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Asymptotic Analysis of a Parabolic Problem in a Thick Two-Level Junction

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We consider an initial boundary value problem for the heat equation in a plane two-level junction Ω_ε, which is the union of a domain and a large number $2N$ of thin rods with the variable thickness of order $\varepsilon = \mathcal{O}(N^{-1})$. The thin rods are divided into two levels depending on boundary conditions given on their sides. In addition, the boundary conditions depend on the parameters $\alpha \geq 1$ and $\beta \geq 1$, and the thin rods from each level are ε-periodically alternated. The asymptotic analysis of this problem for different values of α and β is made as $\varepsilon \to 0$. The leading terms of the asymptotic expansion for the solution are constructed, the asymptotic estimate in the Sobolev space $L^2(0,T;H^1(\Omega_\varepsilon))$ is obtained and the convergence theorem is proved with minimal conditions for the right-hand sides.

Key words: homogenization, thick junctions, parabolic problems, anisotropic Sobolev spaces.

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Introduction

It is an interesting problem to study the asymptotic behaviour of solutions of boundary value problems when the domain is perturbed. There are many kinds of the domain perturbations and we need different asymptotic methods to study

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boundary value problems in perturbed domains (see, e.g., [1–9] and the references therein).

In recent years the interest to the boundary value problems in domains with rapidly oscillating boundaries is quickened due to the development of technologies of porous, composite and other microinhomogeneous materials and biological structures. In the following three items we present a short review showing the main qualitative results obtained for the boundary value problems in domains with rapidly oscillating boundaries.

In [7, Sect. 5] the heat equation is studied in a plane bounded domain whose boundary is a wave surface of the curve $n = \varepsilon F(s/\varepsilon)$, where ε is a small parameter and $F(\cdot)$ is some 1-periodic function. On this waved surface the following boundary condition $\partial_\nu u_\varepsilon + k_0 u_\varepsilon = 0$ is given. This condition is classical in some problems of heat transfer. From physical point of view, it is natural to expect that the wave surface will radiate more heat than a smooth (homogenized) one. This is the reason why the radiators are waved. It is shown that in the limit passage as $\varepsilon \to 0$ we obtain the initial boundary value problem for the heat equation in a domain with homogenized surface and with the following boundary condition. $\partial_\nu u_0 + k_0|\Gamma|u_0 = 0$, where $|\Gamma|$ is the "waving coefficient" of the initial boundary.

The paper [10] deals with the homogenization of an elliptic equation of the second order with quickly oscillating coefficients in a thin perforated domain with rapidly varying thickness. The following inhomogeneous Neumann condition $\sum_{i,j=1}^{n} a_{ij}(x/\varepsilon)\partial_{x_j} u_\varepsilon \nu_i = \varepsilon g(\widehat{x}, x/\varepsilon)$ is given on the oscillating boundary. It is proved that this condition is transformed as $\varepsilon \to 0$ in the "waving" summand of the right-hand side of the homogenized equation.

In paper [11] the authors studied a boundary value problem for the Poisson equation with the inhomogeneous Fourier boundary condition

$$\partial_\nu u_\varepsilon + \varepsilon^\beta p(\widehat{x}, \widehat{x}/\varepsilon^\alpha) u_\varepsilon \varepsilon^{\alpha-1} = g(\widehat{x}, \widehat{x}/\varepsilon^\alpha)$$

on the very rapidly oscillating part ($x_n = \varepsilon F(\widehat{x}, \widehat{x}/\varepsilon^\alpha)$, $\alpha > 1$) of the boundary. Depending on the relation between β and $\alpha - 1$, different limiting boundary conditions as $\varepsilon \to 0$ were obtained for the Poisson equation in the corresponding smooth domain.

From this small review it follows that asymptotic results are very sensitive to the type of the oscillating boundary and boundary conditions.

We have a completely different situation for the boundary value problems in thick junctions (sometimes these domains are called domains perforated by narrow parallel channels or sheets (see [3, 4, 12–17], or thick junctions [18–23], or domains with highly oscillating boundary (see [24, 25])). It is because of special character of the connectedness of thick junctions: there are points in a thick junction, which are at a short distance of order $\mathcal{O}(\varepsilon)$, but the length of all curves connecting these points in the junction is of order $\mathcal{O}(1)$. As a result, there appear many new specific effects and difficulties in asymptotic study of boundary value problems in thick junctions: the loss of coercivity of differential operators in the limit passage as $\varepsilon \to 0$ (for a spectral problem it means the loss of compactness); the absence of extension operators that would be bounded uniformly in ε in the Sobolev space W_2^1; the power behavior of junction-layer solutions at infinity.

The aim of the paper is to continue the asymptotic analysis of boundary value problems in thick multilevel junctions studied in [26–30], where elliptic boundary value problems and spectral problems were considered. First, we deal with initial boundary value parabolic problems. These problems in thick multilevel junctions have not been studied in full. The idea to deal with them resulted from fruitful discussions with the specialists in radioelectronic, where these thick junctions are in common practice as radiators (heat radiators, microstrip radiators, tubular radiators, ferrite-filled rod radiators, folded core radiators, waveguide radiators and so on). Furthermore, we consider the inhomogeneous Fourier boundary conditions $\partial_\nu u_\varepsilon + \varepsilon k_1 u_\varepsilon = \varepsilon^\beta g_\varepsilon$ on the sides of the rods from the first level and the following ones $\partial_\nu u_\varepsilon + \varepsilon^\alpha k_2 u_\varepsilon = \varepsilon^\beta g_\varepsilon$ on the sides of the rods from the second level. These conditions depend on three parameters $\varepsilon > 0$, $\alpha \geq 1$, $\beta \geq 1$, and we study their influence on the asymptotic behaviour of the solution as $\varepsilon \to 0$.

The outline of the paper is the following. In Section 1 the statement of the problem is reported. The auxiliary uniform estimates for the solution are proved in Sect. 2. The leading terms of the asymptotic expansion for the solution of the problem are constructed in Sect. 3 for every analyzed case. The corresponding estimates are deduced in Sect. 4 and the convergence theorem is proved in Sect. 5. Finally, we discuss the obtained results.

1. Statement of the Problem

Let a, d_1, d_2, b_1, b_2 be positive real numbers and let $d_1 \geq d_2$, $0 < b_1 < b_2 < 1$. Consider two positive piecewise smooth functions h_1 and h_2 on the segments $[-d_1, 0]$ and $[-d_2, 0]$, respectively. Suppose the functions h_1 and h_2 satisfy the following conditions:

$$\exists\ \delta_0 \in (b_1, b_2) \quad \forall\ x_2 \in [-d_1, 0]: \quad 0 < b_1 - h_1(x_2)/2, \qquad b_1 + h_1(x_2)/2 < \delta_0;$$

$$\forall\ x_2 \in [-d_2, 0]: \quad \delta_0 < b_2 - h_2(x_2)/2, \qquad b_2 + h_2(x_2)/2 < 1.$$

It follows from these assumptions that there exist the positive constants m_0, M_0 such that

$$0 < m_0 \leq h_1(x_2) < \delta_0 \quad \text{and} \quad |h_1'(x_2)| \leq M_0 \quad \text{a.e. in } [-d_1, 0];$$

$$0 < m_0 \leq h_2(x_2) < 1 - \delta_0 \quad \text{and} \quad |h_2'(x_2)| \leq M_0 \quad \text{a.e. in } [-d_2, 0].$$

We also assume that h_1 and h_2 are locally constant functions in a neighborhood of the point $x_2 = 0$, i.e., there exists some small enough positive number τ_0 such that h_1 and h_2 are constant on $[-\tau_0, 0]$.

Let us divide a segment $[0, a]$ into N equal segments $[\varepsilon j, \varepsilon(j+1)]$, $j = 0, \ldots, N-1$. Here N is a large integer, therefore the value $\varepsilon = a/N$ is a small discrete parameter.

A model plane thick two-level junction Ω_ε (see figure) consists of junction's body $\Omega_0 = \{x \in \mathbb{R}^2 : 0 < x_1 < a,\ 0 < x_2 < \gamma(x_1)\}$, where $\gamma \in C^1([0,a])$, $\min_{[0,a]} \gamma > 0$, $\gamma(0) = \gamma(a) =: \gamma_0$, and of a large number of thin rods

$$G_j^{(1)}(\varepsilon) = \{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + b_1)| < \varepsilon h_1(x_2)/2, \quad x_2 \in (-d_1, 0]\},$$

$$G_j^{(2)}(\varepsilon) = \{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + b_2)| < \varepsilon h_2(x_2)/2, \quad x_2 \in (-d_2, 0]\},$$

$j = 0, 1, \ldots, N-1$, i.e., $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$. Here $G_\varepsilon = G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$,

$$G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} G_j^{(1)}(\varepsilon), \quad G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon).$$

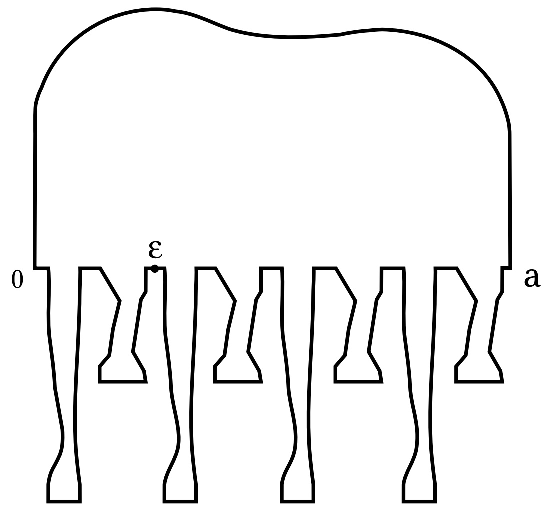

Figure.

We see that the number of thin rods is equal to $2N$ and they are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ depending on their lengths, namely, d_1 and d_2. The parameter ε characterizes the distance between the thin neighboring rods and their thickness. The thickness of the rods from the first level is equal to εh_1, and it is equal to εh_2 for the rods from the second level. These thin rods from each level are ε-periodically alternated along the segment $I_0 = \{x : x_1 \in [0, a], \quad x_2 = 0\}$.

Denote by $\Upsilon_j^{(i,\pm)}(\varepsilon)$ the lateral surfaces of the thin rod $G_j^{(i)}(\varepsilon)$; the sign "+" and "−" indicate the right and left surfaces, respectively. The base of $G_j^{(i)}(\varepsilon)$ is denoted by $\Theta_j^{(i)}(\varepsilon)$. We also introduce the following notation $(i = 1, 2)$:

$$\Upsilon_\varepsilon^{(i,\pm)} := \cup_{j=0}^{N-1}\Upsilon_j^{(i,\pm)}(\varepsilon), \quad \Theta_\varepsilon^{(i)} := \cup_{j=0}^{N-1}\Theta_j^{(i)}(\varepsilon), \quad \Upsilon_\varepsilon^{(i)} := \Upsilon_\varepsilon^{(i,+)} \cup \Upsilon_\varepsilon^{(i,-)} \cup \Theta_\varepsilon^{(i)}.$$

In $\Omega_\varepsilon \times (0,T)$ we consider the following initial boundary value problem

$$\begin{aligned}
\partial_t u_\varepsilon(x,t) &= \Delta_x u_\varepsilon + f_0(x,t), && (x,t) \in \Omega_0 \times (0,T);\\
\partial_t u_\varepsilon(x,t) &= \Delta_x u_\varepsilon(x,t), && (x,t) \in G_\varepsilon \times (0,T);\\
\partial_{x_1}^p u_\varepsilon(0,x_2,t) &= \partial_{x_1}^p u_\varepsilon(a,x_2,t), && (x_2,t) \in (0,\gamma_0) \times (0,T),\ p = 0,1;
\end{aligned}$$

$$\begin{aligned}
[u_\varepsilon]_{|x_2=0} = [\partial_{x_2} u_\varepsilon]_{|x_2=0} &= 0, && (x,t) \in \Theta_\varepsilon^{(0)} \times (0,T),\\
\partial_\nu u_\varepsilon(x,t) + \varepsilon k_1 u_\varepsilon(x,t) &= \varepsilon^\beta g_\varepsilon(x,t), && (x,t) \in \Upsilon_\varepsilon^{(1,\pm)} \times (0,T);\\
\partial_\nu u_\varepsilon(x,t) + \varepsilon^\alpha k_2 u_\varepsilon(x,t) &= \varepsilon^\beta g_\varepsilon(x,t), && (x,t) \in \Upsilon_\varepsilon^{(2)} \times (0,T);\\
\partial_\nu u_\varepsilon(x,t) + k_1 u_\varepsilon(x,t) &= 0, && (x,t) \in \Theta_\varepsilon^{(1)} \times (0,T);\\
\partial_\nu u_\varepsilon(x,t) &= 0, && (x,t) \in \Gamma_\varepsilon \times (0,T);\\
u_\varepsilon(x,0) &= 0, && x \in \Omega_\varepsilon \times \{t = 0\},
\end{aligned} \tag{1}$$

where $\partial_\nu = \partial/\partial_\nu$ is the outward normal derivative; $\partial_{x_1} = \partial/\partial x_1$; the constants k_1, k_2 are positive; the parameters $\alpha \geq 1$ and $\beta \geq 1$; the brackets denote the jump of the enclosed quantities, and $\Theta_\varepsilon^{(0)} := I_0 \cap \Omega_\varepsilon$.

