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Spherical Principal Series of Quantum Harish-Chandra Modules

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The nondegenerate spherical principal series of quantum Harish–Chandra modules is constructed. These modules appear in the theory of quantum bounded symmetric domains.

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1. Introduction

In [3] the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is considered as the Poincare model of hyperbolic plane. The Plancherel formula for $\mathbb{D}$ is one of the most profound results of noncommutative harmonic analysis, and q-analogs of this result are known for $q \in (0, 1)$ being the deformation parameter (see, e.g., [19]). It is important to note that the representations of spherical principal series are the crucial tools in decomposing quasiregular representation both in a classical and a quantum cases.

The unit disc is the simplest bounded symmetric domain. It is known that there is a Plancherel formula for any bounded symmetric domain in the classical setting. On the other hand, it is absent in the quantum case. One of the obstacles here is that there are some difficulties in producing the nondegenerate spherical principal series of Harish–Chandra modules over a quantum universal enveloping algebra. In this paper we overcome these difficulties. A geometrical approach to the representation theory is used instead of the traditional construction of principal series, which is inapplicable in the quantum case. Hence, we generalize the results of [17].

The Casselman theorem claims that any simple Harish–Chandra module can be embedded in a module of the nondegenerate spherical principal series. Thus, one has another class of applications for the constructions made in this paper, beyond the harmonic analysis.

2. A Quantum Analog of the Open K-Orbit in $B\backslash G$

Let $(a_{ij})_{i,j=1,\dots,l}$ be a Cartan matrix of positive type, $\mathfrak{g}$ the corresponding simple complex Lie algebra. So the Lie algebra can be defined by the generators e_i, f_i, h_i, $i=1,\dots,l$, and the well-known relations (see [4]). Let $\mathfrak{h}$ be the linear span of h_i, $i=1,\dots,l$. The simple roots $\{\alpha_i\in\mathfrak{h}^*|i=1,\dots,l\}$ are given by $\alpha_i(h_j)=a_{ji}$. Also, let $\{\varpi_i|i=1,\dots,l\}$ be the fundamental weights, hence $P=\bigoplus_{i=1}^l\mathbb{Z}\varpi_i=\{\lambda=(\lambda_1,\dots,\lambda_l)|\,\lambda_j\in\mathbb{Z}\}$ is the weight lattice and $P_+=\bigoplus_{i=1}^l\mathbb{Z}_+\varpi_i=\{\lambda=(\lambda_1,\dots,\lambda_l)|\,\lambda_j\in\mathbb{Z}_+\}$ is the set of integral dominant weights.

Fix $l_0\in\{1,\dots,l\}$, together with the Lie subalgebra $\mathfrak{k}\subset\mathfrak{g}$ generated by

$$e_i,f_i,\ \ i\neq l_0;\qquad h_i,\ \ i=1,\dots,l.$$

Define $h_0\in\mathfrak{h}$ by

$$\alpha_i(h_0)=0,\ i\neq l_0;\qquad \alpha_{l_0}(h_0)=2.$$

We restrict ourselves by Lie algebras $\mathfrak{g}$ that can be equipped with a $\mathbb{Z}$-grading as follows:

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{+1},\quad \mathfrak{g}_j=\{\xi\in\mathfrak{g}|\,[h_0,\xi]=2j\xi\}. \tag{1}$$

Let δ be the maximal root, and $\delta=\sum_{i=1}^l c_i\alpha_i$. (1) holds if and only if $c_{l_0}=1$. In this case $\mathfrak{g}_0=\mathfrak{k}$ and the pair $(\mathfrak{g},\mathfrak{k})$ is called a Hermitian symmetric pair.

Fix a Hermitian symmetric pair $(\mathfrak{g},\mathfrak{k})$. Let G be a complex algebraic affine group with $\mathrm{Lie}\,(G)=\mathfrak{g}$ and $K\subset G$ the connected subgroup with $\mathrm{Lie}\,(K)=\mathfrak{k}$. Consider the Lie subalgebra $\mathfrak{b}\subset\mathfrak{g}$ generated by e_i, h_i, $i=1,\dots,l$, together with the corresponding connected subgroup $B\subset G$. The homogeneous space G/B has a unique open K-orbit $\Omega\subset X$. It is an affine algebraic variety which is a crucial tool in producing nondegenerate principal series of Harish–Chandra modules (see [16]).

We introduce a q-analog of the algebra of regular functions on the open orbit.

Recall some background and introduce the notations. First of all, recall some notions from the quantum group theory [4]. In the sequel the ground field is $\mathbb{C}$, $q\in(0,1)$, and all the algebras are associative and unital.

Denote by d_i, $i = 1, \ldots, l$, such positive coprime integers that the matrix $(d_i a_{ij})_{i,j=1,\ldots,l}$ is symmetric. Recall that the quantum universal enveloping algebra $U_q\mathfrak{g}$ is a Hopf algebra defined by the generators K_i, K_i^{-1}, E_i, F_i, $i = 1, \ldots, l$, and the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\,\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} E_i^{1-a_{ij}-m}E_jE_i^m = 0,$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} F_i^{1-a_{ij}-m}F_jF_i^m = 0,$$

where $q_i = q^{d_i}$, $1 \le i \le l$, and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad [n]_q! = [n]_q \cdot \ldots \cdot [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The comultiplication Δ, the counit ε, and the antipode S are defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,\ \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

A representation $\rho : U_q\mathfrak{g} \to \mathrm{End}V$ is called *weight* (and V is called a weight module, respectively), if V admits a decomposition into the sum of weight subspaces

$$V = \bigoplus_\lambda V_\lambda, \qquad V_\lambda = \{v \in V \mid \rho(K_j^{\pm 1})v = q_j^{\pm\lambda_j}v, j = 1, \ldots, l\}.$$

The subspace V_λ is called a weight subspace of weight λ.

It is convenient to define the linear operators H_i in a weight module V by

$$H_iv = jv \qquad \text{iff} \qquad K_iv = q_i^jv, \qquad v \in V.$$

Let $U_q\mathfrak{k} \subset U_q\mathfrak{g}$ be a Hopf subalgebra generated by E_i, F_i, $i = 1, \ldots, l$, $i \ne l_0$, and $K_j^{\pm 1}$, $j = 1, \ldots, l$. A finitely generated weight $U_q\mathfrak{g}$-module V is called a *quantum Harish–Chandra module* if V is a sum of finite dimensional simple

$U_q\mathfrak{k}$-modules and $\dim \mathrm{Hom}_{U_q\mathfrak{k}}(W, V) < \infty$ for every finite dimensional simple $U_q\mathfrak{k}$-module W.

We restrict our consideration to quantum Harish–Chandra modules only.

