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Some Multidimensional Inverse Problems of Memory Determination in Hyperbolic Equations

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The local existence and the uniqueness of some multidimensional inverse problems for the second-order hyperbolic integro-differential equations in the class of functions having certain smoothness on time variable and analyticity on a part of spatial variables are proven.

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In this paper the local existence and the uniqueness of some multidimensional inverse problems for the second-order hyperbolic integro-differential equations in the class of functions having certain smoothness on time variable and analyticity on a part of spatial variables are proven. Unlike in paper [1], where the memory is multiplied by the solution, we consider the equations in which the memory is multiplied by the second derivative of time solution or the first-order differential operator with analytical coefficients. Problems of this type often arise in applications. A distinctive feature of problems of memory determination is the dependence of unknown function both on time and on spatial variables (a multidimensional problem). Among the problems of finding memory in hyperbolic integro-differential equations there should be mentioned papers [2, 3], where the problem with the sources distributed over the whole region is considered. In [4] the questions of uniqueness of memory determination in the wave equation on the measurement of diffused wave at the location of the point source are studied.

1. We consider an initial-boundary problem for the wave equation with memory

$$u_{tt} - u_{zz} - \triangle u = \int_0^t k(x,\tau) u_{tt}(x,z,t-\tau)d\tau, \ (x,t) \in R^{n+1}, \ z \in R_+, \qquad (1.1)$$

$$u|_{t<0} \equiv 0, u_z|_{z=0} = -\delta'(t) + g(x,t)\theta(t),\ (x,t) \in R^{n+1}. \tag{1.2}$$

Here $\triangle$ is the Laplace operator in variables $(x_1,\dots,x_n) := x$; $\delta'(t)$ is a derivative of the Dirac delta-function; $\theta(t)$ is the Heavyside function, $R_+ := \{z \in R | z > 0\}$; g is a given smooth function. For the given function $k(x,t)$ finding of the function $u(x,t)$ satisfying equations (1,1),(1,2) is a well-posed problem in the space of generalized functions. We formulate the inverse problem: to determine $k(x,t)$ via the trace of solution of problem (1,1), (1,2) on the hyperplane $z = 0$ for all $x \in R^n$, $t < T$, $T > 0$, i.e.

$$u|_{z=0} = F(x,t),\ x \in R^n,\ t < T. \tag{1.3}$$

In equation (1.1) by integrating by parts we "relocate" time derivative and introducing a new function v according to the formula

$$u = \rho(x,t)v(x,z,t), \rho(x,t) := \exp[k_0(x)t/2], k_0(x) := k(x,0)$$

from equalities (1.1), (1.2), we have

$$v_{tt} - v_{zz} = \triangle v + t\nabla k_0 \nabla v + H(x,t)v + \int_0^t h(x,t-\tau)v(x,z,\tau)d\tau,$$

$$(x,t) \in R^{n+1},\ z \in R_+, \tag{1.4}$$

$$v|_{t<0} \equiv 0, v_z|_{z=0} = -\delta'(t) - [k_0(x)/2]\delta(t) + G(x,t)\theta(t),\ (x,t) \in R^{n+1}, \tag{1.5}$$

where the equation $\exp[-k_0(x)t/2]\delta'(t) = \delta'(t) + [k_0(x)/2]\delta(t)$ is used as well as the notations are introduced

$$H(x,t) := k_{0t}(x) + k_0^2(x)/4 + t\triangle k_0(x)/2 + (t^2/2)\sum_{i=1}^n k_{0x_i}^2(x),$$

$$h(x,t) := \exp[-k_0(x)t/2]k_{tt}(x,t), \tag{1.6}$$

$$G(x,t) = \exp[-k_0(x)t/2]g(x,t).$$

From the theory of hyperbolic equations we conclude that $v \equiv 0$, $t < z$, $x \in R^n$, $z \in R_+$. We represent the solution of (1.4), (1.5) as

$$v(x,z,t) = \delta(t-z) + \tilde{v}(x,z,t)\theta(t-z).$$

Using the method of separation of variables, it is not difficult to find

$$v(x,z,z+0) = (1/2)\left[k_0(x) + \int_0^z H(x,\xi)d\xi\right] =: \beta(x,z).$$

That is why the function $F(x,t)$ in (1.3) should be represented as

$$F(x,t) = \delta(t) + f(x,t)\theta(t), (x,t) \in R^{n+1}. \tag{1.7}$$

It is clear that $\tilde{v} = v$ when $t > z$. For the regular part of function $v(x,z,t)$ in the region $t > z$, $x \in R^n$, the inverse problem (1.4), (1.5), (1.3) is equivalent to the problem