Our main assumptions are as follows. For any $T > 0$ the given function f_0 belongs to $L^2(\Omega_0 \times (0,T))$ and the function g_ε belongs to $L^2(0,T;H^1(D_1))$, where $D_1 = \{x :\ 0 < x_1 < a,\ \ -d_1 < x_2 < 0\}$ is a rectangle that is filled up by the thin rods from the first level in the limit passage as $\varepsilon \to 0$. In addition,

(i) for any $T > 0$ there exist constants c_0, ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\|g_\varepsilon\|_{L^2(0,T;H^1(D_1))} \leq c_0; \tag{2}$$

(ii) moreover, if $\beta = 1$, then

$$g_\varepsilon \to g_0 \ \text{ in } \ L^2(D_1 \times (0,T)) \quad \text{as } \ \varepsilon \to 0. \tag{3}$$

Recall that a function $u_\varepsilon \in L^2\big(0,T;\mathcal{H}_\varepsilon\big)$, where $\mathcal{H}_\varepsilon = \{u \in H^1(\Omega_\varepsilon) : u(0,x_2) = u(a,x_2),\ x_2 \in (0,\gamma_0)\}$, is a weak solution to problem (1) if for any function $\psi \in H^1\big(\Omega_\varepsilon \times (0,T)\big)$ such that $\psi(0,x_2,t) = \psi(a,x_2,t)$ $(x_2,t) \in (0,\gamma_0) \times (0,T)$, and $\psi(x,T) = 0$ $x \in \Omega_\varepsilon$, the following integral identity

$$\int_0^T \Big(-\int_{\Omega_\varepsilon} u_\varepsilon\, \partial_t \psi\ dx + \int_{\Omega_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \psi\, dx$$

$$+ \ \varepsilon k_1 \int_{\Upsilon_\varepsilon^{(1,\pm)}} u_\varepsilon \, \psi \, dl_x \ + \ k_1 \int_{\Theta_\varepsilon^{(1)}} u_\varepsilon \, \psi \, dx_2 \ + \ \varepsilon^\alpha k_2 \int_{\Upsilon_\varepsilon^{(2)}} u_\varepsilon \, \psi \, dl_x \Big) \ dt$$

$$= \int_0^T \Big(\int_{\Omega_0} f_0 \, \psi \, dx \ + \ \varepsilon^\beta \int_{\Upsilon_\varepsilon^{(1,\pm)} \cup \Upsilon_\varepsilon^{(2)}} g_\varepsilon \, \psi \, dl_x \Big) \ dt \tag{4}$$

holds. It follows from the theory of boundary value problems (see, for instance, [31, 32]) that for any fixed value $\varepsilon > 0$ there exists a unique weak solution to problem (1).

Our aim is to study the asymptotic behavior of the weak solution to problem (1) as $\varepsilon \to 0$, i.e., when the number of the attached thin rods from each level infinitely increases and their thickness tends to zero. It should be noted that the limit process as $\varepsilon \to 0$ is accompanied by the perturbed coefficients in the boundary conditions on the lateral sides of thin rods.

2. Auxiliary Uniform Estimates

To homogenize boundary value problems in thick junctions with the nonhomogeneous Neumann or Fourier conditions on the boundaries of the thin attached domains, the method of special integral identities was suggested in [22]. Let us prove the corresponding integral identity for our problem. For this we define the following function

$$Y(t) = \begin{cases} -t + b_1, & t \in [0, \delta_0), \\ -t + b_2, & t \in [\delta_0, 1), \end{cases} \tag{5}$$

and then periodically extend it into $\mathbb{R}$; δ_0 was defined in the previous section. Integrating by parts the integral $\varepsilon \int_{G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}} Y\big(x_1/\varepsilon\big) \, \partial_{x_1} v \ dx$ and taking into account that the outward normal to the lateral surfaces $\Upsilon_j^{(i,\pm)}(\varepsilon)$ of the thin rod $G_j^{(i)}(\varepsilon)$, except a set of zero measure, has the view

$$\nu_\pm^{(i)}(\varepsilon) = \frac{1}{\sqrt{1 + \varepsilon^2 4^{-1} |h_i'(x_2)|^2}} \Big(\pm 1 \, , \, -\varepsilon \, \frac{h_i'(x_2)}{2} \Big), \tag{6}$$

$i = 1, 2, \quad j = 0, \ldots, N-1,$ we get the identity

$$\varepsilon \sum_{i=1}^{2} \int\limits_{\Upsilon_\varepsilon^{(i,\pm)}} \frac{h_i(x_2)}{2\sqrt{1 + \varepsilon^2 4^{-1} |h_i'(x_2)|^2}} \, v \, dl_x$$

$$= \int_{G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}} v \, dx \ - \ \varepsilon \int_{G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}} Y\big(\frac{x_1}{\varepsilon}\big) \partial_{x_1} v \, dx, \qquad \forall \, v \in H^1(\Omega_\varepsilon). \tag{7}$$

By the same arguments as in the proof of Lem. 1 in [29], it is easy to prove the following lemma.

Lemma 1.1. *The norms* $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ *and*

$$\|u\|_{\Theta_\varepsilon^{(1)}} := \Big(\int_{\Omega_\varepsilon} |\nabla u|^2\,dx + \varepsilon k_1 \int_{\Upsilon_\varepsilon^{(1,\pm)}} v^2\,dl_x + k_1 \int_{\Theta_\varepsilon^{(1)}} u^2\,dx_2 + \varepsilon^\alpha k_2 \int_{\Upsilon_\varepsilon^{(2)}} v^2\,dl_x\Big)^{\frac{1}{2}}$$

are uniformly equivalent with respect to ε *small enough and any* $\alpha \ge 1$.

By using the identity (7), Lem. 1.1 and the fact that $\beta \ge 1$, we prove in a standard way (see, for instance, [31, Sect. 7] or [32, Sect. 3]) the following *a priori* estimate for the solution to problem (1):

$$\|u_\varepsilon\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))} + \max_{t\in[0,T]} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)}$$

$$\le C_1\big(\|f_0\|_{L^2(\Omega_0\times(0,T))} + \varepsilon^{\beta-\frac{1}{2}}\|g_\varepsilon\|_{L^2((\Upsilon_\varepsilon^{(1,\pm)}\cup\Upsilon_\varepsilon^{(2,\pm)})\times(0,T))} + \varepsilon^{\beta}\|g_\varepsilon\|_{L^2(\Theta_\varepsilon^{(2)}\times(0,T))}\big). \tag{8}$$

Taking into account (2), with the help of the identity (7) we deduce from (8) that

$$\|u_\varepsilon\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))} + \max_{t\in[0,T]} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)} \le C_2. \tag{9}$$

Remark 1. In (8) and (9) and in what follows all constants $\{C_i\}$ and $\{c_i\}$ in asymptotic inequalities are independent of the parameter ε.

3. Formal Asymptotic Expansions for the Solution

Here the leading terms of outer expansions both in the junction's body and in each thin rod as well as the leading terms of an inner expansion in a neighborhood of the joint zone for the solution u_ε are constructed. Then, using the method of matched asymptotic expansions, we derive the corresponding limit problem and prove the existence and uniqueness of its solution. In this section, by reason of (3), we take g_0 instead of g_ε in the right-hand side of the boundary conditions on $\Gamma_\varepsilon^{(i,\pm)}$ in problem (1) and assume that g_0 is smooth.

3.1. Outer Expansions. We seek the leading terms for the solution u_ε, restricted to $\Omega_0 \times (0,T)$, in the form

$$u_\varepsilon(x,t) \approx v_0^+(x,t) + \sum_{k=1}^{\infty} \varepsilon^k v_k^+(x,t), \tag{10}$$

and, restricted to the thin rod $G_j^{(i)}(\varepsilon) \times (0,T)$, $j = 0, \ldots, N-1$, $i = 1,2$, in the form

$$u_\varepsilon(x,t) \approx v_0^{i,-}(x,t) + \sum_{k=1}^{\infty} \varepsilon^k v_k^{i,-}(x, \xi_1 - j, t), \qquad \xi_1 = \varepsilon^{-1} x_1. \tag{11}$$

The expansions (10) and (11) are usually called *outer expansions.*

Plugging the series (10) into the first equation of problem (1) and into the boundary conditions on $\partial\Omega_0 \setminus I_0$ and collecting coefficients of the same powers of ε, we get the following relations for the function v_0^+:

$$\begin{array}{rcll} \partial_t v_0^+(x,t) & = & \Delta_x \, v_0^+(x,t) + f_0(x,t), & (x,t) \in \Omega_0 \times (0,T); \\ \partial_{x_1}^p v_0^+(0,x_2,t) & = & \partial_{x_1}^p v_0^+(a,x_2,t), & (x_2,t) \in (0,\gamma_0) \times (0,T), \ p = 0,1; \\ \partial_\nu v_0^+(x,t) & = & 0, & (x,t) \in \Gamma_\gamma \times (0,T), \end{array} \tag{12}$$

where $\Gamma_\gamma := \{x : \ x_2 = \gamma(x_1), \ x_1 \in I_0\}$.

Now let us find limit relations in the rectangle $D_i = (0,a) \times (-d_i, 0)$, which is filled up by the thin rods from $i-$level in the limit passage as $\varepsilon \to 0$; the index $i \in \{1,2\}$ is fixed.

Assume for a moment that the functions $v_k^{i,-}$ in (11) are smooth. We write their Taylor series with respect to x_1 at the point $x_1 = \varepsilon(j + b_i)$ and pass to the "fast" variable $\xi_1 = \varepsilon^{-1} x_1$. Then (11) takes the form

$$u_\varepsilon(x,t) \approx v_0^{i,-}\big(\varepsilon(j+b_i), x_2, t\big) + \sum_{k=1}^{+\infty} \varepsilon^k \, V_k^{i,j}(\xi_1, x_2, t), \quad (x,t) \in G_j^{(i)}(\varepsilon) \times (0,T), \tag{13}$$

where

$$\begin{array}{rcl} V_k^{i,j}(\xi_1, x_2, t) & = & v_k^{i,-}\big(\varepsilon(j+b_i), x_2, \xi_1 - j, t\big) \\ & + & \displaystyle\sum_{m=1}^{k} \frac{(\xi_1 - j - b_i)^m}{m!} \, \frac{\partial^m v_{k-m}^{i,-}}{\partial x_1^m}\big(\varepsilon(j+b_i), x_2, \xi_1 - j, t\big). \end{array} \tag{14}$$

Let us plug (13) into (1) instead of u_ε. Since the Laplace operator takes the form $\Delta = \varepsilon^{-2} \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial x_2^2}$, the collection of coefficients of the same power of ε gives us one-dimensional boundary value problems with respect to ξ_1.

The first problem is the following:

$$\partial^2_{\xi_1 \xi_1} V_1^{i,j}(\xi_1, x_2, t) = 0, \quad \xi_1 \in I_{h_i(x_2)}(b_i), \qquad \partial_{\xi_1} V_1^{i,j}(b_i \pm h_i/2, x_2, t) = 0, \tag{15}$$

where $\partial_{\xi_1} = \frac{\partial}{\partial \xi_1}$, $\partial^2_{\xi_1 \xi_1} = \frac{\partial^2}{\partial \xi_1^2}$ and $I_{h_i(x_2)}(b_i) = \big(b_i - \frac{h_i(x_2)}{2}, b_i + \frac{h_i(x_2)}{2}\big)$; the variable x_2 is regarded as a parameter in this problem.

From (15) it follows that function $V_1^{i,j}$ does not depend on ξ_1. Therefore, $V_1^{i,j}$ is equal to some function $\varphi_1^{(i)}\big(\varepsilon(j+b_i), x_2, t\big)$, $(x_2,t)\in[-d_i,0]\times[0,T]$, which will be defined later. Then, due to (14), we have

$$v_1^{i,-}\big(\varepsilon(j+b_i), x_2, \xi_1-j, t\big) = \varphi_1^{(i)}\big(\varepsilon(j+b_i), x_2, t\big) - \big(\xi_1-j-b_i\big)\,\partial_{x_1} v_0^{i,-}\big(\varepsilon(j+b_i), x_2, t\big). \tag{16}$$

The problem for the function $V_2^{i,j}$ is as follows

$$-\partial^2_{\xi_1\xi_1} V_2^{i,j} \;=\; \partial^2_{x_2x_2} v_0^{i,-}(\varepsilon(j+b_i), x_2, t) - \partial_t v_0^{i,-}(\varepsilon(j+b_i), x_2, t), \quad \xi_1\in I_{h_i(x_2)}(b_i), \tag{17}$$

$$\begin{aligned}\partial_{\xi_1} V_2^{i,j}\big(b_i\pm h_i/2, x_2, t\big) \;&=\; \pm 2^{-1} h'(x_2)\,\partial_{x_2} v_0^{i,-}(\varepsilon(j+b_i), x_2, t)\\ &\mp \delta_{\alpha,1} k_i v_0^{i,-}\big(\varepsilon(j+b_i), x_2, t\big) \;\pm\; \delta_{\beta,1} g_0\big(\varepsilon(j+b_i), x_2, t\big),\end{aligned} \tag{18}$$

where $\delta_{\alpha,1}, \delta_{\beta,1}$ are Kronecker's symbols (recall that $\alpha\ge 1$ and $\beta\ge 1$).