Let $\lambda \in P_+$. $L(\lambda)$ denotes the simple $U_q\mathfrak{g}$-module with a single generator $v(\lambda)$ and the defining relations (see [4])

$$K_j^{\pm} v(\lambda) = q^{\pm\lambda_j} v(\lambda), \quad E_j v(\lambda) = 0, \quad F_j^{\lambda_j+1} v(\lambda) = 0.$$

Recall the notion of quantum analog of the algebra $\mathbb{C}[G]$ of regular functions on G [4]. Denote by $\mathbb{C}[G]_q \subset (U_q\mathfrak{g})^*$ the Hopf subalgebra of all matrix coefficients of weight finite dimensional $U_q\mathfrak{g}$-representations. Denote by $U_q^{op}\mathfrak{g}$ a Hopf algebra that differs from $U_q\mathfrak{g}$ by the opposite multiplication. Equip $\mathbb{C}[G]_q$ with a structure of $U_q^{op}\mathfrak{g} \otimes U_q\mathfrak{g}$-module algebra in the following way: $(\xi \otimes \eta) f = (L_{\mathrm{reg}}(\xi) \otimes R_{\mathrm{reg}}(\eta)) f$, where

$$(R_{\mathrm{reg}}(\xi) f)(\eta) = f(\eta\xi), \qquad (L_{\mathrm{reg}}(\xi) f)(\eta) = f(\xi\eta), \qquad \xi, \eta \in U_q\mathfrak{g}, f \in \mathbb{C}[G]_q.$$

The algebra $\mathbb{C}[G]_q$ is called the algebra of regular functions on the quantum group G.

Introduce special notation for some elements of $\mathbb{C}[G]_q$ [8]. Consider the finite dimensional simple $U_q\mathfrak{g}$-module $L(\lambda)$ with highest weight $\lambda \in P_+$. Equip it with an invariant scalar product $(\cdot,\cdot)$, given by $(v(\lambda), v(\lambda)) = 1$ (as usual in the compact quantum group theory [8]). Choose an orthonormal basis of the weight vectors $\{v_{\mu,j}\} \in L(\lambda)_\mu$ for all weights μ. Let

$$c^{\lambda}_{\mu,i,\mu',j}(\xi) = (\xi v_{\mu',j}, v_{\mu,i}).$$

We will omit the indices i, j, if they do not lead to an ambiguity. Introduce an auxiliary $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\widehat{X}]_q$.

Let $\lambda \in P_+$. For all $\lambda', \lambda'' \in P_+$

$$\dim \mathrm{Hom}_{U_q\mathfrak{g}}(L(\lambda' + \lambda''), L(\lambda') \otimes L(\lambda'')) = 1.$$

Hence, the following $U_q\mathfrak{g}$-morphisms are well-defined:

$$m_{\lambda',\lambda''} : L(\lambda') \otimes L(\lambda'') \to L(\lambda' + \lambda''), \qquad m_{\lambda',\lambda''} : v(\lambda') \otimes v(\lambda'') \mapsto v(\lambda' + \lambda'').$$

Therefore, the vector space

$$\mathbb{C}[\widehat{X}]_q \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \in P_+} L(\lambda)$$

is equipped with a $U_q\mathfrak{g}$-module algebra structure as follows:

$$f' \cdot f'' \stackrel{\mathrm{def}}{=} m_{\lambda',\lambda''}(f' \otimes f''), \qquad f' \in L(\lambda'),\ f'' \in L(\lambda'').$$

This is a well-known quantum analog of the homogeneous coordinate ring of the flag manifold $X = B\backslash G$.

The Peter–Weyl theorem claims that

$$\mathbb{C}[G]_q = \bigoplus_{\lambda\in P_+} (L(\lambda)\otimes L(\lambda)^*).$$

Hence, there exists an embedding of $U_q\mathfrak{g}$-module algebras

$$i : \mathbb{C}[\widehat{X}]_q \hookrightarrow \mathbb{C}[G]_q, \qquad i : v(\lambda) \mapsto c^{\lambda}_{\lambda,\lambda}, \quad \lambda \in P_+,$$

where $c^{\lambda}_{\lambda,\lambda}$ are the matrix coefficients of the representations introduced above, so we have

Proposition 1. *$\mathbb{C}[\widehat{X}]_q$ is an integral domain.*

$\mathbb{C}[\widehat{X}]_q$ is a P-graded algebra by obvious reasons:

$$\mathbb{C}[\widehat{X}]_q = \bigoplus_{\lambda\in P_+} \mathbb{C}[\widehat{X}]_{q,\lambda}, \qquad \mathbb{C}[\widehat{X}]_{q,\lambda} = L(\lambda).$$

A simple weight finite dimensional $U_q\mathfrak{g}$-module $L(\lambda)$ is called *spherical*, if it has a nonzero $U_q\mathfrak{k}$-invariant vector [3]. This $U_q\mathfrak{k}$-invariant vector is unique up to a nonzero constant, $\dim L(\lambda)^{U_q\mathfrak{k}} = 1$. Moreover, the set $\Lambda_+ \subset P_+$ of all highest weights of the spherical $U_q\mathfrak{g}$-modules $L(\lambda)$ coincides with the similar set in the classical case, so $\Lambda_+ = \bigoplus_{i=1}^{r} \mathbb{Z}_+\mu_i$, with $\mu_1, \mu_2, \ldots, \mu_r$ being the fundamental spherical weights, and r being the real rank of the bounded symmetric domain $\mathbb{D}$ (see [3]). Choose nonzero elements

$$\psi_i \in L(\mu_i)^{U_q\mathfrak{k}}, \qquad i = 1, 2, \ldots, r.$$

Proposition 2. *$\psi_1, \psi_2, \ldots, \psi_r \in \mathbb{C}[\widehat{X}]_q$ pairwise commute.*

P r o o f. $\psi_i\psi_j = \mathrm{const}(i,j)\psi_j\psi_i$, with $\mathrm{const}(i,j) \neq 0$, since $\dim(L(\mu_i + \mu_j)^{U_q\mathfrak{k}}) = 1$. Prove that $\mathrm{const}(i,j) = 1$. It follows from Appendix, Lemma 5 that we can equip $\mathbb{C}[\widehat{X}]_q$ with an involution $\star$, and without loss of generality we can assume $\psi_j = \psi_j^\star$ for all $j = 1, \ldots, r$. Therefore, $\psi_i\psi_j = \pm\psi_j\psi_i$. The involution $\star$ is a morphism of a continuous vector bundle $\mathcal{E}_\lambda$ over $(0, 1]$ with the fibers $L(\lambda)_q$ defined in Appendix. In the classical case $\psi_i\psi_j = \psi_j\psi_i$, so the same holds in the quantum case. ■

Consider a multiplicative subset

$$\Psi = \{\psi_1^{j_1}\psi_2^{j_2}\cdots\psi_r^{j_r} \,|\, j_1, j_2, \ldots, j_r \in \mathbb{Z}_+\}$$

of the algebra $\mathbb{C}[\widehat{X}]_q$.

