^{\mathrm{rad}}(t)\,\mathrm{d}t}{t^\rho}\right)\\ &+ \frac{a}{M}\left(\left(\frac{2}{3}\right)^{\rho-1}\frac{\nu_T^{\mathrm{rad}}(3r/2)}{r^{\rho-1}} + (\rho-1)\int\limits_0^{3r/2}\frac{\nu_T^{\mathrm{rad}}(t)\,\mathrm{d}t}{t^\rho}\right)\\ &= \frac{3aB}{2M}\left(1+\frac{1}{b^\rho}\right)\cdot r, \quad r \geqslant 0.\end{aligned}$$

We choose $M > 0$ such that the multiplier in front of r is so large that it does not exceed $\varepsilon/6$, i. e.,

$$\sum_{|z_j|\leqslant r} t_j < \frac{\varepsilon}{6}\cdot r \quad \text{for all} \quad r \geqslant 1. \tag{4.35}$$

Further, for short, we call the set E constructed here an *exceptional set*, and the points from $\mathbb{C} \setminus E$ *normal points*.

It is significant that the "screening-out of a part" of the measures ν and ν_T under *increasing* R in **1** does not change conclusions of this item, since the constants B and M, as well as the disks $D(z_j, t_j)$ are not changed for all $R \geqslant 1$. Besides, repeating word by word standard reasonings from the finishing part of [23, item 1)], we can conclude that *for any point $z' \in \mathbb{C}$, $|z'| \geqslant r_0$, there exists a number $\epsilon(z') \in (0, \varepsilon)$ such that the circumference $\partial D\big(z', \epsilon(z')|z'|\big)$ contains only normal points.*

5. *An estimate of the integral $L_0(z)$ from* (4.31)**.** First, using (4.26) for l_T, we estimate the *upper bound*. For $|\zeta - z| < |z|/2$, under the condition (4.20), we have $|T\zeta| \geqslant 8|\zeta|/9$. Hence

$$l_T(z,\zeta) \leqslant \log\left(1 + \frac{|z||\zeta|}{|\zeta - z||T\zeta|}\left|1 - \frac{T\zeta}{\zeta}\right|\right) \leqslant \log\left(1 + \frac{9|z|/8}{|\zeta - z|}\left|1 - \frac{T\zeta}{\zeta}\right|\right).$$

For M and c_ρ from (4.34), we choose a parameter $c > 0$ so small that

$$Mc_\rho\left(c + c\log\Bigl(1 + \frac{1}{2c}\Bigr)\right) \leqslant \frac{\varepsilon}{8} \quad \textit{and simultaneously} \quad c\Bigl(\frac{5}{3}\Bigr)^\rho \frac{B}{b^\rho} \leqslant \frac{\varepsilon}{8}. \tag{4.36}$$

Then we write down a previous estimate of l_T in a somewhat weakened form

$$\begin{aligned} l_T(z,\zeta) &\leqslant \log\left(1 + \frac{2|z|c}{|\zeta - z|}\cdot\frac{1}{c}\left|1 - \frac{T\zeta}{\zeta}\right|\right) \\ &\leqslant \log\left(1 + \frac{2|z|c}{|\zeta - z|}\right) + \log\left(1 + \frac{1}{c}\left|1 - \frac{T\zeta}{\zeta}\right|\right) \\ &\leqslant \log\left(1 + \frac{2|z|c}{|\zeta - z|}\right) + \frac{1}{c}\left|1 - \frac{T\zeta}{\zeta}\right|. \end{aligned} \tag{4.37}$$

Integrating this inequality with respect to ν over $D(z,|z|/2)$, by (4.31) we obtain

$$\begin{aligned} L_0(z) &\leqslant \int\limits_{D(z,|z|/2)} \log\left(1 + \frac{2|z|c}{|\zeta - z|}\right) \mathrm{d}\nu(\zeta) + \frac{1}{c}\int\limits_{D(z,|z|/2)} \left|1 - \frac{T\zeta}{\zeta}\right| \mathrm{d}\nu(\zeta) \\ &= \int\limits_0^{|z|/2} \log\left(1 + \frac{2|z|c}{t}\right) \mathrm{d}\nu(z,t) + \frac{1}{c}\int\limits_{D(z,|z|/2)} |\zeta|^\rho\frac{1}{|\zeta|^\rho}\left|1 - \frac{T\zeta}{\zeta}\right| \mathrm{d}\nu(\zeta) \\ &\leqslant \int\limits_0^{|z|/2} \log\left(1 + \frac{2|z|c}{t}\right) \mathrm{d}\nu(z,t) + \frac{2^\rho}{c}\,|z|^\rho\alpha(R), \end{aligned} \tag{4.38}$$

