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Some Comparison Theorems in Finsler–Hadamard Manifolds

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We give upper and lower bounds for the ratio of the volume of metric ball to the area of metric sphere in Finsler–Hadamard manifolds with the pinched S-curvature. We apply these estimates to find the limit at infinity for this ratio. The estimates derived are the generalization of the result well known in Riemannian geometry. We also estimate the volume growth entropy for the balls in these manifolds.

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1. Introduction

Finsler geometry is an important generalization of Riemannian geometry. It was introduced by P. Finsler in 1918 from the point of view of regular problems in the calculus of variations. In Finsler geometry the metric is not to be quadratic on tangent spaces, thus the structure of Finsler spaces is much more complicated than the structure of Riemannian spaces. But many notions and theorems were generalized to Finsler geometry from Riemannian geometry.

In [1, 2] the following result was proved.

Theorem 1. *Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold with the sectional curvature K such that $-k_2^2 \leqslant K \leqslant -k_1^2$, $k_1, k_2 > 0$. Let Ω be a compact λ-convex domain in M^{n+1} (i.e., the domain, whose boundary is a regular hypersurface with all normal curvatures that are greater or equal λ) with $\lambda \leqslant k_2$. Then there exist the functions $\alpha(r)$ of the inradius and $\beta(R)$ of the circumradius such that $\alpha(r) \to 1/(nk_1)$ and $\beta(R) \to 1/(nk_2)$, as well as r and R, go to infinity and that*

$$\alpha(r)\frac{\lambda}{k_2} \leqslant \frac{Vol(\Omega)}{Vol(\partial\Omega)} \leqslant \beta(R).$$

As a consequence, for a family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ *of compact* λ*-convex domains with* $\lambda \leqslant k_2$ *expanding over the whole space we obtain*

$$\frac{\lambda}{nk_2^2} \leqslant \lim_{t\to\infty}\inf \frac{Vol(\Omega(t))}{Vol(\partial\Omega(t))} \leqslant \lim_{t\to\infty}\sup \frac{Vol(\Omega(t))}{Vol(\partial\Omega(t))} \leqslant \frac{1}{nk_1}.$$

Our goal is to generalize this theorem for Finsler manifolds. We consider metric balls as the family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$. We shall also need bounds for one of non-Riemannian curvatures, namely S-curvature. As a result we prove the following theorem.

Theorem 2. *Let* (M^{n+1}, F) *be an* $(n+1)$*-dimensional Finsler–Hadamard manifold that satisfies the following conditions:*

1. *Flag curvature satisfies the inequalities* $-k_2^2 \leqslant K \leqslant -k_1^2$, $k_1, k_2 > 0$.
2. *S-curvature satisfies the inequalities* $n\delta_1 \leqslant S \leqslant n\delta_2$ *such that* $\delta_i < k_i$.

Let $B_r^{n+1}(p)$ *be the metric ball of radius* r *in* M^{n+1} *with the center at point* $p \in M^{n+1}$, $S_r^n(p) = \partial B_r^{n+1}(p)$ *be the metric sphere. Let* $Vol = \int dV_F$ *be the measure of Busemann–Hausdorff,* $Area = \int dA_F$ *the induced measure on* $S_r^n(p)$*. Then there exist functions* $f(r)$ *and* $\mathcal{F}(r)$ *such that* $f(r) \to 1/(n(k_2-\delta_2))$ *and* $\mathcal{F}(r) \to 1/(n(k_1-\delta_1))$ *as* r *goes to infinity and that*

$$f(r) \leqslant \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \mathcal{F}(r).$$

Here

$$f(r) = \frac{1}{(1-e^{-2k_2r})^n}\left(\frac{1}{n(k_2-\delta_2)} - \frac{n}{n(k_2-\delta_2)-2k_2}(e^{-2k_2r} - e^{-nr(k_2-\delta_2)})\right)$$

$$\mathcal{F}(r) = \frac{1}{n(k_1-\delta_1)}(1-e^{-nr(k_1-\delta_1)}).$$

As a consequence, for a family $\{B_r^{n+1}(p)\}_{r\geqslant 0}$ *we have*

$$\frac{1}{n(k_2-\delta_2)} \leqslant \lim_{r\to\infty}\inf \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \lim_{r\to\infty}\sup \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \frac{1}{n(k_1-\delta_1)}.$$

If (M^{n+1}, F) *is a space of constant flag curvature* $K = -k^2$ *and S-curvature* $S = n\delta$, $\delta < k$, *we have*

$$\lim_{r\to\infty} \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} = \frac{1}{n(k-\delta)}$$

For a Riemannian space $S = 0$ and thus Th. 2 turns to be a special case of Th. 1.