$$v_{tt} - v_{zz} = \triangle v + t\nabla k_0 \nabla v + H(x,t)v + \int_0^t h(x,t-\tau)v(x,z,\tau)d\tau, \tag{1.8}$$

$$v|_{z=0} = f(x,t), v_z|_{z=0} = G(x,t), \tag{1.9}$$

$$v|_{t=z+0} = \beta(x,z). \tag{1.10}$$

In equations (1.8)–(1.10) we replace the variables z,t by z_1,t_1 by the formulas

$$z_1 = t + z, t_1 = t - z.$$

Then $v(x,z,t) = v(x,(z_1-t_1)/2,(z_1+t_1)/2) := v_1(x,z_1,t_1)$. The problem (1.8)–(1.10) in new variables is rewritten as the problem of finding the functions v,k from the equations

$$\frac{\partial^2 v_1(x,z_1,t_1)}{\partial t_1 \partial z_1}$$

$$= -\frac{1}{4}\Big[\triangle v_1 + (t_1+z_1)\nabla k_0 \nabla v_1/2 + H(x,(t_1+z_1)/2)v_1 + h(x,t_1)$$

$$+ \int_0^{t_1} h(x,\tau)v_1(x,z_1-\tau,t_1-\tau)d\tau\Big],$$

$$(x,z_1,t_1) \in \{(x,z_1,t_1) \mid x \in R^n, 0 \le t_1 \le z_1\} := D, \tag{1.11}$$

$$v_1 \mid_{t_1=z_1} = f(x,z_1), \frac{\partial}{\partial z_1} v_1 \mid_{t_1=z_1} = \frac{1}{2} f_t(x,t) \mid_{t=z_1} + \frac{1}{2} G(x,z_1), z_1 \in R_+, x \in R^n, \tag{1.12}$$

$$v_1 \mid_{t_1=0} = \beta[x,(1/2)z_1], z_1 \in R_+, x \in R^n. \tag{1.13}$$

We recall that h is related to k according to the second formula of (1.6).

We introduce the function

$$w(x,z_1,t_1) = \frac{\partial}{\partial z_1} v_1(x,z_1,t_1), t_1 < z_1. \tag{1.14}$$

Demanding the continuity of functions $v(x,z,t), w(x,z,t)$, when $z_1 = t_1 = 0, x \in R^n$, from (1.12), (1.13) it is not difficult to express $k_0(x), k_{0t}(x)$ by the known functions:

$$k_0(x) = 2f(x,0), k_{0t}(x) = 2f_t(x,t)|_{t=0} - f^2(x,0) + 2g(x,0).$$

Further we will assume that in the equalities for $H(x,z), G(x,t), \beta(x,z)$ instead of functions $k_0(x), k_{0t}(x)$ their expressions by means of the latter equations are used. For simplicity we will omit index 1 in z_1, t_1, v_1.

Following [5, p. 92], we introduce the Banach space $A_s(s > 0, r > 0)$ of analytical functions $\nu(x), x \in R^n$, with the norm

$$\|\nu\|_s := \sup_{|x|\le r} \sum_{|\alpha|=0}^{\infty} \frac{s^{|\alpha|}}{\alpha!} \left| \frac{\partial^{|\alpha|}\nu(x)}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}} \right|, \alpha := (\alpha_1, \ldots, \alpha_n),$$

$$|\alpha| := \alpha_1 + \ldots + \alpha_n, \alpha! := (\alpha_1)! \ldots (\alpha)!.$$

The following properties are obvious: if $\nu(x) \in A_s$, then $\nu(x) \in A_{s'}$ for all $s' \in (0,s)$, therefore, $A_s \subset A_{s'}$ if $s' < s$. Besides, if $\nu(x) \in A_s$, then $\left\|\partial^{|\alpha|}\nu(x)/\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}\right\|_{s'} \le C_\alpha ||\nu(x)||_s/(s-s')^{|\alpha|}$, where the constant C_α depends only on α. We denote by $C_z^i\,(A_s, G)$ a class of functions with values in A_s which are continuously differentiable i-times in z and continuous in t in the region G. For fixed (z,t) the norm of the function $\omega(x,z,t)$ in A_{s_0} we denote by $\|\omega\|_{s_0}(z,t)$. The norm of the function ω in $C\,(A_s, G)$ is defined by the equality