The solvability condition for problem (17)–(18) is given by the differential equation

$$\begin{aligned} h_i(x_2)\,\partial_t v_0^{i,-}(\varepsilon(j+b_i), x_2, t) \;&=\; \partial_{x_2}\Big(h_i(x_2)\,\partial_{x_2} v_0^{i,-}\big(\varepsilon(j+b_i), x_2\big)\Big)\\ &-\; 2\,\delta_{\alpha,1} k_i\, v_0^{i,-}\big(\varepsilon(j+b_i), x_2\big) \;+\; 2\,\delta_{\beta,1} g_0\big(\varepsilon(j+b_i), x_2, t\big).\end{aligned} \tag{19}$$

Plugging (13) into the Fourier condition on the bases $\Theta_\varepsilon^{(i)}$, $i=1,2$, we get

$$\partial_{x_2} v_0^{1,-}\big(\varepsilon(j+b_1), -d_1, t\big) = k_1\, v_0^{1,-}\big(\varepsilon(j+b_1), -d_1, t\big), \partial_{x_2} v_0^{2,-}\big(\varepsilon(j+b_2), -d_2, t\big) = 0. \tag{20}$$

To find the conditions in points of the joint zone I_0, we use the method of matched asymptotic expansions for the outer expansions (10), (11) and an inner expansion which is constructed in the following subsection.

3.2. Inner Expansion. In a neighborhood of the joint zone I_0 we introduce the "rapid" coordinates $\xi=(\xi_1,\xi_2)$, where $\xi_1=\varepsilon^{-1}x_1$ and $\xi_2=\varepsilon^{-1}x_2$. Passing to $\varepsilon=0$, we see that the rods $G_0^{(1)}(\varepsilon)$ and $G_0^{(2)}(\varepsilon)$ transform into the semi-infinite strips $\Pi^-_{h_1} = I_{h_1(0)}(b_1)\times(-\infty,0]$, $\Pi^-_{h_2} = I_{h_2(0)}(b_2)\times(-\infty,0]$, respectively; the domain Ω_0 transforms into the first quadrant $\{\xi : \xi_1>0,\, \xi_2>0\}$. Taking into account the periodicity of thin rods, we can regard that the union Π of semi-strips $\Pi^-_{h_1}$, $\Pi^-_{h_2}$ and $\Pi^+=(0,1)\times(0,+\infty)$ is the base domain in which the junction-layer problems should be considered. Obviously, the solutions of these junction-layer problems must be 1-periodic in ξ_1, i.e.,

$$\partial^p_{\xi_1} Z(\xi)|_{\xi_1=0} = \partial^p_{\xi_1} Z(\xi)|_{\xi_1=1}\,, \qquad \xi\in\partial\Pi^+\,,\;\; \xi_2>0, \quad p=0,1.$$

So, we seek the leading terms of the inner expansion in a neighborhood of the joint zone I_0 in the form

$$u_\varepsilon(x) \approx v_0^+(x_1,0,t) \,+\, \varepsilon\Big(Z_1\big(x/\varepsilon\big)\,\partial_{x_1}v_0^+(x_1,0,t)$$
$$+\big(\eta(x_1,t)\,\Xi_1(x/\varepsilon) \,+\, (1-\eta(x_1,t))\,\Xi_2(x/\varepsilon)\big)\,\partial_{x_2}v_0^+(x_1,0,t)\Big) + \ldots\,, \qquad (21)$$

where $Z_1(\xi)$, $\Xi_1(\xi)$, $\Xi_2(\xi)$, $\xi\in\Pi$, are 1-periodic with respect to ξ_1 solutions to junction-layer problems; the function η will be defined from matching conditions.

Plugging (21) into the differential equation of problem (1) and into the corresponding boundary conditions, taking into account that the Laplace operator takes the form $\varepsilon^{-2}\Delta_\xi$ in the coordinates ξ and collecting the coefficients of the same power of ε, we get junction-layer problems for the functions Z_1, Ξ_1, Ξ_2. So, the functions Ξ_1 and Ξ_2 are the solution to the following homogeneous problem

$$\begin{array}{rcll} -\Delta_\xi\,\Xi(\xi) &=& 0, & \xi\in\Pi,\\ \partial_{\xi_2}\Xi(\xi_1,0) &=& 0, & \xi_1\in(0,1)\setminus\big(I_{h_1(0)}(b_1)\cup I_{h_2(0)}(b_2)\big),\\ \partial_{\xi_1}\Xi(\xi) &=& 0, & \xi\in\Big(\partial\Pi_{h_1}^-\setminus I_{h_1(0)}(b_1)\Big)\cup\Big(\partial\Pi_{h_2}^-\setminus I_{h_2(0)}(b_2)\Big),\\ \partial_{\xi_1}^p\,\Xi(0,\xi_2) &=& \partial_{\xi_1}^p\,\Xi(1,\xi_2), & \xi_2>0,\quad p=0,1. \end{array} \qquad (22)$$

The main asymptotic relations for the functions Ξ_1, Ξ_2 can be obtained from general results on the asymptotic behaviour of solutions to elliptic problems in domains with different exits to infinity (see, for instance, [33]). The proofs simplify substantially if the polynomial property of the corresponding sesquilinear forms is employed (see [34]). However, for the domain Π, we can define more exactly the asymptotic relations and detect other properties of the junction-layer solutions Ξ_1, Ξ_2 in the same way as in the papers [19, 20].

Proposition 3.1. *There exist two solutions Ξ_1, $\Xi_2\in H^1_{\sharp,loc}(\Pi)$ to the problems (22), which have the following differentiable asymptotics:*

$$\Xi_1=\left\{\begin{array}{ll} \xi_2+\mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2\to+\infty,\ \xi\in\Pi^+,\\ h_1^{-1}(0)\,\xi_2+\alpha_1^{(1)}+\mathcal{O}(\exp(\pi h_1^{-1}(0)\xi_2)), & \xi_2\to-\infty,\ \xi\in\Pi_{h_1}^-,\\ \alpha_1^{(2)}+\mathcal{O}(\exp(\pi h_2^{-1}(0)\xi_2)), & \xi_2\to-\infty,\ \xi\in\Pi_{h_2}^-, \end{array}\right. \qquad (23)$$

$$\Xi_2=\left\{\begin{array}{ll} \xi_2+\mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2\to+\infty,\ \xi\in\Pi^+,\\ \alpha_2^{(1)}+\mathcal{O}(\exp(\pi h_1^{-1}(0)\xi_2)), & \xi_2\to-\infty,\ \xi\in\Pi_{h_1}^-,\\ h_2^{-1}(0)\,\xi_2+\alpha_2^{(2)}+\mathcal{O}(\exp(\pi h_2^{-1}(0)\xi_2)), & \xi_2\to-\infty,\ \xi\in\Pi_{h_2}^-. \end{array}\right. \qquad (24)$$

Here $H^1_{\sharp,loc}(\Pi)=\{u:\Pi\to\mathbb{R}\ |u(0,\xi_2)=u(1,\xi_2)$ for any $\xi_2>0$, $u\in H^1(\Pi_R)$ for any $R>0\}$, where $\Pi_R=\Pi\cap\{\xi:\ -R<\xi_2<R\}$; $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, $i=1,2$, are some fixed constants.

Any other solution to the homogeneous problem (22), which has a polynomial growth at infinity, can be presented as a linear combination $c_0 + c_1\Xi_1 + c_2\Xi_2$.

The function Z_1 is a solution to the following problem:

$$\begin{array}{rcll} -\Delta_\xi \, Z_1(\xi) &=& 0, & \xi \in \Pi, \\ \partial_{\xi_2} Z_1(\xi_1, 0) &=& 0, & \xi_1 \in (0,1) \setminus \big(I_{h_1(0)}(b_1) \cup I_{h_2(0)}(b_2)\big), \\ \partial_{\xi_1} Z_1(\xi) &=& -1, & \xi \in \Big(\partial\Pi^-_{h_1} \setminus I_{h_1(0)}(b_1)\Big) \cup \Big(\partial\Pi^-_{h_2} \setminus I_{h_2(0)}(b_2)\Big), \\ \partial^p_{\xi_1} Z_1(0, \xi_2) &=& \partial^p_{\xi_1} Z_1(1, \xi_2), & \xi_2 > 0, \quad p = 0, 1. \end{array}$$

Similarly to [19, 20, 34], it is easy to verify that there exists the unique solution $Z_1 \in H^1_{\sharp,loc}(\Pi)$ with the following asymptotics:

$$Z_1 = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty, \ \xi \in \Pi^+, \\ -\xi_1 + b_1 + \alpha_3^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(0)\xi_2)), & \xi_2 \to -\infty, \ \xi \in \Pi^-_{h_1}, \\ -\xi_1 + b_2 + \alpha_3^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(0)\xi_2)), & \xi_2 \to -\infty, \quad \xi \in \Pi^-_{h_2}. \end{cases} \tag{25}$$

Now let us verify matching conditions for the outer expansions (10), (11) and the inner expansion (21), namely, the leading terms of the asymptotics of the outer expansions as $x_2 \to \pm 0$ must coincide with the leading terms of the inner expansion as $\xi_2 \to \pm\infty$. Near the point $(\varepsilon(j+b_i), 0) \in I_0$ at the fixed value of t, the function v_0^+ has the following asymptotics:

$$v_0^+(\varepsilon(j+b_i), 0, t) + \varepsilon \, \xi_2 \, \partial_{x_2} v_0^+(\varepsilon(j+b_i), 0, t) + \mathcal{O}(\varepsilon^2 \xi_2^2), \quad x_2 \to 0 + 0.$$

Taking into account the asymptotics of Z_1 and Ξ_1, Ξ_2 as $\xi_2 \to +\infty$, we see that the matching conditions are satisfied for the expansion (10) and (21).

The asymptotics of (11) are equal to

$$\begin{gathered} v_0^{i,-}(\varepsilon(j+b_i), 0, t) \ + \ \varepsilon\Big(\varphi_1^{(i)}(\varepsilon(j+b_i), 0, t) \\ + \big(-\xi_1 + b_i + j\big) \, \partial_{x_1} v_0^{i,-}(\varepsilon(j+b_i), 0, t) + \xi_2 \, \partial_{x_2} v_0^{i,-}(\varepsilon(j+b_i), 0, t)\Big) + \ldots \\ \text{as } \ x_2 \to 0 - 0, \quad (x,t) \in G_j^{(i)}(\varepsilon) \times (0,T), \quad i = 1, 2. \end{gathered} \tag{26}$$

The first terms of asymptotics of (21) in $G_j^{(1)}(\varepsilon)$ are

$$\begin{gathered} v_0^+(\varepsilon(j+b_1), 0, t) \ + \ \varepsilon\Big(\big(-\xi_1 + j + b_1 + \alpha_3^{(1)}\big) \, \partial_{x_1} v_0^+(\varepsilon(j+b_1), 0, t) \\ + \big\{\eta(\varepsilon(j+b_1), t)\big(\frac{\xi_2}{h_1(0)} + \alpha_1^{(1)}\big) + \big(1 - \eta(\varepsilon(j+b_1, t))\big)\alpha_2^{(1)}\big\} \partial_{x_2} v_0^+(\varepsilon(j+b_1), 0, t)\Big) \\ \text{as } \ \xi_2 \to -\infty, \quad \xi \in \Pi^-_{h_1}, \end{gathered} \tag{27}$$

and in $G_j^{(2)}(\varepsilon)$ are

$$v_0^+(\varepsilon(j+b_2),0,t) \ + \ \varepsilon\Big(\big(-\xi_1+j+b_2+\alpha_3^{(2)}\big)\,\partial_{x_1}v_0^+(\varepsilon(j+b_2),0,t)$$
$$+\big\{\big(1-\eta(\varepsilon(j+b_2),t)\big)\big(\frac{\xi_2}{h_2(0)}+\alpha_2^{(2)}\big)+\eta(\varepsilon(j+b_2),t)\alpha_1^{(2)}\big\}\partial_{x_2}v_0^+(\varepsilon(j+b_2),0,t)\Big)$$
$$\text{as } \ \xi_2\to-\infty, \quad \xi\in\Pi_{h_2}^-. \tag{28}$$

Comparing the first terms of (26), (27) and (28), we get

$$v_0^+(\varepsilon(j+b_i),0,t)=v_0^{i,-}(\varepsilon(j+b_i),0,t), \quad j=0,1,\dots,N-1, \quad i=1,2. \tag{29}$$

Comparing the second terms of (26) and (27), and (26) and (28), we find

$$\varphi_1^{(i)}(\varepsilon(j+b_i),0,t)=\alpha_3^{(i)}\partial_{x_1}v_0^{i,-}(\varepsilon(j+b_i),0,t), \quad i=1,2,$$

and the following relations

$$\eta(\varepsilon(j+b_1),t)\,\partial_{x_2}v_0^+(\varepsilon(j+b_1),0,t)=h_1(0)\,\partial_{x_2}v_0^{1,-}(\varepsilon(j+b_1),0,t), \tag{30}$$
$$\big(1-\eta(\varepsilon(j+b_2),t)\big)\,\partial_{x_2}v_0^+(\varepsilon(j+b_2),0,t)=h_2(0)\,\partial_{x_2}v_0^{2,-}(\varepsilon(j+b_2),0,t), \tag{31}$$

for $j=0,1,\dots,N-1$.

