

The Conductivity Measure for the Anderson Model

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We study the ac-conductivity in linear response theory for the Anderson tight-binding model. We define the electrical ac-conductivity and calculate the linear-response current at zero temperature for arbitrary Fermi energy. In particular, the Fermi energy may lie in a spectral region where extended states are believed to exist.

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Dedicated to Leonid A. Pastur on the occasion of his 70th birthday

1. Introduction

In this article we study the ac-conductivity in linear response theory for the Anderson tight-binding model. We define the electrical ac-conductivity and calculate the linear-response current at temperature $T = 0$ for arbitrary Fermi energy μ .

At temperature $T = 0$, if the Fermi energy μ is either in the region of localization or outside the spectrum of the random Schrödinger operator, this was already done in [KILM] by a careful mathematical analysis of the ac-conductivity in linear response theory, following the approach of [BoGKS], and the introduction of a new concept, the conductivity measure. This approach can be easily extended to the nonzero temperature case, $T > 0$, with μ (here the chemical

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potential) arbitrary. The conductivity measure $\Sigma_\mu^T(d\nu)$, with ν the frequency of the applied electric field, is a finite positive even Borel measure on the real line. If $\Sigma_\mu^T(d\nu)$ was known to be an absolutely continuous measure, the in-phase or active conductivity $\operatorname{Re} \sigma_\mu^T(\nu)$ would then be well-defined as its density. The conductivity measure $\Sigma_\mu^T(d\nu)$ is thus an analogous concept to the density of states measure $\mathcal{N}(dE)$, whose formal density is the density of states $n(E)$. Given a spatially homogeneous, time-dependent electric field $\mathbf{E}(t)$, the in-phase linear-response current at time t , $J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathbf{E})$, has a simple expression in terms of this conductivity measure:

$$J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathbf{E}) = \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) e^{i\nu t} \widehat{\mathbf{E}}(\nu). \quad (1.1)$$

This procedure is conjectured to break down at $T = 0$ for, say, Fermi energies μ in the region of extended states. In this case there has been no suitable derivation of the in-phase linear-response current. In this paper we define the conductivity measure $\Sigma_\mu^0(d\nu)$ and the in-phase linear-response current for *arbitrary* Fermi energy μ . We give an explicit expression for $\Sigma_\mu^0(d\nu)$, and justify the definition by proving that

$$\Sigma_\mu^0(d\nu) = \lim_{T \downarrow 0} \Sigma_\mu^T(d\nu) \quad \text{weakly for Lebesgue-a.e. } \mu \in \mathbb{R}. \quad (1.2)$$

The in-phase linear-response current is then defined by (1.1), and justified by

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathbf{E}) = \lim_{T \downarrow 0} J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathbf{E}) \quad \text{for Lebesgue-a.e. } \mu \in \mathbb{R}. \quad (1.3)$$

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2. Definitions and Results

The Anderson tight-binding model is described by the random Schrödinger operator H , a measurable map $\omega \mapsto H_\omega$ from a probability space (Ω, \mathbb{P}) (with expectation \mathbb{E}) to bounded selfadjoint operators on $\ell^2(\mathbb{Z}^d)$, given by

$$H_\omega := -\Delta + V_\omega. \quad (2.1)$$

Here Δ is the centered discrete Laplacian,

$$(\Delta\varphi)(x) := - \sum_{y \in \mathbb{Z}^d; |x-y|=1} \varphi(y) \quad \text{for } \varphi \in \ell^2(\mathbb{Z}^d), \quad (2.2)$$

and the random potential V consists of independent, identically distributed random variables $\{V(x); x \in \mathbb{Z}^d\}$ on (Ω, \mathbb{P}) , such that the common single site probability distribution has a bounded density ρ with compact support.

The *Anderson Hamiltonian* H given by (2.1) is \mathbb{Z}^d -ergodic, and hence its spectrum, as well as its spectral components in the Lebesgue decomposition, are given by nonrandom sets \mathbb{P} -almost surely [KiM, CL, PF]. This nonrandom spectrum will be denoted by \mathfrak{S} , with \mathfrak{S}_\varkappa , $\varkappa = \text{pp, ac, sc}$, denoting its nonrandom spectral components.

We now outline the derivation of electrical ac-conductivities within linear response theory for the Anderson model. We refer to [BoGKS] and [KILM] for mathematical details, generalizations and proofs.

At the reference time $t = -\infty$, the system is assumed to be in thermal equilibrium at absolute temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$. On the single-particle level, this equilibrium state is given by the random operator $f_\mu^T(H)$, where

$$f_\mu^T(E) := \begin{cases} \left(e^{\frac{E-\mu}{T}} + 1 \right)^{-1} & \text{if } T > 0 \\ \chi_{]-\infty, \mu]}(E) & \text{if } T = 0 \end{cases} \quad (2.3)$$

stands for the Fermi function. By χ_B we denote the indicator function of the set B . A spatially homogeneous, time-dependent electric field $\mathbf{E}(t)$ is then introduced adiabatically: Starting at time $t = -\infty$, we switch on the (adiabatic) electric field $\mathbf{E}_\eta(t) := e^{\eta t} \mathbf{E}(t)$ with $\eta > 0$, and then let $\eta \rightarrow 0$.

On account of isotropy we assume without restriction that the electric field is pointing in the x_1 -direction: $\mathbf{E}(t) = \mathcal{E}(t)\hat{x}_1$, where $\mathcal{E}(t)$ is the (real-valued) amplitude of the electric field, and \hat{x}_1 is the unit vector in the x_1 -direction. Our precise requirements for the real-valued, time-dependent amplitude $\mathcal{E}(t)$ are stated in the following assumption, which we assume valid from now on.

Assumption (E). *The time-dependent amplitude $\mathcal{E}(t)$ of the electric field is of the form*

$$\mathcal{E}(t) = \int_{\mathbb{R}} d\nu \, e^{i\nu t} \widehat{\mathcal{E}}(\nu), \quad (2.4)$$

where $\widehat{\mathcal{E}} \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\widehat{\mathcal{E}}(\nu) = \overline{\widehat{\mathcal{E}}(-\nu)}$.

For each $\eta > 0$ this procedure results in a time-dependent random Hamiltonian

$$H_\omega(\eta, t) := G(\eta, t)H_\omega G(\eta, t)^*, \quad \text{with } G(\eta, t) := e^{iX_1 \int_{-\infty}^t ds \, e^{\eta s} \mathcal{E}(s)}, \quad (2.5)$$

where X_1 stands for the operator of multiplication by the first coordinate of the electron's position. $H_\omega(\eta, t)$ is, of course, gauge equivalent to $H_\omega + e^{\eta t} \mathcal{E}(t)X_1$.

At time t , the state of the system is described by the random operator $\varrho_\omega(\eta, t)$, the solution to the Liouville equation

$$\begin{cases} i\partial_t \varrho_\omega(\eta, t) = [H_\omega(\eta, t), \varrho_\omega(\eta, t)] \\ \lim_{t \rightarrow -\infty} \varrho_\omega(\eta, t) = f_\mu^T(H_\omega) \end{cases} \quad (2.6)$$

The adiabatic electric field generates a time-dependent electric current. Thanks to reflection covariance in all but the first direction, the current is also oriented along the first coordinate axis. Its amplitude is

$$J_\eta(t; \mu, T, \mathcal{E}) = -\mathcal{T}(\varrho_\omega(\eta, t) \dot{X}_1(t)), \quad (2.7)$$

where \mathcal{T} is the trace per unit volume (see (A.14) and (A.15) in App. A) and \dot{X}_1 is the first component of the velocity operator:

$$\dot{X}_1 := i[H_\omega, X_1] = i[-\Delta, X_1]. \quad (2.8)$$

Note that we are using the Schrödinger picture in (2.7). The time dependence of the velocity operator $\dot{X}_1(t) := G(\eta, t) \dot{X}_1 G(\eta, t)^*$ there results from our particular gauge. Finally, the *adiabatic linear-response current* is defined as

$$J_{\eta, \text{lin}}(t; \mu, T, \mathcal{E}) := \frac{d}{d\alpha} J_\eta(t; \mu, T, \alpha\mathcal{E}) \Big|_{\alpha=0}. \quad (2.9)$$

The detailed analysis in [BoGKS] shows that one can give a mathematical meaning to the formal procedure leading to (2.9), for fixed temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$, if the corresponding thermal equilibrium random operator $f_\mu^T(H)$ satisfies the condition

$$\mathbb{E}\{\|X_1 f_\mu^T(H_\omega) \delta_0\|^2\} < \infty, \quad (2.10)$$

where $\{\delta_a\}_{a \in \mathbb{Z}^d}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}^d)$: $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. (This is the condition originally identified in [BES].)