Proposition 3. *$\Psi \subset \mathbb{C}[\widehat{X}]_q$ is an Ore set.*

P r o o f. Consider the decomposition of the $U_q\mathfrak{g}$-module $L(\lambda)$ into a sum of its $U_q\mathfrak{k}$-isotypic components $L(\lambda) = \bigoplus_{\mu} L(\lambda)_\mu$. Fix a subspace $L(\lambda)_\mu$ and a nonzero $U_q\mathfrak{k}$-invariant vector $\psi \in L(\lambda')$. For all $j \in \mathbb{N}$,

$$L(\lambda + j\lambda')_\mu \supset L(\lambda + (j-1)\lambda')_\mu \, \psi.$$

It follows from Prop. 1 that $L(\lambda+j\lambda')_\mu = L(\lambda+(j-1)\lambda')_\mu \, \psi$ for all large enough $j \in \mathbb{N}$, since

$$\dim \operatorname{Hom}_{U_q\mathfrak{k}}(L(\mathfrak{k},\mu), L(\lambda)) \le \dim L(\mathfrak{k},\mu) \tag{2}$$

for any simple weight finite dimensional $U_q\mathfrak{k}$-module $L(\mathfrak{k},\mu)$ and any simple weight finite dimensional $U_q\mathfrak{g}$-module $L(\lambda)$. The inequality (2) is known in the classical case [7, p. 206], and the quantum case follows from the classical case. Therefore,

$$\psi^j \, L(\lambda)_\mu \subset L(\lambda + (j-1)\lambda')_\mu \, \psi$$

for all large enough $j \in \mathbb{N}$, since $\psi^j \, L(\lambda)_\mu \subset L(\lambda + j\lambda')_\mu$. Similarly, $L(\lambda)_\mu \, \psi^j \subset \psi \, L(\lambda + (j-1)\lambda')_\mu$ for all large enough $j \in \mathbb{N}$. Hence, for all $f \in \mathbb{C}[\widehat{X}]_q$, $\psi \in \Psi$

$$\Psi f \cap \mathbb{C}[\widehat{X}]_q \, \psi \ne \varnothing, \qquad f\,\Psi \cap \psi\,\mathbb{C}[\widehat{X}]_q \ne \varnothing,$$

which is just the Ore condition for Ψ. ■

Consider the localization $\mathbb{C}[\widehat{X}]_{q,\Psi}$ of the algebra $\mathbb{C}[\widehat{X}]_q$ with respect to the multiplicative set Ψ.

The P-grading can be extended to $\mathbb{C}[\widehat{X}]_{q,\Psi}$, since the elements of Ψ are homogeneous. The subalgebra

$$\mathbb{C}[\Omega]_q = \{f \in \mathbb{C}[\widehat{X}]_{q,\Psi} | \ \deg f = 0\}$$

is a quantum analog of the algebra $\mathbb{C}[\Omega]$ of regular functions on the open K-orbit $\Omega \subset X = B\backslash G$.

3. $U_q\mathfrak{g}$-Module Algebra $\mathbb{C}[\widehat{X}]_{q,\Psi}$

In this section we equip $\mathbb{C}[\widehat{X}]_{q,\Psi}$ with a $U_q\mathfrak{g}$-module algebra structure. Start with some auxiliary facts.

Consider the vector space L of sequences $\{x_n\}_{n\in\mathbb{Z}_+}$

$$x_n = \sum_{i,j\in\mathbb{Z}_+} a_{ij}\lambda_i^n n^j, \tag{3}$$

where the number of nonzero terms is finite, $a_{ij} \in \mathbb{C}[\widehat{X}]_{q,\Psi}$, λ_i are nonzero and pairwise different. Sequences x'_n and x''_n are called *asymptotically equal*, $x'_n \underset{\text{as}}{=} x''_n$, if there exists $N \in \mathbb{Z}_+$ such that for all $n \geq N$ one has $x'_n = x''_n$.

Lemma 1. *Let $\{x_n\}_{n \in \mathbb{Z}_+} \in L$ and $x_n \underset{\text{as}}{=} 0$. Then $x_n = 0$ for all $n \in \mathbb{Z}_+$.*

P r o o f. Let $x_n \underset{\text{as}}{=} 0$ and $x_n \not\equiv 0$. Without loss of generality, one assumes that $x_0 \neq 0$, since L is invariant under T, with

$$T : \{u_0, u_1, u_2, \ldots\} \mapsto \{u_1, u_2, \ldots\}, \qquad u_j \in \mathbb{C}[\widehat{X}]_{q,\Psi}.$$

Let $\mathcal{M}$ be the smallest T-invariant subspace containing all nonzero terms $\{a_{ij}\lambda_i^n n^j\}_{n \in \mathbb{Z}_+}$ from (3). Then $\dim \mathcal{M} < \infty$ and $0 \notin \operatorname{spec}(T|_{\mathcal{M}})$. However, $(T|_{\mathcal{M}})^N(x_n) = 0$ for all large enough N, since $x_n \underset{\text{as}}{=} 0$. So, $\operatorname{spec}(T|_{\mathcal{M}}) = \{0\}$. This contradiction completes the proof. ∎

Lemma 2. *Let $k \in \mathbb{Z}_+$ and $\lambda_0 \in \mathbb{C}$. The sequence*

$$\left\{ \frac{d^k}{d\lambda^k} (1 + \lambda + \lambda^2 + \cdots + \lambda^n)_{|\lambda = \lambda_0} \right\}_{n \in \mathbb{Z}_+} \tag{4}$$

belongs to L.

P r o o f. Let $\lambda_0 = 1$. (4) is a sequence of values of polynomials at $n \in \mathbb{Z}_+$, since

$$\sum_{j=0}^{n-1} j^{k-1} = \frac{B_k(n) - B_k(0)}{k},$$

with $B_k(z)$ being the Bernoulli polynomials [14, Ch. 3]. For $\lambda_0 \neq 1$,

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} = \frac{\lambda^n - 1}{\lambda - 1},$$

and the statement is now obvious. ∎

Let $\psi_0 = \prod_{j=1}^{r} \psi_j$. Note that for any $f \in \mathbb{C}[\widehat{X}]_{q,\Psi}$, $\xi \in U_q\mathfrak{g}$, the element $\xi(f\psi_0^n)$ is already defined and belongs to $\mathbb{C}[\widehat{X}]_q$ for large enough $n \in \mathbb{Z}_+$.

Proposition 4. *For any $\xi \in U_q\mathfrak{g}$ there exists a unique linear operator R_ξ in $\mathbb{C}[\widehat{X}]_{q,\Psi}$ such that for any $f \in \mathbb{C}[\widehat{X}]_{q,\Psi}$:*

1. *$R_\xi(f\psi_0^n) \underset{\text{as}}{=} \xi(f\psi_0^n)$;*
2. *the sequence $x_n = R_\xi(f\psi_0^n) \cdot \psi_0^{-n}$ belongs to L.*

P r o o f. The uniqueness follows from Lem. 1. Prove the existence.

Consider a set of all $\xi \in U_q\mathfrak{g}$ such that there exists R_ξ with the required properties. The set forms a subalgebra. Indeed,

$$\xi(f\psi_0^n) \underset{\text{as}}{=} R_\xi(f\psi_0^n) = \left(\sum_{i,j\in\mathbb{Z}_+} a_{ij}\lambda_i^n n^j\right)\psi_0^n,$$

$$\eta(a_{ij}\psi_0^n) \underset{\text{as}}{=} R_\eta(a_{ij}\psi_0^n) = \left(\sum_{k,l\in\mathbb{Z}_+} b_{ijkl}\mu_k^n n^l\right)\psi_0^n,$$

so

$$\eta\xi(f\psi_0^n) \underset{\text{as}}{=} \left(\sum_{i,j,k,l\in\mathbb{Z}_+} b_{ijkl}(\lambda_i\mu_k)^n n^{(j+l)}\right)\psi_0^n,$$

and one can put $R_{\eta\xi}(f) = \sum_{i,k} b_{i0k0}$. From Lemma 1, $R_{\eta\xi}(f)$ is a linear operator in $\mathbb{C}[\widehat{X}]_{q,\Psi}$. The sequence $R_{\eta\xi}(f\psi_0^n)$ has all required properties.