Since the segments $\{x:\ x_1=\varepsilon(j+b_i),\ x_2\in[-d_i,0]\}$, $j=0,1,\dots,N-1$, fill in the rectangle $\overline{D}_i$ in the limit passage as $\varepsilon\to0$ ($N\to+\infty$) for $i=1$ and $i=2$, we can extend the equation (19) into the whole rectangle $D_1=I_0\times(-d_1,0)$ for $i=1$ and into rectangle D_2 for $i=2$. On the basis of the same arguments, we extend the relations (20), (29), (30) and (31) into the whole interval I_0.

From the limiting relations (30) and (31) it follows that

$$\partial_{x_2}v_0^+(x_1,0,t)=h_1(0)\partial_{x_2}v_0^{1,-}(x_1,0,t)+h_2(0)\partial_{x_2}v_0^{2,-}(x_1,0,t),\ (x_1,t)\in I_0\times(0,T),$$

and

$$\eta(x_1,t)=\frac{h_1(0)\,\partial_{x_2}v_0^{1,-}(x_1,0,t)}{h_1(0)\,\partial_{x_2}v_0^{1,-}(x_1,0,t)\ +\ h_2(0)\,\partial_{x_2}v_0^{2,-}(x_1,0,t)}, \quad (x_1,t)\in I_0\times(0,T).$$

3.3. Existence and Uniqueness of the Solution to the Limit Problem. Using the first terms v_0^+, $v_0^{1,-}v_0^{2,-}$ of asymptotic expansions (10) and (11), we define the following vector function

$$\mathbf{v}_0(x,t)=\begin{cases} v_0^+(x,t), & (x,t)\in\Omega_0\times(0,T),\\ v_0^{1,-}(x,t), & (x,t)\in D_1\times(0,T),\\ v_0^{2,-}(x,t), & (x,t)\in D_2\times(0,T).\end{cases} \tag{32}$$

As it follows from the foregoing, the components of this vector function must satisfy the relations

$$
\begin{array}{rcll}
\partial_t v_0^+(x,t) &=& \Delta_x\ v_0^+(x,t) + f_0(x,t), & (x,t)\in\Omega_0\times(0,T);\\
\partial_{x_1}^p v_0^+(0,x_2) &=& \partial_{x_1}^p v_0^+(a,x_2), \quad p=0,1, & (x_2,t)\in(0,\gamma_0)\times(0,T);\\
\partial_\nu v_0^+(x,t) &=& 0, & (x,t)\in\Gamma_\gamma\times(0,T);\\
h_1(x_2)\partial_t v_0^{1,-}(x,t) &=& \partial_{x_2}\big(h_1(x_2)\,\partial_{x_2}v_0^{1,-}(x,t)\big) & \\
&& -2k_1 v_0^{1,-} + 2\delta_{\beta,1}g_0(x,t), & (x,t)\in D_1\times(0,T);\\
\partial_{x_2}v_0^{1,-}(x_1,-d_1,t) &=& k_1 v_0^{1,-}(x_1,-d_1,t), & (x_1,t)\in(0,a)\times(0,T);\\
h_2(x_2)\partial_t v_0^{2,-}(x,t) &=& \partial_{x_2}\big(h_2(x_2)\,\partial_{x_2}v_0^{2,-}(x,t)\big) & \\
&& -2k_2\delta_{\alpha,1}v_0^{2,-} + 2\delta_{\beta,1}g_0(x,t), & (x,t)\in D_2\times(0,T);\\
\partial_{x_2}v_0^{2,-}(x_1,-d_2,t) &=& 0, & (x_1,t)\in(0,a)\times(0,T);\\
v_0^+(x_1,0,t) &=& v_0^{1,-}(x_1,0,t) = v_0^{2,-}(x_1,0,t), & (x_1,t)\in(0,a)\times(0,T);\\
\partial_{x_2}v_0^+(x_1,0,t) &=& h_1(0)\,\partial_{x_2}v_0^{1,-}(x_1,0,t) & \\
&& +\,h_2(0)\,\partial_{x_2}v_0^{2,-}(x_1,0,t), & (x_1,t)\in(0,a)\times(0,T);\\
\mathbf{v}_0|_{t=0} &=& \mathbf{0}. &
\end{array}
\tag{33}
$$

These relations form *the limit problem* for problem (1).

Let us show that there exists a unique weak solution to problem (33). For this we introduce the following anisotropic Sobolev spaces. Denote by $\mathcal{V}_0$ the vector space $L^2(\Omega_0)\times L^2(D_1)\times L^2(D_2)$ with the following scalar product

$$
\big(\mathbf{v},\mathbf{u}\big)_{\mathcal{V}_0} = \int_{\Omega_0} u_0 v_0\,dx \ + \sum_{i=1}^{2}\int_{D_i} h_i(x_2) v_i u_i\,dx,
$$

where $\mathbf{v}=(v_0,v_1,v_2)$ and $\mathbf{u}=(u_0,u_1,u_2)$ belong to $\mathcal{V}_0$. We also define the anisotropic Sobolev vector space $\mathcal{H}_0=\{\mathbf{u}\in\mathcal{V}_0:\ u_0\in H^1(\Omega_0),\ u_0(0,x_2)=u_0(a,x_2)$ for $x_2\in(0,\gamma_0)$; $\exists\ \partial_{x_2}u_1\in L^2(D_1)$; $\exists\ \partial_{x_2}u_2\in L^2(D_2)$; $u_0(x_1,0)=u_1(x_1,0)=u_2(x_1,0)$, $x_1\in I_0\}$ with the following scalar product

$$
\begin{aligned}
\big(\mathbf{v},\mathbf{u}\big)_{\mathcal{H}_0} = {} & \int_{\Omega_0}\nabla v_0\cdot\nabla u_0\,dx + \sum_{i=1}^{2}\int_{D_i} h_i(x_2)\partial_{x_2}v_i\,\partial_{x_2}u_i\,dx + 2k_1\int_{D_1} v_1u_1\,dx\\
+ {} & k_1h_1(-d_1)\int_0^a v_1(x_1,-d_1)u_1(x_1,-d_1)\,dx_1 + 2k_2\delta_{\alpha,1}\int_{D_2} v_2u_2\,dx.
\end{aligned}
$$

Obviously, the space $\mathcal{H}_0$ continuously embeds in $\mathcal{V}_0$.

We say that the vector function $\mathbf{v}_0\in L^2\big(0,T;\mathcal{H}_0\big)$ is a weak solution to the initial boundary value problem (33) if for any vector function $\mathbf{u}\in L^2\big(0,T;\mathcal{H}_0\big)$,

$\partial_t \mathbf{u} \in L^2\big(0,T;\mathcal{V}_0\big)$, $\mathbf{u}(x,T)=0$, the following integral identity holds:

$$\int_0^T \Big(-\big(\mathbf{v}_0,\partial_t\mathbf{u}\big)_{\mathcal{V}_0} + \big(\mathbf{v}_0,\mathbf{u}\big)_{\mathcal{H}_0}\Big)\,dt$$

$$= \int_0^T \left(\int_{\Omega_0} f_0(x,t)\,u_0(x,t)\;dx + 2\delta_{\beta,1}\sum_{i=1}^{2}\int_{D_i} g_0(x,t)\,u_i(x,t)\,dx \right) dt. \quad (34)$$

Taking into account the properties of the functions h_1 and h_2, with the help of the standard scheme (see [31, Sect. 7] or [32, Sect. 3]), it is easy to prove the existence and uniqueness of a weak solution to problem (33).

Lemma 3.1. *There exists a unique weak solution $\mathbf{v}_0 \in \mathcal{H}_0$ to problem (33) such that*

$$\|\mathbf{v}_0\|_{L^2(0,T;\mathcal{H}_0)} + \max_{t\in[0,T]}\|\mathbf{v}_0(\cdot,t)\|_{\mathcal{V}_0} \;\le\; C_1\left(\|f_0\|_{L^2(\Omega_0\times(0,T))} + \delta_{\beta,1}\|g_0\|_{L^2(D_1\times(0,T))}\right).$$

4. Approximation and Asymptotic Estimates

Let $\mathbf{v}_0 \in L^2(0,T;\mathcal{H}_0)$ be a unique weak solution to problem (33). With the help of $\mathbf{v}_0$ and the junction-layer solutions Z_1, Ξ_1, Ξ_2 defined in Subsect. 3.2, we construct the leading terms in (10), (11) and (21). Then matching these expansions, we define an asymptotic approximation R_ε belonging to Hilbert space $L^2\big(0,T;H^1(\Omega_\varepsilon)\big)$. It is equal to

$$R_\varepsilon(x,t) := R_\varepsilon^+(x,t) = v_0^+(x,t) + \varepsilon\chi_0(x_2)\,\mathcal{N}^+\big(\xi,x_1,t\big)|_{\xi=\frac{x}{\varepsilon}}, \quad (x,t)\in\Omega_0\times(0,T), \quad (35)$$

$$R_\varepsilon := R_\varepsilon^{i,-} = v_0^{i,-}(x,t) \;+\; \varepsilon\Big(Y_1(\xi_1)\,\partial_{x_1}v_0^{i,-}(x,t) + \chi_0(x_2)\mathcal{N}^-\big(\xi,x_1,t\big)\Big)|_{\xi=\frac{x}{\varepsilon}},$$

$$(x,t)\in G_\varepsilon^{(i)}\times(0,T), \quad i=1,2. \quad (36)$$

Here

$$\mathcal{N}^+ = Z_1\partial_{x_1}v_0^+(x_1,0,t) + \big(\eta(x_1,t)\,\Xi_1(\xi) + (1-\eta(x_1,t))\big(\Xi_2(\xi)-\xi_2\big)\,\partial_{x_2}v_0^+(x_1,0,t),$$

$$\mathcal{N}^-(\xi,x_1,t) = \big(Z_1(\xi) - Y_1(\xi_1)\big)\,\partial_{x_1}v_0^+(x_1,0,t)$$

$$+\big(\eta(x_1,t)\,\Xi_1(\xi) \;+\; (1-\eta(x_1,t))\,\Xi_2(\xi) \;-\; Y_2(\xi_2,x_1,t)\big)\,\partial_{x_2}v_0^+(x_1,0,t),$$

where Y_1 and Y_2 are 1-periodic functions with respect to ξ_1 and on the corresponding cells of periodicity they are equal to

$$Y_1 = \begin{cases} -\xi_1 + b_1 + \alpha_3^{(1)}, & \xi_1\in[0,\delta_0), \\ -\xi_1 + b_2 + \alpha_3^{(2)}, & \xi_1\in[\delta_0,1), \end{cases} \quad Y_2 = \begin{cases} \eta(x_1,t)h_1^{-1}(0)\xi_2, & \xi\in\Pi_{h_1}^-, \\ (1-\eta(x_1,t))h_2^{-1}(0)\xi_2, & \xi\in\Pi_{h_2}^-; \end{cases}$$

the function χ_0 is a smooth cutoff function such that $\chi_0(x_2) = 1$ for $|x_2| \le \tau_0/2$, and $\chi_0(x_2) = 1$ for $|x_2| \ge \tau_0$, where τ_0 was defined in Sect. 1.

Theorem 4.1. *Suppose that functions $f_0(x,t)$, $(x,t) \in \Omega_0 \times [0,+\infty)$, and $g_0(x,t)$, $(x,t) \in \overline{D}_1 \times [0,+\infty)$, are smooth; the support of f_0 with respect to x is concentrated in Ω_0 for any $t \ge 0$; $f(x,0) = 0$ for any $x \in \Omega_0$; g_0 and $\partial_{x_2} g_0$ vanish on I_0 for any $t \ge 0$ and $g_0(x,0) = 0$ for any $x \in \overline{D}_1$.*

Then for any $T > 0$, $\alpha \ge 1$, $\beta \ge 1$ and $\rho \in (0,1)$ there exist positive constants C_0, ε_0 such that for all values $\varepsilon \in (0,\varepsilon_0)$ the difference between the solution u_ε to problem (1) and the approximation function R_ε defined by (35) and (36) satisfies the following estimate

$$\|u_\varepsilon - R_\varepsilon\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))} + \max_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - R_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)}$$
$$\le C_0 \Big(\varepsilon + \varepsilon^{1-\rho} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + \varepsilon^{\delta_{\beta,1}(2-\beta)+\beta-1} \|g_0 - g_\varepsilon\|^{\delta_{\beta,1}}_{L^2(D_1\times(0,T))} \Big). \quad (37)$$

P r o o f. **Discrepancies in the domain Ω_0.** Taking into account the properties of functions Z_1, Ξ_1, Ξ_2 and v_0^+, we conclude that R_ε^+ is a-periodic with respect to x_1 and satisfies all boundary conditions on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$ for problem (2).