The derivation of a Kubo formula for the ac-conductivity [BES, SB, BoGKS] requires normed spaces of measurable covariant operators. The required mathematical framework is described in App. A; here we will be somewhat informal. \mathcal{K}_2 is the Hilbert space of measurable covariant operators A on $\ell^2(\mathbb{Z}^d)$, i.e., measurable, covariant maps $\omega \mapsto A_\omega$ from the probability space (Ω, \mathbb{P}) to operators on $\ell^2(\mathbb{Z}^d)$, with inner product

$$\langle\langle A, B \rangle\rangle := \mathbb{E}\{\langle A_\omega \delta_0, B_\omega \delta_0 \rangle\} = \mathcal{T}\{A_\omega^* B_\omega\} \quad (2.11)$$

and norm $\|A\|_2 := \sqrt{\langle\langle A, A \rangle\rangle}$. Here \mathcal{T} , given by $\mathcal{T}(A) := \mathbb{E}\{\langle \delta_0, A_\omega \delta_0 \rangle\}$, is the trace per unit volume. The Liouvillian \mathcal{L} is the (bounded in the case of the Anderson model) selfadjoint operator on \mathcal{K}_2 given by the commutator with H :

$$(\mathcal{L}A)_\omega := [H_\omega, A_\omega]. \quad (2.12)$$

We also introduce operators \mathcal{H}_L and \mathcal{H}_R on \mathcal{K}_2 given by left and right multiplication by H :

$$(\mathcal{H}_L A)_\omega := H_\omega A_\omega \quad \text{and} \quad (\mathcal{H}_R A)_\omega := A_\omega H_\omega. \quad (2.13)$$

Note that \mathcal{H}_L and \mathcal{H}_R are commuting, bounded (for the Anderson Hamiltonian), selfadjoint operators on \mathcal{K}_2 , anti-unitarily equivalent (see (A.10)), and $\mathcal{L} = \mathcal{H}_L - \mathcal{H}_R$. It follows from the Wegner estimate for the Anderson Hamiltonian that in this case the operators \mathcal{H}_L and \mathcal{H}_R have purely absolutely continuous spectrum (see Lem. 1 in Sect. 3). For each $T \geq 0$ and $\mu \in \mathbb{R}$ we consider the bounded selfadjoint operator \mathcal{F}_μ^T in \mathcal{K}_2 given by

$$\mathcal{F}_\mu^T := f_\mu^T(\mathcal{H}_L) - f_\mu^T(\mathcal{H}_R), \quad \text{i.e.,} \quad (\mathcal{F}_\mu^T A)_\omega = [f_\mu^T(H_\omega), A_\omega]. \quad (2.14)$$

In this setting the key condition (2.10) may be rewritten as

$$Y_\mu^T := i[X_1, f_\mu^T(H)] \in \mathcal{K}_2. \quad (2.15)$$

Note that condition (2.15) is always true for $T > 0$ with arbitrary $\mu \in \mathbb{R}$, since in this case $f_\mu^T(H) = g(H)$ for some $g \in \mathcal{S}(\mathbb{R}^d)$ (cf. [BoGKS, Remark 5.2(iii)]). We set

$$\Xi_0 := \{\mu \in \mathbb{R}; \quad Y_\mu^0 \in \mathcal{K}_2\}. \quad (2.16)$$

For the same reason as when $T > 0$, we have $\mu \in \Xi_0$ if either $\mu \notin \mathfrak{S}$ or μ is the left edge of a spectral gap for H . Moreover, letting Ξ^{cl} denote the region of complete localization, defined as the region of validity of the multiscale analysis, or equivalently, of the fractional moment method, we have (cf. [AG, GK4])

$$\Xi^{\text{cl}} \subset \Xi_0. \quad (2.17)$$

A precise definition of the region of complete localization is given in App. B. Note that we included the complement of the spectrum \mathfrak{S} in Ξ^{cl} for convenience, and that Ξ^{cl} is an open set by its definition. Note also that for $\mu \in \Xi^{\text{cl}}$ the Fermi projection $f_\mu^0(H)$ satisfies a much stronger condition than (2.10), namely exponential decay of its kernel [AG, Th. 2] (see (B.2)). Conversely, fast enough polynomial decay of the kernel of the Fermi projection for all energies in an interval implies complete localization in the interval [GK4, Th. 3].

If $Y_\mu^T \in \mathcal{K}_2$, we proceed as in [KILM], with a slight variation to include also the case when $T > 0$. An inspection of the proof of [BoGKS, Th. 5.9] shows that the adiabatic linear-response current (2.9) is well defined for every time $t \in \mathbb{R}$, and given by

$$J_{\eta, \text{lin}}(t; \mu, T, \mathcal{E}) = \mathcal{T} \left\{ \int_{-\infty}^t ds \, e^{\eta s} \mathcal{E}(s) \dot{X}_1 e^{-i(t-s)\mathcal{L}} Y_\mu^T \right\}. \quad (2.18)$$

It is convenient to rewrite (2.18) in terms of the conductivity measure Σ_μ^T , which we now introduce if either $T > 0$ or $\mu \in \Xi_0$.

Definition 1. *If either $T > 0$ or $\mu \in \Xi_0$, the ac-conductivity measure (x_1 - x_1 component) at temperature T and chemical potential μ is defined by*

$$\Sigma_\mu^T(B) := \pi \langle \langle \dot{X}_1, \chi_B(\mathcal{L}) Y_\mu^T \rangle \rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (2.19)$$

This definition is justified by the following theorem, whose proof, as the proofs of all other results in this section, is postponed to Sect. 3. $\mathcal{M}(\mathbb{R})$ will denote the vector space of complex Borel measures on \mathbb{R} , with $\mathcal{M}_+(\mathbb{R})$ being the cone of finite positive Borel measures, and with $\mathcal{M}_+^{(e)}(\mathbb{R})$ the finite positive even Borel measures. We recall that $\mathcal{M}(\mathbb{R}) = C_0(\mathbb{R})^*$, where $C_0(\mathbb{R})$ denotes the Banach space of complex-valued continuous functions on \mathbb{R} vanishing at infinity with the sup norm. We will use two locally convex topologies on $\mathcal{M}(\mathbb{R})$. The first is the weak* topology, defined by the linear functionals $\{\Gamma \in \mathcal{M}(\mathbb{R}) \mapsto \Gamma(g); g \in C_0(\mathbb{R})\}$. (By $\Gamma(g) := \int_{\mathbb{R}} \Gamma(ds) g(s)$ we denote the integral of a function g with respect to a measure Γ .) The second is the one defined by the similarly defined linear functionals where g is any bounded measurable function on \mathbb{R} . ‘Weak’ will refer to the weak* topology and ‘strong’ to the other topology. We will write w-lim and s-lim to denote the respective limits.

Theorem 1. (i) *If either $T > 0$ or $\mu \in \Xi_0$, the conductivity measure Σ_μ^T is a finite positive even Borel measure on the real line, i.e., $\Sigma_\mu^T \in \mathcal{M}_+^{(e)}(\mathbb{R})$, such that*

$$\Sigma_\mu^T(\mathbb{R}) = -\pi \mathbb{E} \{ \langle \delta_{\hat{x}_1} + \delta_{-\hat{x}_1}, f_\mu^T(H) \delta_0 \rangle \} \leq \sqrt{2} \pi. \quad (2.20)$$

(ii) *For every $\mu \in \Xi_0$ we have*

$$\Sigma_\mu^0(B) = \pi \langle \langle Y_\mu^0, \chi_B(\mathcal{L}) (-\mathcal{L}) \mathcal{F}_\mu^0 Y_\mu^0 \rangle \rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (2.21)$$

(iii) *The map $]0, \infty[\ni T \mapsto \Sigma_\mu^T \in \mathcal{M}_+^{(e)}(\mathbb{R})$ is strongly continuous for every $\mu \in \mathbb{R}$.*

(iv) *For every $\mu \in \Xi_0$ we have*

$$\text{s-lim}_{T \downarrow 0} \Sigma_\mu^T = \Sigma_\mu^0. \quad (2.22)$$

(v) *If $\mu \in \Xi^{\text{cl}}$ we also have $\lim_{T \downarrow 0} Y_\mu^T = Y_\mu^0$ in \mathcal{K}_2 .*

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(i) Theorem 1(ii) shows that for $T = 0$ and $\mu \in \Xi_0$ the conductivity measure Σ_μ^0 defined by (2.19) coincides with the one given in [KILM, Def. 3.3].