In Section 4 we give the estimates for the volume growth entropy of the balls.

2. Preliminaries

In this section we recall some basic facts and theorems from Finsler geometry that we need. See [3–5] for details.

2.1. Finsler Metrics. By definition, a Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. A *Minkowski norm* on a vector space V^n is a nonnegative function $F : V^n \to [0, \infty)$ with the following properties:

1. F is positively homogeneous of degree one, i.e., for any $y \in V^n$ and any $\lambda > 0$, $F(\lambda y) = \lambda F(y)$.

2. F is C^∞ on $V^n \backslash \{0\}$ and for any vector $y \in V^n$ the following bilinear symmetric form $g_y : V^n \times V^n \to \mathbb{R}$ is positively definite,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} [F^2(y + su + tv)]|_{s=t=0}.$$

Property 2 is also called a *strong convexity property.*

A Minkowski norm is said to be *reversible* if $F(y) = F(-y)$, $y \in V^n$. In this paper, Minkowski norms are not assumed to be reversible.

By 1. and 2., one can show that $F(y) > 0$ for $y \neq 0$ and $F(u + v) \leqslant F(u) + F(v)$. See [4] for a proof.

A vector space V^n with the Minkowski norm is called a *Minkowski space.* Notice that reversible Minkowski spaces are finite-dimensional Banach spaces.

Let (V^n, F) be the Minkowski space. Then the set $I = F^{-1}(1)$ is called the *indicatrix* in the Minkowski space. It is also called the *unit sphere.*

A set $U \subset V^n$ is said to be *strongly convex* if there exists a function F satisfying 2. such that $\partial U = F^{-1}(1)$. Remark that a strong convexity is equivalent to a positivity of all normal curvatures of ∂U for any Euclidean metric on V^n.

Let M^n be an n-dimensional connected C^∞-manifold. Denote by $TM^n = \bigsqcup_{x \in M^n} T_x M^n$ the tangent bundle of M^n, where $T_x M^n$ is the tangent space at x. A *Finsler metric* on M^n is a function $F : TM^n \to [0, \infty)$ with the following properties:

1. F is C^∞ on $TM^n \backslash \{0\}$.

2. At each point $x \in M^n$, the restriction $F|_{T_x M^n}$ is a Minkowski norm on $T_x M^n$.

The pair (M^n, F) is called *a Finsler manifold.*

Let (M^n, F) be a Finsler manifold. Let (x^i, y^i) be a standard local coordinate system in TM^n, i.e., y^i are determined by $y = y^i \frac{\partial}{\partial x^i}|_x$. For a non-zero vector $y = y^i \frac{\partial}{\partial x^i}$, put $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$. The induced inner product g_y is given by

$$g_y(u, v) = g_{ij}(x, y) u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$.

By the homogeneity of F, we have $F(x,y) = \sqrt{g_y(y,y)} = \sqrt{g_{ij}(x,y)y^i y^j}$.

In the Riemannian case g_{ij} are the functions of $x \in M^n$ only, and in the Minkowski case g_{ij} are the functions of $y \in T_xM^n = V^n$ only.

2.2. Measuring of Area. The notions of length and area are also generalized to Finsler geometry.

Given a Finsler metric F on a manifold M^n.

Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM^n and $\{\theta^i\}_{i=1}^n$ a dual basis for $T_x^*M^n$. The set

$$B_x^n = \big\{(y^i) \in \mathbb{R}^n : F(x, y^i e_i) < 1\big\}$$

is an open strongly convex subset in $\mathbb{R}^n$, bounded by the indicatrix in T_xM^n. Then define

$$dV_F = \sigma_F(x)\theta^1 \wedge \ldots \wedge \theta^n,$$

where

$$\sigma_F(x) := \frac{Vol_E(\mathbb{B}^n)}{Vol_E(B_x^n)}.$$

Here $Vol_E(A)$ denotes the Euclidean volume of A, and $\mathbb{B}^n$ is the standard unit ball in $\mathbb{R}^n$.

The volume form dV_F determines a regular measure $Vol_F = \int dV_F$ and is called the *Busemann–Hausdorff volume form.*

For any Riemannian metric $g_{ij}(x)u^i v^j$ the Busemann–Hausdorff volume form is the standard Riemannian volume form

$$dV_g = \sqrt{\det(g_{ij})}\theta^1 \wedge \ldots \wedge \theta^n.$$

Let $\varphi : N^{n-1} \to M^n$ be a hypersurface in (M^n, F).