$$\|\omega\|_{C(A_s,G)} = \sup_{(z,t)\in G} \|\omega\|_{s_0}(z,t).$$

Theorem 1. *Let $f(x,0)$, $f_t(x,t)|_{t=0}$, $f_{x_i}(x,0)$, $i = 1, \ldots, n$, $\triangle f(x,0)$, $g(x,0)$ belong to A_{s_0}, $s_o > 0$, and $f(x,t)$, $f_t(x,t)$, $f_{tt}(x,t)$, $g(x,t)$, $g_t(x,t)$ belong to $C(A_{s_0}, [0,T])$, and $\max[\|f(x,t)\|_{s_0}(t), \|w_0(x,z)\|_{s_0}(z), \|h_0(x,z)\|_{s_0}(z), \|k_0(x)\|_{s_0}]$ $= R$, for $(z,t) \in G_T := \{(z,t)|0 \le t \le z \le T\}$. Then for any $\chi > 0$ we can find the number $a = a(s_0, T, R, n)$, $as_0 < T$ so that for any $s \in (0, s_0)$ there exists a unique solution of the problem (1.11)–(1.13) $v(x,z,t) \in C_z^1\,(A_{s_0}, D_s)\,, k(x,t) \in C_t^2\,(A_{s_0}, [0, a(s_0 - s)])\,,$ where D_s is the region on the plane $z,t : D_s := \{(z,t)|0 \le t \le z < a(s_0 - s)\}$, and solution satisfies the following inequalities:*

$$\|v - v_0\|_s(z,t) \le \chi, \|k - k^0\|_s(z) \le \frac{2\chi}{(s_0 - s)}, \tag{1.15}$$

where

$$h_0(x,z) := 2f(x,0)\exp[-f(x,0)z]g(x,z) - 2\exp[-f(x,0)z]g_t(x,z)|_{t=z}$$

$$+(1/2)\triangle f(x,0) - H(x,z)f(x,z) - \triangle f(x,z) - 2z\nabla f(x,0)\nabla f(x,z)$$
$$-2f_{tt}(x,t)|_{t=z} + (z/2)\sum_{i=1}^{n} f_{x_i}^2(x,0),$$
$$v_0 := f(x,z), k^0 := k_0(x) + zk_{0t}, (z,t) \in D_s.$$

P r o o f. In the beginning the problem (1.11)–(1.13) is reduced to the closed system of the Volterra type integro-differential equations in the area $(x,z,t) \in D$. The equation for v is rewritten with the help of equation (1.14) while the equation for k is rewritten with the help of the second equation of (1.6):

$$v(x,z,t) = v_0(x,z) + \int_t^z w(x,\xi,t)d\xi, \tag{1.16}$$

$$k(x,z) = k^0(x) + \int_0^z (z-\xi)\exp\left[f(x,0)\xi\right]h(x,\xi)d\xi. \tag{1.17}$$

For fixed $x \in R^n$ by integrating equality (1.11) on the plane (τ,ξ) along the line $\xi = z$ from the point (t,z) to the point (z,z) and using the second condition in (1.12), we can get the equation for $w(x,z,t)$:

$$w(x,z,t) = w_0(x,z) + \frac{1}{4}\int_t^z [\triangle v(x,z,\tau) + (z+\tau)\nabla f(x,0)\nabla v(x,z,\tau)$$

$$+H[x,(z+\tau)/2]v(x,z,\tau) + h(x,\tau) + \int_0^\tau h(x,\eta)v(x,z-\eta,\tau-\eta)d\eta]d\tau, \tag{1.18}$$

where $w_0(x,z) = (1/2)f_t(x,t)|_{t=z} + (1/2)G(x,z)$.

Using equation (1.13) differentiated by z, when $t = 0$, from (1.18), we find

$$\int_0^z [\triangle v(x,z,\tau) + (z+\tau)\nabla f(x,0)\nabla v(x,z,\tau)$$

$$+H[x,(z+\tau)/2]v(x,z,\tau) + h(x,\tau) + \int_0^\tau h(x,\eta)v(x,z-\eta,\tau-\eta)d\eta]d\tau = 2f_t(x,0)$$

$$+2g(x,0)+(z/2)\triangle f(x,0)+(z^2/4)\sum_{i=1}^{n} f_{x_i}^2(x,0)-2f_t(x,z)-2\exp[-f(x,0)z]g(x,z).$$

The equation for $h(x,t)$ can be easily found by differentiation of the latter equation on z and by using the first condition from (1.12):

$$h(x,t) = h_0(x,t) - \int_0^z [\triangle w(x,z,\tau) + (z+\tau)\nabla f(x,0)\nabla w(x,z,\tau)$$

$$+\tau\nabla f(x,0)\nabla v(x,z,\tau) + H_z[x,(z+\tau)/2]v(x,z,\tau) + H[x,(z+\tau)/2]w(x,z,\tau)$$

$$+h(x,\tau)f(x,z-\tau) + \int_0^\tau h(x,\eta)w(x,z-\eta,\tau-\eta)d\eta]d\tau, \qquad (1.19)$$

where the function $h_0(x,t)$ is determined in Th. 1.