Now we have to construct the linear operators

$$R_{K_i^{\pm1}}, \quad R_{E_i}, \quad R_{K_iF_i}, \qquad i = 1, 2, \ldots, l,$$

that satisfy the conditions of Prop. 4. The case $\xi = K_i^{\pm1}$ is trivial, while the two others are very similar.

Prove the existence of R_{E_i}. Let $N \in \mathbb{Z}_+$ be the smallest number such that $f\psi_0^N \in \mathbb{C}[\widehat{X}]_q$. For any $n \in \mathbb{Z}_+$

$$\begin{gathered} E_i\left(f\psi_0^{n+N}\right)\psi_0^{-(n+N)} \\ = E_i\left(f\psi_0^N\right)\psi_0^{-N} + \left(K_i\left(f\psi_0^N\right)\psi_0^{-N}\right)\left(\psi_0^N E_i\left(\psi_0^n\right)\psi_0^{-(n+N)}\right). \end{gathered}$$

It is enough to prove that $x_n = \psi_0^N \cdot E_i(\psi_0^n) \cdot \psi_0^{-(n+N)}$ looks like (3), since then one can put

$$E_i(f\psi_0^n) \underset{\text{as}}{=} \left(\sum_{i,j\in\mathbb{Z}_+} a_{ij}\lambda_i^n n^j\right)\psi_0^n,$$

and $R_{E_i}(f) = a_{00}$. The linearity of R_{E_i} and all required properties reduce to the special case $f \in \mathbb{C}[\widehat{X}]_q$ by Lem. 1, as before.

Turn back to the proof. Let A be the linear operator in $\mathbb{C}[\widehat{X}]_{q,\Psi}$, defined by

$$Af = \psi_0 f\psi_0^{-1}, \qquad f \in \mathbb{C}[\widehat{X}]_{q,\Psi}.$$

The key observation is that the linear span of $\{A^k(E_i\psi_0\}_{k\in\mathbb{Z}_+}$ is finite dimensional. Indeed, all vectors $A^k(E_i\psi_0)$ belong to the same homogeneous component of the P-graded vector space $\mathbb{C}[\widehat{X}]_{q,\Psi}$, and, moreover, to the same its $U_q\mathfrak{k}$-isotypic component that corresponds to the highest weight $\mu = \sum\limits_{j=1}^{r} j\mu_j + \alpha_i$. But the dimension of intersection $\mathcal{E}$ of these components is at most than $\dim L(\mathfrak{k},\mu)^2$ by (2).

Now one can use the equality

$$x_n = \psi_0^N \left(\sum_{j=0}^{n-1} \psi_0^j (E_i\psi_0)\psi_0^{n-1-j}\right) \psi_0^{-(n+N)} = \psi_0^N \left(\sum_{j=0}^{n-1} (A|_{\mathcal{E}})^j\, (E_i\psi_0)\right) \psi_0^{-(N+1)}$$

and Lem. 2 to compute the function $1+z+z^2+\cdots+z^{n-1}$ of the finite dimensional linear operator $A|_{\mathcal{E}}$ reduced to the Jordan canonical form [13]. ■

Corollary 1. *R_ξ provides a $U_q\mathfrak{g}$-module structure in $\mathbb{C}[\widehat{X}]_{q,\Psi}$ which extends the $U_q\mathfrak{g}$-module structure from $\mathbb{C}[\widehat{X}]_q$ to $\mathbb{C}[\widehat{X}]_{q,\Psi}$.*

Now prove that $\mathbb{C}[\widehat{X}]_{q,\Psi}$ is a $U_q\mathfrak{g}$-module algebra. Let $\widetilde{L}$ be the vector space of functions on $\mathbb{Z}^2$ that takes values in $\mathbb{C}[\widehat{X}]_{q,\Psi}$, such that

$$x(m,n) = \sum_{i',i'',j',j''\in\mathbb{Z}_+} a_{i'i''j'j''}\lambda_{i'}^n\, n^{j'}\, \mu_{i''}^m\, m^{j''},$$

where the sum is finite, $a_{i'i''j'j''} \in \mathbb{C}[\widehat{X}]_{q,\Psi}$, and $\lambda_{i'}, \mu_{i''} \in \mathbb{C}$. Suppose that all $\lambda_{i'}$, $\mu_{i''}$ are pairwise different and nonzero. One can easily expand Lem. 1 to the functions $x(m,n) \in \widetilde{L}$. Namely, if $x(m,n) = 0$ for any $m \geq M$, $n \geq N$, then $x(m,n) = 0$ for all $m,n \in \mathbb{Z}$.

It is important to note that either in the statement or in the proof of Prop. 4 one can replace the conditions on R_ξ by the following conditions:

1. $R_\xi(\psi_0^m f\psi_0^n) = \xi(\psi_0^m f\psi_0^n)$, if $m,n \in \mathbb{Z}_+$ and $m+n$ is large enough;
2. the function $x(m,n) = \psi_0^{-m} R_\xi(\psi_0^m f\psi_0^n)\psi_0^{-n}$ belongs to $\widetilde{L}$.

One gets the same representation R_ξ of $U_q\mathfrak{g}$ in the vector space $\mathbb{C}[\widehat{X}]_{q,\Psi}$.

Proposition 5. *Let $\xi \in U_q\mathfrak{g}$ and $\triangle\xi = \sum\limits_i \xi_i' \otimes \xi_i''$. Then for any $f_1, f_2 \in \mathbb{C}[\widehat{X}]_{q,\Psi}$ one has*

$$\xi(f_1 f_2) = \sum_i (\xi_i' f_1)(\xi_i'' f_2).$$

P r o o f. Let
$$x'(m,n) = \psi_0^{-m}\xi(\psi_0^m f_1 f_2 \psi_0^n)\psi_0^{-n},$$
and
$$x''(m,n) = \sum_i \psi_0^{-m}\xi_i'(\psi_0^m f_1)\xi_i''(f_2\psi_0^n)\psi_0^{-n}.$$

They are equal for large enough m,n, since $\psi_0^m f_1, f_2\psi_0^n \in \mathbb{C}[\widehat{X}]_q$, and $\mathbb{C}[\widehat{X}]_q$ is a $U_q\mathfrak{g}$-module algebra. Also, both belong to $\widetilde{L}$. Therefore, $x'(m,n) = x''(m,n)$ for any $m,n \in \mathbb{Z}$. Now put $m = n = 0$ in the last equality. ∎

R e m a r k 1. It should be noted that V. Lunts and A. Rosenberg described another approach to the extension of the $U_q\mathfrak{g}$-module algebra structure in [11, 10]. Their approach is more general but more intricate.

4. Degenerate and Nondegenerate Spherical Principal Series

For simplicity and more clear presentation, start with the certain degenerate spherical principal series. The same approach is used in producing nondegenerate spherical principal series.