where $\alpha(R)$ is a notation for the integral from (4.11). We estimate the integral in the right-hand side of (4.38) only for *normal points z*. Integration by parts gives

$$\begin{aligned} \int\limits_0^{|z|/2} \log\left(1 + \frac{2|z|c}{t}\right) \mathrm{d}\nu(z,t) &= \log(1 + 4c)\nu(z,|z|/2) + 2c|z|\int\limits_0^{|z|/2} \frac{\nu(z,t)\,\mathrm{d}t}{t(t + 2c|z|)} \\ &\leqslant \log(1 + 4c)\nu(z,|z|/2) + 2c|z|\int\limits_0^{|z|/2} \frac{\nu(z,t)\,\mathrm{d}t}{t(t + 2c|z|)} \leqslant 4c\cdot Mc_\rho|z|^\rho/2 \end{aligned}$$

$$+2c|z| \int\limits_0^{|z|/2} \frac{Mc_\rho |z|^{\rho-1}\,\mathrm{d}t}{t+2c|z|} = 2Mc_\rho |z|^\rho \left(c + c\log\Bigl(1+\frac{1}{4c}\Bigr)\right). \tag{4.39}$$

Hence, in view of (4.36), (4.38) implies

$$L_0(z) \leqslant \frac{\varepsilon}{4}\,|z|^\rho + \frac{2^\rho}{c}\,|z|^\rho \alpha(R), \quad z \in \mathbb{C}\setminus E, \tag{4.40}$$

where E is an exceptional set from item **4**.

Now we establish a *lower bound*.

The identity (4.29) with the parameter c, as in (4.37), gives

$$\begin{aligned}
-l_T(z,\zeta) &\leqslant \log\left(1+\frac{|z|}{|T\zeta - z|}\cdot\left|1-\frac{T\zeta}{\zeta}\right|\right)\\
&\leqslant \log\left(1+\frac{|z|c}{|T\zeta - z|}\right) + \log\left(1+\frac{1}{c}\left|1-\frac{T\zeta}{\zeta}\right|\right)\\
&\leqslant \log\left(1+\frac{c|z|}{|T\zeta - z|}\right) + \frac{1}{c}\left|1-\frac{T\zeta}{\zeta}\right|
\end{aligned}$$

Integrating this inequality with respect to ν over $D(z,|z|/2)$, similarly to (4.38), we obtain

$$-L_0(z) \leqslant \int\limits_{D(z,|z|/2)} \log\left(1+\frac{c|z|}{|T\zeta - z|}\right)\mathrm{d}\nu(\zeta) + \frac{2^\rho}{c}\,|z|^\rho\alpha(R). \tag{4.41}$$

If $\zeta \in D(z,|z|/2)$, then by (4.20) we have $|T\zeta - \zeta| < |\zeta|/9 \leqslant |z|/6$ and

$$|T\zeta - z| \leqslant |T\zeta - \zeta| + |\zeta - z| \leqslant \frac{1}{6}\,|z| + \frac{1}{2}\,|z| \leqslant \frac{2}{3}\,|z|.$$

It means that inclusion $D(z,|z|/2) \subset T^{-1}D(z,2|z|/3)$ is fulfilled. Therefore, for the integral from the right-hand side of (4.41) we can get

$$\int\limits_{D(z,|z|/2)} \log\left(1+\frac{c|z|}{|T\zeta - z|}\right)\mathrm{d}\nu(\zeta) \leqslant \int\limits_{T^{-1}D(z,2|z|/3)} \log\left(1+\frac{c|z|}{|T\zeta - z|}\right)\mathrm{d}\nu(\zeta).$$