The equations (1.16)–(1.19) represent a closed system of integro-differential equations for the functions v, k, w, h in the area $D_T = G_T \times R^n$.

For convenience we introduce the function vector

$$\phi(x,z,t) = (\phi_1,\phi_2,\phi_3,\phi_4) := (v,k,w,h).$$

We rewrite the system (1.16)–(1.19) as an operator equation

$$\phi = M\phi, \qquad (1.20)$$

where $M = (M_1,\dots,M_4)$ is defined by the right sides of equations (1.16)–(1.19), and

$$\phi^0 = M(0), \phi^0 = (\phi_1^0,\phi_2^0,\phi_3^0\phi_4^0) := (v_0,k^0,w_0,h_0).$$

We define the iterations for equation (1.20):

$$\phi^{i+1} = M\phi^i, i = 0,1,2,\dots, \phi^i = (\phi_1^i,\dots,\phi_4^i) \qquad (1.21)$$

and

$$\phi^{i+1} - \phi^i := \psi^i, i = 0,1,2,\dots, \psi^i = (\psi_1^i,\dots,\psi_4^i).$$

Let the sequence of numbers $a_0, a_1, \dots, a_i, \dots$ be determined by the expressions $a_{i+1} = a_i/(1+(i+1)^{-2})$, $i = 0,1,2,\dots$. Here a_0 is a fixed positive number. The number $a_0 < T/s_0$ will be chosen later. With the numerical sequence a_σ we link the sequence of enclosed fields $F_i = \{(z,t,s)|0 < s < s_0, 0 \le t \le z < a_i(s_0 - s)\}$.

The following lemma is valid.

Lemma. *If the conditions of Th. 1 for any fixed $\chi > 0$ and any $i = 0,1,2,\dots$ are fulfilled, then there exist $a_0 \in (0,T/s_0)$, $a_0 = a_0(R,s_0,\chi,n)$ and $\lambda_i = \lambda_i(R,s_0,\chi,n) > 0$, such that for each $s \in (0,s_0)$ $\left(\psi_1^i,\psi_3^i\right) \in C(A_s,D_{si})$,*

$\left(\psi_2^i, \psi_4^i\right) \in C\left(A_s, [0, a_i(s_0 - s)]\right)$, $D_{si} := \{(z,t)|0 \le t \le z < a_i(s_0 - s)\}$, *and the following inequalities are valid:*

$$\|\psi_1^i\|_s(z,t) \le \frac{\lambda_i z}{a_i(s_0 - s) - z}, \ \|\psi_j^\sigma\|_s(z,t) \le \frac{\lambda_i a_i z}{[a_i(s_0 - s) - z]^2}, \ j = 2, 3,$$

$$\|\psi_4^i\|_s(z) \le \frac{\lambda_i a_i^2 z}{[a_i(s_0 - s) - z]^3}, \ (z,t,s) \in F_i; \tag{1.22}$$

$$\|\phi_1^{i+1} - \phi_1^0\|_s(z,t) \le \chi, \ \|\phi_j^{i+1} - \phi_j^0\|_s(z,t) \le \frac{2\chi}{s_0 - s} \ j = 2, 3,$$

$$\|\phi_4^{i+1} - \phi_4^0\|_s(z) \le \frac{4\chi}{(s_0 - s)^2}, \ (z,t,s) \in F_{i+1}. \tag{1.23}$$

We use the Nirenberg method and its modification, developed in [5] to prove the lemma. Using estimate [5, p. 92]

$$\left\| \frac{\partial^{|\alpha|}\nu(x)}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}} \right\|_{s'} \le C_\alpha \frac{\|\nu(x)\|_s}{(s - s')^{|\alpha|}}, \tag{1.24}$$

it is not difficult to check that inequalities (1.22), (1.23) are satisfied when $i = 0$, besides λ_0 is proportional to a_0. We determine the validity of these inequalities for any i using the induction method. We assume that if the statement of the lemma is valid for $i \le \sigma$ and prove that, then it is valid for $i = \sigma + 1$, as well. Using the inductive assumption, we find that $\psi^{\sigma+1} \in C(A_s, D_{s(\sigma+1)})$. Besides, from (1.20) we find

$$\|\psi_1^{\sigma+1}\|_s(z,t) \le \int\limits_t^z \|\psi_3^\sigma\|_s(\xi, t) d\xi$$

