Journal of Mathematical Physics, Analysis, Geometry
2007, vol. 3, No. 4, pp. 424–447

Scattering Scheme with Many Parameters and Translational Models of Commutative Operator Systems

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Received February 27, 2006

The scattering scheme with many parameters for a commutative system of linear bounded operators $\{T_1, T_2\}$, when T_1 is a contraction, is built. Using this construction of the scattering scheme, the translation model of the semigroup with two parameters $T(n) = T_1^{n_1} T_2^{n_2}$, $n = (n_1, n_2) \in \mathbb{Z}_+^2$ is obtained. Description of characteristic properties of the dilation U of the contraction T_1, that follows from the commutative property of the operators T_1 and T_2, in terms of external parameters lies in the basis of the method of the construction of the translational models for $T(n)$.

Key words: scattering scheme with many parameters, translational model, commutative operator system.

Mathematics Subject Classification 2000: 47A45.

The construction of the functional and translational models for a contracting operator T ($\|T\| \leq 1$) and its unitary dilation U [3, 6] is based on the study of the basic properties of the wave operators $W_\pm$ and scattering operator S [6]. Immediate generalization of these constructions for the case of the commutative operator system $\{T_1, T_2\}$, $[T_1, T_2] = 0$ is not trivial and not always possible.

In this paper, a new method generalizing the scattering scheme on the case of many parameters by P. Lax and R. Fillips is presented. This method uses the isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_1^2$ [7] based on isometric dilation for the commutative operator system $\{T_1, T_2\}$ of the class $C(T_1)$ [7, 8] constructed in [8]. Unlike in a one-variable situation, two scattering operators $S(p,k)$ and $\tilde{S}(p,k)$, p, $k \in \mathbb{Z}_+$ appear here. These operators have the property $S^*(0,0) = \tilde{S}(0,0)$. Using the method presented in [6], the generalization of construction of translational models for one operator T_1 for the commutative operator system $\{T_1, T_2\}$, when one of the operators, e. g., T_1, is a contraction, is presented in this paper.

1. Isometric Dilations of Commutative Operator System

I. Consider the commutative system of linear bounded operators $\{T_1, T_2\}$, $[T_1, T_2] = T_1T_2 - T_2T_1 = 0$ in the separable Hilbert space H. Hereinafter, we will suppose that one of the operators of the system $\{T_1, T_2\}$, e.g., T_1, is a contraction, $\|T_1\| \leq 1$. Following [4, 7, 8], define the commutative unitary expansion for the system $\{T_1, T_2\}$.

Definition 1. *Let the commutative system of the linear bounded operators $\{T_1, T_2\}$ be given in the Hilbert space H, where T_1 is a contraction, $\|T_1\| \leq 1$. The set of mappings*

$$\begin{array}{ll} V_1 = \begin{bmatrix} T_1 & \Phi \\ \Psi & K \end{bmatrix}; \quad V_2 = \begin{bmatrix} T_2 & \Phi N \\ \Psi & K \end{bmatrix}: & H \oplus E \to H \oplus \tilde{E}; \\ \overset{+}{V}_1 = \begin{bmatrix} T_1^* & \Psi^* \\ \Phi^* & K^* \end{bmatrix}; \quad \overset{+}{V}_2 = \begin{bmatrix} T_2^* & \Psi^*\tilde{N}^* \\ \Phi^* & K^* \end{bmatrix}: & H \oplus \tilde{E} \to H \oplus E, \end{array} \tag{1.1}$$

where E and $\tilde{E}$ are Hilbert spaces, is said to be the commutative unitary expansion of the commutative system of operators T_1, T_2 in H, $[T_1, T_2] = 0$, if there are such operators σ, τ, N, Γ and $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{N}$, $\tilde{\Gamma}$ in the Hilbert spaces E and $\tilde{E}$, where σ, τ, $\tilde{\sigma}$, $\tilde{\tau}$ are selfadjoint, that the following relations take place:

$$\begin{array}{lll} 1) & \overset{+}{V}_1 V_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; & V_1 \overset{+}{V}_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \\ 2) & V_2^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V_2 = \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix}; & V_2^* \begin{bmatrix} I & 0 \\ 0 & \tau \end{bmatrix} \overset{+}{V}_2 = \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau} \end{bmatrix}; \\ 3) & T_2\Phi - T_1\Phi N = \Phi\Gamma; & \Psi T_2 - \tilde{N}\Psi T_1 = \tilde{\Gamma}\Psi; \\ 4) & \tilde{N}\Psi\Phi - \Psi\Phi N = K\Gamma - \tilde{\Gamma}K; & \\ 5) & \tilde{N}K = KN. & \end{array} \tag{1.2}$$

Consider the following class of commutative systems of linear operators $\{T_1, T_2\}$.

Definition 2. *The commutative system of operators T_1, T_2 belongs to the class $C(T_1)$ and is said to be the contracting operator system for T_1 if:*

1) T_1 *is a contraction,* $\|T_1\| \leq 1$;
2) $E \overset{def}{=} \overline{\tilde{D}_1 H} \supseteq \overline{\tilde{D}_2 H}$; $\quad \tilde{E} \overset{def}{=} \overline{D_1 H} \supseteq \overline{D_2 H}$;
3) $\dim \overline{T_2 \tilde{D}_1 H} = \dim E$; $\quad \dim \overline{D_1 T_2 H} = \dim \tilde{E}$;
4) *operators* $D_1|_{\tilde{E}}$, $\tilde{D}_1\Big|_E$, $\tilde{D}_1 T_2^*\Big|_{\overline{T_2\tilde{D}_1 H}}$, $\tilde{D}_1\Big|_E$, $T_2^* D_1|_{\overline{D_1 T_2 H}}$
are boundedly invertible, where $D_s = T_s^* T_s - I$, $\tilde{D}_s = T_s T_s^* - I$, $s = 1, 2$. (1.3)

It is easy to see that if $\{T_1, T_2\} \in C(T_1)$, then the unitary expansion (1) always exists.

Let E and $\tilde{E}$ be Hilbert spaces defined in 2) (1.3). Choose the unitary operators V and $\tilde{V}$, V: $\overline{T_2\tilde{D}_1 H} \to \overline{\tilde{D}_1 H}$; $\tilde{V}$: $\overline{T_2^* D_1 H} \to \overline{D_1 H}$, what is always possible in view of 3) (1.3). Define now the invertible operators $N_1 = \tilde{D}_1 T_2^* V^*$ and $\tilde{N}_1 = \tilde{V} T_2^* D_1$ in E and $\tilde{E}$ (see 4) (1.3)). It is easy to see that the operators $\sigma_1 = -N_1^{*-1}\tilde{D}_1^{-1}N_1$ in E and $\tilde{\sigma}_1 = -D_1$ in $\tilde{E}$ are invertible, selfadjoint and nonnegative in view of 1), 4) (1.3). Consider the following set of operators

$$N = \sqrt{\sigma_1} N_1^{-1}\tilde{D}_2 T_1^* \sqrt{\sigma_1^{-1}}; \quad \tilde{N} = \sqrt{\tilde{\sigma}_1}\tilde{N}_1^{-1} T_1^* D_2 \sqrt{\tilde{\sigma}_1^{-1}};$$

$$\Gamma = \sqrt{\sigma_1} N_1^{-1}\left(\tilde{D}_1 - \tilde{D}_2\right)\sqrt{\sigma_1^{-1}}; \quad \tilde{\Gamma} = \sqrt{\tilde{\sigma}_1}\tilde{N}_1^{-1}\left(D_1 - D_2\right)\sqrt{\tilde{\sigma}_1^{-1}};$$

$$\sigma = -\sqrt{\sigma_1^{-1}} T_1 \tilde{D}_2 T_1^* \sqrt{\sigma_1^{-1}}; \quad \tilde{\sigma} = -\sqrt{\tilde{\sigma}_1^{-1}} D_2 \sqrt{\tilde{\sigma}_1^{-1}};$$

$$\tau = -\sqrt{\sigma_1} N_1^{-1} D_2 N_1^{*-1}\sqrt{\sigma_1}; \quad \tilde{\tau} = -\sqrt{\sigma_1}\tilde{N}_1^{-1} T_1^* D_2 T_1 \tilde{N}_1^{*-1}\sqrt{\tilde{\sigma}_1};$$

$$\varphi = P_E N_1 \sqrt{\sigma_1^{-1}}; \quad \psi = \sqrt{\tilde{\sigma}_1} P_{\tilde{E}}; \quad K = \sqrt{\tilde{\sigma}_1} T_1^* T_2^* \sqrt{\sigma_1^{-1}},$$

where P_E and $P_{\tilde{E}}$ are orthoprojectors on E and $\tilde{E}$, respectively. It is easy to prove that in this case relations 1.2 are true for $\left\{V_s, \overset{+}{V}_s\right\}_1^2$ (1.1). Thus for the commutative operator system $\{T_1, T_2\}$ of the class C)(T_1) there always exists the unitary isometric expansion (1.1), (1.2).

Note that the conditions 1) and 2) (1.2) for the expansions $\left\{V_s \, \overset{+}{V}_s\right\}_1^2$ (1.1) have a standard nature and play an important role in the construction of isometric (unitary) dilations [3, 6, 7]. One should consider relations 3)–5) (1.2) as the conditions of concordance of these expansions which follow from the commutative property of the operator system $\{T_1, T_2\}$.

II. Remind the construction of the unitary dilation [3, 6] for a contraction T_1. As usually [6, 7], we will denote by $l_M^2(G)$ the Hilbert space of G-valued functions u_k which assume a value in the Hilbert space G, $u_k \in G$, where $k \in M$ and $M \subseteq \mathbb{Z}$ are such that $\sum\limits_{k\in M}\|u_k\|^2 < \infty$. Let $\mathcal{H}$ be the Hilbert space of the following type

$$\mathcal{H} = D_- \oplus H \oplus D_+, \tag{1.4}$$

where $D_- = l_{\mathbb{Z}_-}^2(E)$ and $D_+ = l_{\mathbb{Z}_+}^2(\tilde{E})$. Specify the dilation U on the vector-functions $f = (u_k, h, v_k)$ from $\mathcal{H}$ (1.4) in the following way:

$$Uf = \left(P_{D_-}u_{k-1}, \tilde{h}, \tilde{v}_k\right), \tag{1.5}$$

where $\tilde{h} = T_1 h + \Phi u_{-1}$, $\tilde{v}_0 = \Psi h + K u_{-1}$, $\tilde{v}_k = v_{k-1}$ $(k = 1, 2, \ldots)$, and P_{D_-} is the operator of contraction on D_-. The unitary property of U (1.5) in $\mathcal{H}$ follows from 1) (1.2).