By (2.2) and (4.14), for *normal points* z we obtain

$$\begin{aligned}
&\int\limits_{D(z,|z|/2)} \log\left(1+\frac{c|z|}{|T\zeta - z|}\right)\mathrm{d}\nu(\zeta) \leqslant \int\limits_{D(z,2|z|/3)} \log\left(1+\frac{c|z|}{|\zeta - z|}\right)\mathrm{d}\nu_T(\zeta)\\
&= \int\limits_0^{2|z|/3} \log\left(1+\frac{c|z|}{t}\right)\mathrm{d}\nu_T(z,t) = \left(\int\limits_0^{|z|/2} + \int\limits_{|z|/2}^{2|z|/3}\right)\log\left(1+\frac{c|z|}{t}\right)\mathrm{d}\nu_T(z,t)
\end{aligned}$$

$$\leqslant \int_0^{|z|/2} \log\left(1+\frac{c|z|}{t}\right) \mathrm{d}\nu_T(z,t) + \log(1+2c)\nu_T^{\mathrm{rad}}(5|z|/3)$$

$$\leqslant \log(1+2c)\nu_T(z,|z|/2) + c|z| \int_0^{|z|/2} \frac{\nu_T(z,t)\,\mathrm{d}t}{t(t+c|z|)} + 2c\Big(\frac{5}{3}\Big)^\rho \frac{B}{b^\rho}\,|z|^\rho.$$

Hence, for normal point $z \in \mathbb{C}\setminus E$, in view of (4.34), in the same way as in (4.39) we get

$$\int_{D(z,|z|/2)} \log\left(1+\frac{c|z|}{|T\zeta - z|}\right) \mathrm{d}\nu(\zeta)$$

$$\leqslant Mc_\rho|z|^\rho \left(c + c\log\Big(1+\frac{1}{2c}\Big)\right) + 2c\Big(\frac{5}{3}\Big)^\rho \frac{B}{b^\rho}\,|z|^\rho.$$

Then, in view of (4.36), (4.41) implies

$$\begin{aligned} -L_0(z) &\leqslant Mc_\rho|z|^\rho \left(c + c\log\Big(1+\frac{1}{2c}\Big)\right) + 2c\Big(\frac{5}{3}\Big)^\rho \frac{B}{b^\rho}\,|z|^\rho + \frac{2^\rho}{c}\,|z|^\rho\alpha(R) \\ &\leqslant \frac{\varepsilon}{8}\,|z|^\rho + \frac{\varepsilon}{4}\,|z|^\rho + \frac{2^\rho}{c}\,|z|^\rho\alpha(R) = \frac{3\varepsilon}{8}\,|z|^\rho + \frac{2^\rho}{c}\,|z|^\rho\alpha(R). \end{aligned}$$

The above and (4.40) give the final estimate

$$|L_0(z)| \leqslant \frac{3\varepsilon}{8}\,|z|^\rho + \frac{2^\rho}{c}\,|z|^\rho\alpha(R), \quad z \in \mathbb{C}\setminus E.$$

Thus, by (4.30), we have

$$|I(z)| \leqslant \alpha(R)\left(19\cdot 4^\rho + \frac{2^\rho}{c}\right)|z|^\rho + \frac{3\varepsilon}{8}\,|z|^\rho.$$

As it was said at the end of item **4**, we can increase the number $R \geqslant 1$ from **1** without any limits . Considering (4.11), we can choose R to be so large that $\alpha(R)\big(19\cdot 4^\rho + 2^\rho/c\big) \leqslant \varepsilon/8$. Then $|I(z)| \leqslant \dfrac{\varepsilon}{2}\,|z|^\rho$ for $z \in \mathbb{C}\setminus E$. But remembering the arrangements given at the end of item 2, we say that $I(z)$ is $I_{1/9}(z)$ from (4.18). For the initial integral $I(z)$ from (4.15), in view of (4.18) and (4.19), we obtain $|I(z)| \leqslant \varepsilon|z|^\rho$ for all normal points $z \in \mathbb{C}\setminus(E\cup E_0)$ where E_0 is a C_0-set. By (4.35), the set $E_\varepsilon := E\cup E_0$ has the upper density $< \varepsilon/6$. Then the concluding remark of item **4** remains in force in the following form: *for any point $z' \in \mathbb{C}$, $|z'| \geqslant 1$, there exists a number $\epsilon(z') \in (0,\varepsilon)$ such that $\partial D\big(z', \epsilon(z')|z'|\big) \cap E_\varepsilon = \varnothing$.*