$$\le \int\limits_t^z \frac{\lambda_\sigma a_\sigma \xi d\xi}{[a_\sigma(s_0 - s) - \xi]^2} \le \frac{\lambda_\sigma a_\sigma z}{[a_\sigma(s_0 - s) - z]} \le a_0 \frac{\lambda_\sigma z}{a_{\sigma+1}(s_0 - s) - z},$$

$$\|\psi_2^{\sigma+1}\|_s(z) \le \int\limits_0^z \|\psi_4^\sigma\|_s(\xi) d\xi \le a_0 \frac{\lambda_\sigma a_{\sigma+1} z}{[a_{\sigma+1}(s_0 - s) - z]^2},$$

$$\|\psi_3^{\sigma+1}\|_s(z,t) \le \frac{1}{4} \int\limits_t^z \{\|\triangle \psi_1^\sigma\|_s(z,\xi) + 2TR \sum_{j=1}^n \|\frac{\partial}{\partial x_j} \psi_1^\sigma\|_s(z,\xi)$$

$$+R\|\psi_1^\sigma\|_s(z,\xi) + \|\psi_4^\sigma\|_s(\xi) + \int\limits_0^\xi [\|\psi_4^\sigma\|_s(\tau)\|\phi_1^{\sigma+1}|_s(z - \tau, \xi - \tau)$$

$$+\|\phi_4^\sigma\|_s(\tau)\|\psi_1^\sigma\|_s(z-\tau,\xi-\tau)]d\tau\}d\xi.$$

Using inequality (1.24) to estimate $\|\triangle\psi_1^\sigma\|_s, \|\frac{\partial}{\partial x_j}\psi_1^\sigma\|_s, j = 1,2,\dots,n$, and assuming that $s' = s'(\xi) = (s+s_0-\xi/a_n)/2$, we get

$$\|\psi_3^{\sigma+1}\|_s(z,t) \leq \frac{1}{4}\int\limits_t^z\{\frac{16na_\sigma^2}{[a_\sigma(s_0-s)-\xi]^2}\cdot\frac{2\lambda_\sigma z}{a_\sigma(s_0-s)-z}$$

$$+\frac{8TRna_\sigma}{a_\sigma(s_0-s)-\xi}\cdot\frac{2\lambda_\sigma z}{a_\sigma(s_0-s)-z}+\frac{2R\lambda_\sigma z}{a_\sigma(s_0-s)-z}+\frac{\lambda_\sigma a_\sigma^2\xi}{[a_\sigma(s_0-s)-\xi]^3}$$

$$+\int\limits_0^\xi[\frac{\lambda_\sigma a_\sigma^2\tau(\chi+R)}{(a_\sigma(s_0-s)-\tau)^3}+\frac{4\chi+Rs_0^2}{(s_0-s)^2}\cdot\frac{2\lambda_\sigma z}{a_\sigma(s_0-s)-z}]d\tau\}d\xi$$

$$\leq c_1a_0\frac{a_{\sigma+1}\lambda_\sigma z}{[a_{\sigma+1}(s_0-s)-z]^2},$$

where

$$c_1 = c_1(s_0,n,a_0,\chi,R,T).$$

In the latter inequality it is used that $a_\sigma^2 \leq a_{\sigma+1}a_0$ are valid for any $\sigma \geq 1$. Moreover, $a_\sigma^3 \leq a_{\sigma+1}^2a_0$ for $\sigma \geq 1$. These inequalities can be easily checked by the method of mathematical induction. The latter inequalities are used to estimate the function $\psi_4^{\sigma+1}$. Similarly, for $\psi_4^{\sigma+1}$ we obtain

$$\|\psi_4^{\sigma+1}\|_s(z) \leq c_2a_0\frac{a_{\sigma+1}^2\lambda_\sigma z}{[a_{\sigma+1}(s_0-s)-z]^3},$$

where

$$c_2 = c_2(s_0,n,a_0,\chi,R,T).$$

It is obvious that $c_i, i = 1,2$, in the latter estimates are monotonically nondecreasing functions of the s_0, a_0 parameters. From the estimations made above it follows that (1.22) is valid when $i = \sigma+1$, if we assume that

$$\lambda_{\sigma+1} = a_0c\lambda_\sigma,$$

where

$$c = \max(1,s_0,c_1,c_2). \tag{1.25}$$

Now we show that inequalities (1.23) are also satisfied when $i = \sigma+1$, if the number a_0 is chosen properly. For $(x,z,t) \in F_{\sigma+2}$