The Finsler metric F determines a local normal vector field as follows. A vector n_x is called the *normal vector* to N^{n-1} at $x \in N^{n-1}$ if $n_x \in T_{\varphi(x)}M^n$ and $g_{n_x}(y, n_x) = 0$ for all $y \in T_xN^{n-1}$. It was proved in [4] that such vector exists. Notice that in general nonsymmetric case the vector $-n_x$ is not a normal vector.

Define now an induced volume form on N^{n-1}. Let n be a unit normal vector field along N^{n-1}. Let $\overline{F} = \varphi^* F$ be the induced Finsler metric on N^{n-1} and $dV_{\overline{F}}$ be the Busemann–Hausdorff volume form of $\overline{F}$. For $x \in N^{n-1}$ we define

$$\zeta(x, n_x) := \frac{Vol_E(\mathbb{B}^n)}{Vol_E(B_x^n)} \frac{Vol_E(B_x^{n-1}(n_x))}{Vol_E(\mathbb{B}^{n-1})}.$$

Here $B_x^n = \big\{(y^i) \in \mathbb{R}^n : F(y^i e_i) < 1\big\}$. To define $B_x^{n-1}(n_x)$ we take a basis $\{e_i\}_{i=1}^n$ for $T_{\varphi(x)}M^n$ such that $e_1 = n_x$ and $\{e_i\}_{i=2}^n$ is a basis for T_xN^{n-1}. Then $B_x^{n-1}(n_x)$ $= \big\{(y^j) \in \mathbb{R}^{n-1} : F(y^j e_j) < 1\big\}$, where the index j passes from 2 to n.

Note that if F is a Riemannian metric, then $\zeta \equiv 1$.

Set

$$dA_F := \zeta(x, n_x) dV_{\overline{F}}.$$

The form dA_F is called the *induced volume form* of dV_F with respect to n [4].

The sense of defining such volume form is given by the *co-area formula* [4]. We shall need the co-area formula in one simple case for metric balls:

$$Vol(B(r,p)) = \int_0^r Vol(S(t,p))dt. \tag{1}$$

Here $Vol(S(t,p))$ is the induced volume on $S(t,p)$.

2.3. Geodesics, Connections and Curvature. Locally minimizing curves in a Finsler space are determined by a system of second order differential equations (geodesic equations).

Let (M^n, F) be a Finsler space, and $c : [a,b] \to M^n$ be a constant speed piecewise C^∞ curve $F(c, \dot{c}) = const$. Denote the local functions $G^i(x,y)$ by

$$G^i(x,y) = \frac{1}{4} g^{il}(x,y) \left\{ 2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y) \right\} y^j y^k.$$

We call $G^i(x,y)$ the *geodesic coefficients* [4]. Notice that in Riemannian case $G^i(x,y) = \frac{1}{2}\Gamma^j_{ik}(x) y^i y^k$.

Consider the functions $N^i_j(x,y) = \frac{\partial G^i}{\partial y^j}(x,y)$. They are called the *connection coefficients*. At each point $x \in M^n$, define a mapping

$$D : T_x M^n \times C^\infty(TM^n) \to T_x M^n$$

by

$$D_y U := \{dU^i(y) + U^j N^i_j(x,y)\} \frac{\partial}{\partial x^i}|_x,$$

where $y \in T_x M^n$ and $U \in C^\infty(TM^n)$. We call $D_y U(x)$ the *covariant derivative* of U at x in the direction y.

If c is a solution of the system $D_{\dot{c}}\dot{c} = 0$, then it is called *geodesic*.

Next, we introduce a notion of curvature in Finsler geometry. At first, we consider the generalization of Riemann curvature. In 1926, L. Berwald extended the Riemann curvature to Finsler metrics.

Let (M^n, F) be a Finsler space. For a vector $y \in T_x M^n \backslash \{0\}$ consider the functions

$$R^k_i(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For every vector $y \in T_xM^n \backslash \{0\}$, define a linear transformation

$$R_y = R_i^k(y) \frac{\partial}{\partial x^i} \otimes dx^k|_x.$$

Then the family of transformations

$$R = \{R_y : T_xM^n \to T_xM^n, y \in T_xM^n \backslash \{0\}, x \in M^n\}$$

is called the *Riemann curvature* [4].

Let $P \subset T_xM^n$ be a tangent plane. For a vector $y \in P \backslash \{0\}$, define

$$K(P, y) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)^2},$$

where $u \in P$ such that $P = span\{y, u\}$. $K(P, y)$ is independent of $u \in P$. The number $K(P, y)$ is called a *flag curvature* of the flag (P, y) in T_xM^n.