To construct the isometric dilation [8] of a commutative operator system $\{T_1, T_2\} \in C(T_1)$, continue the incoming D_- and outgoing D_+ subspaces

$$D_- = l^2_{\mathbb{Z}_-}(E); \quad D_+ = l^2_{\mathbb{Z}_+}(\tilde{E}) \tag{1.6}$$

by the second variable "n_2". At first, continue functions $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$ from the semiaxis $\mathbb{Z}_-$ into the domain

$$\tilde{\mathbb{Z}}^2_- = \mathbb{Z}_- \times (\mathbb{Z}_- \cup \{0\}) = \left\{ n = (n_1; n_2) \in \mathbb{Z}^2 : n_1 < 0; n_2 \le 0 \right\} \tag{1.7}$$

using the following Cauchy problem [7, 8]:

$$\begin{cases} \tilde{\partial}_2 u_n = \left(N \tilde{\partial}_1 + \Gamma \right) u_n; & n = (n_1, n_2) \in \tilde{\mathbb{Z}}^2_-; \\ u_n|_{n_2 = 0} = u_{n_1} \in l^2_{\mathbb{Z}_-}(E), \end{cases} \tag{1.8}$$

where $\tilde{\partial}_1 u_n = u_{(n_1 - 1, n_2)}$, $\tilde{\partial}_2 u_n = u_{(n_1, n_2 - 1)}$. As a result, we obtain the Hilbert space $D_-(N, \Gamma)$ which is formed by u_n, the solutions of (1.8), at the same time the norm in $D_-(N, \Gamma)$ is induced by the norm of initial data $\|u_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$.

Similarly, continue functions $v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E})$ from the semiaxis $\mathbb{Z}_+$ into the domain $\mathbb{Z}^2_+ = \mathbb{Z}_+ \times \mathbb{Z}_+$ using the Cauchy problem

$$\begin{cases} \tilde{\partial}_2 v_n = \left(\tilde{N} \tilde{\partial}_1 + \tilde{\Gamma} \right) v_n; & n = (n_1, n_2) \in \mathbb{Z}^2_+; \\ v_n|_{n_2 = 0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(E). \end{cases} \tag{1.9}$$

Thus we obtain the Hilbert space $D_+(\tilde{N}, \tilde{\Gamma})$ that is made of solutions v_n (1.9), besides $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}_+}(\tilde{E})}$. Unlike the evident recurrent scheme (1.8) of the layer-to-layer calculation of $n_2 \to n_2 - 1$ for u_n, in this case, while constructing v_n in $\mathbb{Z}^2_+$, we are dealing with the implicit linear system of equations for layer-to-layer calculation of $n_2 \to n_2 + 1$ for the function v_n.

Hereinafter, the following lemma plays an important role. The proof of the lemma is given in [8].

Lemma 1.1. *Suppose the commutative unitary expansion V_s, $\overset{+}{V}_s$ (1.1) is such that*

$$\operatorname{Ker} \Phi = \operatorname{Ker} \Psi^* = \{0\}. \tag{1.10}$$

Then $\operatorname{Ker} N \cap \operatorname{Ker} \Gamma = \{0\}$ *given* $\operatorname{Ker} K^* = \{0\}$, *and respectively* $\operatorname{Ker} \tilde{N}^* \cap \operatorname{Ker} \tilde{\Gamma}^* = 0$ *given* $\operatorname{Ker} K = \{0\}$.

The solvability of the Cauchy problem (1.9) easily follows [8] from the given lemma.

Statement 1.1. *Let* $\dim \tilde{E} < \infty$ *and the assumptions of Lem. 1.1 be true, then the solution* v_n *of the Cauchy problem (1.9) exists and is unique in the domain* $\mathbb{Z}_+^2$ *for all initial data* v_{n_1} *from* $l^2_{\mathbb{Z}_+}(\tilde{E})$.

Consider now the operator-function of discrete argument

$$\tilde{\sigma}_\Delta = \begin{cases} I: & \Delta = (1;0); \\ \tilde{\sigma}; & \Delta = (0,1). \end{cases} \tag{1.11}$$

Let L_0^n be the nonincreasing polygon that connects points $O = (0,0)$ and $n = (n_1, n_2) \in \mathbb{Z}_+^2$ and linear segments of which are parallel to the axes OX ($n_2 = 0$) and OY ($n_1 = 0$). Denote by $\{P_k\}_0^N$ all integer-valued points from $\mathbb{Z}_+^2$, $P_k \in \mathbb{Z}_+^2$ ($N = n_1 + n_2$) that lie on L_0^n, beginning with $(0,0)$ and finishing with the point (n_1, n_2), that are numbered in nondescending order (of one of the coordinates of P_k). Assuming that $P_{-1} = (-1, 0)$, define the quadratic form

$$\langle \tilde{\sigma} v_k \rangle^2_{L_0^n} = \sum_{k=0}^{N} \left\langle \tilde{\sigma}_{P_k - P_{k-1}} v_{P_k}, v_{P_k} \right\rangle \tag{1.12}$$

on the vector-functions $v_k \in D_+(\tilde{N}, \tilde{\Gamma})$.

Similarly, consider the nondecreasing polygon L_m^{-1} in $\tilde{\mathbb{Z}}_-^2$ (1.7) that connects points $m = (m_1, m_2) \in \tilde{\mathbb{Z}}_-^2$ and $(-1, 0)$, the straight segments of which are parallel to OX and OY. Let $\{Q_s\}_M^{-1}$ ($M = m_1 + m_2$) be all integer-valued points on L_m^{-1}, beginning with $m = (m_1, m_2)$ and finishing with $(-1, 0)$, that are numbered in nondescending order (of one of the coordinates of Q_s). Define the metric in $D_-(N, \Gamma)$,

$$\langle \sigma u_k \rangle^2_{L_m^{-1}} = \sum_{s=M}^{-1} \left\langle \sigma_{Q_s - Q_{s-1}} u_{Q_s}, u_{Q_s} \right\rangle, \tag{1.13}$$

besides $Q_M - Q_{M-1} = (1, 0)$, and the operator-function σ_Δ is defined similarly to $\tilde{\sigma}_\Delta$ (1.11). Denote by $\tilde{L}_{-n}^{-1}$ the polygon in $\tilde{\mathbb{Z}}_-^2$ that is obtained from the curve L_0^n in $\mathbb{Z}_+^2$ ($n \in \mathbb{Z}_+^2$) using the shift by "n"

$$\tilde{L}_{-n}^{-1} = \left\{ Q_s = (l_1, l_2) \in \tilde{\mathbb{Z}}_-^2 : (l_1 + n_1 + 1, l_2 + n_2) = P_k \in L_0^n \right\}. \tag{1.14}$$

III. Having now the Hilbert space $D_-(N, \Gamma)$, that is formed by the solutions of the Cauchy problem (1.8) and the space $D_+(\tilde{N}, \tilde{\Gamma})$, that is formed by the solutions of (1.9), we can define the Hilbert space

$$\mathcal{H}_{N,\Gamma} = D_-(N, \Gamma) \oplus H \oplus D_+(\tilde{N}, \tilde{\Gamma}), \tag{1.15}$$

the norm in which is defined by the norm of the initial space $\mathcal{H} = D_- \oplus H \oplus D_+$ (1.4). Denote by $\hat{\mathbb{Z}}_+^2$ the subset in $\mathbb{Z}_+^2$,

$$\hat{\mathbb{Z}}_+^2 = \mathbb{Z}_+^2 \backslash (\{0\} \times \mathbb{N}) = \{(0,0)\} \cup (\mathbb{N} \times \mathbb{Z}_+), \tag{1.16}$$

that is obviously an additional semigroup.

For every $n \in \hat{\mathbb{Z}}_+^2$ (1.16), define an operator-function $U(n)$ that acts on the vectors $f = (u_k, h, v_k) \in \mathcal{H}_{n,\Gamma}$ (1.15) in the following way:

$$U(n)f = f(n) = (u_k(n), h(n), v_k(n)), \tag{1.17}$$

where $u_k(n) = P_{D_-(N,\Gamma)} u_{k-n}$ ($P_{D_-(N,\Gamma)}$ is an orthoprojector that corresponds to the restriction on $D_-(N,\Gamma)$); $h(n) = y_0$, besides $y_k \in H$ ($k \in \mathbb{Z}_+^2$) is a solution of the Cauchy problem

$$\begin{cases} \tilde{\partial}_1 y_k = T_1 y_k + \Phi u_{\tilde{k}}; \\ \tilde{\partial}_2 y_k = T_2 y_k + \Phi N u_{\tilde{k}}; \\ y_n = h; \quad k = (k_1, k_2) \in \mathbb{Z}_+^2 \quad (0 \le k_1 \le n_1 - 1, \quad 0 \le k_2 \le n_2); \end{cases} \tag{1.18}$$

at the same time $\tilde{k} = k - n$ when $0 \le k_1 \le n_1 - 1$, $0 \le k_2 \le n_2$, and finally

$$v_k(n) = \hat{v}_k + v_{k-n} \tag{1.19}$$

and $\hat{v}_k = K u_{\tilde{k}} + \Psi y_k$, where y_k is a solution of the Cauchy problem (1.18).

It is easy to see that the operator-function $U(n)$ (1.17) maps the space $\mathcal{H}_{N,\Gamma}$ (1.15) into itself for all $n \in \hat{\mathbb{Z}}_+^2$ (1.16), moreover, the following theorem takes place [8].

Theorem 1.1. *Suppose* $\dim \tilde{E} < \infty$ *and the suppositions of Lem. 1.1 take place, then the following conservation law is true for the vector-function $f(n) = U(n)f$ (1.17):*

$$\|h(n)\|^2 + \langle \tilde{\sigma} v_k(n) \rangle^2_{L_0^{\hat{n}}} = \|h\|^2 + \langle \sigma u_k \rangle^2_{\tilde{L}_{-n}^{-1}} \tag{1.20}$$

for all $n \in \hat{\mathbb{Z}}_+^2$ (1.16) and for all nondecreasing polygons $\hat{L}_0^{\hat{n}}$ that connect points $O = (0,0)$ and $\hat{n} = (n_1 - 1, n_2) \in \mathbb{Z}_+^2$, where $\tilde{L}_{-\hat{n}}^{-1}$ is a polygon obtained from L_0^n by the shift (1.14) by "n", at the same time the corresponding σ-forms in (1.20) have the form of (1.12) and (1.13). The operator-function $U(n)$ (1.17) is a semigroup, $U(n) \cdot U(m) = U(n+m)$, for all n, $m \in \hat{\mathbb{Z}}_+^2$ (1.16).