(ii) If the Fermi energy μ is above or below the almost-sure spectrum \mathfrak{S} of H , we have $Y_\mu^0 = 0$, and hence also $\Sigma_\mu^0 = 0$. If $]a, b[$ is a spectral gap, we clearly have $Y_\mu^0 = Y_a^0$, and hence $\Sigma_\mu^0 = \Sigma_a^0$, for all $\mu \in]a, b[$. Moreover, it is shown in [KILM, Prop. 3.7] that the measure Σ_μ^0 can be expressed in terms of a measure Ψ_μ on \mathbb{R}^2 , supported by the set \mathfrak{S}_μ given in [KILM, Eq. (3.41)]. Since Ψ_μ depends on μ only through Y_μ^0 , we have $\Psi_\mu = \Psi_a$ for all $\mu \in]a, b[$, and hence Ψ_a is supported by the set

$$\bigcap_{\mu \in]a, b[} \mathfrak{S}_\mu = \{]-\infty, a[\times]b, \infty[\} \cup \{]b, \infty[\times]-\infty, a[\}. \quad (2.23)$$

It then follows from [KILM, Eq. (3.40)] that for all $\mu \in]a, b[$ we have

$$\Sigma_\mu^0([-\nu, \nu]) = \Sigma_a^0([-\nu, \nu]) = 0 \quad \text{for all } \nu \in]0, b - a[. \quad (2.24)$$

(iii) If $\mu \in \Xi_0$, as shown in [N, BoGKS], the direct-current conductivity vanishes at zero temperature:

$$\sigma_{\mu, \text{dc}}^0 := \lim_{\eta \downarrow 0} \left\langle \left\langle \dot{X}_1, \frac{1}{i\mathcal{L} + \eta} Y_\mu^0 \right\rangle \right\rangle = 0. \quad (2.25)$$

(iv) For $\mu \in \Xi^{\text{cl}}$, the region of complete localization, the Mott-type bound

$$\limsup_{\nu \downarrow 0} \frac{\frac{1}{\nu} \Sigma_\mu^0([0, \nu])}{\nu^2 \left(\log \frac{1}{\nu} \right)^{d+2}} \leq \text{constant} \quad (2.26)$$

for the ac-conductivity measure was established in [KILM].

We may now rewrite (2.18) in terms of the conductivity measure as follows. If either $T > 0$ or $\mu \in \Xi_0$, the same argument leading to [KILM, Eq. (3.30) and Th. 3.4] gives

$$J_{\eta, \text{lin}}(t; \mu, T, \mathcal{E}) = e^{\eta t} \int_{\mathbb{R}} d\nu e^{i\nu t} \sigma_\mu^T(\eta, \nu) \widehat{\mathcal{E}}(\nu), \quad (2.27)$$

where $\sigma_\mu^T(\eta, \cdot)$ is the Stieltjes transform of the conductivity measure Σ_μ^T :

$$\sigma_\mu^T(\eta, \nu) := -\frac{i}{\pi} \int_{\mathbb{R}} \Sigma_\mu^T(d\lambda) \frac{1}{\lambda + \nu + i\eta}. \quad (2.28)$$

The adiabatic in-phase linear-response current is now defined by

$$J_{\eta, \text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) := e^{\eta t} \int_{\mathbb{R}} d\nu e^{i\nu t} (\text{Re } \sigma_\mu^T(\eta, \nu)) \widehat{\mathcal{E}}(\nu). \quad (2.29)$$

Turning off the adiabatic switching, we obtain a simple expression for the in-phase linear-response current in terms of the conductivity measure, as in [KILM, Cor. 3.5], given by

$$J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) := \lim_{\eta \downarrow 0} J_{\eta, \text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) = \int_{\mathbb{R}} \Sigma_{\mu}^T(d\nu) e^{i\nu t} \widehat{\mathcal{E}}(\nu). \quad (2.30)$$

This gives a derivation of the in-phase linear-response current (1.1), and (2.30) is valid as long as either $T > 0$ or $\mu \in \Xi_0$. Moreover, it follows from (2.30) and Theorem 1(iv) that

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathcal{E}) = \lim_{T \downarrow 0} J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) \quad \text{for all } \mu \in \Xi_0. \quad (2.31)$$

We have so far constructed the conductivity measure and the in-phase linear-response current at $T = 0$ if $\mu \in \Xi_0$. But what if, say, there is absolutely continuous spectrum and $\mu \in \mathfrak{S}_{ac}$? In this case there is no reason to expect $\mu \in \Xi_0$. In view of Remark 1 (iii) we conjecture that $\mu \notin \Xi_0$ for most $\mu \in \mathfrak{S}_{ac}$.

In this article we show that the conductivity measure at zero temperature can be constructed for *arbitrary* Fermi energy μ in a physically sensible way as the weak limit of the finite-temperature conductivity measures as $T \downarrow 0$, with the corresponding in-phase linear-response current given by (2.31).

To motivate our construction, we take $T > 0$ and decompose Σ_{μ}^T as

$$\Sigma_{\mu}^T = \Sigma_{\mu}^T(\{0\}) \delta_0 + (\Sigma_{\mu}^T - \Sigma_{\mu}^T(\{0\}) \delta_0), \quad (2.32)$$

where the Dirac measure δ_0 is the Borel measure on \mathbb{R} concentrated at 0 with total measure one. The details of this decomposition, presented in the following theorem, will lead to a natural definition of Σ_{μ}^0 for arbitrary μ . We recall that the Anderson model satisfies the Wegner estimate [W], and hence the density of states measure $\mathcal{N} \in \mathcal{M}_+(\mathbb{R})$, defined by

$$\mathcal{N}(B) := \mathcal{T}(\chi_B(H)) = \mathbb{E}\{\langle \delta_0, \chi_B(H_{\omega}) \delta_0 \rangle\} \quad \text{for all Borel sets } B \subset \mathbb{R}, \quad (2.33)$$

supported by the spectrum \mathfrak{S} of H , is absolutely continuous with density n satisfying $\|n\|_{\infty} \leq \|\rho\|_{\infty}$.

We will use the following convention: If $\Gamma \in \mathcal{M}_+(\mathbb{R})$ is absolutely continuous and supported by the closed set $F \subset \mathbb{R}$, we always assume that its density γ is also supported by F .

We set

$$\mathcal{Q}_0 := \chi_{\{0\}}(\mathcal{L}) \quad \text{and} \quad \mathcal{Q}_{\perp} := I - \mathcal{Q}_0, \quad (2.34)$$

the orthogonal projections onto the kernel of \mathcal{L} in \mathcal{K}_2 and its orthogonal complement. Note that \mathcal{Q}_0 and \mathcal{Q}_\perp commute with \mathcal{H}_L and \mathcal{H}_R , and we have

$$g(\mathcal{H}_L)\mathcal{Q}_0 = g(\mathcal{H}_R)\mathcal{Q}_0 \quad \text{for all bounded Borel functions } g. \quad (2.35)$$

For each $T \geq 0$ and $\mu \in \mathbb{R}$, the bounded selfadjoint operator \mathcal{F}_μ^T , defined in (2.14), satisfies

$$\mathcal{Q}_0\mathcal{F}_\mu^T = \mathcal{F}_\mu^T\mathcal{Q}_0 = 0 \quad \text{and} \quad \mathcal{F}_\mu^T = \mathcal{F}_\mu^T\mathcal{Q}_\perp = \mathcal{Q}_\perp\mathcal{F}_\mu^T. \quad (2.36)$$

We let \mathcal{L}_\perp^{-1} denote the pseudoinverse to \mathcal{L} , that is,

$$\mathcal{L}_\perp^{-1} := g(\mathcal{L}) \quad \text{with } g(t) := \frac{1}{t} \text{ if } t \neq 0 \text{ and } g(0) = 0. \quad (2.37)$$

In particular,

$$\mathcal{L}_\perp^{-1}\mathcal{L} = \mathcal{Q}_\perp. \quad (2.38)$$

Moreover, we have $-\mathcal{L}\mathcal{F}_\mu^T \geq 0$ and

$$-\mathcal{L}_\perp^{-1}\mathcal{F}_\mu^T = F_\mu^T(\mathcal{H}_L, \mathcal{H}_R), \quad (2.39)$$

where

$$F_\mu^T(\lambda_1, \lambda_2) := \begin{cases} -\frac{f_\mu^T(\lambda_1) - f_\mu^T(\lambda_2)}{\lambda_1 - \lambda_2} = \left| \frac{f_\mu^T(\lambda_1) - f_\mu^T(\lambda_2)}{\lambda_1 - \lambda_2} \right| & \text{if } \lambda_1 \neq \lambda_2 \\ 0 & \text{otherwise} \end{cases}. \quad (2.40)$$

We write $\mathcal{D}(\mathcal{A})$ for the domain of an unbounded operator \mathcal{A} in \mathcal{K}_2 .