The flag curvature is a generalization of the sectional curvature in Riemannian geometry. It can be defined in another way. For a vector $y \in T_xM^n \backslash \{0\}$ consider the Riemannian metric $\hat{g}(u, v) = g_Y(u, v)$. Here the vector field Y is an arbitrary extension of the vector y. Then the flag curvature $K(P, y)$ of the flag (P, y) in the Finsler metric F is equal to the sectional curvature of the plane P in the metric $\hat{g}(u, v)$. If we change y, then $\hat{g}(u, v)$ and $K(P, y)$ will also change [3].

Define the *Ricci curvature* by

$$Ric(y) = \sum_{i=1}^{n} R_i^i(y).$$

A simply-connected Finsler space with nonpositive flag curvature is called a *Finsler–Hadamard space*. In these spaces the generalization of Cartan–Hadamard's theorem holds [6].

The notions of exponential map, completeness, cut-locus, conjugate and focal points in Finsler geometry are defined in the same way as in Riemannian geometry. For details, see [4].

Finally, we introduce some more functions which are called *non-Riemannian curvatures*. These curvatures all vanish for Riemannian spaces. We shall need only one of these curvatures, which is closely connected with the volume form.

Let (M^n, F) be a Finsler space. Consider the Busemann–Hausdorff volume form dV_F with the density σ_F. We define

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)}, \ y \in T_xM^n.$$

τ is called the *distortion* of (M^n, F). The condition $\tau \equiv const$ implies that F is a Riemannian metric [4].

To measure the rate of changes of distortion along geodesics, we define

$$S(x,y) = \frac{d}{dt}\left[\tau(c(t),\dot{c}(t))\right]|_{t=0},\ y \in T_xM^n,$$

where $c(t)$ is a geodesic with $\dot{c}(0) = y$. S is called the *S-curvature*. It is also called the *mean covariation* and *mean tangent curvature*. A local formula for the S-curvature is

$$S(x,y) = N^m_m(x,y) - \frac{y^m}{\sigma_F(x)}\frac{\partial\sigma_F}{\partial x^m}(x).$$

One can easily show that $S = 0$ for any Riemannian metric.

A Finsler metric F is said to be of *constant S-curvature* δ if

$$S(x,y) = \delta F(x,y)$$

for all $y \in T_xM^n\backslash\{0\}$ and $x \in M^n$. The upper and lower bounds of S-curvature are defined in the same way.

2.4. Geometry of Hypersurfaces and Comparison Theorems. Let (M^n,F) be a Finsler manifold and $\varphi : N^{n-1} \to M^n$ be a hypersurface. Let $\overline{F} = \varphi^*F$ denote the induced Finsler metric on N^{n-1}. Let ρ be a C^∞-distance function on an open subset $U \subset M^n$ such that $\rho^{-1}(s) = N^{n-1} \cap U$ for some s. Let dV_F denote the Busemann-Hausdorff volume form of F, dA_t denote the induced volume form of $N^{n-1}_t = \rho^{-1}(t)$. Let $c(t)$ be an integral curve of $\nabla\rho$ with $c(0) \in N^{n-1}_s$. We have $\rho(c(t)) = t$, hence $c(\varepsilon) \in N^{n-1}_{s+\varepsilon}$ for small $\varepsilon > 0$. By definition, the flow ϕ_ε of $\nabla\rho$ satisfies

$$\phi_\varepsilon(c(s)) = c(s+\varepsilon).$$

$$\phi_\varepsilon : N^{n-1} \cap U = N^{n-1}_s \to N^{n-1}_{s+\varepsilon}.$$

The $(n-1)$-form $\phi^*_\varepsilon dA_{s+\varepsilon}$ is a multiply of dA_s. Thus there is a function $\Theta(x,\varepsilon)$ on N^{n-1} such that

$$\phi^*_\varepsilon dA_{s+\varepsilon}|_x = \Theta(x,\varepsilon)dA_s|_x,,\ \forall x \in N^{n-1},$$

$$\Theta(x,0) = 1,\ \forall x \in N^{n-1}.$$

Set

$$\Pi_{n_x} = \frac{\partial}{\partial\varepsilon}\left(\ln\Theta(x,\varepsilon)\right)|_{\varepsilon=0}.$$

Π_{n_x} is called the mean curvature of N^{n-1} at x with respect to $n_x := \nabla\rho_x$ [4].