Fix $k \in \{1, 2, \ldots, r\}$. Consider
$$\mathbb{C}[\widehat{X}_k]_q \stackrel{\mathrm{def}}{=} \bigoplus_{j\in\mathbb{Z}_+} L(j\mu_k).$$

As in the previous section, one equips $\mathbb{C}[\widehat{X}_k]_q$ with a $U_q\mathfrak{g}$-module algebra structure. $\mathbb{C}[\widehat{X}_k]_q$ naturally embeds in the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\widehat{X}]_q$ and has a $\mathbb{Z}_+$-grading:
$$\deg f = j, \qquad \text{iff} \qquad f \in L(j\mu_k).$$

It follows from Prop. 3 that $\Psi_k = \psi_k^{\mathbb{Z}_+}$ is an Ore set, and the localization $\mathbb{C}[\widehat{X}_k]_{q,\Psi_k}$ is a $\mathbb{Z}$-graded $U_q\mathfrak{g}$-module algebra. Consider the subalgebra
$$\mathbb{C}[\Omega_k]_q = \{f \in \mathbb{C}[\widehat{X}_k]_{q,\Psi_k} |\ \deg(f) = 0\}.$$

Evidentially, $\mathbb{C}[\Omega_k]_q \subset \mathbb{C}[\Omega]_q$ is a $U_q\mathfrak{g}$-module algebra.

For $u \in \mathbb{Z}$ denote
$$\pi_{k,u}(\xi)f \stackrel{\mathrm{def}}{=} \xi\,(f\psi_k^u)\cdot\psi_k^{-u}, \qquad \xi \in U_q\mathfrak{g}, \quad f \in \mathbb{C}[\Omega_k]_q. \tag{5}$$

The representations $\pi_{k,u}$ are the representations of degenerate spherical principal series. Now we are going to introduce $\pi_{k,u}$ for arbitrary $u \in \mathbb{C}$. We need some auxiliary constructions.

Let

$$\mathbb{C}[\Omega_k]_q = \bigoplus_{\lambda \in P_+^{\mathcal{S}}} \mathbb{C}[\Omega_k]_{q,\lambda}$$

be the decomposition of $\mathbb{C}[\Omega_k]_q$ into a sum of its $U_q\mathfrak{k}$-isotypic components. $P_+^{\mathcal{S}}$ denotes the set of all integral dominant weights of $\mathfrak{k}$. By considerations from Appendix, $\mathbb{C}[\Omega_k]_{q,\lambda}$ are fibers of a continuous vector bundle $\mathcal{F}_\lambda$ over $(0,1]$ that is analytic on $(0,1)$. We identify the morphisms of such vector bundles over $(0,1]$ with the corresponding continuous in $(0,1]$ and analytic in $(0,1)$ "operator valued functions".

It is easy to prove that the operator valued function

$$A_{k,\lambda}(q) : \mathbb{C}[\Omega_k]_{q,\lambda} \to \mathbb{C}[\Omega_k]_{q,\lambda}, \qquad A_{k,\lambda}(q) : f \mapsto \psi_k f \psi_k^{-1}$$

is well-defined, invertible, continuous in $(0,1]$ and analytic in $(0,1)$.

Lemma 3. *All eigenvalues of $A_{k,\lambda}(q)$ are positive, and, moreover, rational powers of q.*

P r o o f. Let a be an eigenvalue of $A_{k,\lambda}(q)$, thus there exists a nonzero $f \in L(j\mu_k) \subset \mathbb{C}[\widehat{X}_k]_q$, such that $\psi_k f = a\, f \psi_k$.

It is easy to show that $aq^{-j(\mu_k,\mu_k)}$ is an eigenvalue of the linear operator $R_{L(j\mu_k)L(\mu_k)}$ corresponding to the universal R-matrix of $U_q\mathfrak{g}$. Here $(\cdot,\cdot)$ is fixed by $(\alpha_i,\alpha_j) = d_i a_{ij}$.

It remains to prove that the eigenvalues of $R_{L(\lambda),L(\lambda')}$ are rational powers of q for all $\lambda, \lambda' \in P_+$. There exists a suitable basis of tensor products of weight vectors such that the matrix of $R_{L(\lambda),L(\lambda')}$ is upper-triangular and its diagonal elements belong to the set $\{q^{-(\mu',\mu'')} \mid \mu',\mu'' \in P\}$, hence, they are rational powers of q. ■

$\pi_{k,u}(K_i^{\pm1}), \pi_{k,u}(E_i), \pi_{k,u}(F_i)$ are defined for $u \in \mathbb{Z}$. We are going to extend these operator valued functions to the complex plane. Evidentially,

$$\pi_{k,u}(K_i^{\pm1})f = K_i^{\pm1}(f\psi_k^u)\psi_k^{-u} = K_i^{\pm1} f K_i^{\pm1}(\psi_k^u)\psi_k^{-u},$$
$$\pi_{k,u}(E_i)f = E_i f + K_i f E_i(\psi_k^u)\psi_k^{-u},$$
$$\pi_{k,u}(K_iF_i)f = K_iF_i f + K_i f K_i F_i(\psi_k^u)\psi_k^{-u}, \qquad f \in \mathbb{C}[\Omega_k]_q.$$

Denote by P_λ a projection in $\mathbb{C}[\Omega_k]_q$ with $\mathrm{Im}\,\mathrm{P}_\lambda = \mathbb{C}[\Omega_k]_{q,\lambda}$, $\mathrm{Ker}\,\mathrm{P}_\lambda = \bigoplus_{\lambda' \neq \lambda} \mathbb{C}[\Omega_k]_{q,\lambda'}$. In the sequel we deal with the operator valued functions

$$P_{\lambda_2}\pi_{k,u}(K_i^{\pm1})|_{\mathbb{C}[\Omega_k]_{q,\lambda_1}}, \qquad P_{\lambda_2}\pi_{k,u}(E_i)|_{\mathbb{C}[\Omega_k]_{q,\lambda_1}}, \qquad P_{\lambda_2}\pi_{k,u}(K_iF_i)|_{\mathbb{C}[\Omega_k]_{q,\lambda_1}}.$$

Consider a sequence $\{a_u\}_{u \in \mathbb{Z}_+}$

$$a_u = E_i \psi_k^u = \left(\sum_{j=0}^{u-1} A_k^j (E_i \psi_k) \right) \psi_k^{u-1}, \qquad u \in \mathbb{Z}_+,$$

where $A_k : \mathbb{C}[\Omega_k]_q \to \mathbb{C}[\Omega_k]_q$ and $A_k|_{\mathbb{C}[\Omega_k]_{q,\lambda}} = A_{k,\lambda}(q)$.

As in the previous section, show that $\{E_i \psi_k^u\}_u \in L$.

It is clear that $E_i \psi_k$ belongs to $V = \oplus_{\lambda \in \mathcal{M}} \mathbb{C}[\Omega_k]_{q,\lambda}$, with $\mathcal{M} \subset P_+^{\mathcal{S}}$ being a finite set. Consider the restriction A_k to V, $A_k V \subset V$. Using the Jordan canonical form of $A_k|_V$ and the equality

$$E_i \psi_k^u = \left(\sum_{j=0}^{u-1} A_k^j (E_i \psi_k) \right) \psi_k^{u-1}, \qquad u \in \mathbb{Z}_+,$$

one has $\{E_i \psi_k^u\}_u \in L$.