Theorem 2. (i) *Let*

$$\Psi(B) := \pi\langle\langle \dot{X}_1, \mathcal{Q}_0 \chi_B(\mathcal{H}_L) \dot{X}_1 \rangle\rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (2.41)$$

Then $\Psi \in \mathcal{M}_+(\mathbb{R})$ is absolutely continuous with respect to the density of states measure \mathcal{N} , and its density with respect to Lebesgue measure, ψ , satisfies $\psi(E) \leq 4\pi n(E) \leq 4\pi \|\rho\|_\infty$ for Lebesgue-a.e. $E \in \mathbb{R}$. Moreover, we have $\text{supp } \Psi \subset \overline{\mathbb{R} \setminus \Xi_0} \subset \mathbb{R} \setminus \Xi^{\text{cl}}$.

(ii) *For each $T \geq 0$ and $\mu \in \mathbb{R}$ we have $\dot{X}_1 \in \mathcal{D}\left((-\mathcal{L}_\perp^{-1}\mathcal{F}_\mu^T)^{\frac{1}{2}}\right)$. Setting*

$$\Gamma_\mu^T(B) := \pi\langle\langle (-\mathcal{L}_\perp^{-1}\mathcal{F}_\mu^T)^{\frac{1}{2}} \dot{X}_1, \chi_B(\mathcal{L})(-\mathcal{L}_\perp^{-1}\mathcal{F}_\mu^T)^{\frac{1}{2}} \dot{X}_1 \rangle\rangle \quad (2.42)$$

for all Borel sets $B \subset \mathbb{R}$, we have $\Gamma_\mu^T \in \mathcal{M}_+^{(e)}(\mathbb{R})$ with $\Gamma_\mu^T(\{0\}) = 0$.

(iii) *If either $T > 0$ or $\mu \in \Xi_0$, we have $\mathcal{F}_\mu^T \dot{X}_1 \in \mathcal{D}(\mathcal{L}_\perp^{-1})$ and*

$$\Gamma_\mu^T(B) = \pi\langle\langle \dot{X}_1, \chi_B(\mathcal{L})(-\mathcal{L}_\perp^{-1}\mathcal{F}_\mu^T) \dot{X}_1 \rangle\rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (2.43)$$

(iv) For all $T > 0$ and $\mu \in \mathbb{R}$ we have

$$\Sigma_\mu^T(\{0\}) = \Psi((-f_\mu^T)'), \quad (2.44)$$

$$\Sigma_\mu^T(B \setminus \{0\}) = \Gamma_\mu^T(B) \quad \text{for all Borel sets } B \subset \mathbb{R}, \quad (2.45)$$

yielding the following decomposition of the conductivity measure into mutually singular measures:

$$\Sigma_\mu^T = \Psi((-f_\mu^T)')\delta_0 + \Gamma_\mu^T. \quad (2.46)$$

(v) For all $\mu \in \Xi_0$ we have

$$\Sigma_\mu^0 = \Gamma_\mu^0. \quad (2.47)$$

R e m a r k 2. On account of Theorem 2(i) we assume without loss of generality that $\psi(\mu) = 0$ for all $\mu \in \Xi_0$.

R e m a r k 3. The measure Γ_μ^T given in (2.42) can be expressed in terms of the velocity-velocity correlation measure $\Phi \in \mathcal{M}_+(\mathbb{R}^2)$, defined by (cf. [KILM, Eq. (3.46)])

$$\Phi(C) := \langle \dot{X}_1, \chi_C(\mathcal{H}_L, \mathcal{H}_R)\dot{X}_1 \rangle \quad \text{for all Borel sets } C \subset \mathbb{R}^2. \quad (2.48)$$

It follows from (2.39) that for each $T \geq 0$ and $\mu \in \mathbb{R}$ the measure Γ_μ^T can be written as

$$\Gamma_\mu^T(B) = \pi \int_{\mathbb{R}^2} \Phi(d\lambda_1 d\lambda_2) F_\mu^T(\lambda_1, \lambda_2) \chi_B(\lambda_1 - \lambda_2). \quad (2.49)$$

We are thus led to the following definition.

Definition 2. The ac-conductivity measure (x_1 - x_1 component) at $T = 0$ and $\mu \in \mathbb{R}$ is the finite positive even Borel measure Σ_μ^0 on the real line given by

$$\Sigma_\mu^0 := \psi(\mu)\delta_0 + \Gamma_\mu^0. \quad (2.50)$$

The corresponding in-phase linear-response current is defined by

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathcal{E}) := \int_{\mathbb{R}} \Sigma_\mu^0(d\nu) e^{i\nu t} \widehat{\mathcal{E}}(\nu). \quad (2.51)$$

R e m a r k 4. In view of Theorem 2(v) and Remark 2, Definition 2 agrees with Definition 1 on the common domain of definition, i.e., we have a unique definition for Σ_μ^0 for all $\mu \in \mathbb{R}$.

R e m a r k 5. In the absence of randomness, i.e., $H = -\Delta$, we may still carry out the above procedure and define Σ_μ^0 by (2.50) with Ψ as in (2.41) and

Γ_μ^0 as in (2.42). In this case \dot{X}_1 commutes with H , and hence $\mathcal{Q}_0 \dot{X}_1 = \dot{X}_1$. Thus $\Gamma_\mu^0 = 0$ and, for a Borel set $B \subset \mathbb{R}$,

$$\Psi(B) = \pi \langle \langle \dot{X}_1, \chi_B(-\Delta) \dot{X}_1 \rangle \rangle = \pi \langle (\delta_{\hat{x}_1} - \delta_{-\hat{x}_1}), \chi_B(-\Delta) (\delta_{\hat{x}_1} - \delta_{-\hat{x}_1}) \rangle. \quad (2.52)$$

It follows that Ψ has a density given by a continuous function ψ , the limit in (3.38) holds for every μ , and (recall $\sigma(-\Delta) = [-2d, 2d]$)

$$\Sigma_\mu^0 = \psi(\mu) \delta_0 \quad \text{with} \quad \psi(\mu) \begin{cases} > 0 & \text{if } \mu \in]-2d, 2d[\\ = 0 & \text{otherwise} \end{cases}. \quad (2.53)$$

Since the in-phase conductivity $\text{Re } \sigma_\mu^0(\nu)$ is formally the density of Σ_μ^0 , (2.53) is formally equivalent to the usual statement that for $H = -\Delta$ we have

$$\text{Re } \sigma_\mu^0(\nu) = \psi(\mu) \delta(\nu), \quad (2.54)$$

with $\delta(\nu)$ the formal Dirac delta function.

Remark 6. The picture described in Remark 5 changes in the presence of any amount of randomness. Let us introduce a disorder parameter in the Anderson Hamiltonian by setting $H_\omega^{(\lambda)} := -\Delta + \lambda V_\omega$, where $\lambda \in \mathbb{R}$ is the disorder parameter. Although the velocity operator \dot{X}_1 does not depend on λ , any amount of randomness (i.e., $\lambda \neq 0$) implies $\mathcal{Q}_0^{(\lambda)} \dot{X}_1 \neq \dot{X}_1$ since then $[\dot{X}_1, H_\omega^{(\lambda)}] = \lambda [\dot{X}_1, V_\omega] \neq 0$ for a.e. ω . In the region of complete localization we know $\psi^{(\lambda)}(\mu) = 0$ by Theorem 2 (i), and hence the conductivity measure has no atom at 0 and we have (2.47). At high disorder it is known that the region of complete localization (we include the complement of the spectrum) is the whole real line, in which case we can conclude that $\mathcal{Q}_0^{(\lambda)} \dot{X}_1 = 0$, i.e., $\mathcal{Q}_\perp^{(\lambda)} \dot{X}_1 = \dot{X}_1$.

What happens if the Fermi energy μ lies in a spectral region where extended states are believed to exist is an open question. Common belief says that the conductivity is nonzero in the region of extended states, but it is finite for all Fermi energies. The latter seems to rule out the existence of an atom of Σ_μ^0 at 0 for all Fermi energies, which is equivalent to having $\mathcal{Q}_0^{(\lambda)} \dot{X}_1 = 0$. That would mean that any amount of disorder would have a very strong effect on the kernel of the Liouvillian, since we would have $\mathcal{Q}_\perp^{(\lambda)} \dot{X}_1 = \dot{X}_1$ for all $\lambda \neq 0$ although we know that $\mathcal{Q}_0^{(0)} \dot{X}_1 = \dot{X}_1$.