Putting R_ε^+ into the corresponding equation of problem (1), we get

$$\begin{aligned} \partial_t R_\varepsilon^+(x,t) - \Delta_x R_\varepsilon^+(x,t) - f_0(x,t) &= \varepsilon \chi_0(x_2) \partial_t \mathcal{N}^+(\xi, x_1, t) \\ &- \chi_0'(x_2) \big(\partial_{\xi_2} \mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon} \\ - \chi_0(x_2)\big(\partial^2_{x_1\xi_1} \mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon} &- \varepsilon \partial_{x_2}\big(\chi_0'(x_2) \mathcal{N}^+(x/\varepsilon, x_1, t)\big) \\ - \varepsilon \chi_0(x_2) \partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}\big), &\quad x \in \Omega_0. \end{aligned} \quad (38)$$

Further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion. We multiply (38) by a test function $\psi \in H^1\big(\Omega_\varepsilon \times (0,T)\big)$ such that $\psi(0,x_2,t) = \psi(a,x_2,t)$ $(x_2,t) \in (0,\gamma_0) \times (0,T)$, and $\psi(x,T) = 0$ $x \in \Omega_\varepsilon$, and integrate by parts in $\Omega_0 \times (0,T)$:

$$\int_0^T \Big(-\int_{\Omega_0} R_\varepsilon^+ \,\partial_t\psi \, dx - \int_{\Theta_\varepsilon^{(0)}} \partial_{x_2} R_\varepsilon^+(x_1,0)\, \psi \, dx_1 + \int_{\Omega_0} \nabla_x R_\varepsilon^+ \cdot \nabla_x \psi \, dx$$
$$- \int_{\Omega_0} f_0 \psi \, dx \Big)\, dt = I_0^+(\varepsilon,\psi) + \ldots + I_4^+(\varepsilon,\psi), \quad (39)$$

where

$$I_0^+(\varepsilon,\psi) := \varepsilon \int_{\Omega_0\times(0,T)} \chi_0(x_2) \partial_t \mathcal{N}^+(\xi,x_1,t)\, \psi \, dx\, dt,$$

$$I_1^+(\varepsilon,\psi) := -\int_{\Omega_0\times(0,T)} \chi_0'(x_2)\big(\partial_{\xi_2}\mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}\,\psi\,dx\,dt,$$
$$I_2^+(\varepsilon,\psi) := -\int_{\Omega_0\times(0,T)} \chi_0(x_2)\big(\partial^2_{x_1\xi_1}\mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}\,\psi\,dx\,dt,$$
$$I_3^+(\varepsilon,\psi) := \varepsilon\int_{\Omega_0\times(0,T)} \chi_0'(x_2)\mathcal{N}^+(x/\varepsilon,x_1,t)\,\partial_{x_2}\psi\,dx\,dt,$$
$$I_4^+(\varepsilon,\psi) := \varepsilon\int_{\Omega_0\times(0,T)} \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}^+(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}\big)\,\partial_{x_1}\psi\,dx\,dt.$$

Discrepancies in the thin rods. It is easy to calculate that $\partial_{x_2}R_\varepsilon^{1,-}(x_1,-d_1,t) = k_1R_\varepsilon^{1,-}(x_1,-d_1,t)$ on $\Theta_\varepsilon^{(1)}\times(0,T)$, $\partial_{x_2}R_\varepsilon^{2,-}(x_1,-d_2)=0$ on $\Theta_\varepsilon^{(2)}\times(0,T)$,

$$\partial_{x_2}R_\varepsilon^{i,-}(x_1,0,t) = \varepsilon Y_1\big(\frac{x_1}{\varepsilon}\big)\partial^2_{x_2x_1}v_0^{i,-}(x_1,0,t)+\partial_{x_2}R_\varepsilon^+(x_1,0,t),\ x\in I_0\cap G_\varepsilon^{(i)};\quad (40)$$

$$\partial_\nu R_\varepsilon^{i,-} = \frac{1}{\sqrt{1+\frac{\varepsilon^2|h_i'|^2}{4}}}\Big(\pm\varepsilon\big(Y_1(\frac{x_1}{\varepsilon})\partial^2_{x_1x_1}v_0^{i,-}(x,t) + \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}^-(\xi,x_1,t)\big)|_{\xi=\frac{x}{\varepsilon}}\big)$$
$$-\varepsilon 2^{-1}h_i'(x_2)\,\partial_{x_2}\big(v_0^{i,-} + \varepsilon Y_1(\frac{x_1}{\varepsilon})\partial_{x_1}v_0^{i,-}\big)\Big),\quad (x,t)\in\Upsilon_\varepsilon^{(i,\pm)},\quad i=1,2.\quad (41)$$

Putting $R_\varepsilon^{i,-}$ into the differential equation of problem (1), we obtain

$$\partial_t R_\varepsilon^{i,-} - \Delta_x R_\varepsilon^{i,-} = \varepsilon\Big(Y_1(\xi_1)\,\partial^2_{x_1t}v_0^{i,-}(x,t) + \chi_0(x_2)\partial_t\mathcal{N}^-\big(\xi,x_1,t\big)\Big)|_{\xi=\frac{x}{\varepsilon}}$$
$$-\chi_0'(x_2)\big(\partial_{\xi_2}\mathcal{N}^-(\xi,x_1,t)\big)|_{\xi=x/\varepsilon} - \chi_0(x_2)\big(\partial^2_{x_1\xi_1}\mathcal{N}^-(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}$$
$$-\varepsilon\partial_{x_2}\big(\chi_0'(x_2)\mathcal{N}^-(x/\varepsilon,x_1,t)\big) - \varepsilon\chi_0(x_2)\partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}^-(\xi,x_1,t)\big)|_{\xi=x/\varepsilon}\big)$$
$$-\varepsilon\partial_{x_1}\Big(Y_1\big(\frac{x_1}{\varepsilon}\big)\partial^2_{x_1x_1}v_0^{i,-}\Big) - \varepsilon\partial_{x_2}\Big(Y_1\big(\frac{x_1}{\varepsilon}\big)\partial^2_{x_2x_1}v_0^{i,-}\Big)$$
$$+\partial_{x_2}\big(\ln h_i(x_2)\big)\,\partial_{x_2}v_0^{i,-}(x,t) - 2k_ih_i^{-1}(x_2)v_0^{i,-}(x,t),$$
$$(x,t)\in G_\varepsilon^{(i)}\times(0,T),\quad i=1,2.\quad (42)$$

Using (7) and taking into account the boundary values of $\partial_\nu R_\varepsilon^{i,-}$ (see (40), (41)), we multiply (42) by a test function $\psi\in H^1\big(\Omega_\varepsilon\times(0,T)\big)$ such that $\psi(0,x_2,t)=\psi(a,x_2,t)$ on $(0,\gamma_0)\times(0,T)$, $\psi(x,T)=0$ and integrate by parts in $G_\varepsilon^{(i)}\times(0,T)$, $i=1,2$. This yields

$$\int_0^T\Big(-\int_{G_\varepsilon^{(1)}} R_\varepsilon^{1,-}\,\partial_t\psi\,dx + \int_{I_0\cap\partial G_\varepsilon^{(1)}}\partial_{x_2}R_\varepsilon^+(x_1,0,t)\,\psi\,dx_1$$
$$+\int_{G_\varepsilon^{(1)}}\nabla_xR_\varepsilon^{1,-}\cdot\nabla_x\psi\,dx + \varepsilon k_1\int_{\Upsilon_\varepsilon^{(1,\pm)}}R_\varepsilon^{1,-}\psi\,dl_x + k_1\int_{\Theta_\varepsilon^{(1)}}R_\varepsilon^{1,-}\psi\,dx_1$$

$$- \varepsilon^{\beta} \int_{\Upsilon_\varepsilon^{(1,\pm)}} g_\varepsilon \psi \, dl_x \Big) \, dt \; = \sum_{j=0}^{7} I_j^{1,-}(\varepsilon, \psi), \tag{43}$$

$$\int_0^T \Big(- \int_{G_\varepsilon^{(2)}} R_\varepsilon^{2,-} \, \partial_t \psi \, dx + \int_{I_0 \cap \partial G_\varepsilon^{(2)}} \partial_{x_2} R_\varepsilon^{+}(x_1, 0, t) \, \psi \, dx_1 + \int_{G_\varepsilon^{(2)}} \nabla_x R_\varepsilon^{2,-} \cdot \nabla_x \psi \, dx$$

$$+ \varepsilon^{\alpha} k_2 \int_{\Upsilon_\varepsilon^{(2)}} R_\varepsilon^{2,-} \psi \, dl_x \; - \; \varepsilon^{\beta} \int_{\Upsilon_\varepsilon^{(2)}} g_\varepsilon \psi \, dl_x \Big) \, dt \; = \; \sum_{j=0}^{7} I_j^{2,-}(\varepsilon, \psi), \tag{44}$$

where

$$I_0^{i,-}(\varepsilon, \psi) = \varepsilon \int_{G_\varepsilon^{(i)} \times (0,T)} \Big(Y_1(\xi_1) \, \partial^2_{x_1 t} v_0^{i,-}(x,t) + \chi_0(x_2) \partial_t \mathcal{N}^-\big(\xi, x_1, t\big) \Big)\big|_{\xi = \frac{x}{\varepsilon}} \, \psi \, dx \, dt,$$

$$I_1^{i,-}(\varepsilon, \psi) = - \int_{G_\varepsilon^{(i)} \times (0,T)} \chi_0'(x_2) \big(\partial_{\xi_2} \mathcal{N}^-(\xi, x_1, t) \big)\big|_{\xi = x/\varepsilon} \, \psi \, dx \, dt,$$

$$I_2^{i,-}(\varepsilon, \psi) = - \int_{G_\varepsilon^{(i)} \times (0,T)} \chi_0(x_2) \big(\partial^2_{x_1 \xi_1} \mathcal{N}^-(\xi, x_1, t) \big)\big|_{\xi = x/\varepsilon} \, \psi \, dx \, dt,$$

$$I_3^{i,-}(\varepsilon, \psi) = \varepsilon \int_{G_\varepsilon^{(i)} \times (0,T)} \chi_0'(x_2) \mathcal{N}^-(x/\varepsilon, x_1, t) \, \partial_{x_2} \psi \, dx \, dt,$$

$$I_4^{i,-}(\varepsilon, \psi) = \varepsilon \int_{G_\varepsilon^{(i)} \times (0,T)} \chi_0(x_2) \big(\partial_{x_1} \mathcal{N}^-(\xi, x_1, t) \big)\big|_{\xi = x/\varepsilon} \big) \, \partial_{x_1} \psi \, dx \, dt,$$

$$I_5^{i,-}(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon^{(i)} \times (0,T)} Y_1\big(\frac{x_1}{\varepsilon}\big) \Big(\nabla_x \big(\partial_{x_1} v_0^{i,-} \big) \cdot \nabla_x \psi + \partial_{x_1} \big(\psi \, \partial_{x_2} (\ln h_i) \, \partial_{x_2} v_0^{i,-} \big) \Big) \, dx \, dt,$$

$$I_6^{1,-}(\varepsilon, \psi) = - k_1 \, \varepsilon \int_{\Upsilon_\varepsilon^{(1,\pm)} \times (0,T)} \frac{v_0^{1,-} \, \psi}{\sqrt{1 + \varepsilon^2 4^{-1} |h_1'(x_2)|^2}} \; dl_x dt$$

$$+ \, k_1 \varepsilon \int_{\Upsilon_\varepsilon^{(1,\pm)} \times (0,T)} R_\varepsilon^{1,-} \psi \, dl_x \, dt \; - \; 2 k_1 \, \varepsilon \int_{G_\varepsilon^{(1)} \times (0,T)} Y\big(\frac{x_1}{\varepsilon}\big) \, \frac{\partial_{x_1}(v_0^{1,-} \psi)}{h_1(x_2)} \, dx \, dt,$$

$$I_6^{2,-}(\varepsilon, \psi) = - \varepsilon \, \delta_{\alpha,1} k_2 \int_{\Upsilon_\varepsilon^{(2,\pm)} \times (0,T)} \frac{v_0^{2,-} \, \psi}{\sqrt{1 + \varepsilon^2 4^{-1} |h_2'(x_2)|^2}} \; dl_x dt$$

$$+ \; \varepsilon^{\alpha} k_2 \int_{\Upsilon_\varepsilon^{(2)} \times (0,T)} R_\varepsilon^{2,-} \psi \, dl_x \, dt \; - \; 2 \delta_{\alpha,1} k_2 \, \varepsilon \int_{G_\varepsilon^{(2)} \times (0,T)} Y\big(\frac{x_1}{\varepsilon}\big) \, \frac{\partial_{x_1}(v_0^{2,-} \psi)}{h_2(x_2)} \, dx \, dt,$$

$$I_7^{i,-}(\varepsilon, \psi) = \varepsilon \delta_{\beta,1} \int_{\Upsilon_\varepsilon^{(i,\pm)} \times (0,T)} \frac{g_0 \, \psi}{\sqrt{1 + \varepsilon^2 4^{-1} |h_i'|^2}} \, dl_x dt - \, \varepsilon^{\beta} \int_{\Upsilon_\varepsilon^{(i,\pm)} \times (0,T)} g_\varepsilon \psi \, dl_x dt$$

$$- \; \varepsilon^{\beta} \delta_{i,2} \int\limits_{\Theta_\varepsilon^{(2)} \times (0,T)} g_\varepsilon \psi \, dx_2 \, dt \; + + 2 \varepsilon \delta_{\beta,1} \int\limits_{G_\varepsilon^{(i)} \times (0,T)} Y\big(\frac{x_1}{\varepsilon}\big) \, \frac{\partial_{x_1}(g_0 \, \psi)}{h_i(x_2)} \, dx dt; \;\; i = 1, 2.$$

Asymptotic estimates. Summing (39), (43) and (44), we see that the function R_ε constructed by formulas (35) and (36) satisfies the following integral identity

$$\int_0^T \Big(-\int_{\Omega_\varepsilon} R_\varepsilon\, \partial_t \psi\, dx + \int_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi\, dx + \varepsilon k_1 \int_{\Upsilon_\varepsilon^{(1,\pm)}} R_\varepsilon\, \psi\, dl_x + k_1 \int_{\Theta_\varepsilon^{(1)}} R_\varepsilon\, \psi\, dx_2$$
$$+ \varepsilon^\alpha k_2 \int_{\Upsilon_\varepsilon^{(2)}} R_\varepsilon\, \psi\, dl_x - \int_{\Omega_0} f_0\, \psi\, dx - \varepsilon^\beta \int_{\Upsilon_\varepsilon^{(1,\pm)} \cup \Upsilon_\varepsilon^{(2)}} g_\varepsilon\, \psi\, dl_x \Big)\, dt = F_\varepsilon(\psi) \quad (45)$$

for any function $\psi \in H^1\left(\Omega_\varepsilon \times (0,T)\right)$, $\psi(x,T) = 0$. Here $F_\varepsilon(\psi) = I_0^\pm(\varepsilon,\psi) + \ldots + I_4^\pm(\varepsilon,\psi) + I_5^-(\varepsilon,\psi) + \ldots + I_7^-(\varepsilon,\psi)$; $I_j^\pm(\varepsilon,\psi) = I_j^+(\varepsilon,\psi) + I_j^-(\varepsilon,\psi)$, $j = 0,\ldots,4$; $I_j^-(\varepsilon,\psi) = I_j^{1,-}(\varepsilon,\psi) + I_j^{2,-}(\varepsilon,\psi)$, $j = 0,\ldots,7$.