We also need some estimates on the mean curvature of metric sphere. The following theorem gives these estimates. For a given real λ, put

$$s_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}},\ \lambda > 0,$$

$$s_\lambda(t) = t,\ \lambda = 0,$$

$$s_\lambda(t) = \frac{\sinh(\sqrt{-\lambda}t)}{\sqrt{-\lambda}},\ \lambda < 0.$$

Theorem 3 [4]. *Let (M^n, F) be an n-dimensional positively complete Finsler space. Let Π_t denote the mean curvature of $S(p,t)$ in the cut-domain of p with respect to the outward-pointing normal vector.*

1. *Suppose that*

$$K \leqslant \lambda,\ S \leqslant (n-1)\delta.$$

Then

$$\Pi_t \geqslant (n-1)\frac{s'_\lambda(t)}{s_\lambda(t)} - (n-1)\delta. \tag{2}$$

2. *Suppose that*

$$Ric \geqslant n\lambda,\ S \geqslant -(n-1)\delta.$$

Then

$$\Pi_t \leqslant (n-1)\frac{s'_\lambda(t)}{s_\lambda(t)} + (n-1)\delta. \tag{3}$$

Theorem 4 [4]. *Let (M^n, F) be an n-dimensional positively complete Finsler space. Suppose that for constants $\lambda \leqslant 0$ and $\delta \geqslant 0$ with $\sqrt{-\lambda} - \delta > 0$, the flag curvature and the S-curvature satisfy the inequalities*

$$K \leqslant \lambda,\ S \leqslant (n-1)\delta.$$

Then for any regular domain $\Omega \subset M^n$,

$$Vol(\Omega) \leqslant \frac{Vol(\partial\Omega)}{(n-1)(\sqrt{-\lambda} - \delta)}.$$

Remark that the right-hand asymptotic estimate in Th. 2 is proved in Th. 4.

3. Relation between Area and Volume for the Balls in Finsler–Hadamard Manifolds

In this section we prove Th. 2.

P r o o f o f T h e o r e m 2. Let S_pM^{n+1} denote the unit sphere in T_pM^{n+1}. Fix a vector $y \in S_pM^{n+1}$. Let $\{e_i\}_{i=1}^{n+1}$ be a basis for T_pM^{n+1} such that

$$e_1 = y,\ g_y(y, e_i) = 0,\ i = 2, \dots, n+1.$$

Extend $\{e_i\}_{i=1}^{n}$ to a global frame on T_pM^{n+1} in a natural way. Let $\{\theta^i\}_{i=1}^{n+1}$ denote the basis for $T_x^*M^{n+1}$ dual to $\{e_i\}_{i=1}^{n+1}$. Express dV_F at p by

$$dV_F(p) = \sigma_F(p)\theta^1 \wedge ... \wedge \theta^{n+1},$$

$$\sigma_F(p) = \frac{Vol_E(\mathbf{B}^{n+1})}{Vol_E(\{(y^i) \in \mathbb{R}^{n+1} : F(y^i e_i) < 1\})}.$$

Thus we obtain the volume form dV_p on T_pM^{n+1}. Denote by dA_p the induced volume form by dV_p on S_pM^{n+1}.

Define the diffeomorfism $\varphi_t : S_pM^{n+1} \to S_t^n(p)$ [4] by

$$\varphi_t(y) = \exp_p(ty), \; y \in S_pM^{n+1}, \; t \geqslant 0.$$

Let dA_t denote the induced volume form on $S_t^n(p)$ by dV_F. Define

$$\eta_t : S_pM^{n+1} \to [0, \infty)$$

by

$$\varphi_t^* dA_t|_{\varphi_t(y)} = \eta_t(y) dA_p|_y. \tag{4}$$

Integrating (4) over S_pM^{n+1}, we have

$$Area(S_t^n(p)) = \int\limits_{S_pM^{n+1}} \eta_t(y) dA_p.$$

Applying the co-area formula (1), we obtain

$$Vol(B_r^{n+1}(p)) = \int\limits_0^t \left(\int_{S_pM^{n+1}} \eta_s(y) dA_p \right) ds.$$

Remark that in the Riemannian case η_t is the Jacobian of the exponential map, and the explicit expression for the Jacobian gives us all the necessary estimates. Unfortunately, the integration of these estimates only leads to the "coarse" estimates for Finsler geometry.

Now, let us estimate η_t. For a small number $\varepsilon > 0$ define the flow

$$\phi_\varepsilon(x) = \varphi_{t+\varepsilon} \circ \varphi_t^{-1}(x), \; x \in S_t^n(p). \tag{5}$$

For a point $x \in S_t^n(p)$, there is an open neighborhood U of x such that ϕ_ε is defined on U. The Cartan–Hadamard theorem guarantees the non-existence of conjugate points in all M^{n+1}, i.e., the existence of metric balls of arbitrary radii.