So one can extend the operator valued function $\pi_{k,u}(E_i)$ to the complex plane, since the eigenvalues of $A_{k,\lambda}(q)$ are positive (Lem. 3). Similarly, one can extend the operator valued function $\pi_{k,u}(K_i F_i)$. The extensions of the operator valued functions $\pi_{k,u}(K_i^{\pm 1})$, $i = 1, 2, \ldots, l$, exist by obvious reasons. At last,

$$\pi_{k,u}(F_i) = \pi_{k,u}(K_i^{-1})\, \pi_{k,u}(K_i F_i), \qquad i = 1, 2, \ldots, l.$$

Now we have to check whether the map

$$E_i \mapsto \pi_{k,u}(E_i), \quad F_i \mapsto \pi_{k,u}(F_i), \quad K_i \mapsto \pi_{k,u}(K_i)$$

can be extended to an algebra homomorphism.

Introduce an auxiliary algebra of analytic functions $F(u;q)$ on $\mathbb{C} \times (0,1)$ that take values in the space of linear operators in $\mathbb{C}[\Omega_k]_q = \bigoplus_{\lambda \in P_+^{\mathcal{S}}} \mathbb{C}[\Omega_k]_{q,\lambda}$. In other words, we assume the analyticity of all operator valued functions $P_{\lambda_2} F(u;q)_{|\mathbb{C}[\Omega_k]_{q,\lambda_1}}$, where $\lambda_1, \lambda_2 \in P_+^{\mathcal{S}}$.

The vector bundle $\mathcal{F}_\lambda$ is equipped with a Hermitian metric, so the operator norm $\|F(u;q)\|$ is well-defined. Consider a subalgebra of operator valued functions satisfying the following condition:

$$\|F(u;q)\| \le a_F(q) \exp(b_F(q)|u|), \tag{6}$$

for some $a_F(q) > 0$ and $b_F(q) > 0$ such that $\lim_{q \to 1} b_F(q) = 0$. Note that the subalgebra does not depend on the choice of metrics.

The operator valued functions

$$\pi_{k,u}(E_i),\ \pi_{k,u}(F_i),\ \pi_{k,u}(K_i^{\pm 1}), \qquad i = 1, 2, \ldots, l,$$

are analytic and satisfy (6). Prove it. Consider $A_k|_V$, where V is the finite dimensional $U_q\mathfrak{k}$-invariant subspace and $E_i\psi_k \in V$. One has

$$\|(A_k|_V)^u(E_i\psi_k)\| = \|\sum_{i,j\in\mathbb{Z}_+} a_{ij}\lambda_i^u u^j\|.$$

The number of terms in the r.h.s. of expression is at most $(\dim V)^2$ (obviously, $0 \le i,j \le \dim V - 1$). Hence,

$$\|(A_k|_V)^u(E_i\psi_k)\| = \|\sum_{i,j\in\mathbb{Z}_+} a_{ij}\lambda_i^u u^j\| \le (\dim V)^2 \max\|a_{ij}\| \max|\lambda_i|^u u^j.$$

Proposition 6. *([2]). Let $f(z)$ be continuous in $\{z \in \mathbb{C}|\operatorname{Re} z \ge 0\}$ and holomorphic in $\{z \in \mathbb{C}|\operatorname{Re} z > 0\}$. Assume also that*

1. $|f(iy)| \le \mathrm{const}\cdot\exp\{(\pi-\varepsilon)|y|\}, \quad y \in \mathbb{R},$
2. $|f(z)| \le M\exp(a|z|), \qquad \operatorname{Re} z \ge 0,$

for some real M, a and $\varepsilon > 0$.

If $f(n) = 0$ for all $n \in \mathbb{N}$, then $f(z) \equiv 0$.

Corollary 2. *The extension of $\pi_{k,u}(E_i), \pi_{k,u}(F_i), \pi_{k,u}(K_i)$ is unique.*

Using (6), one can prove easily that the Drinfeld–Jimbo relations hold for these operator valued functions. So $\pi_{k,u}$ is a representation. It is a q-analog of a representation of degenerate spherical principal series of Harish–Chandra modules.

Now turn back to the nondegenerate spherical principal series. For $\mathbf{u} = (u_1, u_2, \ldots, u_r) \in \mathbb{Z}^r$ define (Cf. (5))

$$\pi_{\mathbf{u}}(\xi)f \stackrel{\mathrm{def}}{=} \xi\left(f\prod_{j=1}^r \psi_j^{u_j}\right)\cdot\prod_{j=1}^r \psi_j^{-u_j}, \qquad \xi \in U_q\mathfrak{g}, \quad f \in \mathbb{C}[\Omega]_q.$$

We describe the extension of $\pi_{\mathbf{u}}$ to $\mathbb{C}^r$. As before, consider the decomposition of $\mathbb{C}[\Omega]_q$ into a sum of its $U_q\mathfrak{k}$-isotypic components

$$\mathbb{C}[\Omega]_q = \bigoplus_{\lambda\in P_+^{\mathcal{S}}} \mathbb{C}[\Omega]_{q,\lambda}$$

and operator valued functions

$$\mathcal{A}_{j,\lambda}(q) : \mathbb{C}[\Omega]_{q,\lambda} \to \mathbb{C}[\Omega]_{q,\lambda}, \qquad \mathcal{A}_{j,\lambda}(q) : f \mapsto \psi_j f \psi_j^{-1}.$$

The construction of

$$\pi_{\mathbf{u}}(E_i), \quad \pi_{\mathbf{u}}(F_i), \quad \pi_{\mathbf{u}}(K_i^{\pm 1}), \qquad i = 1, 2, \ldots, l, \tag{7}$$

essentially reduces to an analytic continuation of the vector-valued functions

$$\sum_{j=1}^{u_k - 1} \mathcal{A}_{k,\lambda}^{j}(E_i \psi_k), \qquad k = 1, 2, \ldots, r.$$

The Drinfeld–Jimbo relations for the operator valued functions (7) can be proved in the same way. Namely, consider an algebra of analytic operator valued functions $F(u_1, u_2, \ldots, u_r; q)$ such that

$$\|F(u_1, u_2, \ldots, u_r; q)\| \le a_{F}(q) \exp(b_{F}(q) \sum_{k=1}^{r} |u_k|)$$

(Cf. (6)). Now one can prove the uniqueness of the interpolation of $\pi_{\mathbf{u}}$ in this subalgebra.

5. Appendix

We present some auxiliary statements on certain vector bundles over $(0, 1]$. Start with the well-known facts on Verma modules. Let $U_q\mathfrak{b}^+$ be a Hopf subalgebra generated by $E_i, K_i^{\pm 1}$. Let $\lambda \in P_+$, and $\mathbb{C}_\lambda$ be a one dimensional $U_q\mathfrak{b}^+$-module defined by its generator 1_λ and the relations

$$K_i^{\pm 1} 1_\lambda = q_i^{\pm \lambda_i} 1_\lambda, \qquad E_i 1_\lambda = 0, \qquad i = 1, 2, \ldots, l.$$

As usual, a Verma module over $U_q\mathfrak{g}$ can be defined as follows

$$M(\lambda)_q \stackrel{\mathrm{def}}{=} U_q\mathfrak{g} \otimes_{U_q\mathfrak{b}^+} \mathbb{C}_\lambda .$$

Fix $v_\lambda = 1 \otimes 1_\lambda$. It is known that v_λ is a generator, and $M(\lambda)_q$ can be defined by the relations

$$E_i v_\lambda = 0, \qquad K_i^{\pm 1} v_\lambda = q_i^{\pm \lambda_i} v_\lambda, \qquad i = 1, 2, \ldots, l.$$