Subtracting the integral identity (4) from (45), we get

$$\int_0^T \Big(-\int_{\Omega_\varepsilon} (R_\varepsilon - u_\varepsilon)\, \partial_t \psi\, dx + \int_{\Omega_\varepsilon} \nabla_x (R_\varepsilon - u_\varepsilon) \cdot \nabla_x \psi\, dx + k_1 \int_{\Theta_\varepsilon^{(1)}} (R_\varepsilon - u_\varepsilon)\, \psi\, dx_2$$
$$+ \varepsilon k_1 \int_{\Upsilon_\varepsilon^{(1,\pm)}} (R_\varepsilon - u_\varepsilon)\, \psi\, dl_x + \varepsilon^\alpha k_2 \int_{\Upsilon_\varepsilon^{(2)}} (R_\varepsilon - u_\varepsilon)\, \psi\, dl_x \Big)\, dt = F_\varepsilon(\psi). \quad (46)$$

Now we are going to estimate the value $F_\varepsilon(\psi)$. Using the Cauchy–Schwartz–Bunyakovskii inequality, it is easy to verify that $|I_0^\pm(\varepsilon,\psi)| \le C_0 \varepsilon \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. The summands $I_1^\pm, \ldots, I_4^\pm$ are estimated by using the same technics as in [29]. As a result, we obtain that $|I_1^\pm(\varepsilon,\psi) + I_3^\pm(\varepsilon,\psi)| \le \varepsilon C_1 \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$, $|I_2^\pm(\varepsilon,\psi)| \le \varepsilon^{1-\rho} C_2 \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$, and $|I_4^\pm(\varepsilon,\psi)| \le \varepsilon^{3/2} C_4 \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$, where ρ is the arbitrary fixed positive number.

Remark 2. The constant C_0 depends on

$$\|\partial^2_{tx_1} v_0^{i,-}\|_{L^2(D_i \times (0,T))}, \quad i = 1,2, \quad \text{and} \quad \sup_{(x,t) \in I_0 \times (0,T)} \left|\partial^2_{tx_j} v_0^+(x,t)\right|, \quad j = 1,2.$$

The constant C_4 depends on the following quantities $\sup_{(x,t) \in I_0 \times (0,T)} \left|\mathcal{D}^\alpha\big(v_0^+(x,t)\big)\right|$, $|\alpha| = \alpha_1 + \alpha_2 \le 2$. Due to the assumptions for f_0 and g_0 and by virtue of classical results on the smoothness of solutions to boundary value problems, these quantities are bounded.

Since $\partial_{x_1} g_0 \in L^2(0,T;H^1(D_1))$, $|I_5^{i,-}(\varepsilon,\psi)| \le \varepsilon C_5 \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

To estimate I_6^-, we consider more complex summand $I_6^{2,-}$. First, let $\alpha = 1$. It is obvious that the third summand in $I_6^{2,-}$ is not greater than $C\varepsilon \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. The sum of the first and second summands is equal to

$$\varepsilon^3 4^{-1} k_2 \int_{\Upsilon_\varepsilon^{(2,\pm)} \times (0,T)} \frac{|h_2'|^2\, v_0^{2,-}\, \psi}{\big(1 + \sqrt{1 + \varepsilon^2 4^{-1} |h_2'|^2} + \varepsilon^2 4^{-1} |h_2'|^2\big)}\, dl_x dt$$

$$+\,\varepsilon^2 k_2 \int_{\Upsilon_\varepsilon^{(2,\pm)}\times(0,T)} \Big(Y\big(\frac{x_1}{\varepsilon}\big)\partial_{x_1} v_0^{2,-}(x,t) + \chi_0(x_2)\mathcal{N}^-\Big)\,\psi\, dl_x dt$$

$$+\,\varepsilon\, k_2 \int_{\Theta_\varepsilon^{(2)}\times(0,T)} \Big(v_0^{2,-}(x_1,-d_2,t) + Y_1(\xi_1)|_{\xi_1=\frac{x_1}{\varepsilon}}\, \partial_{x_1} v_0^{2,-}(x_1,-d_2,t)\Big)\psi\, dx_1 dt$$

$$=: J_1(\varepsilon,\psi) + J_2(\varepsilon,\psi) + J_3(\varepsilon,\psi).$$

With the help of the following inequality $u^2(0) \le 2\varepsilon^{-1}\int_0^\varepsilon u^2(t)\,dt + 2\varepsilon\int_0^\varepsilon (u')^2(t)\,dt$, we deduce that $|J_1(\varepsilon,\psi) + J_2(\varepsilon,\psi)| \le C\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Taking into account the boundedness of the trace operator and that $g_0 \in H^1(D_1)$, we have

$$|J_3(\varepsilon,\psi)| \le c_1\varepsilon\|\psi\|_{L_2(\Theta_\varepsilon^{(2)}\times(0,T))} \le c_2\varepsilon\|\psi\|_{L^2(0,T;H^1(G_\varepsilon^{(2)}))}.$$

Thus in this case $|I_6^-(\varepsilon,\psi)| \le \varepsilon\, C_6\, \|\psi\|_{H^1(L^2(0,T;H^1(\Omega_\varepsilon))}$.

If $\alpha > 1$, then $I_6^{2,-}(\varepsilon,\psi) = \varepsilon^\alpha k_2 \int_{\Upsilon_\varepsilon^2\times(0,T)} R_\varepsilon^{2,-}\psi\, dl_x\, dt$, and with the help of the identity (7) we derive that $|I_6^{2,-}(\varepsilon,\psi)| \le \varepsilon^{\alpha-1}C_6\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

By the same arguments as for $I_6^{2,-}$, we can estimate I_7^-. But for this we should use the assumptions for the functions g_ε and g_0. Thus

$$|I_7^-(\varepsilon,\psi)| \;\le\; C_7\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\begin{cases} \varepsilon\|g_0 - g_\varepsilon\|_{L^2(D_1\times(0,T))}, & \text{if } \beta = 1,\\ \varepsilon^{\beta-1}, & \text{if } \beta > 1.\end{cases}$$

Regarding to the inequalities obtained above, we conclude that for the right-hand side in (46) the following inequality holds

$$|F_\varepsilon(\psi)| \le \Big(C_8\varepsilon + \varepsilon^{1-\rho}\, C_2(\rho) + C_6\varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + C_7\varepsilon^{\delta_{\beta,1}(2-\beta)+\beta-1}\|g_0 - g_\varepsilon\|^{\delta_{\beta,1}}_{L^2(D_1\times(0,T))}\Big)\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}, \quad (47)$$

where ρ is an arbitrary positive fixed number from $(0,\frac{1}{2})$.

Due to Lemma 1.1, we deduce from (46) and (47) with the standard scheme (see, for example, Ref. [32, Sect. 3]) the asymptotic estimate (37). ■

Corollary 4.2. *From (37) it follows that*

$$\|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon\times(0,T))} + \max_{t\in[0,T]}\|u_\varepsilon(\cdot,t) - v_0(\cdot,t)\|_{L^2(\Omega_\varepsilon)} \le C_1\Big(\varepsilon + \varepsilon^{1-\rho} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + \varepsilon^{\delta_{\beta,1}(2-\beta)+\beta-1}\|g_0 - g_\varepsilon\|^{\delta_{\beta,1}}_{L^2(D_1\times(0,T))}\Big),$$

where v_0 coincides with the solution to the limit problem (33) by the following way: v_0 is the restriction of v_0^+ on Ω_0, v_0 coincides with $v_0^{1,-}$ on the thin rods $G_\varepsilon^{(1)}$ and with $v_0^{2,-}$ on the thin rods $G_\varepsilon^{(2)}$.

5. Convergence Theorem

As it was shown in [18–22], thick multistructures are not strong or weak connected domains, i.e., there is not any sequence of extension operators $\{ \mathbf{P}_\varepsilon : H^1(\Omega_\varepsilon) \mapsto H^1(\mathbb{R}^n) \}_{\varepsilon>0}$ whose norms are uniformly bounded in ε. This fact creates one of the main difficulties in the proofs of convergence theorems.

There are different methods to prove such convergence theorems. The first convergence theorems for the solutions to boundary value problems in thick junctions of different types were proved in [18–20], where there were used special extension operators whose H^1-norms were uniformly bounded in ε only for the solutions. This approach allows to prove the convergence theorems if the boundaries of thin domains of thick junctions are not smooth and rectilinear with respect to some variables and in the case of different boundary conditions on the boundaries of thin domains; in the last case the method of special integral identities is used in addition (see [22, 35, 27]).

Later, in [24], where a homogeneous Neumann boundary value problem was studied in a thick junction, it was shown that if the boundaries of thin rods were rectilinear, then the solution could be extended by zero. This is explained by the fact that this extension preserves the generalized derivative with respect to x_2 due to the rectilinearity of the boundaries of the rods along the Ox_2-axis. This approach was used to prove the convergence theorem for nonlinear problems in [25]. Also, in [24], the homogeneous Neumann problem was considered in a bounded plane domain whose boundary was waved by the function $x_2 = h(x_1/\varepsilon)$, where h had to be a continuously differentiable periodic function, and the reciprocal functions of h on some intervals had to exist for a special extension operator to be constructed. But this extension does not preserve the space class of the solution (only in $H^1_{loc}(\Omega_1^+)$, where $\Omega_1^+ \subset \mathbb{R}^2$ is a domain that is filled up by the oscillating boundary in the limit) and this extension was constructed under the assumption that the right-hand side $f \in H^1$. In this section we prove the convergence theorem for the solution to problem (1) with minimal conditions for the functions f_0 and g_ε.

In addition to the assumptions made in Sect. 1, we suppose that for any $T > 0$ there exist positive constants C_1, ε_0 such that for the whole value $\varepsilon \in (0, \varepsilon_0)$

$$\int_0^T \int_{\Omega_0} \mathbf{f}_\varepsilon^2(x,t)\ dxdt \le C_1, \quad \mathbf{f}_\varepsilon(x,t) = \varepsilon^{-1}(\, f_0(x_1+\varepsilon, x_2, t) - f_0(x,t)\,). \tag{48}$$

We regard that f_0 and g_ε are $a-$periodic with respect to x_1. In fact, every function from the space $L^2(\Omega_0 \times (0,T))$ is continuous with respect to the L^2-norm, but in (48) we need little more.

Theorem 5.1. *If the conditions (2), (3) and (48) hold, then for any $T > 0$ there exist extension operators $\mathbf{P}_\varepsilon^{(1)} :\ L^2\big(0,T; H^1(\Omega_0 \cup G_\varepsilon^{(1)})\big) \mapsto L^2\big(0,T; H^1(\Omega_1)\big)$*

and $\mathbf{P}^{(2)}_\varepsilon: \ L^2\big(0,T;H^1(\Omega_0\cup G^{(2)}_\varepsilon)\big)\mapsto L^2\big(0,T;H^1(\Omega_2)\big)$ *such that for the solution* u_ε *to problem (1) we have*

$$\parallel \mathbf{P}^{(1)}_\varepsilon u_\varepsilon \parallel_{L^2(0,T;\,H^1(\Omega_1))} + \parallel \mathbf{P}^{(2)}_\varepsilon u_\varepsilon \parallel_{L^2(0,T;\,H^1(\Omega_2))} \le C_2. \tag{49}$$

P r o o f. From the beginning we show that the scattering of values of solution u_ε on thin rods is small in a sense.