$$\|\phi_1^{\sigma+2}-\phi_1^0\|_s(z,t) \leq \sum_{j=0}^{\sigma+1}\|\psi_1^j\|_s(z,t) \leq \sum_{j=0}^{\sigma+1}\frac{\lambda_j z}{a_j(s_0-s)-z}$$

$$\leq \lambda_0 \sum_{j=0}^{\sigma+1} (a_0 c)^j \left(\frac{a_j}{a_{j+1}} - 1 \right) \leq \lambda_0 \sum_{j=0}^{\sigma+1} (a_0 c)^j (j+1)^2,$$

$$\|\phi_\gamma^{\sigma+2} - \phi_\gamma^0\|_s(z,t) \leq \sum_{j=0}^{\sigma+1} \|\psi_\gamma^j\|_s(z,t) \leq \frac{2\lambda_0}{s_0 - s} \sum_{j=0}^{\sigma+1} (a_0 c)^j (j+1)^4, \gamma = 2, 3,$$

$$\|\phi_4^{\sigma+2} - \phi_4^0\|_s(z) \leq \sum_{j=0}^{\sigma+1} \|\psi_4^j\|_s(z) \leq \frac{4\lambda_0}{(s_0 - s)^2} \sum_{j=0}^{\sigma+1} (a_0 c)^j (j+1)^6.$$

That is why inequalities (1.23) are satisfied when $i = \sigma + 1$, if the number a_0 is chosen so that

$$a_0 c < 1, \lambda_0 \sum_{j=0}^{\infty} (a_0 c)^j (j+1)^6 \leq \chi. \tag{1.26}$$

It is clear that a_0 can be always chosen to be so small that inequalities (1.26) are satisfied. Thus the validity of the lemma is proven. Further we assume that the number a_0 is chosen from the conditions (1.26).

In order to complete the proof of Th. 1, we should note that under the chosen value of a_0 the ϕ^i sequence uniformly converges in the norm of space $C(A_s, D_s), a = lim a_i$, the limiting function belongs to $C(A_s, D_s)$ and provides the solution of the operator equation (1.20). The limit transition in the first two inequalities (1.23), when $(x,z,t) \in F = \{(z,t,s) | 0 < s < s_0, 0 \leq t \leq z < a(s_0 - s)\}$ leads to the values coinciding with (1.15). The uniqueness of the constructed solution is established by using the described above technics based on the standard method [5, p. 103].

We indicate a few settings of inverse problems of memory determination which could be studied by the above methods.

2. Find the functions $u(x,z,t), k(x,t)$ that satisfy the following equalities:

$$u_{tt} - u_{zz} - \triangle u = \int_0^t k(x,\tau) u_t(x,z,t-\tau) d\tau, (x,t) \in R^{n+1}, z \in R_+, \tag{2.1}$$

$$u|_{t<0} \equiv 0, u_z|_{z=0} = -g(x)\delta'(t), (x,t) \in R^{n+1}, \tag{2.2}$$

$$u|_{z=0} = g(x)\delta(t) + f(x,t)\theta(t), (x,t) \in R^{n+1}, \tag{2.3}$$

where $g(x), f(x,t)$ are given smooth functions.

The solution of problem (2.1),(2.2) is represented as

$$u(x,z,t) = g(x)\delta(t-z) + \tilde{u}(x,z,t)\theta(t-z). \tag{2.4}$$

We introduce the function $\tilde{u}(x,(z-t)/2,(z+t)/2) := v(x,z,t)$. As in Part 1, the problem (2.1)–(2.3) can be replaced by the equivalent problem of finding the functions $v(x,z,t), k(x,t)$ in the region $z > t > 0, x \in R^n$ from the equations

$$\frac{\partial^2 v(x,z,t)}{\partial t \partial z}$$

$$= -\frac{1}{4}\left[\triangle v + k_0(x)v - g(x)h(x,t) - \int_0^t h(x,\tau)v(x,z-\tau,t-\tau)d\tau\right],$$

$$(x,z,t) \in \{(x,z,t) \mid x \in R^n, 0 \le t \le z\} := D, \tag{2.5}$$

$$v\mid_{t=z}= f(x,z), \frac{\partial}{\partial z}v\mid_{t=z}= \frac{1}{2}f_t(x,t)|_{t=z}, z \in R_+, x \in R^n, \tag{2.6}$$

$$v\mid_{t=0}= \frac{1}{2}k_0(x)z, z \in R_+, x \in R^n, \tag{2.7}$$

in which $k_0 := k(x,0) = 1/g(x)[f_t(x,t)|_{t=0} + \triangle g(x)], h(x,t) := k_t(x,t), t > 0.$

With respect to problem (2.5)–(2.7) the analogue of Th. 1 is valid.