Define $\Theta(x, \varepsilon)$ by

$$\phi_\varepsilon^* dA_{s+\varepsilon}|_x = \Theta(x, \varepsilon) dA_s|_x.$$

Using (4), (5), we get

$$\Theta(x,\varepsilon) = \frac{\eta_{t+\varepsilon}(y)}{\eta_t(y)},\ x = \varphi_t(y). \tag{6}$$

Let Π_t denote the mean curvature or $S_t^n(p)$ at x with respect to the outward-pointing normal vector. From the definition of mean curvature and (6), we have

$$\Pi_t = \frac{\partial}{\partial\varepsilon}(\ln\Theta(x,\varepsilon))\,|_{\varepsilon=0} = \frac{d}{dt}(\ln\eta_t(y)). \tag{7}$$

Define $\chi_i(t)$ by

$$\chi_i(t) = \left(e^{-\delta_i t}\frac{\sinh(k_i t)}{k_i}\right)^n.$$

Then we have

$$\frac{d}{dt}(\ln\chi_i(t)) = nk_i\coth(k_i t) - n\delta_i. \tag{8}$$

Taking into account the restrictions on curvature we can apply Th. 3. Then using (2), (3), we get

$$nk_1\coth(k_1 t) - n\delta_1 \leqslant \Pi_t \leqslant nk_2\coth(k_2 t) - n\delta_2.$$

This implies

$$\frac{d}{dt}\left(\frac{\eta_t(y)}{\chi_2(t)}\right) \leqslant 0,\ \frac{d}{dt}\left(\frac{\eta_t(y)}{\chi_1(t)}\right) \geqslant 0,$$

and

$$\eta_{t_2}(y)\chi_1(t_1) \geqslant \eta_{t_1}(y)\chi_1(t_2),$$

$$\eta_{t_2}(y)\chi_2(t_1) \leqslant \eta_{t_1}(y)\chi_2(t_2),\ 0 < t_1 \leqslant t_2.$$

Integrating over S_pM^{n+1} with respect to dA_p, we obtain

$$Area(S_{t_2}^n(p))\chi_1(t_1) \geqslant Area(S_{t_1}^n(p))\chi_1(t_2),$$

$$Area(S_{t_2}^n(p))\chi_2(t_1) \leqslant Area(S_{t_1}^n(p))\chi_2(t_2),\ 0 < t_1 \leqslant t_2.$$

Integrating from 0 to t_2 with respect to t_1, we obtain

$$Area(S_{t_2}^n(p))\int_0^{t_2}\chi_1(t)dt \geqslant Vol(B_{t_2}^{n+1}(p))\chi_1(t_2),$$

$$Area(S_{t_2}^n(p))\int_0^{t_2}\chi_2(t)dt \leqslant Vol(B_{t_2}^{n+1}(p))\chi_2(t_2),\ 0 < t_2.$$

Hence, we get

$$\frac{\chi_1(r)}{\int_0^r \chi_1(t)dt} \leqslant \frac{Area(S_r^n(p))}{Vol(B_r^{n+1}(p))} \leqslant \frac{\chi_2(r)}{\int_0^r \chi_2(t)dt},$$

or

$$\frac{\int_0^r \left(e^{-\delta_2 t}\sinh(k_2 t)\right)^n dt}{\left(e^{-\delta_2 r}\sinh(k_2 r)\right)^n} \leqslant \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \frac{\int_0^r \left(e^{-\delta_1 t}\sinh(k_1 t)\right)^n dt}{\left(e^{-\delta_1 r}\sinh(k_1 r)\right)^n}, \ r>0$$

Let us estimate these integrals.

$$\frac{\int_0^r \left(e^{-\delta_1 t}\sinh(k_1 t)\right)^n dt}{\left(e^{-\delta_1 r}\sinh(k_1 r)\right)^n} = \frac{1}{(e^{-\delta_1 r})^n}\int\limits_0^r \left(e^{-\delta_1 t}\frac{e^{k_1 t}-e^{-k_1 t}}{e^{k_1 r}-e^{-k_1 r}}\right)^n dt$$

$$\leqslant \frac{1}{(e^{-\delta_1 r})^n}\int\limits_0^r \left(e^{-\delta_1 t + k_1(t-r)}\right)^n dt = \frac{e^{n\delta_1 r}}{n(k_1-\delta_1)}\left(e^{-n\delta_1 r}-e^{-nk_1 r}\right)$$

$$= \frac{1}{n(k_1-\delta_1)}\left(1-e^{-nr(k_1-\delta_1)}\right) := \mathcal{F}(r).$$

We can estimate the following integral by using the fact that $(1-a)^n \geqslant 1-na$ for $0 \leqslant a \leqslant 1$.