It follows from [8] and from this theorem that the operator-function $U(n)$ (1.17) is an isometric dilation of the semigroup

$$T(n) = T_1^{n_1} T_2^{n_2}, \quad n = (n_1, n_2) \in \mathbb{Z}_+^2. \tag{1.21}$$

IV. Make the similar continuation of the subspaces D_+ and D_- (1.6) from the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ by the second variable "n_2", corresponding to the dual situation. Denote by $D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$ the Hilbert space generated by solutions $\tilde{v}_n$ of the Cauchy problem

$$\begin{cases} \partial_2 \tilde{v}_n = \left(\tilde{N}^* \partial_1 + \tilde{\Gamma}^*\right) \tilde{v}_n; \quad n = (n_1, n_2) \in \mathbb{Z}_+^2; \\ \tilde{v}_n\big|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E}), \end{cases} \tag{1.22}$$

in which the norm is induced by the norm of the initial data $\|\tilde{v}_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}_+}(E)}$, besides $\partial_1 \tilde{v}_n = \tilde{v}_{(n_1+1, n_2)}$, $\partial_2 \tilde{v}_n = \tilde{v}_{(n_1, n_2+1)}$.

Continue now every function $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$ into the domain $\tilde{\mathbb{Z}}^2_-$ (1.7) using the Cauchy problem

$$\begin{cases} \partial_2 \tilde{u}_n = \left(N^* \partial_1 + \Gamma^*\right) \tilde{u}_n; \quad n = (n_1, n_2) \in \tilde{\mathbb{Z}}_-^2; \\ \tilde{u}_n\big|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}_-}(E). \end{cases} \tag{1.23}$$

As a result, we obtain the Hilbert space $D_-\left(N^*, \Gamma^*\right)$ generated by $\tilde{u}_n$, solutions of (1.23), besides $\|\tilde{u}_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$. Using now Lem. 1.1, we can formulate an analogue of St. 1 [8].

Statement 1.2. *Let* $\dim E < \infty$ *and the suppositions of Lem. 1.1 be true, then the solution* $\tilde{u}_n$ *of the Cauchy problem (1.23) exists and is unique in the domain* $\tilde{\mathbb{Z}}^2_-$ *(1.7) for all initial data* $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$.

O b s e r v a t i o n 1.1. The sufficient condition for the simultaneous existence of solutions of the Cauchy problems (1.9) and (1.23), in view of the reversibility of operators K and K^*, according to Lem. 1.1, is the following: all hypotheses of Lem. 1.1 are met and $\dim E = \dim \tilde{E} < \infty$.

Hence we come to the Hilbert space

$$\mathcal{H}_{N^*, \Gamma^*} = D_-\left(N^*, \Gamma^*\right) \oplus H \oplus D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right), \tag{1.24}$$

where the metric is induced by the norm of the initial space $\mathcal{H} = D_- \oplus H \oplus D_+$ (1.4). Note that the dual feature of the spaces $\mathcal{H}_{N,\Gamma}$ (1.15) and $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) is that differential operators of the Cauchy problems (18) and (1.23) and operators (1.9) and (1.22) also are adjoint with each other respectively in the metric l^2.

Define now the operator-function $\overset{+}{U}(n)$ for $n \in \hat{\mathbb{Z}}^2_+$ (1.16) in the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24), which acts on $\tilde{f} = \left(\tilde{u}_k, \tilde{h}, \tilde{v}_k\right) \in \mathcal{H}_{N^*,\Gamma^*}$ in the following way:

$$\overset{+}{U}(n) \tilde{f} = \tilde{f}(n) = \left(\tilde{u}_k(n), \tilde{h}(n), \tilde{v}(n)\right), \tag{1.25}$$

where $\tilde{v}_k(n) = P_{D_+(\tilde{N}^*, \tilde{\Gamma}^*)} \tilde{v}_{k+n}$ ($P_{D_+(\tilde{N}^*, \tilde{\Gamma}^*)}$ is an orthoprojector onto $D_+(\tilde{N}^*, \tilde{\Gamma}^*)$);

$\tilde{h}(n) = \tilde{y}_{(-1;0)}$, besides $\tilde{y}_k$ $(k \in \tilde{\mathbb{Z}}^2_-)$ satisfies the Cauchy problem

$$\begin{cases} \partial_1 \tilde{y}_k = T_1^* \tilde{y}_k + \Psi^* \tilde{v}_{\tilde{k}}; \\ \partial_2 \tilde{y}_k = T_2^* \tilde{y}_k + \Psi^* \tilde{N}^* \tilde{v}_{\tilde{k}}; \\ \tilde{y}_{(-n_1;-n_2)} = h;\ k = (k_1;k_2) \in \tilde{\mathbb{Z}}^2_-\ (-n_1 \le k_1 \le -1;\ -n_2 \le k_2 \le 0); \end{cases} \tag{1.26}$$

besides $\tilde{k} = k + n$ ш $(-n_1 \le k_1 \le -1;\ -n_2 \le k_2 \le 0)$; and finally

$$\tilde{u}_k(n) = \hat{u}_k + \tilde{u}_{k+n}, \tag{1.27}$$

and $\hat{u}_k = K^* \tilde{v}_{\tilde{k}} + \Phi^* \tilde{y}_k$, where $\tilde{y}_k$ is a solution of the system (1.26).

Similarly to (1.11), define the operator-function

$$\tau_\Delta = \begin{cases} I; & \Delta = (-1,0); \\ \tau; & \Delta = (0,-1). \end{cases} \tag{1.28}$$

Denote by L_m^{-1} a nondecreasing polygon in $\tilde{\mathbb{Z}}^2_-$ (1.7) with the linear segments that are parallel to the axes OX and OY which connects the points $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$ and $(-1,0)$. Choose now all the points $\{Q_s\}_M^{-1}$ $(M = m_1 + m_2)$ on L_m^{-1} that are numerated in nonascending order (of one of the coordinates Q_s) beginning with the point $(-1,0)$ and finishing with $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$. Define the quadratic form

$$\langle \tau \tilde{u}_k \rangle^2_{L_m^{-1}} = \sum_{s=M}^{-1} \left\langle \tau_{Q_s - Q_{s+1}} \tilde{u}_{Q_s}, \tilde{u}_{Q_s} \right\rangle \tag{1.29}$$

in the space $D_-(N^*, \Gamma^*)$, where $Q_0 = (0,0)$. For the polygon L_0^n in $\mathbb{Z}^2_+$, $n = (n_1, n_2) \in \mathbb{Z}^2_+$, of the similar type with points $\{P_k\}_0^N$ $(N = n_1 + n_2)$ on L_0^n which are also chosen in nonascending order, define the quadratic form for the functions $\tilde{v}_k \in D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$

$$\langle \tilde{\tau} \tilde{v}_k \rangle^2_{L_0^n} = \sum_{k=0}^{N} \left\langle \tilde{\tau}_{P_k - P_{k+1}} \tilde{v}_{P_k}, \tilde{v}_{P_k} \right\rangle, \tag{1.30}$$

where $P_N - P_{N+1} = (-1,0)$ and $\tilde{\tau}_\Delta$ is defined similarly to τ_Δ (1.28). Denote by $\tilde{L}_0^m$ the polygon in $\mathbb{Z}^2_+$ obtained from the curve L_m^{-1} from $\tilde{\mathbb{Z}}^2_-$ using the shift by "m"

$$\tilde{L}_0^m = \left\{ P_k = (l_1, l_2) \in \mathbb{Z}^2_+ : (l_1 + m_1, l_2 + m_2) = Q_s \in L_m^{-1} \right\}, \tag{1.31}$$

where $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$. Similarly to Th. 1.1, the following statement [8] takes place.

Theorem 1.2. *Suppose that* $\dim E < \infty$ *and that the hypotheses of Lem. 1.1 take place, then for the vector-function* $\tilde{f}(n) = \overset{+}{U}(n)\tilde{f}$ *(1.25) the equality*

$$\|\tilde{h}(n)\|^2 + \langle \tau \tilde{u}_k(n) \rangle^2_{L^{-1}_{-n}} = \|h\|^2 + \langle \tilde{\tau} \tilde{v}_k \rangle_{\tilde{L}_0^{-n}} \tag{1.32}$$

takes place for all $n \in \hat{\mathbb{Z}}^2_+$ *(1.16) and for all polygons* L^{-1}_{-n} *connecting points* $-n = (-n_1, -n_2) \in \tilde{\mathbb{Z}}^2_-$ *and* $(-1, 0)$*, where* $\tilde{L}_0^{-n}$ *is a curve in* $\mathbb{Z}^2_+$ *obtained from* L^{-1}_{-n} *using the shift (1.31) by "*$-n$*", and corresponding* τ*-forms in (1.32) have the form of (1.29) and (1.30). The operator-function* $\overset{+}{U}(n)$ *(1.25) has the semigroup property,* $\overset{+}{U}(n)\overset{+}{U}(m) = \overset{+}{U}(n+m)$ *for all* n*,* $m \in \hat{\mathbb{Z}}^2_+$ *(1.16).*

The fact that the semigroup $\overset{+}{U}(n)$ (1.25) is the isometric dilation of the semigroup $T^*(n)$, where $T(n)$ has the form of (1.21), is proved in [8].

In the conclusion of this paragraph, note that the dilations $U(n)$ (1.17) and $\overset{+}{U}(n)$ (1.25) are unitary linked, i.e., $U^*(n_1, 0) f = \overset{+}{U}(n_1, 0) f$ for all $f \in \mathcal{H}$ (1.4) and for all $n_1 \in \mathbb{Z}_+$, and the narrowing $U(n_1, 0)$ onto $\mathcal{H}$ is a unitary semigroup.

2. Scattering Scheme with Many Parameters and Translational Models

I. As it is known [3, 6], a translational (as well as a functional) model of the contraction T and its dilation U (1.5) is based on the study of the main properties of the wave operators $W_\pm$ and scattering operator S.

In order to construct the wave operators $W_\pm$ in the case of many parameters, it is necessary also to continue the vector-functions from $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ and $l^2_{\mathbb{Z}}(E)$ from the axis $\mathbb{Z}$ into the domain $\mathbb{Z}^2$. Continue every function $u_{n_1} \in l^2_{\mathbb{Z}}(E)$ to the function u_n, where $n = (n_1, n_2) \in \mathbb{Z}^2$, using the Cauchy problem

$$\begin{cases} \tilde{\partial}_2 u_n = \left(N\tilde{\partial}_1 + \Gamma\right) u_n; & n \in \mathbb{Z}^2; \\ u_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}}(E); \end{cases} \tag{2.1}$$

besides $\|u_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}}(E)}$. Note that this continuation into the lower semiplane ($n_2 \in \mathbb{Z}_-$), $u(n_1, n_2) \to u(n_1, n_2 - 1)$, has a recurrent nature and continuation into the upper semiplane $u(n_1, n_2) \to u(n_1, n_2 + 1)$ may be carried out in a non-explicit way, certainly, in the context of suppositions of Lem. 1.1 and $\dim E < \infty$. As a result, we obtain the Hilbert space $l^2_{N,\Gamma}(E)$ the norm of which is induced by the norm of the initial data.