The justification for Def. 2 is given in the following theorem.

Theorem 3. (i) For all $T \geq 0$ the map $\mu \in \mathbb{R} \mapsto \Sigma_\mu^T \in \mathcal{M}_+^{(e)}(\mathbb{R})$ is strongly measurable, and for every $T > 0$ and $\mu \in \mathbb{R}$ we have

$$\begin{aligned} \Sigma_\mu^T &= \left((-f_0^T)' * \Sigma_\bullet^0 \right) (\mu), \quad \text{that is,} \\ \Sigma_\mu^T(B) &= \int_{\mathbb{R}} dE (-f_\mu^T)'(E) \Sigma_E^0(B) \quad \text{for all Borel sets } B \subset \mathbb{R}. \end{aligned} \quad (2.55)$$

(ii) We have

$$\Sigma_\mu^0 = \begin{cases} \text{s-lim}_{T \downarrow 0} \Sigma_\mu^T & \text{for all } \mu \in \Xi_0 \\ \text{w-lim}_{T \downarrow 0} \Sigma_\mu^T & \text{for a.e. } \mu \in \mathbb{R} \setminus \Xi_0 \end{cases}. \quad (2.56)$$

(iii) We have

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathcal{E}) = \lim_{T \downarrow 0} J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) \begin{cases} \text{for all } \mu \in \Xi_0 \\ \text{for a.e. } \mu \in \mathbb{R} \setminus \Xi_0 \end{cases}. \quad (2.57)$$

3. Proofs

In this section we prove Ths. 1, 2 and 3. We refer to App. A for the mathematical framework and basic notation.

We start with a consequence of the Wegner inequality [W].

Lemma 1. \mathcal{H}_L and \mathcal{H}_R have purely absolutely continuous spectrum.

P r o o f. In view of (A.10) it suffices to prove that \mathcal{H}_L has purely absolutely continuous spectrum. Given \mathcal{K}_2 , let $\eta_A \in \mathcal{M}_+(\mathbb{R})$ be defined by

$$\eta_A(B) := \langle A, \chi_B(\mathcal{H}_L)A \rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (3.1)$$

Since \mathcal{K}_∞ is dense in \mathcal{K}_2 , to prove the lemma it suffices to show that η_A is absolutely continuous for all $A \in \mathcal{K}_\infty$. In this case, using (A.6) and (2.33), we get

$$\eta_A(B) = \|\chi_B(H)A\|_2^2 = \|A^* \chi_B(H)\|_2^2 \leq \|A\|_\infty^2 \|\chi_B(H)\|_2^2 = \|A\|_\infty^2 \mathcal{N}(B). \quad (3.2)$$

Since \mathcal{N} is absolutely continuous, we conclude that η_A is also absolutely continuous. ■

Lemma 2. For all $g \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{Q}_0[X_1, g(H)] = i g'(\mathcal{H}_L) \mathcal{Q}_0 \dot{X}_1. \quad (3.3)$$

P r o o f. The lemma is proved by means of the Helffer-Sjöstrand formula for smooth functions of selfadjoint operators (cf. [HS, App. B]). If $g \in \mathcal{S}(\mathbb{R})$, then for any selfadjoint operator K we have

$$g(K) = \int_{\mathbb{R}^2} d\tilde{g}(z) (K - z)^{-1}, \quad (3.4)$$

$$g'(K) = - \int_{\mathbb{R}^2} d\tilde{g}(z) (K - z)^{-2}, \quad (3.5)$$

where the integrals converge absolutely in operator norm. Here $z = x + iy$, $\tilde{g}(z)$ is an *almost analytic extension* of g to the complex plane, and $d\tilde{g}(z) := \frac{1}{2\pi} \partial_z \tilde{g}(z) dx dy$ with $\partial_z = \partial_x + i\partial_y$.

Thus, for $g \in \mathcal{S}(\mathbb{R})$ we have, with $R_\omega(z) = (H_\omega - z)^{-1}$, $\mathcal{R}_L(z) = (\mathcal{H}_L - z)^{-1}$, $\mathcal{R}_R(z) = (\mathcal{H}_R - z)^{-1}$,

$$\begin{aligned} [X_1, g(H)] &= \int_{\mathbb{R}^2} d\tilde{g}(z) [X_1, R(z)] = -i \int_{\mathbb{R}^2} d\tilde{g}(z) R(z) \dot{X}_1 R(z) \\ &= -i \int_{\mathbb{R}^2} d\tilde{g}(z) \mathcal{R}_L(z) \mathcal{R}_R(z) \dot{X}_1. \end{aligned} \tag{3.6}$$

We recall $[X_1, g(H)], [X_1, R(z)] \in \mathcal{K}_2$, and the integrals converge absolutely in operator norm in \mathcal{K}_2 (see [BoGKS, Prop. 2.4] and its proof). It follows, using (2.35), that

$$\mathcal{Q}_0[X_1, g(H)] = -i \int_{\mathbb{R}^2} d\tilde{g}(z) \mathcal{R}_L(z)^2 \mathcal{Q}_0 \dot{X}_1 = i g'(\mathcal{H}_L) \mathcal{Q}_0 \dot{X}_1. \tag{3.7}$$

■

The following lemma plays an important role in our analysis.

Lemma 3. (i) *If either $T > 0$ or $\mu \in \Xi_0$, we have*

$$\mathcal{F}_\mu^T \dot{X}_1 = -\mathcal{L} Y_\mu^T. \tag{3.8}$$

In particular, we conclude that $\mathcal{F}_\mu^T \dot{X}_1 \in \mathcal{D}(\mathcal{L}_\perp^{-1})$.

(ii) *Let $T > 0$. Then for all $\mu \in \mathbb{R}$ we have*

$$Y_\mu^T = (-f_\mu^T)'(\mathcal{H}_L) \mathcal{Q}_0 \dot{X}_1 - \mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T \dot{X}_1. \tag{3.9}$$

P r o o f. Let either $T > 0$ or $\mu \in \Xi_0$, so $Y_\mu^T \in \mathcal{K}_2$. Given $\varphi \in \ell^2(\mathbb{Z}^d)$ with compact support, we have

$$\begin{aligned} \mathcal{F}_\mu^T \dot{X}_1 \varphi &= i \left\{ f_\mu^T(H) \circ_L [H, X_1] - [H, X_1] \circ_R f_\mu^T(H) \right\} \varphi \\ &= -i \left\{ H \circ_L [X_1, f_\mu^T(H)] - [X_1, f_\mu^T \circ_R(H)] H \right\} \varphi \\ &= -(\mathcal{H}_L - \mathcal{H}_R) Y_\mu^T \varphi = -\mathcal{L} Y_\mu^T \varphi, \end{aligned} \tag{3.10}$$

since $f_\mu^T(H)\phi \in \mathcal{D}(X_1)$ for $\phi \in \ell^2(\mathbb{Z}^d)$ with compact support by (2.10). Thus (3.8) follows, and, in view of (2.36), we have $\mathcal{F}_\mu^T \dot{X}_1 \in \mathcal{D}(\mathcal{L}_\perp^{-1})$.

We now let $T > 0$, and note that (3.9) follows from (3.8) since Lem. 2 gives

$$\mathcal{Q}_0 Y_\mu^T = (-f_\mu^T)'(\mathcal{H}_L) \mathcal{Q}_0 \dot{X}_1. \tag{3.11}$$

■

Lemma 4. *The map $]0, \infty[\ni T \mapsto Y_\mu^T \in \mathcal{K}_2$ is norm continuous for every $\mu \in \mathbb{R}$.*

P r o o f. If $g \in \mathcal{S}(\mathbb{R})$, it follows from [BoGKS, Prop. 2.4] and (A.7) that

$$\| [X_1, g(H)] \|_2 \leq \| [X_1, g(H)] \|_\infty \leq C \{ \{g\} \}_3, \quad (3.12)$$

where C is a constant depending only on H and

$$\{ \{g\} \}_3 := \sum_{r=0}^3 \int_{\mathbb{R}} du |g^{(r)}(u)| (1 + |u|^2)^{\frac{r-1}{2}}. \quad (3.13)$$