6. *From the integral $I(z)$ to the functions u and u_T*. Let u be a function from Theorem 3. By item **1**, we can assume that $0 \notin \nu$ and $0 \notin \nu_T$. Since the Riesz measure ν has a finite type with respect to the order ρ, the function u admits the Weierstrass–Hadamard representation of genus $p = [\rho]$ with a *harmonic* addition h_u (see [22, 4.2]):

$$u(z) = \int\limits_{\mathbb{C}} e_p(z,\zeta)\,\mathrm{d}\nu(\zeta) + h_u(z), \quad z \in \mathbb{C}. \tag{4.42}$$

Here q from (4.1) is connected with p by the rule

$$q = \begin{cases} p = [\rho], & \text{if } \rho \text{ is noninteger}, \\ \rho - 1 = p - 1, & \text{if } \rho \text{ is integer}. \end{cases}$$

Now we set

$$u_T(z) := \int\limits_{\mathbb{C}} e_p(z,\zeta)\,\mathrm{d}\nu_T(\zeta) + h_u(z) + v_\rho(z), \quad z \in \mathbb{C}, \tag{4.43}$$

where, by definition, $v_\rho(z) \equiv 0$ *if* ρ *is noninteger*, and

$$v_\rho(z) := \operatorname{Re} \frac{z^\rho}{\rho} \int\limits_{\mathbb{C}} \Bigl(\frac{1}{\zeta^\rho} - \frac{1}{(T\zeta)^\rho}\Bigr)\,\mathrm{d}\nu(\zeta) \quad \textit{if} \quad \rho \textit{ is integer}, \quad z \in \mathbb{C}. \tag{4.44}$$

The function u_T is well-defined. First, the integral from (4.43) is a Weierstrass–Hadamard potential of the measure ν_T of finite type with respect to the order ρ (see (4.14) and [22, 4.2]), and, second, for the *integer* ρ the integral from (4.44) is finite. Indeed, it follows from (4.23) that

$$\Bigl|\int\limits_{|\zeta|\geqslant R} \Bigl(\frac{1}{\zeta^\rho} - \frac{1}{(T\zeta)^\rho}\Bigr)\,\mathrm{d}\nu(\zeta)\Bigr| \leqslant \rho 2^\rho \int\limits_{|\zeta|\geqslant R} \Bigl|1 - \frac{T\zeta}{\zeta}\Bigr| \frac{1}{|\zeta|^\rho}\,\mathrm{d}\nu(\zeta),$$

where, by (4.6), the right-hand side tends to 0 as $R \to +\infty$. By construction (4.44), the function v_ρ is harmonic. Therefore, by construction (4.43), u_T is a subharmonic function with the Riesz measure ν_T.

From the form (4.2) of the subharmonic Weierstrass kernels of genus q and p and the representations (4.42)–(4.44) for u and u_T it follows that

$$u_T(z) - u(z) \equiv \int\limits_{\mathbb{C}} e_q(z,\zeta)\,\mathrm{d}(\nu_T - \nu)(\zeta) = I(z),$$

where $I(z)$ is the integral from (4.15). Therefore, by item **5**, for any $\varepsilon > 0$ we get

$$\bigl|u_T(z) - u(z)\bigr| = |I(z)| \leqslant \varepsilon |z|^\rho \quad \textit{for all} \quad z \in \mathbb{C} \setminus E_\varepsilon, \tag{4.45}$$

where the exceptional set E_ε has the upper density $\leqslant \varepsilon$. Thereby, the main part of Th. 3 is proved.

By principle of the maximum for subharmonic functions, the relations (4.8) follow from (4.45) and concluding remark of item **5** on circumferences outside E_ε. Finally, the last assertion on the type and the indicator function of u_T is an evident consequence of (4.8).

This completes the proof of Th. 3.

R e m a r k 3. By representations (4.43)–(4.44), our construction of the function u_T by u is completely constructive.

R e m a r k 4. If we replace the first condition from (4.6) by its nonasymptotic analog $\inf\limits_{\zeta\in\mathbb{C}} |T\zeta|/|\zeta| > 0$, then there implies a condition "*... the preimage of any bounded set under T is bounded*".

In conclusion, we give a version of Th. 3 for entire functions.