Define now the shift operator $V(p)$

$$V(p) u_n = u_{n-p}, \tag{2.2}$$

where $u_n \in l^2_{N,\Gamma}(E)$ for all $p \in \mathbb{Z}^2$. Obviously, this operator $V(p)$ (2.2) is an isometric one.

Knowing the perturbed $U(n)$ (1.17) and the free $V(n)$ (2.2) operator semigroups, it is natural to define the wave operator $W_-(n)$,

$$W_-(k) = s - \lim_{n\to\infty} U(n,k)P_{D_-(N,\Gamma)}V(-n,-k) \tag{2.3}$$

for every fixed $k \in \mathbb{Z}_+$, where $P_{D_-(N,\Gamma)}$ is the orthoprojector of the narrowing onto the component u_n^- from $l^2_{N,\Gamma}(E)$ obtained as a result of continuation into $\tilde{\mathbb{Z}}^2_-$ (1.7) from the semiaxis $\mathbb{Z}_-$ using the Cauchy problem (2.1). It is obvious that $W_-(0) = W_-$, where the wave operator W_- corresponds with the dilation U (1.5) and the shift operator V in $l^2_{\mathbb{Z}}(E)$ [6]. Thus, $W_-(k)$ (2.3) is a natural continuation of the wave operator W_- onto the "k-th" horizontal line in $\mathbb{Z}^2$ for $k \in \mathbb{Z}_+$.

Denote by $L^\infty_{0,k}$ the polygon in $\mathbb{Z}^2_+$ consisting of two linear segments: the first one is a vertical segment connecting points $O = (0,0)$ and $(0,k)$, where $k \in \mathbb{Z}_+$, and the second segment is a horizontal semiline from the point $(0,k)$ to (∞,k). Similarly, choose the polygon $\tilde{L}^{-1}_{-\infty,p}$ in $\tilde{\mathbb{Z}}^2_-$ (1.7) that consists also of two linear segments, the first of which is a semiline from $(-\infty,-p)$ to the point $(-1,-p)$, where $p \in \mathbb{Z}_+$, and the second one is a vertical segment from the point $(-1,-p)$ to $(-1,0)$. In the space $\mathcal{H}_{N,\Gamma}$ (1.15), specify the following quadratic forms:

$$\begin{aligned}
\langle f\rangle^2_{\sigma(p,k)} &= \langle \sigma u_n\rangle^2_{\tilde{L}^{-1}_{-\infty,p}} + \|h\|^2 + \langle\tilde{\sigma} v_n\rangle^2_{L^\infty_{0,k}}\,;\\
\langle f\rangle^2_{\tilde{\sigma}(k)} &= \|u_n\|^2_{l^2} + \|h\|^2 + \langle\tilde{\sigma} v_n\rangle^2_{L^\infty_{0,k}}\,;\\
\langle f\rangle^2_{\sigma(p)} &= \langle \sigma u_n\rangle^2_{\tilde{L}^{-1}_{-\infty,p}} + \|h\|^2 + \|v_n\|^2_{l^2}\,,
\end{aligned} \tag{2.4}$$

where corresponding σ and $\tilde{\sigma}$ forms in (2.4) are understood in the sense of (1.12) and (1.13). It is easy to see that $\langle f\rangle^2_{\sigma(0,0)} = \langle f\rangle^2_{\tilde{\sigma}(0)} = \langle f\rangle^2_{\sigma(0)} = \|f\|^2_{\mathcal{H}_{N,\Gamma}}$ and $\langle f\rangle^2_{\sigma(0,k)} = \langle f\rangle^2_{\tilde{\sigma}(k)}$, $\langle f\rangle^2_{\sigma(p,0)} = \langle f\rangle^2_{\sigma(p)}$.

Similarly to (2.4), specify in $l^2_{N,\Gamma}(E)$ the following σ-forms:

$$\begin{aligned}
\langle u_n\rangle^2_{\sigma(p,k)} &= \langle \sigma u_n^-\rangle^2_{\tilde{L}^{-1}_{-\infty,p}} + \langle\sigma u_n^+\rangle_{L^\infty_{0,k}}\,;\\
\langle u_n\rangle^2_{\sigma_+(k)} &= \|u_n^-\|^2_{l^2} + \langle\sigma u_n^+\rangle_{L^\infty_{0,k}}\,;\\
\langle u_n\rangle^2_{\sigma_-(p)} &= \langle \sigma u_n^-\rangle^2_{\tilde{L}^{-1}_{-\infty,-p}} + \|u_n^+\|^2_{l^2}\,,
\end{aligned} \tag{2.5}$$

where $u_n^\pm$ are the continuations of $l^2_{\mathbb{Z}_\pm}(E)$ from the semiaxes using the Cauchy problem (2.1). Note that $\langle u_n\rangle^2_{\sigma(0,k)} = \langle u_n\rangle^2_{\sigma_+(k)}$; $\langle u_n\rangle^2_{\sigma(p,0)} = \langle u_n\rangle^2_{\sigma_-(p)}$ and finally $\langle u_n\rangle^2_{\sigma(0,0)} = \langle u_n\rangle^2_{\sigma_+(0)} = \langle u_n\rangle^2_{\sigma_-(0)} = \|u_n\|^2_{l^2}$.

Theorem 2.1. *The wave operator $W_-(k)$ (2.3) mapping $l^2_{N,\Gamma}(E)$ into the space $\mathcal{H}_{N,\Gamma}$ (1.15) exists for all $k \in \mathbb{Z}_+$, and it is an isometry*

$$\langle W_-(k)u_n\rangle^2_{\sigma(p,k)} = \langle u_n\rangle^2_{\sigma(p,k)} \tag{2.6}$$

in metrics (2.4), (2.5) for all $p \in \mathbb{Z}_+$. Moreover, the wave operator $W_-(k)$ (2.3) meets the conditions

$$\begin{array}{ll} 1) & U(1,s)W_-(k) = W_-(k+s)V(1,s); \\ 2) & W_-(k)P_{D_-(N,\Gamma)} = P_{D_-(N,\Gamma)} \end{array} \tag{2.7}$$

for all k, $s \in \mathbb{Z}_+$, where $P_{D_-(N,\Gamma)}$ is an orthoprojector onto $D_-(N,\Gamma)$.

P r o o f. Relation 2) (2.7) is proved exactly in the same way as for W_- [6]. The isometric property (2.6) for $W_-(k)$ (2.3) follows from Th. 1.1. In order to prove 1) (2.7), consider the identity

$$U(1,s)U(n,k)P_{D_-(N,\Gamma)}V(-n,-k)$$
$$= U(n+1,k+s)P_{D_-(N,\Gamma)}V(-n-1,-k-s)V(1,s),$$

where the limit process leads us to equality 1) when $n \to \infty$. And since

$$W_-(s)V(1,s) = U(1,s)W_-(0),$$

then $W_-(s)$ existence follows from the existence of $W_-(0) = W_-$ [6] for all $s \in \mathbb{Z}_+$. ∎

Note that the equalities

$$\begin{array}{ll} 1) & U(1,0)W_-(k) = W_-(k)V(1,0); \\ 2) & U(1,k)W_-(0) = W_-(k)V(1,k) \end{array} \tag{2.8}$$

for all $k \in \mathbb{Z}_+$ follow from 1) (2.7).

II. Consider now the continuation of the vector-functions v_{n_1} from $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ into the domain $\mathbb{Z}^2$ using the Cauchy problem

$$\begin{cases} \tilde{\partial}_2 v_n = \left(\tilde{N}\tilde{\partial}_1 + \tilde{\Gamma}\right)v_n; & n = (n_1,n_2) \in \mathbb{Z}^2; \\ v_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}}\left(\tilde{E}\right). \end{cases} \tag{2.9}$$

As in the case of problem (2.1), in the semiplane $n_2 \in \mathbb{Z}_-$ we have a recurrent way of the continuation from the axis $n_2 = 0$, $n_2 \to n_2 - 1$ and, when $n_2 \in \mathbb{Z}_+$, this continuation may be carried out in the context of Supposition 1.1. The Hilbert space obtained in this way may be denoted by $l_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$, besides $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}}(\tilde{E})}$.

Similarly to $V(p)$ (2.2), introduce the shift operator

$$\tilde{V}(p)v_n = v_{n-p} \tag{2.10}$$

for all $p \in \mathbb{Z}^2$ and all $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$. Define the wave operator $W_+(p)$ from $\mathcal{H}_{N,\Gamma}$ into the space $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$,

$$W_+(p) = s - \lim_{n\to\infty} \tilde{V}(-n,-p)P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)}U(n,p) \tag{2.11}$$

for all $p \in \mathbb{Z}_+$, where $U(n)$ has the form of (1.17). It is obvious that $W_+(0) = W_+^*$, where W_+ is a traditional wave operator [6] corresponding to U (1.5) and to the shift $\tilde{V}$ in $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$. Similarly to Th. 2.1, the following statement is true.

Theorem 2.2. *For all $p \in \mathbb{Z}_+$, the wave operator $W_+(p)$ (2.11) acting from the space $\mathcal{H}_{N,\Gamma}$ into $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ exists and satisfies the relations*

$$\begin{aligned} &1) \quad W_+(p)U(1,s) = \tilde{V}(1,s)W_+(p+s);\\ &2) \quad W_+(p)P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)} = P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)} \end{aligned} \tag{2.11}$$

for all p, $s \in \mathbb{Z}_+$, where $P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)}$ is an orthoprojector onto $D_+\left(\tilde{N},\tilde{\Gamma}\right)$.

The proof of this statement is similar to the proof of Th. 2.1. ■

The equalities

$$\begin{aligned} &1) \quad W_+(p)U(1,0) = \tilde{V}(1,0)W_+(p);\\ &2) \quad W_+(0)U(1,p) = \tilde{V}(1,p)W_+(p) \end{aligned} \tag{2.12}$$

for all $p \in \mathbb{Z}_+$ easily follow from 1) (2.11).

Knowing the wave operators $W_-(k)$ (2.3) and $W_+(p)$ (2.11), define the scattering operator in a traditional way [6]:

$$S(p,k) = W_+(p)W_-(k) \tag{2.13}$$

for all p, $k \in \mathbb{Z}_+$. It is obvious that, when $p = k = 0$, we have $S(0,0) = S$, where S is a standard scattering operator $S = W_+^*W_-$ for the dilation U (1.5) [6]. The following statement results from Ths. 2.1 and 2.2.