Recall that the Weyl group W acts on the root system R of Lie algebra $\mathfrak{g}$ and is generated by simple reflections $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Fix the reduced expression of the longest element $w_0 = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_M} \in W$. One can associate to it a total order on the set of positive roots of $\mathfrak{g}$, and then a basis in the vector space $U_q\mathfrak{g}$. We use the following total order on the set of positive roots:

$$\beta_1 = \alpha_1, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \quad \ldots \quad \beta_M = s_{i_1} \ldots s_{i_{M-1}}(\alpha_{i_M}).$$

Following G. Lusztig [12, 15, 1], introduce the elements $E_{\beta_s}, F_{\beta_s} \in U_q\mathfrak{g}$ for $s = 1, \ldots, M$. As a direct consequence of definitions, E_{β_s}, (resp. F_{β_s}), is a linear combination of $E_{j_1}^{m_1} \cdot \ldots \cdot E_{j_l}^{m_l}$ (resp. $F_{i_1}^{n_1} \cdot \ldots \cdot F_{i_k}^{n_k}$) with the coefficients in the expansion being rational functions of q without poles in $(0, 1]$.

Proposition 7. *The set* $\{F_{\beta_M}^{j_M} \cdot F_{\beta_{M-1}}^{j_{M-1}} \cdot \ldots \cdot F_{\beta_1}^{j_1} \cdot K_1^{i_1} \cdot K_2^{i_2} \cdot \ldots \cdot K_l^{i_l} \cdot E_{\beta_1}^{j_1} \cdot E_{\beta_2}^{j_2} \cdot \ldots \cdot E_{\beta_M}^{j_M} | k_1, k_2, \ldots, k_M, j_1, j_2, \ldots, j_M \in \mathbb{Z}_+, i_1, i_2, \ldots, i_l \in \mathbb{Z}\}$ *is a basis in the vector space* $U_q\mathfrak{g}$.

Hence, the weight vectors

$$v_J(\lambda) = F_{\beta_M}^{j_M} F_{\beta_{M-1}}^{j_{M-1}} \ldots F_{\beta_1}^{j_1} v_\lambda, \qquad j_1, j_2, \ldots, j_M \in \mathbb{Z}_+, \tag{8}$$

form a basis of $M(\lambda)_q$.

Equip $U_q\mathfrak{g}$ with a $*$-Hopf algebra structure as follows:

$$(K_j^{\pm 1})^\star = K_j^{\pm 1}, \quad E_j^\star = K_j F_j, \quad F_j^\star = E_j K_j^{-1}, \quad j = 1, 2, \ldots, l.$$

Lemma 4. *There exists a unique Hermitian form in* $M(\lambda)_q$ *such that:*

- $(\xi v', v'') = (v', \xi^\star v''), \qquad v', v'' \in M(\lambda)_q, \quad \xi \in U_q\mathfrak{g}$;
- $(v_\lambda, v_\lambda) = 1$.

The kernel $K(\lambda)_q$ of the form $(\cdot, \cdot)$ is the largest proper submodule of $M(\lambda)_q$.

In this section we write $L(\lambda)_q$ instead of $L(\lambda)$ to make the dependence on q explicit.

Proposition 8. *1.* $L(\lambda)_q \simeq M(\lambda)_q / K(\lambda)_q$.

2. The form $(\cdot, \cdot)$ *is nondegenerate in* $L(\lambda)_q$.

Introduce a morphism $p_\lambda : M(\lambda)_q \to L(\lambda)_q$, $v_\lambda \mapsto v(\lambda)$.

The first statement on special vector bundles over $(0, 1]$ is as follows. Let $\lambda \in P_+$. There exists the continuous over $(0, 1]$ and analytic in $(0, 1)$ vector bundle $\mathcal{E}_\lambda$ with fibers isomorphic to $L(\lambda)_q$. Of course, E_i, F_i, H_i, $i = 1, 2, \ldots, l$, are the endomorphisms of $\mathcal{E}_\lambda$.*

Describe the construction of $\mathcal{E}_\lambda$. Recall that to any reduced expression of w_0 we assign the basis of $M(\lambda)_q$ (see (8)).

Fix $q_0 \in (0, 1]$. Choose a subset $\{v_j\}_{j=1,\ldots,\dim L(\lambda)}$ of one of the mentioned bases in such a way that the matrix $((v_i, v_j))_{i,j=1,\ldots,\dim L(\lambda)}$ is nondegenerate. It is nondegenerate for all q that are close enough to q_0. Hence, in a neighborhood

*I.e. E_i, F_i, H_i correspond to continuous in $(0, 1]$ and analytic in $(0, 1)$ operator valued functions.

of q_0, the set $\{p_\lambda(v_j)\}_{j=1,\dots,\dim L(\lambda)}$ is a basis, since $\dim L(\lambda)_q$ does not depend on $q \in (0,1]$. One gets a trivial vector bundle with the required properties over the neighborhood of q_0.

The elements of the matrix $((v_i, v_j))$ are the continuous in $(0,1]$ and analytic in $(0,1)$ functions. Therefore, the matrices of E_i, F_i, H_i in the basis $\{p_\lambda(v_j)\}_{j=1,\dots,\dim L(\lambda)}$ are continuous in $(0,1]$ and analytic in $(0,1)$. Indeed, any function

$$(p_\lambda(E_{j_1}^{m_1} \cdot \ldots \cdot E_{j_l}^{m_l} F_{i_1}^{n_1} \cdot \ldots \cdot F_{i_k}^{n_k} v_\lambda), p_\lambda(v_\lambda))$$

is continuous in $(0,1]$ and analytic in $(0,1)$, since to calculate the value one should just use the commutation relations, which are "well-dependent" on q. Therefore, the functions

$$(E_i p_\lambda(F_{i_1}^{n_1} \cdot \ldots \cdot F_{i_k}^{n_k} v_\lambda), p_\lambda(F_{j_1}^{m_1} \cdot \ldots \cdot F_{j_k}^{m_k} v_\lambda))$$

and

$$(E_i p_\lambda(F_{\beta_1}^{n_1} \cdot \ldots \cdot F_{\beta_M}^{n_M} v_\lambda), p_\lambda(F_{\beta_1}^{m_1} \cdot \ldots \cdot F_{\beta_M}^{m_M} v_\lambda))$$

are "well-dependent" on q. Hence, the matrix elements of the operator E_i are "well-dependent" on q, since the matrix is nondegenerate in the neighborhood of q_0. The same holds for F_i, H_i, and the transition matrices defined on intersection of the neighborhoods. Finally, the vector bundle over $(0,1]$ which we obtain in this way does not depend on the choices made above. So, the vector bundle $\mathcal{E}_\lambda$ is constructed.

Now proceed to the construction of a subbundle of $\mathcal{E}_\lambda$ corresponding to the fixed $U_q\mathfrak{k}$-type μ. Consider the decomposition of $L(\lambda)_q = \bigoplus_\mu L(\lambda)_{q,\mu}$ into a sum of its $U_q\mathfrak{k}$-isotypic components. Then one has the following statement on special vector bundles. For any $\lambda \in P_+$ and $\mu \in P_+^{\mathcal{S}}$, $L(\lambda)_{q,\mu}$ is a fiber of a continuous vector bundle over $(0,1]$, analytic in $(0,1)$.