The lemma follows in view of (2.15). ■

We are ready to prove Th. 1. Note that for all $T \geq 0$ and $\mu \in \mathbb{R}$ we have

$$0 \leq (\mathcal{F}_\mu^T)^2 \leq 1. \quad (3.14)$$

Moreover, for all $\mu \in \mathbb{R}$ the operator $(\mathcal{F}_\mu^0)^2$ is an orthogonal projection in \mathcal{K}_2 , and hence

$$(\mathcal{F}_\mu^0)^3 = \mathcal{F}_\mu^0. \quad (3.15)$$

In addition, if $\mu \in \Xi_0$ we have

$$(\mathcal{F}_\mu^0)^2 Y_\mu^0 = Y_\mu^0, \quad (3.16)$$

$$\mathcal{F}_\mu^0 Y_\mu^T = \mathcal{F}_\mu^T Y_\mu^0 \quad \text{for all } T \geq 0. \quad (3.17)$$

P r o o f o f T h e o r e m 1. Let $\mu \in \Xi_0$ and Σ_μ^0 be given by (2.19). Using (3.16) and (3.8), we have

$$\begin{aligned} \Sigma_\mu^0(B) &= \pi \langle \dot{X}_1, \chi_B(\mathcal{L})(\mathcal{F}_\mu^0)^2 Y_\mu^0 \rangle = \pi \langle \mathcal{F}_\mu^0 \dot{X}_1, \chi_B(\mathcal{L}) \mathcal{F}_\mu^0 Y_\mu^0 \rangle \\ &= \pi \langle Y_\mu^0, \chi_B(\mathcal{L})(-\mathcal{L}) \mathcal{F}_\mu^0 Y_\mu^0 \rangle, \end{aligned} \quad (3.18)$$

and hence coincides with [KILM, Eq. (3.31)], a finite positive even Borel measure by [KILM, Th. 3.4].

If $T > 0$ and $\mu \in \mathbb{R}$ arbitrary, we use (3.9) to rewrite Σ_μ^T given by (2.19) as in (2.46), where Ψ , given by (2.41), is clearly in $\mathcal{M}_+(\mathbb{R})$, and Γ_μ^T , given in (2.43), is also seen to be in $\mathcal{M}_+(\mathbb{R})$ by (2.39). We conclude that $\Sigma_\mu^T \in \mathcal{M}_+(\mathbb{R})$. The same argument as in [KILM, Proof of Th. 3.4] shows that the measure Γ_μ^T , and hence also Σ_μ^T , is even.

To prove (2.20), note that for either $T > 0$ or $\mu \in \Xi_0$ it follows from (2.19), the Cauchy–Schwarz inequality and $|f_\mu^T| \leq 1$, that

$$\begin{aligned} \Sigma_\mu^T(\mathbb{R}) &= -\pi \mathbb{E}\{\langle X_1^2 H_\omega \delta_0, f_\mu^T(H_\omega) \delta_0 \rangle\} = -\pi \mathbb{E}\{\langle \delta_{\hat{x}_1} + \delta_{-\hat{x}_1}, f_\mu^T(H_\omega) \delta_0 \rangle\} \\ &\leq \sqrt{2} \pi \|f_\mu^T(H)\|_2 \leq \sqrt{2} \pi \|f_\mu^T(H)\|_\infty \leq \sqrt{2} \pi. \end{aligned} \tag{3.19}$$

We have thus proved parts (i) and (ii). Part (iii) is an immediate consequence of Lem. 4. To prove (iv), given a bounded measurable function g and $T \geq 0$, we write

$$\Sigma_\mu^T(g) = \pi \langle \dot{X}_1, g(\mathcal{L})(\mathcal{F}_\mu^0)^2 Y_\mu^T \rangle + \pi \langle \dot{X}_1, g(\mathcal{L})(1 - (\mathcal{F}_\mu^0)^2) Y_\mu^T \rangle. \tag{3.20}$$

In view of (3.14), the same argument used to prove $\Sigma_\mu^T \in \mathcal{M}_+(\mathbb{R})$ shows that both terms on the right-hand side of (3.20) are integrals of g with respect to finite positive Borel measures on \mathbb{R} . On account of (3.17) we have

$$\langle \dot{X}_1, g(\mathcal{L})(\mathcal{F}_\mu^0)^2 Y_\mu^T \rangle = \langle \dot{X}_1, g(\mathcal{L}) \mathcal{F}_\mu^0 \mathcal{F}_\mu^T Y_\mu^0 \rangle = \langle \mathcal{F}_\mu^T \dot{X}_1, g(\mathcal{L}) \mathcal{F}_\mu^0 Y_\mu^0 \rangle. \tag{3.21}$$

Using the Cauchy–Schwarz inequality, we get

$$\|(\mathcal{F}_\mu^T - \mathcal{F}_\mu^0) \dot{X}_1\|_2 \leq 2 \|\dot{X}_1\|_\infty \|f_\mu^T(H) - f_\mu^0(H)\|_2. \tag{3.22}$$

Recalling (2.33), we have

$$\|f_\mu^T(H) - f_\mu^0(H)\|_2^2 = \int_{\mathbb{R}} \mathcal{N}(dE) |f_\mu^T(E) - f_\mu^0(E)|^2, \tag{3.23}$$

and hence

$$\lim_{T \downarrow 0} \|f_\mu^T(H) - f_\mu^0(H)\|_2 = 0 \tag{3.24}$$

by dominated convergence. It follows that $\lim_{T \downarrow 0} \|(\mathcal{F}_\mu^T - \mathcal{F}_\mu^0) \dot{X}_1\|_2 = 0$. We conclude, using (3.16), that

$$\pi \lim_{T \downarrow 0} \langle \dot{X}_1, g(\mathcal{L})(\mathcal{F}_\mu^0)^2 Y_\mu^T \rangle = \pi \langle \mathcal{F}_\mu^0 \dot{X}_1, g(\mathcal{L}) \mathcal{F}_\mu^0 Y_\mu^0 \rangle = \Sigma_\mu^0(g). \tag{3.25}$$

On the other hand, it follows from (2.20) that

$$\lim_{T \downarrow 0} \Sigma_\mu^T(\mathbb{R}) = \Sigma_\mu^0(\mathbb{R}). \tag{3.26}$$

Combining this with (3.25), where we set $g = 1$, we conclude that

$$\lim_{T \downarrow 0} \langle \dot{X}_1, (1 - (\mathcal{F}_\mu^0)^2) Y_\mu^T \rangle = 0. \tag{3.27}$$

Since $\langle\langle \dot{X}_1, \chi_B(\mathcal{L})(1 - (\mathcal{F}_\mu^0)^2)Y_\mu^T \rangle\rangle$ is a positive measure, it converges to 0 strongly. Part (iv) is proven.

It remains to prove part (v). Let $\mu \in \Xi^{\text{cl}}$, so $Y_\mu^T \in \mathcal{K}_2$ for all $T \geq 0$. We need to prove that

$$\lim_{T \downarrow 0} \|Y_\mu^T - Y_\mu^0\|_2 = 0. \tag{3.28}$$

Standard calculations give

$$\begin{aligned} \|Y_\mu^T - Y_\mu^0\|_2^2 &= \mathbb{E} \left\{ \left\langle (f_\mu^T(H) - f_\mu^0(H))\delta_0, X_1^2 (f_\mu^T(H) - f_\mu^0(H))\delta_0 \right\rangle \right\} \\ &\leq \|f_\mu^T(H) - f_\mu^0(H)\|_2 \left(\mathbb{E} \left\{ \|X_1^2 (f_\mu^T(H) - f_\mu^0(H))\delta_0\|^2 \right\} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.29}$$

In view of (3.24), the desired (3.28) follows if we prove that

$$\limsup_{T \downarrow 0} \mathbb{E} \left\{ \|X_1^2 (f_\mu^T(H) - f_\mu^0(H))\delta_0\|^2 \right\} < \infty. \tag{3.30}$$

To prove (3.30) we use that $\mu \in \Xi^{\text{cl}}$, and hence there exists $\delta > 0$ such that $I_\delta \subset \Xi^{\text{cl}}$, where $I_\eta :=]\mu - \eta, \mu + \eta[$ for $\eta > 0$. We pick functions $g_j \in C_c^\infty(\mathbb{R})$, $j = 1, 2$, such that $0 \leq g_j \leq 1$, $\chi_{\mathfrak{E}} = (g_1 + g_2)\chi_{\mathfrak{E}}$, $\text{supp } g_1 \subset I_\delta$, $\text{supp } g_2 \subset \mathbb{R} \setminus I_{\frac{\delta}{2}}$. Letting $g_\mu^T = f_\mu^T - f_\mu^0$, we have

$$f_\mu^T(H) - f_\mu^0(H) = g_\mu^T(H) = g_\mu^T(H)g_1(H) + g_\mu^T(H)g_2(H). \tag{3.31}$$

Since $\text{supp } g_1 \subset \Xi^{\text{cl}}$ and $|g_\mu^T| \leq 2$ for all $T > 0$, standard estimates [A, AG, GK1, GK4] give

$$\sup_{T > 0} \mathbb{E} \left\{ \|X_1^2 g_\mu^T(H)g_1(H)\delta_0\|^2 \right\} < \infty. \tag{3.32}$$

On the other hand, explicit calculations show that

$$\sup_{T > 0} \left\| (g_\mu^T)^{(k)} \chi_{\mathbb{R} \setminus I_{\frac{\delta}{2}}} \right\|_\infty < \infty \quad \text{for all } k = 0, 1, 2, \dots \tag{3.33}$$

Since $\text{supp } g_2 \subset \mathbb{R} \setminus I_{\frac{\delta}{2}}$, a calculation using [GK2, Th. 2] shows that

$$\sup_{T > 0} \mathbb{E} \left\{ \|X_1^2 g_\mu^T(H)g_2(H)\delta_0\|^2 \right\} < \infty. \tag{3.34}$$

The estimate (3.30) follows. ■

We now turn to Th. 2.