Theorem 2.3. *The scattering operator $S(p,k)$ (2.13) represents the bounded operator from $l^2_{N,\Gamma}(E)$ into $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ satisfying the following relations:*

$$\begin{aligned} &1) \quad S(p,k)V(1,q) = \tilde{V}(1,q)S(p+q,k-q);\\ &2) \quad S(p,k)P_-l^2_{N,\Gamma}(E) \subseteq P_-l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \end{aligned} \tag{2.14}$$

for all p, k, $q \in \mathbb{Z}_+$, $0 \le q \le k$, where P_- is the narrowing orthoprojector onto the solutions of the Cauchy problems (2.1) and (2.9) with the initial data on the semiaxis $\mathbb{Z}_-$ when $n_2 = 0$.

Note that the translational invariability of $S(p,k)$ (2.13) with respect to the shift by the first variable "n_1"

$$S(p,k)V(1,0) = \tilde{V}(1,0)S(p,q) \tag{2.15}$$

for all p, $k \in \mathbb{Z}_+$ follows from the equality 1) (2.14). Moreover, from 1) it follows that

$$\begin{array}{ll} 1) & S(p,k)V(1,k) = \tilde{V}(1,k)S(p+k,0) \quad (k=q); \\ 2) & S(0,k)V(1,k) = \tilde{V}(1.k)S(k,0) \quad (k=q, p=0); \end{array} \tag{2.16}$$

and thus the scattering operator $S(p,k)$ (2.13) is the function of sum (up to the multiplication of $V(1,k)$ and $\tilde{V}(1,k)$) for all p and k from $\mathbb{Z}_+$, and it may be obtained from the operator $S(k,0)$ (or from $S(0,k)$) using the "bordering" by the shift operators $V(1,k)$ and $\tilde{V}(1,k)$.

III. Specify now the mapping $\mathcal{P}_{p,k}$ from $l^2_{N,\Gamma}(E) + l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ into the Hilbert space $\mathcal{H}_{N,\Gamma}$ (1.15)

$$f_{p,k} = \mathcal{P}_{p,k}\left(g_n\right) = \mathcal{P}_{p,k}\begin{pmatrix} v_n \\ u_n \end{pmatrix} = W_+^*(p)v_n + W_-(k)u_n, \tag{2.17}$$

where $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$, $u_n \in l^2_{N,\Gamma}(E)$, besides p, $k \in \mathbb{Z}_+$ and $W_+^*(p)$ adjoined to the operator $W_+(p)$ is understood in the sense of Hilbert metric l^2.

For the commutative operator systems $\{T_1, T_2\} \in C\left(T_1\right)$ (1.3), the simplicity of the expansion V_s, $\overset{+}{V}_s$ (1.1) is guaranteed by the operator T_1 [4, 8]. Therefore in the case of simplicity of the expansion V_s, $\overset{+}{V}_s$ (1.1), the functions $f_{p,k} = \mathcal{P}_{p,k}\left(g_k\right)$ (2.17) form the everywhere dense set in the space $\mathcal{H}_{N,\Gamma}$ when $g_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) + l^2_{n,\Gamma}(E)$ for all fixed p and k from $\mathbb{Z}_+$. And thus "every" function f from the functions of the space $\mathcal{H}_{N,\Gamma}$ (e.g., every finite one) may have various forms $f = f_{p,k}$ or $f = f_{p',k'}$ ($p \ne p'$, $k \ne k'$) when one takes different mappings $\mathcal{P}_{p,k}$ (2.17). It is easy to see that

$$\langle f_{p,k}, f_{p,k}\rangle_{\mathcal{H}_{N,\Gamma}} = \langle W_{p,k}g_n, g_n\rangle_{l^2},$$

when the weight operator-function $W_{p,k}$ has the form

$$W_{p,k} = \begin{bmatrix} W_+(p)W_+^*(p) & S(p,k) \\ S^*(p,k) & W_-^*(k)W_-(k) \end{bmatrix}, \tag{2.18}$$

and the scattering operator $S(p,k)$ is defined by formula (2.13).

O b s e r v a t i o n 2.1. All the elements of the weight operator-function $W_{p,k}$ (2.18) have the translational invariance with respect to the shift by the variable "n_1" in view of 1) (2.8), 1) (2.12) and (2.15), and also the unitarity of the operator $U(1,0)$.

So, the mapping $\mathcal{P}_{k,s}$ (2.17) defines the isomorphism between the spaces $\mathcal{H}_{N,\Gamma}$ (1.15) and

$$l^2\left(W_{p,k}\right) = \left\{ g_n = \begin{pmatrix} v_n \\ u_n \end{pmatrix} : \langle W_{p,k} g_n, g_n \rangle_{l^2} < \infty \right\}, \tag{2.19}$$

where $u_n \in l^2_{N,\Gamma}(E)$, $v_n \in l^2_{N,\Gamma}\left(\tilde{E}\right)$ and the operator $W_{p,k}$ has the form of (2.18). It is obvious that the space $l^2\left(W_{p,k}\right)$ (2.19) coincides with the well-known space $l^2\begin{pmatrix} I & S \\ S & I \end{pmatrix}$ [6] when $p = k = 0$. From the relations 1) (2.8), 1) (2.12) and from the unitarity of $U(1,0)$, it follows that the dilation $U(1,0)$ in every space $l^2\left(W_{p,k}\right)$ (2.19) is carried out by the shift operator

$$\hat{U}(1,0) g_n = \begin{bmatrix} \tilde{V}(1,0) & 0 \\ 0 & V(1,0) \end{bmatrix} g_n \tag{2.20}$$

for all $g_n \in l^2\left(W_{p,k}\right)$.

Study now how the dilation $U(1,s)$ (1.17) acts on the vector-functions $f_{p,k} = \mathcal{P}_{p,k}\left(g_n\right)$ (2.17) when $s \neq 0$. First of all, note that it follows from 1) (2.7) that an application of $U(1,s)$ to the wave operator $W_-(k)$ (2.3) from the left increases the index $k \in \mathbb{Z}_+$ by s, i.e. $k \to k+s$, and it follows from the equality 1) (2.11) that an application of the dilation $U(1,s)$ to the wave operator $W_+(p)$ (2.11) from the right also changes the parameter $p \in \mathbb{Z}_+$, namely, $p \to p+s$. Therefore the dilation $U(1,s)$ maps the element $f_{p,k}$ from $\mathcal{H}_{N,\Gamma}$ to the representative $f_{p-s,k+s}$ in the space $\mathcal{H}_{N,\Gamma}$ (1.15), where $0 \leq s \leq p$. Consider only the case when the dilation $U(1,p)$ (1.17) acts on the vectors of the form $f_{p,0} = \mathcal{P}_{p,0}\left(g_n\right)$ (2.17).

So, in view of above, consider the scalar product

$$\begin{gathered} \left\langle U(1,p) f_{p,0}, \hat{f}_{0,p} \right\rangle_{\tilde{\sigma}(p)} = \left\langle U(1,p) W_+^*(p) v_n, W_+^*(0) \hat{v}_n \right\rangle_{\tilde{\sigma}(p)} \\ + \left\langle U(1,p) W_+^*(p) v_n, W_-(p) \hat{u}_n \right\rangle_{\tilde{\sigma}(p)} + \left\langle U(1,p) W_-(0) u_n, W_+^*(0) \hat{v}_n \right\rangle_{\tilde{\sigma}(p)} \\ + \left\langle U(1,p) W_-(0) u_n, W_-(p) \hat{u}_n \right\rangle_{\tilde{\sigma}(p)}, \end{gathered} \tag{2.21}$$

where $f_{p,0} = \mathcal{P}_{p,0}\left(g_n\right)$, $\hat{f}_{0,p} = \mathcal{P}_{0,p}\left(\hat{g}_n\right)$ (2.17). Simplify every element from the right part in (2.21). It is easy to see that the third and the fourth elements have the form

$$\left\langle U(1,p) W_-(0) u_n, W_+^*(0) \hat{v}_n \right\rangle_{\tilde{\sigma}(p)} = \left\langle S(0,p) V(1,p) u_n, \hat{v}_n \right\rangle_{\tilde{\sigma}_+(p)};$$

$$\langle U(1,p)W_-(0)u_n, W_-(p)\hat{u}_n\rangle_{\tilde{\sigma}(p)} = \langle V(1,p)u_n, \hat{u}_n\rangle_{\sigma_+(p)}$$

taking into account property 2) (2.8), the form of the operator $S(0,p)$ (2.13), and the σ-isometric condition of the wave operator $W_-(p)$ (2.3) by Th. 2.1 and 2) (2.11) used in the first relation. In order to simplify the first elements in (2.21), use relations 2) (2.11) and 2) (2.12) for the wave operator $W_+(p)$ to obtain

$$\left\langle U(1,p)W_+^*(p)v_n, W_+^*(0)\hat{v}_n\right\rangle_{\tilde{\sigma}(p)} = \left\langle \tilde{V}(1,p)W_+(p)W_+^*(p)v_n, \hat{v}_n\right\rangle_{\tilde{\sigma}_+(p)}.$$

Finally, taking into account σ-isometric property of the dilation $U(1,p)$ (Th. 1.1), for the second element we have

$$\left\langle U(1,p)W_+^*(p)v_n, W_-(p)\hat{u}_n\right\rangle_{\tilde{\sigma}(p)}$$

$$= \left\langle U(1,p)W_+^*(p)v_n, U(1,p)W_-(0)V(-1,-p)\hat{u}_n\right\rangle_{\tilde{\sigma}(p)}$$

$$= \left\langle W_+^*(p)v_n, W_-(0)V(-1,p)\hat{u}_n\right\rangle_{\sigma(p)} = \langle S^*(p,0)v_n, V(-1,-p)\hat{u}_n\rangle_{\sigma_-(p)}$$

in view of 2) (2.7). Using now relation 2) (2.16), we obtain that

$$\left\langle U(1,p)W_+^*(p)v_n, W_-(p)\hat{u}_n\right\rangle_{\tilde{\sigma}(p)} = \langle V^*(-1,-p)S^*(p,0)v_n, \hat{u}_n\rangle_{\sigma_+(p)}$$

$$= \left\langle S^*(0,p)\tilde{V}^*(-1,-p)v_n, \hat{u}_n\right\rangle_{\sigma_+(p)}.$$