Theorem 2. *Let* $\{g(x), 1/g(x), (\partial/\partial z)f(x,z)|_{z=0}, (\partial^3/\partial^2 x_i \partial z)f(x,z)|_{z=0}\} \in A_{s_0}$, $s_0 > 0$, $i = 0,1,\dots,n$, $\{f(x,z), (\partial/\partial z)f(x,z), (\partial^2/\partial^2 z)f(x,z)\} \in C(A_{s_0},[0,T])$, *when some fixed* $s_0 > 0$, $T > 0$, *whereas*

$$\max\left[\|g(x)\|_{s_0}, \|f(x,z)\|_{s_0}, \|f_z(x,z)\|_{s_0}, 1/\|g(x)\|_{s_0}, \|\triangle g(x)\|_{s_0}, \|f_{zz}(x,z)\|_{s_0},\right] = R,$$

for $t \in [0,T]$ *and the consistency condition is satisfied:*

$$f_z(x,z)|_{z=0} + \triangle g(x) = g(x)f_z(x,z)|_{z=0}.$$

Then for any $\chi > 0$ *the number* $a \in (0,T/s_0)$ *can be found such that for any* $s \in (0,s_0)$ *there exists the unique solution of problem (2.5)–(2.7)* $v(x,z,t) \in C_z^1(A_{s_0},D_s), k(x,t) \in C_t^1(A_{s_0},[0,a(s_0-s)])$, *where the* D_s *field is determined in Th. 1, and for the solution inequalities (1.15) are valid with the functions*

$$v_0 = f(x,z), k_0(x) = \frac{1}{g(x)}[f_t(x,t)|_{t=0} + \triangle g(x)].$$

3. Find the functions $u(x,z,t), k(x,t)$ that satisfy the following equalities:

$$u_{tt} - u_{zz} - Lu = \int_0^t k(x,\tau)L_0 u(x,z,t-\tau)d\tau, (x,t) \in R^{n+1}, z \in R_+, \tag{3.1}$$

$$u|_{t<0} \equiv 0, u_z|_{z=0} = -g(x)\delta'(t), (x,t) \in R^{n+1}, \tag{3.2}$$

$$u|_{z=0} = g(x)\delta(t) + f(x,t)\theta(t), (x,t) \in R^{n+1}. \tag{3.3}$$

Here

$$L = \sum_{i=1,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + L_0, L_0 = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

a_{ij}, b_i, c, g, f $(1 \le i, j \le n)$ are given smooth functions.

Problem (3.1)–(3.3), when $a_{ij} = \delta_{ij}$, $b_i = 0$, where δ_{ij} is the Kronecker symbol, was studied in paper [1]. In the case of inequality (1.24) let us estimate the differential expressions Lu, L_0u with analytical coefficients. We assume that functions a_{ij}, b_i, $c(x)$, $1 \le i$, $j \le n$, are the elements of A_{s_0}, $s_0 > 0$, and

$$d := \max_{1 \le i,j \le n} \|a_{ij}\|_{s_0}, b_0 := \max_{1 \le i,j \le n} \|b_i\|_{s_0}, c_0 := \|c\| s_0.$$

For the operators L_0, L from (1.24) it follows:

$$\|L_0\nu\|_{s'} \le \frac{c_1}{s-s'}\|\nu\|_s, \ \|L\nu\|_{s'} \le \frac{c_2}{(s-s')^2}\|\nu\|_s, \ s' \in (0,s),$$

$$c_1 := ne^{-1}b_0 + s_0c_0, \ c_2 := 4n^2e^{-2}d + c_1s_0.$$

The solution of problem (3.1), (3.2) is represented as

$$u(x,z,t) = \alpha(x,z)\delta(t-z) + \tilde{u}(x,z,t)\theta(t-z), \tag{3.4}$$

and we make use of the method of separation of variables [5, p. 29]. Denoting $\tilde{u}(x,z,z+0) =: \beta(x,z)$, we put (3.4) in (3.1), (3.2) and find

$$\alpha(x,z) \equiv \alpha(x) = g(x), \beta(x,z) = (z/2)Lg(x), x \in R^n, z \in R_+. \tag{3.5}$$