P r o o f o f T h e o r e m 2. Note that we already proved parts (iii) and (iv) while proving Th. 1. To prove (v), note that it follows from (2.19), (3.16), (2.38), (3.8), and (3.15) that for all Borel sets $B \subset \mathbb{R}$ we have

$$\begin{aligned} \Sigma_\mu^0(B) &= \pi \langle \dot{X}_1, \chi_B(\mathcal{L})(\mathcal{F}_\mu^0)^2 Y_\mu^0 \rangle = \pi \langle \dot{X}_1, \chi_B(\mathcal{L})(\mathcal{F}_\mu^0)^2 \mathcal{L}_\perp^{-1} \mathcal{L} Y_\mu^0 \rangle \\ &= -\pi \langle \dot{X}_1, \chi_B(\mathcal{L})(\mathcal{F}_\mu^0)^2 \mathcal{L}_\perp^{-1} \mathcal{F}_\mu^0 \dot{X}_1 \rangle = \Gamma_\mu^0(B). \end{aligned} \quad (3.35)$$

Now, we turn to part (i). Let Ψ be given by (2.41), it is clearly in $\mathcal{M}_+(\mathbb{R})$. Since

$$\dot{X}_1 \delta_0 = -i(\delta_{\hat{x}_1} - \delta_{-\hat{x}_1}), \quad (3.36)$$

we have, for all Borel sets $B \subset \mathbb{R}$, recalling (2.33),

$$\begin{aligned} \frac{1}{\pi} \Psi(B) &\leq \langle \dot{X}_1, \chi_B(\mathcal{H}_L) \dot{X}_1 \rangle = \mathbb{E} \{ \langle (\delta_{\hat{x}_1} - \delta_{-\hat{x}_1}), \chi_B(H) (\delta_{\hat{x}_1} - \delta_{-\hat{x}_1}) \rangle \} \\ &\leq 2\mathcal{N}(B) + 2 \mathbb{E} \{ \| \chi_B(H) \delta_{\hat{x}_1} \| \| \chi_B(H) \delta_{-\hat{x}_1} \| \} \leq 4\mathcal{N}(B). \end{aligned} \quad (3.37)$$

It follows that Ψ is absolutely continuous with respect to the density of states measure \mathcal{N} , and that its density with respect to Lebesgue measure, ψ , satisfies $\psi(E) \leq 4\pi n(E)$ for Lebesgue-a.e. $E \in \mathbb{R}$. Since the functions $(-f_0^T)'$ form an approximate identity as $T \downarrow 0$, it follows from the absolute continuity of Ψ and the Lebesgue Differentiation Theorem (cf. [Gr, Cor. 2.1.17]) that

$$\lim_{T \downarrow 0} \Psi((-f_\mu^T)') = \psi(\mu) \quad \text{for a.e. } \mu. \quad (3.38)$$

From parts (ii) and (iv) of Th. 1 and (2.44) (which is proved already) we conclude that $\lim_{T \downarrow 0} \Psi((-f_\mu^T)') = 0$ for Lebesgue-almost all $\mu \in \Xi_0$. Th. 2(i) is proven.

To finish, we need to prove part (ii). Let $\Phi \in \mathcal{M}_+(\mathbb{R}^2)$ be the velocity-velocity correlation measure given in (2.48). As a consequence of (2.49), (2.46) and (2.20), we have

$$\int_{\mathbb{R}^2} \Phi(d\lambda_1 d\lambda_2) F_\mu^T(\lambda_1, \lambda_2) \leq \sqrt{2} \quad \text{for all } T > 0 \text{ and } \mu \in \mathbb{R}. \quad (3.39)$$

But for all $\mu \in \mathbb{R}$ we have

$$\lim_{T \downarrow 0} F_\mu^T(\lambda_1, \lambda_2) = F_\mu^0(\lambda_1, \lambda_2) \quad \text{for } \Phi\text{-a.e. } (\lambda_1, \lambda_2) \in \mathbb{R}^2, \quad (3.40)$$

where we used the fact that the two marginals of Φ are absolutely continuous, a consequence of Lem. 1. (More is true: the two marginals are equal to the measure $\eta_{\dot{X}_1}$, and hence have a bounded density, cf. (3.2).) Using Fatou's Lemma and (3.39), we conclude that for all $\mu \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^2} \Phi(d\lambda_1 d\lambda_2) F_\mu^0(\lambda_1, \lambda_2) \leq \liminf_{T \downarrow 0} \int_{\mathbb{R}^2} \Phi(d\lambda_1 d\lambda_2) F_\mu^T(\lambda_1, \lambda_2) \leq \sqrt{2}. \quad (3.41)$$

Theorem 2(ii) follows. ■

It remains to prove Th. 3.

P r o o f o f T h e o r e m 3. To prove part (i), we remark that measurability in μ follows from (2.46) and (2.49) if $T > 0$, respectively from Def. 2 and (2.49) if $T = 0$. Now, Definition 2, Theorem 2(iv) and (i) imply that it suffices to prove (2.55) with Γ_μ^T substituted for Σ_μ^T , that is,

$$\Gamma_\mu^T(B) = \int_{\mathbb{R}} dE (-f_\mu^T)'(E) \Gamma_E^0(B) \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (3.42)$$

But this follows from (2.49) using Fubini's Theorem plus the fact that

$$f_\mu^T(t) = \int_{\mathbb{R}} ds (-f_\mu^T)'(s) f_s^0(t) \quad \text{for all } t \in \mathbb{R}. \quad (3.43)$$

Next we turn to part (ii). As in the proof of (3.38), it follows from (2.55) and the Lebesgue Differentiation Theorem that for each Borel set $B \subset \mathbb{R}$ we have $\lim_{T \downarrow 0} \Gamma_\mu^T(B) = \Gamma_\mu^0(B)$ for Lebesgue-a.e. $\mu \in \mathbb{R}$ (the exceptional set depending on B !). Let $\{I_n\}_{n \in \mathbb{N}}$ denote an enumeration of the bounded intervals with rational endpoints. It follows that for a.e. μ we have $\lim_{T \downarrow 0} \Gamma_\mu^T(I_n) = \Gamma_\mu^0(I_n)$ for all $n \in \mathbb{N}$, and hence we have $\text{w-lim}_{T \downarrow 0} \Gamma_\mu^T = \Gamma_\mu^0$ for a.e. μ . Part (ii) now follows using Th. 1(iv) for $\mu \in \Xi_0$.

Part (iii) is an immediate consequence of part (ii). ■

Appendix A. The Mathematical Framework for Linear Response Theory

In this appendix we recall the mathematical framework for linear response theory, following [BoGKS, Sect. 3] and [KILM, Sect. 3] (see also [BES, SB]). We restrict ourselves to the Anderson model. The Hamiltonian H_ω , given in (2.1), is a measurable map from the probability space (Ω, \mathbb{P}) to the bounded selfadjoint operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. The probability space (Ω, \mathbb{P}) is equipped with an ergodic group $\{\tau_a; a \in \mathbb{Z}^d\}$ of measure preserving transformations, satisfying the covariance relation

$$U(a)H_\omega U(a)^* = H_{\tau_a(\omega)} \quad \text{for all } a \in \mathbb{Z}^d, \quad (\text{A.1})$$

where $U(a)$ denotes translation by a , i.e., $U(a)\delta_b := \delta_{b+a}$ when applied to any member of the canonical orthonormal basis $\{\delta_b; b \in \mathbb{Z}^d\}$ for $\ell^2(\mathbb{Z}^d)$.