Thus, we can write formula (2.21) in the following way:

$$\left\langle U(1,p)f_{p,0}\hat{f}_{0,p}\right\rangle_{\tilde{\sigma}(p)}$$

$$= \left\langle \begin{bmatrix} \tilde{V}(1,p)W_+(p)W_+^*(p)\tilde{V}^*(1,p) & S(0,p) \\ S^*(0,p) & I \end{bmatrix} \right.$$

$$\times \left. \begin{bmatrix} \tilde{V}^*(-1,-p) & 0 \\ 0 & V(1,p) \end{bmatrix} g_n, \hat{g}_n \right\rangle_{\tilde{\sigma}_+(p),\sigma_+(p)}, \tag{2.22}$$

where the bi-linear form in the right part is understood component-wisely in the sense of $\tilde{\sigma}_+(p)$ and $\sigma_+(p)$ (2.5). Let

$$W'_{p,0} = \begin{bmatrix} \tilde{V}(1,p)W_+(p)W_+^*(p)\tilde{V}^*(1,p) & S(0,p) \\ S^*(0,p) & I \end{bmatrix};$$
$$\hat{V}(1,p) = \begin{bmatrix} \tilde{V}^*(-1,-p) & 0 \\ 0 & V(1,p) \end{bmatrix}. \tag{2.23}$$

O b s e r v a t i o n 2.2. Consider the mapping $\mathcal{P}_{p,0}$ (2.17) and let $f'_{p,0} = \mathcal{P}_{p,0}\left(g'_n\right) = W^*_+(p)\tilde{V}^*(1,p)v_n + W_-(0)V(-1,-p)u_n$, where $u_n \in l^2_{N,\Gamma}(E)$ and $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$. Then it is easy to find that

$$\left\langle f'_{p,0}, f'_{p,0}\right\rangle_{\mathcal{H}_{N,\Gamma}} = \left\langle W'_{p,0} g_n, g_n\right\rangle_{l^2}$$

in view of 2) (2.16). Thus the difference between the weight $W_{p,0}$ (2.18) and $W'_{p,0}$ (2.23) is that the components v_n and u_n are shifted by $\tilde{V}^*(1,p)$ and $V(-1,-p)$ respectively after the mapping $\mathcal{P}_{p,0}$ (2.17).

Hence, the dilation $U(1,p)$ (1.7) acts by the shift

$$\hat{U}(1,p)g_n = \hat{V}(1,p)g_n, \tag{2.24}$$

($\hat{V}(1,p)$ has the form of (2.23)) from the Hilbert space

$$l^2\left(W'_{p,0}\right) = \left\{ g_n = \begin{pmatrix} v_n \\ u_n \end{pmatrix} : \left\langle W'_{p,0} g_n, g_n\right\rangle_{l^2} < \infty \right\} \tag{2.19$'$}$$

into the space $l^2\left(W_{p,0}\right)$ (2.19).

It is obvious that the following subspaces

$$\hat{D}_-(N,\Gamma) = \begin{pmatrix} 0 \\ P_- l^2_{N,\Gamma}(E) \end{pmatrix}; \quad \hat{D}_+\left(\tilde{N},\tilde{\Gamma}\right) = \begin{pmatrix} P_+ l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \\ 0 \end{pmatrix}$$

are the prototypes of $D_-(N,\Gamma)$ and $D_+\left(\tilde{N},\tilde{\Gamma}\right)$ from $\mathcal{H}_{N,\Gamma}$ (1.15) for the mapping $\mathcal{P}_{p,k}$ (2.17) (for all p, $k \in \mathbb{Z}_+$). P_- and P_+ are the orthoprojectors onto the subspaces in $l^2_{N,\Gamma}(E)$ and in $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ formed by the solutions of the Cauchy problems (2.1) and (2.9) with the initial data on the semiaxes $\mathbb{Z}_-$ and $\mathbb{Z}_+$, respectively. Therefore the initial space H is isomorphic to the space

$$\hat{H}_p = l^2\left(W_{p,0}\right) \ominus \begin{pmatrix} P_+ l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \\ P_- l^2_{N,\Gamma}(E) \end{pmatrix}. \tag{2.25}$$

Similar constructions for $l^2\left(W'_{p,0}\right)$ (2.19$'$) lead to another space realization of the Hilbert space H

$$\hat{H}'_p = l^2\left(W'_{p,0}\right) \ominus \begin{pmatrix} \tilde{V}^*(-1,-p) P_+ l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \\ V(1,p) P_- l^2_{N,\Gamma}(E) \end{pmatrix} \tag{2.25$'$}$$

in view of Observation 2.2. It is natural that the spaces $\hat{H}_p$ (2.25) and $\hat{H}'_p$ (2.25$'$) are isomorphic one to another. As it is easy to see, the operator $R_p : \hat{H}_p \to \hat{H}'_p$

defining this isomorphism has the form

$$R_p = P_{\hat{H}'_p} \begin{bmatrix} \tilde{V}^*(1,p) & 0 \\ 0 & V(-1,-p) \end{bmatrix} P_{\hat{H}_p}, \tag{2.26}$$

where $P_{\hat{H}_p}$ and $P_{\hat{H}'_p}$ are orthoprojectors onto $\hat{H}_p$ (2.25) and $\hat{H}'_p$ (2.25′) in corresponding spaces. It follows from (2.20) and (2.24) that the operators T_1 and $T(1,p) = T_1 T_2^p$, $p \in \mathbb{Z}_+$ have the form

$$\left(\hat{T}_1 f\right)_n = P_{\hat{H}_p} f_{n-(1,0)}; \quad \left(\hat{T}(1,p) f\right)_n = P_{\hat{H}_p} \hat{V}(1,p) \left(R_p f\right)_n \tag{2.27}$$

for all $f_n \in \hat{H}_p$ (2.25), where $P_{\hat{H}_p}$ is an orthoprojector onto $\hat{H}_p$ (2.25) and the operator R_p has the form (2.26). It is typical that the operator $\hat{T}_1$ has the same form (2.27) in all the spaces $\hat{H}_p$ (2.25) in view of Observation 2.1, and the operator $\hat{T}(1,p)$ has this form (2.27) only in one specific space $\hat{H}_p$ (2.25).

Theorem 2.4. *Consider the simple [8] commutative unitary expansion V_s, $\overset{+}{V}_s$ (2.1) corresponding to the commutative operator system $\{T_1, T_2\}$ from the class $C(T_1)$ (1.3), and let the suppositions of Lem. 1.1 take place, besides $\dim E = dim\tilde{E} < \infty$. Then the isometric dilation $U(1,p)$ (1.17), $p \in \mathbb{Z}_+$, acting in the Hilbert space $\mathcal{H}_{N,\Gamma}$ (1.15), is unitary equivalent to the operator $\hat{U}(1,0)$ (2.20) for $p = 0$ in $l^2(W_{p,0})$ (2.19), and to the operator $\hat{U}(1,p)$ (2.24), for $p \in \mathbb{N}$, mapping the space $l^2\left(W'_{p,0}\right)$ (2.19′) into $l^2(W_{p,0})$ (2.19). Moreover, the operators T_1 and $T(1,p) = T_1 T_2^p$ (1.21) specified in H are unitary equivalent to the shift operator $\hat{T}_1$ (2.27) in $\hat{H}_p$ (2.25) for all $p \in \mathbb{Z}_+$ and to the operator $\hat{T}(1,p)$ (2.27) acting in the specific space $\hat{H}_p$ (2.25) for $p \in \mathbb{N}$.*

IV. Let us now study a dual situation corresponding to the dilation $\overset{+}{U}(n)$ (1.25). Similarly to (2.1), continue every vector-function $v_k \in l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ into the domain $\mathbb{Z}^2$ using the Cauchy problem

$$\begin{cases} \partial_2 v_n = \left(\tilde{N}^* \partial_1 + \tilde{\Gamma}^*\right) v_n; & n = (n_1, n_2) \in \mathbb{Z}^2; \\ v_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}}\left(\tilde{E}\right). \end{cases} \tag{2.28}$$

Besides, we have the recurrent way of the continuation $v(n_1, n_2) \to v(n_1, n_2+1)$ into the upper semiplane ($n_2 \in \mathbb{Z}+$), and when $n_2 \in \mathbb{Z}_-$, the continuation $v(n_1, n_2) \to v(n_1, n_2 - 1)$ has the nonexplicit nature and may be carried out in the context of suppositions of Lem. 1.1 when $\dim \tilde{E} < \infty$. Thus, we obtain the Hilbert space $l^2_{\tilde{N}^*, \tilde{\Gamma}^*}\left(\tilde{E}\right)$ assuming that $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}}(\tilde{E})}$. Define the shift operator $\tilde{V}_+(p)$ in the space $l^2_{\tilde{N}^*, \tilde{\Gamma}^*}\left(\tilde{E}\right)$,

$$\tilde{V}_+(p) v_n = v_{n+p} \tag{2.29}$$

for all $p \in \mathbb{Z}^2$. It is obvious that the operator $\tilde{V}_+(q)$ (2.29) is isometric. Specify now the wave operator $\tilde{W}_+(p)$ mapping the space $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$ into $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) by the following formula:

$$\tilde{W}_+(p) = s - \lim_{n\to\infty} \overset{+}{U}(n,p) P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)} \tilde{V}_+(-n,-p), \tag{2.30}$$

where the number $p \in \mathbb{Z}_+$ is fixed and the operators $\overset{+}{U}(n)$ and $\tilde{V}_+(n)$ are specified by the formulas (1.25) and (2.29), respectively. It is obvious that $\tilde{W}_+(0) = W+$, where the operator W_+ corresponds to the dilation U (1.5), and so the operator $\tilde{W}_+(p)$ (2.30) is a continuation of the wave operator W_+ onto the "$-p$th" horizontal line in $\tilde{\mathbb{Z}}^2_-$ (1.7).