Here, for simplicity we assume that $\gamma\equiv const$. In general case we should use the procedure from the proof of Th. 4.1 ([19]). Thus, the problem (1) is invariant under ε-shift along the axis x_1. This means that the function $\mathbf{U}_\varepsilon(x,t)=\varepsilon^{-1}(u_\varepsilon(x+\varepsilon\overline{e}_1,t)-u_\varepsilon(x,t))$ $(\overline{e}_1=(1,0))$ is $a-$periodic in x_1 solution to the following problem:

$$\begin{aligned}
\partial_t\mathbf{U}_\varepsilon &= \Delta_x\mathbf{U}_\varepsilon+\mathbf{F}_\varepsilon, && (x,t)\in\Omega_0\times(0,T);\\
\partial_t\mathbf{U}_\varepsilon &= \Delta_x\mathbf{U}_\varepsilon, && (x,t)\in G_\varepsilon\times(0,T);\\
\partial_\nu\mathbf{U}_\varepsilon+\varepsilon k_1\mathbf{U}_\varepsilon &= \varepsilon^\beta\mathbf{G}_\varepsilon, && (x,t)\in\Upsilon^{(1,\pm)}_\varepsilon\times(0,T);\\
\partial_\nu\mathbf{U}_\varepsilon+\varepsilon^\alpha k_2\mathbf{U}_\varepsilon &= \varepsilon^\beta\mathbf{G}_\varepsilon, && (x,t)\in\Upsilon^{(2)}_\varepsilon\times(0,T);\\
\partial_\nu\mathbf{U}_\varepsilon+k_1\mathbf{U}_\varepsilon &= 0, && (x,t)\in\Theta^{(1)}_\varepsilon\times(0,T);\\
\partial_\nu\mathbf{U}_\varepsilon &= 0, && (x,t)\in\Gamma_\varepsilon\times(0,T);\\
\mathbf{U}_\varepsilon(x,0) &= 0, && x\in\Omega_\varepsilon\times\{t=0\},
\end{aligned} \tag{50}$$

where $\mathbf{G}_\varepsilon(x,t)=\varepsilon^{-1}(g_\varepsilon(x+\varepsilon\overline{e}_1,t)-g_\varepsilon(x,t))$. By virtue of condition (2), Lem. 1.1, identity (7) and (48), we get the following estimate $\|\mathbf{U}_\varepsilon\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))}\le C_3$.

We extend the solution u_ε by using the "linear matching"

$$\widehat{P}^{(i)}_\varepsilon(u_\varepsilon)=\begin{cases} u_\varepsilon, & \text{in } (\Omega_0\cup G^{(i)}_\varepsilon)\times(0,T),\\ B^\varepsilon_{j,i}+S^\varepsilon_{j,i}\left(x_1-\varepsilon\big(j+b_i+\frac{h_i(x_2)}{2}\big)\right), & \text{in } \widetilde{Q}^{(i)}_j(\varepsilon)\times(0,T),\end{cases} \tag{51}$$

in domain $\Omega_0\cup G^{(i)}_\varepsilon\cup\widetilde{Q}^{(i)}_\varepsilon$. Here

$$B^\varepsilon_{j,i}(x_2,t)=u_\varepsilon\big(\varepsilon(j+b_i+2^{-1}h_i(x_2)),x_2,t\big),$$

$$S^\varepsilon_{j,i}(x_2,t)=\frac{1}{\varepsilon(1-h_i(x_2))}\left(u_\varepsilon\Big(\varepsilon(j+1+b_i-2^{-1}h_i(x_2)),x_2,t\Big)-B^\varepsilon_j(x_2,t)\right),$$

$$\widetilde{Q}^{(i)}_\varepsilon=\bigcup_{j=-1}^{N}\widetilde{Q}^{(i)}_j(\varepsilon),$$

where

$$\widetilde{Q}^{(i)}_j(\varepsilon)=\left\{x:\ \ x_2\in(-d_i,-\varepsilon),\ \ x_1\in\left(\varepsilon\frac{j+b_i+h_i(x_2)}{2},\varepsilon\frac{j+1+b_i-h_i(x_2)}{2}\right)\right\}$$

is between two rods $G_j^{(i)}(\varepsilon)$ and $G_{j+1}^{(i)}(\varepsilon)$; recall that index $i \in \{1,2\}$ is fixed. In the case of extreme rods we perform the $a-$periodic extension of problem (1) with respect to the axis Ox_1.

After that, repeating word for word the steps from the proof of Th. 3.1 ([27]) and using the estimates (2) and (9), we obtain that the norms $\|\widehat{P}_\varepsilon^{(i)}(u_\varepsilon)\|_{L^2\left(0,T;\,H^1(\Omega_0\cup G_\varepsilon^{(i)}\cup\widetilde{Q}_\varepsilon^{(i)})\right)}$, $i=1,2$, are bounded with respect to ε.

Now it remains to extend $\widehat{P}_\varepsilon^{(i)}(u_\varepsilon)$ into each domain

$$T_j^{(i)}(\varepsilon) = \left\{x: \ x_2 \in (-\varepsilon,0), \ x_1 \in \left(\varepsilon\frac{j+b_i+h_i(x_2)}{2}, \varepsilon\frac{j+1+b_i-h_i(x_2)}{2}\right)\right\},$$

$j=-1,0,1,\ldots,N$. Since the domains $T_j^{(i)}(\varepsilon)$, $j=-1,0,1,\ldots,N$, are equal (each of this domain can be obtained from $T_0^{(i)}(\varepsilon)$ by parallel shift along the axis Ox_1), we use the results on the extension operators in perforated domains [6]. It follows from these results that there exists a uniformly bounded in ε extension operator $\mathfrak{P}_\varepsilon^{(i)}: L^2\big(0,T;H^1\big(G^{(i)}(\varepsilon)\cup\widetilde{Q}^{(i)}(\varepsilon)\big)\big) \mapsto L^2\big(0,T;H^1(\Omega_i)\big)$, $i=1,2$.

Thus, the extension operators $\mathbf{P}_\varepsilon^{(i)} := \mathfrak{P}_\varepsilon^{(i)} \circ \widehat{P}_\varepsilon^{(i)}$, $i=1,2$, are constructed and (49) holds. ∎

Theorem 5.2. *If (48) and assumptions made for f_0, g_ε in Sect. 1 hold, then*

$$\big(u_\varepsilon\big)|_{\Omega_0} \to v_0^+, \qquad \Big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\Big)|_{D_1} \to v_0^{1,-}, \qquad \Big(\mathbf{P}_\varepsilon^{(2)}u_\varepsilon\Big)|_{D_2} \to v_0^{2,-} \tag{52}$$

weakly in $L^2\big(0,T;\,H^1(\Omega_0)\big)$, $L^2\big(0,T;\,H^1(D_1)\big)$, $L^2\big(0,T;\,H^1(D_2)\big)$, *respectively, as* $\varepsilon\to 0$, *where the vector function* $\mathbf{v}_0(x,t) = \big(v_0^+, v_0^{1,-}, v_0^{2,-}\big)$ *is the unique weak solution to the limit problem (33).*

P r o o f. We carry out the proof in a more difficult case when $\alpha=\beta=1$. To prove this theorem we should pass to the limit in the integral identity (4). For this we use the identity (7), the extension operators constructed in Th. 2 and the characteristic function $\chi_\varepsilon^{(i)}(x) := \chi^{(i)}\big(\frac{x_1}{\varepsilon}, x_2\big)$ of the set $\overline{G_\varepsilon^{(i)}}$, $i=1,2$. We ε-periodically extend these functions with respect to x_1. In the same way as in Sect. 4 [35], we can prove that $\chi_\varepsilon^{(i)} \to h_i$ weakly in $L_2(D_i)$ as $\varepsilon\to 0$, $i=1,2$. Also, it is easy to verify that $\chi_\varepsilon^{(i)}|_{x_2=\varrho} \to h_i(\varrho)$ weakly in $L_2(0,a)$ as $\varepsilon\to 0$.

In view of inequality (49) and Lem. 3 in [36, Ch. 6], for any $\theta\in L_2(0,T)$ we can choose a subsequence $\{\varepsilon'\}$ (we denote it again by $\{\varepsilon\}$) such that if $\varepsilon\to 0$, then the limits (52) hold and, in addition,

$$\int_0^T u_\varepsilon(\cdot,t)\,\theta(t)\,dt \to \int_0^T v_0^+(\cdot,t)\,\theta(t)\,dt,$$

$$\int_0^T \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)|_{D_i}\, \theta(t)\, dt \to \int_0^T v_0^{i,-}(\cdot,t)\, \theta(t)\, dt, \tag{53}$$

weakly in $H^1(\Omega_0)$, $H^1(D_i)$, and strongly in $L^2(\Omega_0)$, $L^2(D_i)$, $i=1,2$, respectively, and

$$\partial_{x_q} \int_0^T u_\varepsilon(x,t)\, \theta(t)\, dt = \int_0^T \partial_{x_q} u_\varepsilon(x,t)\, \theta(t)\, dt \to \partial_{x_q} \int_0^T v_0^+\, \theta\, dt$$
$$= \int_0^T \partial_{x_q} \big(v_0^+\big)\, \theta\, dt, \tag{54}$$
$$\partial_{x_q} \int_0^T \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)|_{D_i}\, \theta(t)\, dt = \int_0^T \partial_{x_q} \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)|_{D_i} \theta\, dt \to \partial_{x_q} \int_0^T v_0^{i,-}\, \theta\, dt$$
$$= \int_0^T \partial_{x_q} \big(v_0^{i,-}\big)\, \theta\, dt, \quad q=1,2, \tag{55}$$

weakly in $L^2(\Omega_0)$, $L^2(D_i)$, $i=1,2$, respectively.

Consider a set of the following test vector functions $\mathcal{C} = \{\theta(t)\, \mathbf{\Phi}(x) : \theta \in C^1([0,T]),\ \theta(T)=0,\ \mathbf{\Phi}(x) = \big(\varphi_0(x),\ x\in\overline{\Omega}_0;\ \varphi_1(x),\ x\in\overline{D}_1;\ \varphi_2(x),\ x\in\overline{D}_2\big),$ $\varphi_0 \in C^\infty(\overline{\Omega}_0)$, $\varphi_0(0,x_2)=\varphi_0(a,x_2)$, $x_2\in(0,\gamma_0)$, $\varphi_i\in C^\infty(\overline{D}_i)$, $i=1,2$, $\varphi_0|_{I_0} = \varphi_1|_{I_0} = \varphi_2|_{I_0}\}$. The set of these functions is dense in $L^2(0,T;\mathcal{H}_0)$ and the set of their restrictions $\{\theta(t)\,\big(\varphi_0, \varphi_1|_{G_\varepsilon^{(1)}}, \varphi_2|_{G_\varepsilon^{(2)}}\big)\}$ is dense in $L^2(0,T;\mathcal{H}_\varepsilon)$.

By using the extension operators $\mathbf{P}_\varepsilon^{(i)}$, the functions $\chi_\varepsilon^{(i)}$, $i=1,2$, and equality (7), we rewrite the identity (4) with any of the test functions mentioned above in the form

$$-\int_{\Omega_0} \left(\int_0^T u_\varepsilon(x,t)\, \partial_t\theta(t)\, dt\right) \varphi_0\, dx - \sum_{i=1}^2 \int_{D_i} \chi_\varepsilon^{(i)} \left(\int_0^T \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)\, \partial_t\theta(t)\, dt\right) \varphi_i\, dx$$
$$+\int_{\Omega_0} \nabla_x \left(\int_0^T u_\varepsilon\, \theta\, dt\right) \cdot \nabla_x\varphi_0\, dx + \sum_{i=1}^2 \left(\int_{D_i} \chi_\varepsilon^{(i)}\, \nabla_x \left(\int_0^T \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)\, \theta\, dt\right) \cdot \nabla_x \varphi_i\, dx\right.$$
$$+2k_i \int_{D_i} \frac{\sqrt{1+\varepsilon^2 4^{-1} |h_i'(x_2)|^2}}{h_i(x_2)}\, \chi_\varepsilon^{(i)}(x) \left(\int_0^T (\mathbf{P}_\varepsilon^{(i)} u_\varepsilon)(x,t)\theta(t)\, dt\right) \varphi_i(x)\, dx$$
$$-2\varepsilon k_i \int_0^T \int_{G_\varepsilon^{(i)}} Y\big(\frac{x_1}{\varepsilon}\big) \frac{\sqrt{1+\varepsilon^2 4^{-1} |h_i'(x_2)|^2}}{h_i(x_2)}\, \partial_{x_1}\big(u_\varepsilon\, \varphi_i\big)\theta(t)\, dx\, dt$$
$$\left.+\varepsilon^{i-1} k_i \int_0^a \chi_\varepsilon^{(i)} \int_0^T \big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)|_{x_2=-d_i} \theta\, dt\, \varphi_i(x_1,-d_i)\, dx_1\right) = \int_0^T \int_{\Omega_0} f_0\, \theta\, \varphi_0\, dx\, dt$$
$$+2\sum_{i=1}^2 \int_0^T \int_{D_i} \frac{\sqrt{1+\varepsilon^2 4^{-1} |h_i'(x_2)|^2}}{h_i(x_2)}\, \chi_\varepsilon^{(i)}(x)\, g_\varepsilon(x,t)\theta(t)\varphi_i(x)\, dx\, dt$$

$$-2\varepsilon\sum_{i=1}^{2}\int_0^T\int_{G_\varepsilon^{(i)}} Y\big(\frac{x_1}{\varepsilon}\big)\frac{\sqrt{1+\varepsilon^2 4^{-1}|h_i'|^2}}{h_i(x_2)}\,\partial_{x_1}\big(g_\varepsilon\,\varphi_i\big)\,\theta(t)\,dx\,dt$$
$$+\varepsilon\int_0^T\int_{\Theta_\varepsilon^{(2)}} g_\varepsilon\,\varphi_2\,\theta\,dx_2\,dt. \tag{56}$$