Fix $\mu \in P_+^{\mathcal{S}}$ and consider $\mathcal{E}_\lambda^\mu = \{(f,q) | f \in L(\lambda)_{q,\mu}\}$. Prove that it defines a subbundle of $\mathcal{E}_\lambda$. Indeed, consider a fiber $L(\lambda)_q$ together with its decomposition $L(\lambda)_q = \bigoplus_{\nu \in P_+^{\mathcal{S}}} L(\lambda)_{q,\nu}$. Note that the sum consists of finite number of terms. Hence, there exists $c_q \in Z(U_q\mathfrak{k})$, that is polynomial in q, and

$$c_q|_{L(\lambda)_{q,\mu}} = 1, \qquad c_q|_{L(\lambda)_{q,\nu}} = 0, \quad \nu \neq \mu,$$

see [4, p. 125–126]. Hence c_q defines the morphism of vector bundles

$$\begin{tikzcd} \mathcal{E}_\lambda \arrow[r, "c_q"] \arrow[dr, "j"'] & \mathcal{E}_\lambda \arrow[d] \\ & (0,1] \end{tikzcd} .$$

It is an orthogonal projection onto $L(\lambda)_{q,\mu}$ in any fiber $L(\lambda)_q$. Since $\operatorname{rank} \mathrm{c_q}$ is constant, the image of c_q is a vector subbundle.

Now we can construct the last required vector bundle. $\mathbb{C}[\Omega_k]_{q,\lambda}$ are the fibers of a continuous vector bundle $\mathcal{F}_\lambda$ over $(0,1]$, analytic in $(0,1)$.

Consider a map

$$\Psi_q : \mathbb{C}[\widehat{X}_k]_{q,\Psi_k} \to \mathbb{C}[\widehat{X}_k]_{q,\Psi_k}, \qquad f \mapsto \psi_k f.$$

It is easy to prove that Ψ_q is an invertible, continuous operator valued function, analytic in $(0,1)$. Using Ψ_q one can carry the vector bundle structure to $\mathcal{F}_\lambda$ from $\{(f,q) | f \in \mathbb{C}[\Omega_k]_{q,\lambda}\psi_k^N, q \in (0,1]\}$ for large enough N.

R e m a r k 2. Note that the matrix elements of all operator valued functions belong to $\mathbb{Q}(q^{1/s})$ with $s = \mathrm{card(P/Q)}$, and Q being the root lattice.

The next considerations are related to the self-adjointness. Consider an auxiliary algebra

$$\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q = \bigoplus_{\lambda \in \Lambda_+} L(\lambda)_q.$$

Equip $U_q\mathfrak{g}$ and $\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q$ with a "complex conjugation". Recall that $\dim L(\lambda)_q^{U_q\mathfrak{k}} = 1$ for any $\lambda \in \Lambda_+$. Consider the antilinear involutive automorphism $\bar{\cdot}$ of $U_q\mathfrak{g}$ defined by

$$\bar{E}_i = E_i, \ \ \bar{F}_i = F_i, \ \ \bar{K}_i^{\pm 1} = K_i^{\pm 1}, \qquad i = 1, 2, \ldots, l.$$

There exists a unique antilinear involutive operator: $\bar{\cdot}$

$$L(\lambda)_q \to L(\lambda)_q, \qquad \xi v(\lambda) \mapsto \bar{\xi} v(\lambda), \qquad \xi \in U_q\mathfrak{g}.$$

(Indeed, the uniqueness is obvious, while the existence follows from the definition of $L(\lambda)_q$.)

It is easy to see that $\overline{L(\lambda)_q^{U_q\mathfrak{k}}} = L(\lambda)_q^{U_q\mathfrak{k}}$. Hence, there exists a nonzero vector $w_\lambda \in L(\lambda)_q^{U_q\mathfrak{k}}$ such that $w_\lambda = \bar{w_\lambda}$. Let $l(\lambda) = \mathbb{R}w_\lambda$.

Lemma 5. *There exists a unique involution $\star$ of $\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q$ such that $\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q$ is a $(U_q\mathfrak{g}, \star)$-module algebra, and $\star|_{l(\lambda)} = id$.*

P r o o f. Firstly prove that

$$L(\lambda)_q^* \approx L(\lambda)_q, \qquad \lambda \in \Lambda_+. \tag{9}$$

Following [20], introduce a system of strongly orthogonal roots $\gamma_1 > \gamma_2 > \ldots > \gamma_r$ with γ_1 being the maximal root. It is easy to prove that $-w_0(\gamma_j) = \gamma_j$

for $j = 1, 2, \ldots, r$, where $w_0 \in W$ is the longest element. Hence, (9) follows from the fact that the fundamental spherical weights belong to the linear span of $\gamma_1, \gamma_2, \ldots, \gamma_r$ (see [3]).

An involution $\star$ on $L(\lambda)_q$ such that

$$(\xi f)^\star = (S(\xi))^\star f^\star, \qquad \xi \in U_q\mathfrak{g}, \quad f \in L(\lambda)_q,$$

is unique up to ± 1. The uniqueness of the involution now follows from the fact that $w_\lambda = \bar{w_\lambda}$.

Turn to the proof of existence of $\star$. $U_q\mathfrak{g}$ is equipped with the involution as follows:

$$(K_j^{\pm 1})^\star = K_j^{\pm 1}, \quad E_j^\star = K_j F_j, \quad F_j^\star = E_j K_j^{-1}, \quad j = 1, 2, \ldots, l.$$

Consider the $*$-algebra $(\mathbb{C}[G]_q, \star)$ with the involution $\star$ given by

$$f^\star(\xi) \stackrel{\mathrm{def}}{=} \overline{f((S(\xi))^\star)}, \quad \xi \in U_q\mathfrak{g}, \quad f \in \mathbb{C}[G]_q.$$

Let

$$F = \{f \in \mathbb{C}[G]_q \mid L_{\mathrm{reg}}(\xi) f = \varepsilon(\xi) f, \quad \xi \in U_q\mathfrak{k}\}.$$

It follows from the Peter–Weyl expansion that $F \approx \oplus_{\lambda \in \Lambda_+} L(\lambda)_q$ as a $U_q\mathfrak{g}$-module. One can consider Λ_+ with a natural partial order $\leq$, and F can be equipped with a $(U_q\mathfrak{g}, \star)$-invariant filtration $F = \bigcup_{\lambda \in \Lambda_+} F_\lambda$, $F_\lambda = \bigoplus_{\mu \leq \lambda} L(\mu)_q$.

Consider the zonal spherical functions ρ_λ related to the $U_q\mathfrak{g}$-modules $L(\lambda)_q$. In the associated $\mathbb{Z}^r$-graded algebra $\mathrm{Gr}F$ their images pairwise commute. Indeed, the quasicommutativity is evident, while the commutativity follows from the self-adjointness of ρ_λ and the commutativity in the classical case. Now, it follows from [22, Cor. 2.6] that $\mathrm{Gr}F$ is naturally isomorphic to $\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q$. Note that $l(\lambda)$ corresponds to $\mathbb{R}\rho_\lambda$. Carry the involution $\star$ from F to $\mathrm{Gr}F$ and $\mathbb{C}[\widehat{X}^{\mathrm{spher}}]_q$. It can be verified easily that $\star$ is a morphism of the vector bundle with fibers $L(\lambda)_q$. ■

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