Consider now the polygon $L^{-1}_{-\infty,p}$ in $\tilde{\mathbb{Z}}^2_-$ (1.7) formed by the vertical segment connecting points $(-1,0)$ and $(-1,-p)$ and by the horizontal semiline from the point $(-1,-p)$ to $(-\infty,-p)$, where $p \in \mathbb{Z}_+$. And let $\tilde{L}^{\infty}_{0,k}$ be the similar polygon consisting of the rectilinear segments connecting the points $(0,0)$, $(0,k)$ and (∞,k) one-by-one in $\mathbb{Z}^2_+$. Similarly to (2.4), define the quadratic forms

$$\begin{aligned} \left\langle \tilde{f}\right\rangle^2_{\tau(p,k)} &= \langle \tau\tilde{u}_n\rangle^2_{L^{-1}_{-\infty,p}} + \left\|\tilde{h}\right\|^2 + \langle\tilde{\tau}\tilde{v}_n\rangle^2_{\tilde{L}^{\infty}_{0,k}}; \\ \left\langle \tilde{f}\right\rangle^2_{\tilde{\tau}(k)} &= \|\tilde{u}_n\|^2_{l^2} + \left\|\tilde{h}\right\|^2 + \langle\tilde{\tau}v_n\rangle^2_{\tilde{L}^{\infty}_{0,k}}; \\ \langle f\rangle^2_{\tau(p)} &= \langle \tau\tilde{u}_n\rangle^2_{L^{-1}_{-\infty,p}} + \left\|\tilde{h}\right\|^2 + \|v_n\|^2_{l^2} \end{aligned} \tag{2.31}$$

in $\mathcal{H}_{N^*,\Gamma^*}$ (1.24), where $\tilde{f} = \left(\tilde{u}_n, \tilde{h}, \tilde{v}_n\right) \in \mathcal{H}_{N^*,\Gamma^*}$ and respective $\tilde{\tau}$ and τ forms are understood in the sense of (1.29) and (1.30). It is easy to see that $\left\langle \tilde{f}\right\rangle^2_{\tau(0,0)} = \left\langle \tilde{f}\right\rangle^2_{\tilde{\tau}(0)} = \left\langle \tilde{f}\right\rangle^2_{\tau(0)} = \left\|\tilde{f}\right\|^2_{\mathcal{H}_{N^*,\Gamma^*}}$ and $\left\langle \tilde{f}\right\rangle^2_{\tau(0,k)} = \left\langle \tilde{f}\right\rangle^2_{\tilde{\tau}(k)}$, $\langle f\rangle^2_{\tau(p,0)} = \langle f\rangle^2_{\tau(p)}$. As in (2.5), specify the quadratic τ-forms,

$$\begin{aligned} \langle v_n\rangle^2_{\tilde{\tau}(p,k)} &= \left\langle\tilde{\tau}v^-_n\right\rangle^2_{L^{-1}_{-\infty,p}} + \left\langle\tilde{\tau}v^+_n\right\rangle^2_{\tilde{L}^{\infty}_{0,k}}; \\ \langle v_n\rangle^2_{\tilde{\tau}_+(k)} &= \left\|v^-_n\right\|^2_{l^2} + \left\langle\tilde{\tau}v^+_n\right\rangle^2_{\tilde{L}^{\infty}_{0,k}}; \\ \langle v_n\rangle^2_{\tilde{\tau}_-(p)} &= \left\langle\tilde{\tau}v^-_n\right\rangle^2_{L^{-1}_{-\infty,p}} + \left\|v^+_n\right\|^2_{l^2}, \end{aligned} \tag{2.33}$$

in the space $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$, where $v^{\pm}_n$ are the corresponding continuations in the second variable n_2 from the semiaxes $\mathbb{Z}_\pm$ of the functions of $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ obtained by using the Cauchy problem (2.28).

The following statement, similar to Th. 2.1, is true.

Theorem 2.5. *The wave operator* $\tilde{W}_+(p)$ *(2.30) acting from the space* $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$ *into the Hilbert space* $\mathcal{H}_{N^*,\Gamma^*}$ *(1.24) exists for all* $p \in \mathbb{Z}_+$ *and is an isometry*

$$\left\langle \tilde{W}_+(p)v_n \right\rangle^2_{\tau(p,k)} = \langle v_n \rangle^2_{\tilde{\tau}(p,k)} \tag{2.34}$$

in respective metrics (2.32) and (2.33) for all $p \in \mathbb{Z}_+$*. Moreover, for all* $\tilde{W}_+(p)$ *(2.30) the relations*

$$\begin{array}{ll} 1) & \overset{+}{U}(1,s)\tilde{W}_+(p) = \tilde{W}_+(p+s)\tilde{V}_+(1,s); \\ 2) & \tilde{W}_+(p)P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)} = P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)} \end{array} \tag{2.35}$$

are true for all $p,\ s \in \mathbb{Z}_+$*, where* $P_{D_-\left(\tilde{N}^*,\tilde{\Gamma}^*\right)}$ *is an orthoprojector onto the subspace* $D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)$*.*

Select two relations that are an immediate corollary of 1) (2.35) and are similar to (2.8),

$$\begin{array}{ll} 1) & \overset{+}{U}(1,0)\tilde{W}_+(p) = \tilde{W}_+(p)\tilde{V}_+(1,0); \\ 2) & \overset{+}{U}(1,p)\tilde{W}_+(0) = \tilde{W}_+(p)\tilde{V}_+(1,p) \end{array} \tag{2.36}$$

for all $p \in \mathbb{Z}_+$.

Continue now each vector-function u_{n_1} from the space $l^2_{\mathbb{Z}}(E)$ by the second variable n_2 into the domain $\mathbb{Z}^2$ using the Cauchy problem

$$\begin{cases} \partial_2 u_n = \left(N^*\partial_1 + \Gamma^*\right) u_n; & n = (n_1, n_2) \in \mathbb{Z}^2; \\ u_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}}(E). \end{cases} \tag{2.37}$$

As in the case of the Cauchy problem (2.28), the continuation $u\,(n_1,n_2) \to u\,(n_1,n_2+1)$ has the explicit recurrent nature, and the continuation into the lower semiplane $n_2 \in \mathbb{Z}_-$, $u\,(n_1,n_2) \to u\,(n_1,n_2-1)$ may be done under suppositions of Lem. 1.1 and $\dim E < \infty$. The Hilbert space obtained in this way is denoted by $l^2_{N^*,\Gamma^*}(E)$, besides $\|u_n\| \overset{\text{def}}{=} \|u_{n_1}\|_{l^2_{\mathbb{Z}}(E)}$.

Similarly to the operator $\tilde{V}_+(p)$ (2.29), define the shift operator

$$V_+(p)u_n = u_{n+p} \tag{2.38}$$

in the space $l^2_{N^*,\Gamma^*}(E)$ for all $p \in \mathbb{Z}^2$ and for all $u_n \in l^2_{N^*,\Gamma^*}(E)$. Specify now the wave operator $\tilde{W}_-(k)$ from the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) into $l^2_{N^*,\Gamma^*}(E)$

$$\tilde{W}_-(k) = s - \lim_{n\to\infty} V_+(-n,-k)P_{D_-(N^*,\Gamma^*)}\overset{+}{U}(n,k) \tag{2.39}$$

for all fixed $k \in \mathbb{Z}_+$, where $\overset{+}{U}(n)$ and $V_+(n)$ are specified by the formulas (1.25) and (2.38), respectively. It is easy to see that $\tilde{W}_-(0) = W_-^*$, besides W_- has the standard form [6].

Theorem 2.6. *The wave operator $\tilde{W}_-(k)$ (2.39) mapping the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) into $l^2_{N^*,\Gamma^*}(E)$ exists for all $k \in \mathbb{Z}_+$ and has the following properties:*

$$\begin{array}{ll} 1) & V_+(1,s)\tilde{W}_-(k+s) = \tilde{W}_-(k)\overset{+}{U}(1,s); \\ 2) & \tilde{W}_-(k)P_{D_-(N^*,\Gamma^*)} = P_{D_-(N^*,\Gamma^*)} \end{array} \tag{2.40}$$

for all k, $s \in \mathbb{Z}_+$, where $P_{D_-(N^*,\Gamma^*)}$ is an orthoprojector onto $D_-(N^*,\Gamma^*)$.

Select two relations following from equality 1) (2.40):

$$\begin{array}{ll} 1) & V_+(1,0)\tilde{W}_-(k) = \tilde{W}_-(k)\overset{+}{U}(1,0); \\ 2) & V_+(1,k)\tilde{W}_-(k) = \tilde{W}_-(0)\overset{+}{U}(1,k) \end{array} \tag{2.41}$$

for all $k \in \mathbb{Z}_+$.

Similarly to (2.13), define now the scattering operator $\tilde{S}(k,p)$ from $l_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$ into the space $l^2_{N^*,\Gamma^*}(E)$

$$\tilde{S}(k,p) = \tilde{W}_-(k)\tilde{W}_+(p) \tag{2.42}$$

for all k, $p \in \mathbb{Z}_+$, that obviously coincides with S^* when $k = p = 0$.

Theorem 2.7. *The scattering operator $\tilde{S}(k,p)$ (2.42) is a bounded operator from $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$ into the space $l^2_{N^*,\Gamma^*}(E)$, besides the following relations*

$$\begin{array}{ll} 1) & \tilde{S}(k,p)\tilde{V}(1,s) = V_+(1,s)\tilde{S}(k+s,p-s); \\ 2) & \tilde{S}(k,p)P_+l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right) \subseteq P_+l^2_{N^*,\Gamma^*}(E) \end{array} \tag{2.43}$$

take place for all k, p, $s \in \mathbb{Z}_+$, whereas $0 \le s \le p$ and P_+ is an orthoprojector onto the respective subspaces corresponding to the solutions of the Cauchy problems (2.28) and (2.37) with the initial data on the semiaxis $\mathbb{Z}_+$ ($n_2 = 0$).

It is obvious that the invariant property of the operator $\tilde{S}(k,p)$ with respect to the shift by the coordinate "n_1"

$$\tilde{S}(k,p)\tilde{V}_+(1,0) = V_+(1,0)\tilde{S}(k,p) \tag{2.44}$$

follows from 1) (2.43) for all p, $k \in \mathbb{Z}_+$, and

$$\begin{array}{lll} 1) & \tilde{S}(k,p)\tilde{V}_+(1,p) = V_+(1,p)\tilde{S}(k+p,0), & p = s; \\ 2) & \tilde{S}(0,p)\tilde{V}_+(1,p) = V_+(1,p)\tilde{S}(p,0), & p = s, k = 0. \end{array} \tag{2.45}$$

This fact is similar to equalities (2.16).