$$\frac{\int_0^r \left(e^{-\delta_2 t}\sinh(k_2 t)\right)^n dt}{\left(e^{-\delta_2 r}\sinh(k_2 r)\right)^n} = \frac{e^{n\delta_2 r}}{\left(1-e^{-2k_2 r}\right)^n}\int\limits_0^r e^{-n\delta_2 t}\left(1-e^{-2k_2 t}\right)^n e^{k_2 n(t-r)}dt$$

$$\geqslant \frac{e^{n\delta_2 r}}{\left(1-e^{-2k_2 r}\right)^n}\int\limits_0^r e^{-n\delta_2 t}\left(1-ne^{-2k_2 t}\right)e^{k_2 n(t-r)}dt = \frac{e^{n\delta_2 r}}{\left(1-e^{-2k_2 r}\right)^n}$$

$$\times\left[\frac{1}{n(k_2-\delta_2)}\left(e^{-n\delta_2 r}-e^{-nk_2 r}\right) - \frac{n}{n(k_2-\delta_2)-2k_2}\left(e^{-n\delta_2 r-2k_2 r}-e^{-nk_2 r}\right)\right]$$

$$= \frac{1}{\left(1-e^{-2k_2 r}\right)^n}\left[\frac{1}{n(k_2-\delta_2)}\left(1-e^{-n(k_2-\delta_2)r}\right)\right.$$

$$\left. - \frac{n}{(k_2-\delta_2)-2k_2}\left(e^{-2k_2 r}-e^{-n(k_2-\delta_2)r}\right)\right] := f(r).$$

Thus, we have

$$f(r) \leqslant \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \mathcal{F}(r).$$

Using the inequalities $\delta_i < k_i$, we have

$$\lim_{r\to\infty} f(r) = \frac{1}{n(k_2-\delta_2)} ,$$

$$\lim_{r\to\infty} \mathcal{F}(r) = \frac{1}{n(k_1-\delta_1)} .$$

As a consequence, we have

$$\frac{1}{n(k_2-\delta_2)} \leqslant \lim_{r\to\infty} \inf \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \lim_{r\to\infty} \sup \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} \leqslant \frac{1}{n(k_1-\delta_1)} .$$

In the case when $K = -k^2$, $k > 0$, $S = n\delta$, $\delta < k$, by denoting $k_1 = k_2 = k$, $\delta_1 = \delta_2 = \delta$, we have

$$\lim_{r\to\infty} \frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} = \frac{1}{n(k-\delta)} .$$

This completes the proof. ■

E x a m p l e 1. Let U be an open bounded strongly convex domain in $\mathbb{R}^n$. Take a point $x \in U$ and a direction $y \in T_xU\backslash\{0\} \simeq U\backslash\{0\}$. Then the *Funk metric* $F(x,y)$ is a Finsler metric that satisfies the following condition

$$x + \frac{y}{F(x,y)} \in \partial U.$$

The indicatrix at each point for the Funk metric is a domain that is a translate of U.

The *Hilbert metric* is a symmetrized Funk metric:

$$\tilde{F}(x,y) := \frac{1}{2}\left(F(x,y) + F(x,-y)\right) .$$

Note that for the Funk metric $B_x^n = U$. Thus

$$\sigma_F(x) = \frac{Vol_E(\mathbb{B}^n)}{Vol_E(B_x^n)} = \frac{Vol_E(\mathbb{B}^n)}{Vol_E(U)} = const.$$

Let F be the Funk metric and let $\overline{F}$ be the Hilbert metric on a strongly convex domain U in $\mathbb{R}^n$.

Then the geodesics of Funk and Hilbert metrics are straight lines, the Funk metric is of constant flag curvature $-\frac{1}{4}$, the Hilbert metric is of constant flag curvature -1, and the Funk metric is of constant S-curvature $\frac{n+1}{2}$ [4].

Let F be the Funk metric on a strongly convex domain U in $\mathbb{R}^{n+1}$. It is known that the S-curvature is equal to $S = \frac{n+2}{2} = n\delta$, flag curvatures are equal to $-k^2 = -\frac{1}{4}$. Then the condition $\delta < k$ does not hold.

It is known that for the Funk metric

$$\frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} = \frac{\int_0^r \left(e^{-\frac{n+2}{2n}t}\sinh(\frac{t}{2})\right)^n dt}{\left(e^{-\frac{n+2}{2n}r}\sinh(\frac{r}{2})\right)^n},$$

and one can show that

$$\lim_{r\to\infty}\frac{Vol(B_r^{n+1}(p))}{Area(S_r^n(p))} = \infty.$$

Indeed, using Mathematica program, one can compute that

$$\frac{\int_0^r \left(e^{-\frac{n+2}{2n}t}\sinh(\frac{t}{2})\right)^n dt}{\left(e^{-\frac{n+2}{2n}r}\sinh(\frac{r}{2})\right)^n} = \frac{(e^r-1)}{n+1}\left(\frac{e^{-\frac{(n+1)}{n}r}(e^r-1)}{e^{-\frac{(n+r)}{n}r}(e^r-1)}\right)^n.$$

It is clear that such function grows to infinity as r tends to infinity.