Let us pass to the limit in (56). First, we note that the traces of the limit functions are equal, i.e., $v_0^+(x_1,0,t) = v_0^{1,-}(x_1,0,t) = v_0^{2,-}(x_1,0,t)$, $(x_1,t)\in I_0\times(0,T)$, since $\big(u_\varepsilon\big)|_{I_0} = \big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\big)|_{I_0} = \big(\mathbf{P}_\varepsilon^{(2)}u_\varepsilon\big)|_{I_0}$ a.e. in $(0,T)$. Because of (49), the sequences

$$\chi_\varepsilon^{(i)}\,\partial_{x_q}\left(\int_0^T\big(\mathbf{P}_\varepsilon^{(i)}u_\varepsilon\big)(x,t)\theta(t)\right)\,dt,\quad q=1,2, \tag{57}$$

are bounded in $L_2(D_i)$, $i=1,2$. Therefore, we can choose a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and find the weak limits $\sigma_q^{(i)}$ of these sequences in $L_2(D_i)$, $i=1,2$, as $\varepsilon\to 0$. Taking into account all these facts, (53)–(55), (2), (3), in the limit passage we obtain

$$-\int_{\Omega_0}\left(\int_0^T v_0^+(x,t)\,\partial_t\theta\,dt\right)\varphi_0(x)dx-\sum_{i=1}^{2}\int_{D_i}h_i\left(\int_0^T v_0^{i,-}(x,t)\,\partial_t\theta\,dt\right)\varphi_i(x)dx$$
$$+\int_{\Omega_0}\nabla_x\left(\int_0^T v_0^+(x,t)\theta(t)\,dt\right)\cdot\nabla_x\varphi_0\,dx+\sum_{i=1}^{2}\int_{D_i}\sum_{q=1}^{2}\sigma_q^{(i)}(x)\partial_{x_q}\varphi_i\,dx$$
$$+2\sum_{i=1}^{2}k_i\int_{D_i}\int_0^T v_0^{i,-}\,\theta\,dt\,\varphi_i\,dx+k_1\int_0^a\int_0^T h(-d_1)\,v_0^{1,-}(x_1,-d_1,t)\,\theta\,dt\,\varphi_1\,dx_1$$
$$=\int_0^T\int_{\Omega_0}f_0(x,t)\,\theta(t)\varphi_0(x)\,dx\,dt+2\sum_{i=1}^{2}\int_0^T\int_{D_i}g_0(x,t)\theta(t)\varphi_i(x)\,dx\,dt. \tag{58}$$

Next, we should find $\sigma_q^{(i)}$, $q=1,2$, $i=1,2$. In order to determine $\sigma_1^{(i)}$, $i=1,2$, we consider the integral identity (4) with the following test functions:

$$\psi_1=\begin{cases}0, & \text{in } \Omega_0\times[0,T],\\ \varepsilon Y(\frac{x_1}{\varepsilon})\,\phi_1\,\theta, & \text{in } G_\varepsilon^{(1)}\times[0,T],\\ 0, & \text{in } G_\varepsilon^{(2)}\times[0,T],\end{cases}\quad \psi_2=\begin{cases}0, & \text{in } \Omega_0\times[0,T],\\ 0, & \text{in } G_\varepsilon^{(1)}\times[0,T],\\ \varepsilon Y(\frac{x_1}{\varepsilon})\,\phi_2\,\theta, & \text{in } G_\varepsilon^{(2)}\times[0,T],\end{cases}$$

where ϕ_1 and ϕ_2 are arbitrary functions from $C_0^\infty(D_1)$ and $C_0^\infty(D_2)$ respectively, $\theta\in C^1([0,T])$, $\theta(T)=0$. It is obvious that ψ_1,ψ_2 belong to $L^2\big(0,T;\mathcal{H}_\varepsilon\big)$. As a result, we get

$$\int_{D_i}\chi_\varepsilon^{(i)}\,\partial_{x_1}\left(\int_0^T\big(\mathbf{P}_\varepsilon^{(i)}u_\varepsilon\big)(x,t)\theta(t)\right)\,dt\,\phi_i(x)\,dx=\mathcal{O}(\varepsilon),\quad \varepsilon\to 0,\quad i=1,2,$$

whence $\sigma_1^{(1)} \equiv 0$ and $\sigma_1^{(2)} \equiv 0$.

Then let us define $\sigma_2^{(1)}$. Take the arbitrary functions $\phi \in C_0^\infty(D_1)$, $\theta \in C^1([0,T])$, $\theta(T)=0$, and perform the following calculations

$$\begin{aligned}
&\int_{D_1} \chi_\varepsilon^{(1)}(x)\partial_{x_2}\left(\int_0^T \big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\big)(x,t)\,\theta(t)\right) dt\,\phi(x)\,dx \\
&= \int_0^T \theta(t)\sum_{j=0}^{N-1}\int_{G_j^{(1)}(\varepsilon)} \partial_{x_2}u_\varepsilon\,\phi(x)\,dx\,dt \\
&= \int_0^T \theta(t)\sum_{j=0}^{N-1}\Big(\int_{\Upsilon_j^{(1,\pm)}(\varepsilon)} u_\varepsilon\,\phi\,\alpha_2^{(1)}(x_2,\varepsilon)\,dl_x - \int_{G_j^{(1)}(\varepsilon)} u_\varepsilon\,\partial_{x_1}\phi\,dx\Big)\,dt \\
&= -2^{-1}\varepsilon\int_0^T\theta(t)\int_{-d_1}^0 h_1'(x_2)\sum_{j=0}^{N-1}\big(u_\varepsilon\phi\big)|_{x_1=\varepsilon(j+b_1\pm h_1(x_2)/2)}\,dx_2\,dt \\
&- \int_{D_1}\chi_\varepsilon^{(1)}(x)\int_0^T\big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\big)(x,t)\,\theta(t)\,dt\,\partial_{x_2}\phi\,dx \;=: B_1(\varepsilon)+B_2(\varepsilon). \qquad (59)
\end{aligned}$$

Here $\alpha_2^{(1)}(x_2,\varepsilon) = -\varepsilon h_1'(x_2)\Big(2\sqrt{1+\varepsilon^2 4^{-1}(h_1'(x_2))^2}\Big)^{-1}$ is the second coordinate of the outward normal $\nu_\pm^{(1)}$ (see (6)) to the lateral surfaces $\Upsilon_j^{(1,\pm)}(\varepsilon)$ of the thin rod $G_j^{(1)}(\varepsilon)$. Thanks to (53)

$$\lim_{\varepsilon\to 0} B_2(\varepsilon) = -\int_{D_1} h_1(x_2)\int_0^T v_0^{1,-}(x,t)\,\theta(t)\,dt\,\partial_{x_2}\phi(x)\,dx. \qquad (60)$$

To find the limit of $B_1(\varepsilon)$ we rewrite this value in the following way:

$$\begin{aligned}
B_1(\varepsilon) = &-\int_0^T\theta(t)\left(\frac{\varepsilon}{2}\int_{-d_1}^0 h_1'(x_2)\Big(\sum_{j=0}^{N-1}\int_{\varepsilon(j+b_1-h_0(x_2)/2)}^{\varepsilon(j+b_1+h_0(x_2)/2)}\partial_{x_1}\big(u_\varepsilon\phi\big)\,dx_1\Big)\,dx_2\right. \\
&+\left.\varepsilon\int_{-d_1}^0 h_1'(x_2)\Big(\sum_{j=0}^{N-1}\big((u_\varepsilon - v_0^{1,-})\,\phi\big)|_{x_1=\varepsilon(j+b_1-h_0(x_2)/2)}\Big)\,dx_2\right)\,dt \\
&-\int_{-d_1}^0 h_1'(x_2)\left(\sum_{j=0}^{N-1}\Big(\int_0^T v_0^{1,-}\,\theta\,dt\,\phi\Big)|_{x_1=\varepsilon(j+b_1-h_0(x_2)/2)}\,\big(\varepsilon(j+1)-\varepsilon j\big)\right)\,dx_2.
\end{aligned} \qquad (61)$$

The first term in (61) is bounded by $\varepsilon\|u_\varepsilon\|_{L^2(0,T;H^1(G_\varepsilon^{(1)})}\|\phi\|_{H^1(D_1)}$. Due to the estimate $u^2(0) \le 2\varepsilon^{-1}\int_0^\varepsilon u^2(t)\,dt + 2\varepsilon\int_0^\varepsilon (u'(t))^2\,dt$ holding for every $u\in H^1([0,\varepsilon])$, the second term in (61) is estimated by the value

$$c_1\left(\|\mathbf{P}_\varepsilon^{(1)}u_\varepsilon - v_0^{1,-}\|_{L^2(G_\varepsilon^{(1)}\times(0,T))} + \varepsilon^2\|\partial_{x_1}(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon - v_0^{1,-})\|_{L^2(G_\varepsilon^{(1)}\times(0,T))}\right)\|\phi\|_{H^1(D_1)}. \tag{62}$$

Since for almost all points $x_2 \in (-d_1, 0)$ the function $\int_0^T v_0^{1,-}\,\theta(t)\,dt \in H^1(0,a)$, the inner sum of the third term in (61) is the Riemann sum for the integral $\int_0^a\int_0^T v_0^{1,-}\,\theta(t)\,dt\,\phi\,dx_1$. Then, in view of Lebesgue's and Fubini's theorems, the limit of the third term is equal to

$$-\int_{D_1} h_1'(x_2)\int_0^T v_0^{1,-}(x,t)\,\theta(t)\,dt\,\phi(x)\,dx. \tag{63}$$

Passing to the limit in (59) and taking into account (60)-(63), we get

$$\sigma_2^{(1)}(x) = h_1(x_2)\int_0^T \partial_{x_2}v_0^{1,-}(x,t)\,\theta(t)\,dt, \qquad x\in D_1.$$

Similarly, we deduce that $\sigma_2^{(2)}(x) = h_2(x_2)\int_0^T \partial_{x_2}v_0^{2,-}(x,t)\,\theta(t)\,dt, \quad x\in D_2$.

Thus, the vector function $\mathbf{v}_0 = \left(v_0^+, v_0^{1,-}, v_0^{2,-}\right)$ satisfies the following integral identity

$$\int_0^T\left(-\left(\mathbf{v}_0, \mathbf{\Phi}\,\partial_t\theta\right)_{\mathcal{V}_0} + \left(\mathbf{v}_0, \mathbf{\Phi}\,\theta\right)_{\mathcal{H}_0}\right)dt$$
$$= \int_{\Omega_0\times(0,T)} f_0\,\varphi_0\,\theta\,dx\,dt \;+\; 2\sum_{i=1}^2\int_{D_i\times(0,T)} g_0\,\varphi_i\,\theta\,dxdt, \quad \forall\ \theta\,\mathbf{\Phi}\in\mathcal{C},$$

which means that $\mathbf{v}_0$ is a weak solution to the limit problem (33).

Due to the uniqueness of the weak solution of problem (33), the above arguments hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof.

■

Conclusion

As it was stated in [37], the multiscale modelling and computation are rapidly evolving areas of research that will have a fundamental impact on computational science and applied mathematics. They are connected with the prospect of development of more efficient methods that should be symbiosis of a new class of numerical and analytical modelling techniques. One class of multiscale problems is the boundary value problems in perturbed domains. In our paper we presented two asymptotic methods (the asymptotic approximation and the convergence theorem) for the solution to the parabolic problem (1) in the thick multilevel junction Ω_ε. An important problem for the existing multiscale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle applied to the analysis of the efficiency of the multiscale method (see [37]). We proved these estimates in Th. 4.1 and Cor. 4.2. It follows from the results that for the applied problems or for numerical calculations in thick multilevel junctions we can use the corresponding limit problem, which is simpler, instead of the initial problem with the sufficient validity. Due to Th. 5.2 we can use the limit problem (33) with minimal conditions for the right-hand sides of problem (1).

The limit problem (33) possesses a new qualitative property. We see that the local properties of heat conductivity in two levels of Ω_ε are different. But the thin rods from each level are connected through the junction's body and alternate along the joint zone. As a result, the global heat flow described by the limit problem behaves as a "two-phase system" in the region which is filled up by the thin rods from each level in the limit passage as the parameter $\varepsilon \to 0$. Due to our main results, we can state that the initial problem possesses a similar property for the sufficiently small ε.

We considered the perturbed Fourier boundary conditions on the boundaries of thin rods. These conditions mean that there is a flux of heat through these sides. At first sight it seems that there is no difference between these inhomogeneous Fourier conditions and the homogeneous Neumann conditions. As it follows from our results, it is true only if $\alpha > 1,\ \beta > 1$. If $\alpha > 1$ and $\beta = 1$, then these conditions are transformed as $\varepsilon \to 0$ in the special "waving" summands $2g_0(x,t)$ of the right-hand side in the corresponding homogenized differential equation in $D_i \times (0,T),\ i = 1,2$. If $\alpha = 1$, then we get the zeroth-order term $2k_i v_0^{i,-}$ in the corresponding homogenized differential equation in $D_i \times (0,T)$; this term describes the local quantity exhaustion. Thus radiators in the form of thick junctions are better than simple waving radiators (see the beginning of Introduction).

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