Theorem 4 (partial formulations in [24, Corollary 1])**.** *Let $f \not\equiv 0$ be an entire function with zero set $\mathrm{Zero_f} = (\lambda_k) =: \Lambda$ of the finite upper density with respect to the order $\rho > 0$. Let $\Gamma = (\gamma_k) \subset \mathbb{C}$, $k \in \mathbb{N}$, be a sequence. If the series* (4.5) *converges and $\liminf\limits_{k\to\infty} |\gamma_k|/|\lambda_k| > 0$, then there exists an entire function g with $\mathrm{Zero_g} = \Gamma$ such that for any number $\varepsilon > 0$ there is a set $E_\varepsilon \subset \mathbb{C}$ of the upper density $\leqslant \varepsilon$ that*

$$\big|\log|g(z)| - \log|f(z)|\big| \leqslant \varepsilon |z|^\rho \quad \textit{for all} \quad z \in \mathbb{C} \setminus E_\varepsilon.$$

If the function f is of a finite type with respect to the order ρ, then the function g is of the same type, and the indicator function (with respect to ρ) of function f coincides with the indicator function (with respect to ρ) of function g.

P r o o f. By Theorem 1, there exists an entire function $\tilde f$ with the sequence of *simple zeros* $\mathrm{Zero_{\tilde f}} = (\tilde\lambda_k) =: \tilde\Lambda$ such that

$$\big|\log|\tilde f(z)| - \log|f(z)|\big| \leqslant \mathrm{const}^+ \quad \text{for all} \quad z \in \mathbb{C} \setminus \tilde E, \tag{4.46}$$

where the set $\tilde E$ is covered with the disks having a finite sum of radii, and $|\tilde\lambda_k - \lambda_k| \leqslant 1$ for all $k \in \mathbb{N}$ ($\beta \equiv 1$ chosen sufficiently). Then the condition (4.5) holds if we replace Λ by $\tilde\Lambda$, because $|\tilde\lambda_k/\lambda_k| \to 1$ as $k \to +\infty$ if Λ is infinity. Therefore, for sufficiently large $k_0 \in \mathbb{N}$,

$$\sum_{k\geqslant k_0} \frac{1}{|\tilde\lambda_k|^\rho}\left|1 - \frac{\gamma_k}{\tilde\lambda_k}\right| \leqslant \left(\max_{k\geqslant k_0}\left|\frac{\tilde\lambda_k}{\lambda_k}\right|^{\rho-1}\right) \sum_{k\geqslant k_0} \frac{1}{|\lambda_k|^\rho}\, \frac{|\lambda_k - \gamma_k| + |\tilde\lambda_k - \lambda_k|}{|\lambda_k|}$$

$$\leqslant \mathrm{const}^+ \left(\sum_{k\geqslant k_0} \frac{1}{|\lambda_k|^\rho}\left|1 - \frac{\gamma_k}{\lambda_k}\right| + \sum_{k\geqslant k_0} \frac{1}{|\lambda_k|^{\rho+1}}\right); \qquad \liminf_{k\to\infty} \frac{|\gamma_k|}{|\tilde\lambda_k|} > 0. \tag{4.47}$$

Here the penultimate sum converges by condition, and the last sum converges since the sequence Λ has the finite upper density with respect to ρ.

Now we can consider *a mapping* $T\colon \mathbb{C} \to \mathbb{C}$ that is defined by rule $T\tilde{\lambda}_k = \gamma_k$ and $Tz \equiv z$ for $z \notin \tilde{\Lambda}$. Then, by definition (1.1), the integer-valued measure n_Γ is a T-shift of the integer-valued measure $n_{\tilde{\Lambda}}$, i. e., $n_\Gamma = (n_{\tilde{\Lambda}})_T$. The convergence of the first sum in (4.47) and the last relation in the same place mean that the measure $\nu := n_{\tilde{\Lambda}}$ satisfies the first and the second conditions from (4.6). Besides, the last relation in (4.47) guarantees that the preimage of each bounded set is bounded. Therefore, by Th. 3, for the subharmonic function $u := \log|\tilde{f}|$ there exists a subharmonic function u_T with the Riesz measure n_Γ such that (4.7) holds, as well as the rest of conclusions of Th. 3. Hence there is an entire function g with $\mathrm{Zero_g} = \Gamma$ such that $u_T = \log|g|$, as the Riesz measure n_Γ of function u_T is an integer-valued measure. The last together with (4.46) completes the proof of Th. 4.

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