In an $(n+1)$-dimensional Euclidean space such ratio also tends to infinity.

This shows that the restrictions $\delta_i < k_i$ in the hypothesis of the theorem are essential. ∎

4. Estimates on the Volume Growth Entropy

Let (M^{n+1}, F) be a Finsler manifold. Then the exponential speed of the volume growth of a ball of radius $t > 0$ is called the *volume growth entropy* of (M^{n+1}, F). The explicit expression for the volume growth entropy is given by

$$\lim_{t\to\infty}\frac{\ln(Vol(B_t^{n+1}(p))}{t}.$$

In this section we estimate the volume growth entropy of a Finsler–Hadamard manifold with the pinched flag curvature and the S-curvature.

Theorem 5. *Let (M^{n+1}, F) be an $(n+1)$-dimensional Finsler–Hadamard manifold that satisfies the following conditions:*

1. *Flag curvature satisfies the inequalities $-k_2^2 \leqslant K \leqslant -k_1^2$, $k_1, k_2 > 0$.*
2. *S-curvature satisfies the inequalities $n\delta_1 \leqslant S \leqslant n\delta_2$ such that $\delta_i < k_i$.*

Then we have

$$n(k_1-\delta_1) \leqslant \lim_{t\to\infty}\frac{\ln(Vol(B_t^{n+1}(p))}{t} \leqslant n(k_2-\delta_2).$$

If (M^{n+1}, F) is a space of constant flag curvature $K = -k^2$ and S-curvature $S = n\delta$, $\delta < k$, we have

$$\lim_{t\to\infty}\frac{\ln(Vol(B_t^{n+1}(p))}{t} = n(k-\delta).$$

P r o o f o f T h e o r e m 5. Define $\chi_i(t)$ by

$$\chi_i(t) = \left(e^{-\delta_i t}\frac{\sinh(k_i t)}{k_i}\right)^n .$$

It was proved in [3, 4] that under conditions 1 and 2 the volume of a metric ball satisfies

$$Vol_E(\mathbb{S}^n)\int_0^t \chi_1(s)ds \leqslant Vol(B_t^{n+1}(p)) \leqslant Vol_E(\mathbb{S}^n)\int_0^s \chi_2(s)ds. \tag{9}$$

By direct computation, we have

$$\int_0^t \left(e^{-\delta_2 s}\frac{\sinh(k_2 s)}{k_2}\right)^n ds \leqslant \frac{1}{k_2^n}\int_0^t e^{sn(k_2-\delta_2)}ds$$

$$= \frac{1}{n(k_2-\delta_2)k_2^n}\left(e^{tn(k_2-\delta_2)}-1\right).$$

Therefore, we get

$$\lim_{t\to\infty}\frac{\ln(Vol(B_t^{n+1}(p))}{t} \leqslant n(k_2-\delta_2).$$

Next,

$$\int_0^t \left(e^{-\delta_1 s}\frac{\sinh(k_1 s)}{k_1}\right)^n ds \geqslant \frac{1}{k_1^n}\int_0^t e^{-sn\delta_1}(1-ne^{-2k_1 s})e^{k_1 sn}ds$$

$$= \frac{1}{k_1^n}\left[\frac{1}{n(k_1-\delta_1)}(e^{tn(k_1-\delta_1)}-1) + \frac{n}{k_1(n-2)-n\delta_1}(e^{tk_1(n-2)-n\delta_1}-1)\right].$$

This implies

$$\lim_{r\to\infty}\frac{\ln(Vol(B_r^{n+1}(p))}{r} \geqslant n(k_1-\delta_1).$$

And Theorem 5 follows easily. ■

E x a m p l e 2. Let F be the Funk metric on a strongly convex domain U in $\mathbb{R}^{n+1}$. Then the condition $\delta < k$ does not hold.

Then, analogously as in Ex. 1, one can show that

$$\lim_{t\to\infty}\frac{\ln(Vol(B_t^{n+1}(p))}{t} = \infty.$$

In an (n+1)-dimensional Euclidean space such ratio also tends to infinity.

This shows that the restrictions $\delta_i < k_i$ in the hypothesis of the theorem are essential.

In was shown in [7] that for the Hilbert metric F on a strongly convex domain U in $\mathbb{R}^{n+1}$

$$\lim_{t\to\infty} \frac{\ln(Vol(B_t^{n+1}(p))}{t} = n.$$

Recall that n is precisely the volume growth entropy of $\mathbb{H}^{n+1}$. ■

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