V. Define now the mapping $\tilde{\mathcal{P}}_{p,k}$ from the direct sum of the Hilbert spaces $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)+l^2_{N^*,\Gamma^*}(E)$ into the Hilbert space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) in the following way:

$$\tilde{f}_{p,k}=\tilde{\mathcal{P}}_{p,k}\left(g_n\right)=\tilde{\mathcal{P}}_{p,k}\left(\begin{array}{c} v_n \\ u_n \end{array}\right)=\tilde{W}_+(p)v_n+\tilde{W}_-^*(k)u_n, \qquad (2.46)$$

where $v_n\in l_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$, $u_n\in l^2_{N^*,\Gamma^*}(E)$ for all p, $k\in\mathbb{Z}_+$. As it was noted above (see Paragraph III), the vector-functions $\tilde{f}_{p,k}$ form the dense set in the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) in the case of simplicity of expansion V_s, $\overset{+}{V}_s$ (1.1), for the fixed p, $k\in\mathbb{Z}_+$. Therefore every vector from the space $\mathcal{H}_{N^*,\Gamma^*}$ has different realizations $\tilde{f}_{p,k}$ (2.46) for different values of the parameters p and k. It is obvious that

$$\left\langle\tilde{f}_{p,k},\tilde{f}_{p,k}\right\rangle_{\mathcal{H}_{N^*,\Gamma^*}}=\left\langle\tilde{W}_{p,k}g_n,g_n\right\rangle_{l^2},$$

where the weight operator $\tilde{W}_{p,k}$ is equal to

$$\tilde{W}_{p,k}=\left[\begin{array}{cc} \tilde{W}_+^*(p)\tilde{W}_+(p) & \tilde{S}^*(k,p) \\ \tilde{S}(k,p) & \tilde{W}_-(k)\tilde{W}_-^*(k) \end{array}\right], \qquad (2.47)$$

besides $\tilde{S}(k,p)$ has the form of (2.42). Similarly to Observation 2.1, it is obvious that all blocks of the operator $\tilde{W}_{p,k}$ are translational invariant with respect to the shift by the variable "n_1". Thus, the mapping $\tilde{\mathcal{P}}_{p,k}$ (2.46) defines the one-to-one unitary correspondence between the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) and the space

$$l^2\left(\tilde{W}_{p,k}\right)=\left\{g_n=\left(\begin{array}{c} v_n \\ u_n \end{array}\right):\left\langle\tilde{W}_{p,k}g_n,g_n\right\rangle_{l^2}<\infty\right\}, \qquad (2.48)$$

where $v_n\in l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$, $u_n\in l^2_{N^*,\Gamma^*}(E)$. It is easy to see that the given space $l^2\left(\tilde{W}_{p,k}\right)$ coincides with $l^2\left(\begin{array}{cc} I & S \\ S^* & I \end{array}\right)$, when $p=k=0$, as in the case of the space $l^2\left(W_{p,k}\right)$ (2.19). It follows from the relations 1) (2.36) and 1) (2.41) and from the unitarity of the $\overset{+}{U}(1,0)$ that the dilation $\overset{+}{U}(1,0)$ acts in every space $l^2\left(\tilde{W}_{p,k}\right)$ (2.48) by the shift by the variable "n_1"

$$\hat{U}_+(1,0)g_n=\left[\begin{array}{cc} \tilde{V}_+(1,0) & 0 \\ 0 & V_+(1,0) \end{array}\right]g_n \qquad (2.49)$$

for all $g_n\in l^2\left(\tilde{W}_{p,k}\right)$.

Further, study how the dilation $\overset{+}{U}(1,s)$ (1.25) acts on the vectors $\tilde{f}_{p,k} = \tilde{\mathcal{P}}_{p,k}(g_n)$ (2.46). As in the considerations above, study only the case when the dilation $\overset{+}{U}(1,p)$ (1.25) acts on the vectors of the type $\tilde{f}_{0,p} = \tilde{\mathcal{P}}_{0,p}(g_n)$ (2.46).

Similarly to (2.22), it is easy to prove that

$$\left\langle \overset{+}{U}(1,p)\tilde{f}_{0,p}, \tilde{f}'_{p,0} \right\rangle_{\tau(p)}$$

$$= \left\langle \begin{bmatrix} I & \tilde{S}^*(0,p) \\ \tilde{S}(0,p) & V_+(1,p)\tilde{W}_-(p)\tilde{W}_-^*(p)V_+^*(1,p) \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \tilde{V}_+(1,p) & 0 \\ 0 & V_+^*(-1,-p) \end{bmatrix} g_n, g'_n \right\rangle_{\tilde{\tau}_-(p),\tau_-(p)}, \tag{2.50}$$

besides, the bi-linear form in the right part is understood component-wisely in the sense of the metrics $\tilde{\tau}_-(p)$ and $\tau_-(p)$ (2.31). Let

$$\begin{aligned} \tilde{W}'_{0,p} &= \begin{bmatrix} I & \tilde{S}^*(0,p) \\ \tilde{S}(0,p) & V_+(1,p)\tilde{W}_-(p)\tilde{W}_-^*(p)V_+^*(1,p) \end{bmatrix}; \\ \hat{V}_+(1,p) &= \begin{bmatrix} \tilde{V}_+(1,p) & 0 \\ 0 & V_+^*(-1,-p) \end{bmatrix}. \end{aligned} \tag{2.51}$$

O b s e r v a t i o n 2.3. Consider the mapping $\tilde{\mathcal{P}}_{0,p}$ (2.46), denote by $\tilde{f}'_{0,p} = \tilde{\mathcal{P}}_{0,p}(g'_n) = \tilde{W}_+(0)\tilde{V}_+(-1,-p)v_n + \tilde{W}_-^*V_+^*(1,p)u_n$, where $u_n \in l^2_{N^*,\Gamma^*}(E)$, $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$. Then

$$\left\langle \tilde{f}'_{0,p}, \tilde{f}'_{0,p} \right\rangle_{\mathcal{H}_{N^*,\Gamma^*}} = \left\langle \tilde{W}'_{0,p} g_n, g_n \right\rangle_{l^2}$$

follows from 2) (2.45), that is similar to Observation 2.2.

Therefore the dilation $\overset{+}{U}(1,p)$ (1.25) acts as a shift operator

$$\hat{U}_+(1,p) = \hat{V}(1,p)g_n \tag{2.52}$$

from the Hilbert space

$$l^2\left(\tilde{W}'_{0,p}\right) = \left\{ g_n = \begin{pmatrix} v_n \\ u_n \end{pmatrix} : \left\langle \tilde{W}'_{0,p} g_n, g_n \right\rangle_{l^2} < \infty \right\} \tag{2.48'}$$

($v_n \in l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$, $u_n \in l^2_{N^*,\Gamma^*}(E)$) into the space $l^2\left(\tilde{W}_{0,p}\right)$ (2.48).

It is clear that the subspaces

$$\hat{D}_-\left(N^*,\Gamma^*\right) = \begin{pmatrix} 0 \\ P_- l^2_{N^*,\Gamma^*}(E) \end{pmatrix}; \quad \hat{D}_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right) = \begin{pmatrix} P_+ l_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right) \\ 0 \end{pmatrix},$$

where, as usual, P_- and P_+ are orthoprojectors in $l^2_{N^*,\Gamma^*}(E)$ and in $l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)$ onto the subspaces of the solutions of the Cauchy problems (2.37) and (2.28) with the initial data on $\mathbb{Z}_-$ and $\mathbb{Z}_+$, respectively, and are the prototypes of the subspaces $D_-\left(N^*,\Gamma^*\right)$ and $D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)$ from $\mathcal{H}_{N^*,\Gamma^*}$ (1.24) for the mapping $\tilde{\mathcal{P}}_{p,k}$ (for all p, $k\in\mathbb{Z}_+$). Therefore the space H is isomorphic to

$$\hat{H}_{p,+}=l^2\left(\tilde{W}_{0,p}\right)\ominus\left(\begin{array}{c}P_+l_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)\\ P_-l_{N^*,\Gamma^*}(E)\end{array}\right). \tag{2.53}$$

Using similar considerations for $l^2\left(\tilde{W}'_{0,p}\right)$ (2.53), we obtain a different realization

$$\hat{H}'_{p,+}=l^2\left(\tilde{W}'_{p,0}\right)\ominus\left(\begin{array}{c}\tilde{V}_+(1,p)P_+l^2_{\tilde{N}^*,\tilde{\Gamma}^*}\left(\tilde{E}\right)\\ V_+^*(-1,-p)P_-l^2_{N^*,\Gamma^*}(E)\end{array}\right) \tag{2.53'}$$

in view of Observation 2.3. The spaces $\hat{H}_{p,+}$ (2.53) and $\hat{H}'_{p,+}$ (2.53′) are isomorphic, besides, the operator $R_{p,+}$: $\hat{H}_{p,+}\to\hat{H}'_{p,+}$ defining this isomorphism has the form

$$R_{p,+}=P_{\hat{H}'_{p,0}}\left[\begin{array}{cc}\tilde{V}_+(-1,-p) & 0\\ 0 & V_+(1,p)\end{array}\right]P_{\hat{H}_{p,+}}, \tag{2.54}$$

where $P_{\hat{H}_{p,+}}$ and $P_{\hat{H}_{p,+}}$ are orthoprojectors onto $\hat{H}'_{p,+}$ (2.53′) and onto $\hat{H}_{p,+}$ (2.53), respectively. It follows from (2.49) and (2.52) that the operators T_1^* and $T^*(1,p)=T_1^*T_2^{*p}$, $p\in\mathbb{Z}_+$, are

$$\left(\hat{T}_1^*f\right)_n=P_{\hat{H}_{p,+}}f_{n+(1,0)};\quad\left(\hat{T}^*(1,p)f\right)_n=P_{\hat{H}_{p,+}}\hat{V}_+(1,p)\left(R_{p,+}f\right)_n \tag{2.54}$$

for all $f_n\in\hat{H}_{p,+}$ (2.53), $P_{\hat{H}_{p,+}}$ is an orthoprojector onto $\hat{H}_{p,+}$ and the operator $R_{p,+}$ is specified by formula (2.54). As in the previous case, the operator $\hat{T}_1^*$ has the same form (2.54) in all spaces $\hat{H}_{p,+}$, and the operator $\hat{T}^*(1,p)$ has a given form (1.54) only in one space $\hat{H}_{p,+}$ (2.53).

Theorem 2.8. *Let* V_s, $\overset{+}{V}_s$ *(1.1) be the simple [8] commutative unitary expansion of the operator system* $\{T_1,T_2\}$ *from the class* $C(T_1)$ *(1.3) and, moreover, the hypotheses of Lem. 1.1 be met, and* $\dim E=\dim\tilde{E}<\infty$. *Then the isometric dilation* $\overset{+}{U}(1,p)$ *(1.25),* $p\in\mathbb{Z}_+$, *acting in the Hilbert space* $\mathcal{H}_{N^*,\Gamma^*}$ *(1.24), is unitary equivalent to the operator* $\hat{U}_+(1,0)$ *(2.49), for* $p=0$, *in* $l^2\left(\tilde{W}_{0,p}\right)$ *(1.24) and to the operator* $\hat{U}_+(1,p)$ *(2.52), for* $p\in\mathbb{N}$, *mapping the space* $l^2\left(\tilde{W}'_{0,p}\right)$ *(2.48′) into* $l^2\left(\tilde{W}_{0,p}\right)$ *(2.48). Moreover, the operators* T_1^* *and* $T^*(1,p)$ *(1.21) acting in* H *are unitary equivalent to the shift operator* $\hat{T}_1^*$ *(2.54) in* $\hat{H}_{p,+}$ *(2.53) for all* $p\in\mathbb{Z}_+$ *and to the operator* $\hat{T}(1,p)$ *(2.54) acting in the fixed* $\hat{H}_{p,+}$ *(2.53)* $(p\in\mathbb{N})$.

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