It is not difficult, using equalities (3.4), (3.5), to replace the inverse problem (3.1)–(3.3) by initial-characteristic problem with Cauchy data on $z = 0$. It should be noted that when $t > z$ the equality $u = \tilde{u}$ takes place, then the functions $u(x,z,t), k(x,z)$ satisfy the equations

$$u_{tt} - u_{zz} - Lu = L_0g(x)k(x,t-z)$$

$$+ \int_0^{t-z} k(x,\tau)L_0u(x,z,t-\tau)d\tau, (x,z,t) \in G := \{(x,z,t)|x \in R^n, t > z > 0\}, \tag{3.6}$$

$$u|_{z=0} = f(x,t), u_z|_{z=0} = 0, x \in R^n, t \in R_+, \tag{3.7}$$

$$u|_{z=0} = \frac{1}{2}zLg(x), x \in R^n, t \in R_+. \tag{3.8}$$

Further, the problem (3.6)–(3.8) is reduced to the following system of integro-differential equations for u, u_t, k:

$$u(x,z,t) = u_0(x,z,t) + \frac{1}{2} \iint\limits_{\triangle(z,t)} [Lu(x,\xi,\tau) - L_0 g(x) k(x,\tau-\xi)$$

$$+ \int\limits_0^{\tau-\xi} k(x,\gamma) L_0 u(x,\xi,\tau-\gamma) d\gamma] d\tau d\xi, \tag{3.9}$$

$$u_t(x,z,t) = u_{0t}(x,z,t) + \frac{1}{2} \int_{-z}^{z} [Lu(x,z-|\xi|,t+\xi) - L_0 g(x) k(x,t-z+|\xi|+\xi)$$

$$+ \int\limits_0^{t-z+|\xi|+\xi} k(x,\gamma) L_0 u(x,z-|\xi|,t+\xi-\gamma) d\gamma] sgn\xi d\xi, \tag{3.10}$$

$$k(x,z) = k_0(x,z) + \frac{2}{L_0 g(x)} \int\limits_0^{z/2} [Lu_t(x,\xi,z-\xi) + \frac{1}{2}\xi L_0 g(x) k(x,z-2\xi)$$

$$+ \int\limits_0^{z-2\xi} k(x,\gamma) L_0 u_t(x,\xi,z-\xi-\gamma) d\gamma] d\xi, \tag{3.11}$$

where

$$u_0(x,z,t) = \frac{1}{2}[f(x,t+z) + f(x,t-z)],$$

$$k_0(x,z) = \frac{1}{L_0 g(x)}[(1/4) z L^2 g(x) - 2 f_{tt}(x,t)|_{t=z}],$$

$$\triangle(z,t) = \{(\xi,\tau) | 0 \le \xi \le z, t-z+\xi \le \tau \le t+z-\xi, t > z\}.$$

For the system (3.9)–(3.11) the following theorem is valid.

Theorem 3. *Assume that the consistency conditions of* $f(x,0) = 0$, $f_t(x,t)|_{t=0} = (1/2) Lg(x)$, *are satisfied, besides*

$$(a_{ij}, b_i, c, g)(x) \in A_{s_0}, 1 \le i,j \le n,$$

$$[f(x,t), f_t(x,t), f_{tt}(x,t)] \in C(A_{s_0}, [0,T]),$$

$$\max[\|f\|_{s_0}(t), \|f_t\|_{s_0}(t), \|f_{tt}\|_{s_0}(t), \|L_0 g\|_{s_0}, 1/\|L_0 g\|_{s_0}, \|L^2 g\|_{s_0}] = R,$$

for $t \in [0,T]$, $R > 0$. *Then* $a \in (0, T/2), as_0 < T/2$ *can be found such that for any* $s \in (0, s_0)$ *in the region* $G_T \cap \{(x,z,t)|x \in R^n, 0 \le z \le a(s_0 - s)\}$ *there exists the unique solution of the system (3.9)–(3.11), for which*

$$(u(x,z,t), u_t(x,z,t)) \in C(A_{s_0}, P_{sT})$$

$$k(x,z) \in C(A_{s_0}, [0, a(s_0 - s)]), P_{sT} := G_T \cap \{(z,t)|0 \le z \le a(s_0 - s)\},$$

moreover,

$$\|u - u_0\|_s(z,t) \le R_0, \|k - k_0\|_s(t) \le \frac{R_0}{(s_0 - s)^2}, \|u_t - u_{0t}\|_s(z,t) \le \frac{R_0}{(s_0 - s)},$$

$(z,t) \in P_{sT}, R_0 = R_0(R, T, s_0, n)$ *is a constant.*

Theorems 2 and 3 are proved similarly to Th. 1.

R e m a r k 1. *The solution of the inverse problem suggests the unique continuation on the variable* t *from the interval* $[0, a(s_0 - s)]$ *onto the interval* $[0,T]$ *for any* T *(see [1]).*

R e m a r k 2. *Under appropriate conditions similar results hold when the operator* $\triangle$ *from Parts 1,2 is replaced by the operator* L, *defined in Part 3.*

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