Let $\mathcal{H}_c = \ell_c^2(\mathbb{Z}^d)$ be the (dense) subspace of finite linear combinations of the canonical basis vectors. By \mathcal{K}_{mc} we denote the vector space of measurable covariant operators $A: \Omega \rightarrow \text{Lin}(\mathcal{H}_c, \mathcal{H})$, identifying measurable covariant operators

that agree \mathbb{P} -a.e.; all properties stated are assumed to hold for \mathbb{P} -a.e. $\omega \in \Omega$. Here $\text{Lin}(\mathcal{H}_c, \mathcal{H})$ is the vector space of linear operators from \mathcal{H}_c to \mathcal{H} . Recall that A is measurable if the functions $\omega \rightarrow \langle \phi, A_\omega \phi \rangle$ are measurable for all $\phi \in \mathcal{H}_c$, A is covariant if

$$U(a)A_\omega U(a)^* = A_{\tau_a(\omega)} \quad \text{for all } a \in \mathbb{Z}^d. \quad (\text{A.2})$$

It follows (for $\mathcal{H} = \ell^2(\mathbb{Z}^d)$) that $\mathcal{D}(A_\omega^*) \supseteq \mathcal{H}_c$ for $A \in \mathcal{K}_{mc}$, i.e., A is locally bounded. Thus, the operator $A_\omega^\dagger := A_\omega^*|_{\mathcal{H}_c}$ is well defined. Note that $(\mathcal{J}A)_\omega := A_\omega^\dagger$ defines a conjugation in \mathcal{K}_{mc} .

We introduce norms on \mathcal{K}_{mc} given by

$$\begin{aligned} \|A\|_\infty &:= \| \|A_\omega\| \|_{L^\infty(\Omega, \mathbb{P})}, \\ \|A\|_p^p &:= \mathbb{E}\{ \langle \delta_0, |\overline{A_\omega}|^p \delta_0 \rangle \}, \quad p = 1, 2, \end{aligned} \quad (\text{A.3})$$

and consider the normed spaces

$$\mathcal{K}_p := \{A \in \mathcal{K}_{mc}; \|A\|_p < \infty\}, \quad p = 1, 2, \infty. \quad (\text{A.4})$$

It turns out that \mathcal{K}_∞ is a Banach space and \mathcal{K}_2 is a Hilbert space with inner product

$$\langle\langle A, B \rangle\rangle := \mathbb{E}\{ \langle A_\omega \delta_0, B_\omega \delta_0 \rangle \}, \quad (\text{A.5})$$

and we have

$$\langle\langle A, B \rangle\rangle = \langle\langle B^\dagger, A^\dagger \rangle\rangle. \quad (\text{A.6})$$

Since \mathcal{K}_1 is not complete, we introduce its (abstract) completion $\overline{\mathcal{K}_1}$. The conjugation \mathcal{J} is an isometry on each \mathcal{K}_p , $p = 1, 2, \infty$. We also have

$$\|A\|_1 \leq \|A\|_2 \leq \|A\|_\infty \quad \text{and hence} \quad \mathcal{K}_\infty \subset \mathcal{K}_2 \subset \mathcal{K}_1, \quad (\text{A.7})$$

and \mathcal{K}_∞ is dense in \mathcal{K}_p , $p = 1, 2$. Moreover, we have $H, \Delta, \dot{X}_1 \in \mathcal{K}_\infty$.

Given $A \in \mathcal{K}_\infty$, we identify A_ω with its closure $\overline{A_\omega}$, a bounded operator in \mathcal{H} . We may then introduce a product in \mathcal{K}_∞ by pointwise operator multiplication, and \mathcal{K}_∞ becomes a C^* -algebra. (\mathcal{K}_∞ is actually a von Neumann algebra [BoGKS, Subsect. 3.5].) This C^* -algebra acts by left and right multiplication in \mathcal{K}_p , $p = 1, 2$. Given $A \in \mathcal{K}_p$, $B \in \mathcal{K}_\infty$, left multiplication $B \circ_L A$ is simply defined by $(B \circ_L A)_\omega := B_\omega A_\omega$. Right multiplication is more subtle, we set $(A \circ_R B)_\omega := A_\omega^* B_\omega$ (see [BoGKS, Lem. 3.4] for a justification), and note that $(A \circ_R B)^\dagger = B^* \circ_L A^\dagger$. Moreover, left and right multiplication commute:

$$B \circ_L A \circ_R C := B \circ_L (A \circ_R C) = (B \circ_L A) \circ_R C \quad (\text{A.8})$$

for $A \in \mathcal{K}_p$, $B, C \in \mathcal{K}_\infty$. We refer to [BoGKS, Sect. 3] for an extensive set of rules and properties which facilitate calculations in these spaces of measurable covariant operators.

Since $H \in \mathcal{K}_\infty$, we define bounded commuting selfadjoint operators \mathcal{H}_L and \mathcal{H}_R on \mathcal{K}_2 by

$$\mathcal{H}_L A := H \circ_L A \quad \text{and} \quad \mathcal{H}_R A := A \circ_R H; \quad (\text{A.9})$$

note that

$$\mathcal{H}_R = \mathcal{J} \mathcal{H}_L \mathcal{J}. \quad (\text{A.10})$$

The *Liouvilian* is then defined by

$$\mathcal{L} := \mathcal{H}_L - \mathcal{H}_R, \quad (\text{A.11})$$

and hence satisfies

$$\mathcal{L} = -\mathcal{J} \mathcal{L} \mathcal{J}. \quad (\text{A.12})$$

Note that (cf. [BoGKS, argument below Eq. (5.91)])

$$\ker \mathcal{L} = \{A \in \mathcal{K}_2; A \circ_L f(H) = f(H) \circ_R A \text{ for all } f \in \mathcal{S}(\mathbb{R})\}. \quad (\text{A.13})$$

The *trace per unit volume* is given by

$$\mathcal{T}(A) := \mathbb{E}\{\langle \delta_0, A_\omega \delta_0 \rangle\} \quad \text{for } A \in \mathcal{K}_1, \quad (\text{A.14})$$

a well defined linear functional on \mathcal{K}_1 with $|\mathcal{T}(A)| \leq \|A\|_1$, and hence can be extended to $\overline{\mathcal{K}_1}$. Note that \mathcal{T} is indeed the trace per unit volume:

$$\mathcal{T}(A) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \text{tr} \{ \chi_{\Lambda_L} A_\omega \chi_{\Lambda_L} \} \quad \text{for } \mathbb{P}\text{-a.e. } \omega, \quad (\text{A.15})$$

where Λ_L denotes the cube of side L centered at 0 (see [BoGKS, Prop. 3.20]). Moreover,

$$\langle\langle A, B \rangle\rangle = \mathcal{T} \{ A^* B \} \quad \text{for all } A, B \in \mathcal{K}_2. \quad (\text{A.16})$$

Appendix B. The Region of Complete Localization

There is a wealth of localization results for the Anderson model in arbitrary dimension, based either on the multiscale analysis [FS, FMSS, DK], or on the fractional moment method [AM, A]. The spectral region of applicability of both methods turns out to be the same, and in fact it can be characterized by many equivalent conditions [GK3, GK4]. For this reason we call it the *region of complete localization* as in [GK4].

The most convenient definition for this paper is by the conclusions of [GK4, Th. 3]. For convenience we include the complement of the spectrum in the region of complete localization.

Definition 3. *The region of complete localization Ξ^{cl} for the Anderson Hamiltonian H is the set of energies $E \in \mathbb{R}$ for which there is an open interval $I \ni E$ and constants $\zeta > 0$ and $C < \infty$ such that*

$$\mathbb{E} \left\{ \sup_{\mu \in I} |\langle \delta_x, f_{\mu}^0(H_{\omega}) \delta_0 \rangle|^2 \right\} \leq C e^{-|x|^{\zeta}} \quad \text{for all } x \in \mathbb{Z}^d. \quad (\text{B.1})$$

Remark 7. As remarked in the comments below [GK4, Th. 3], it suffices to require fast enough polynomial decay in (B.1); subexponential decay then follows.

Remark 8. For the Anderson model, it follows from [A, AG] that we have exponential decay in (B.1). More precisely, if $E \in \Xi^{\text{cl}}$, there is an open interval $I \ni E$ and constants $m > 0$ and $C < \infty$ such that

$$\mathbb{E} \left\{ \sup_{\mu \in I} |\langle \delta_x, f_{\mu}^0(H_{\omega}) \delta_0 \rangle|^2 \right\} \leq C e^{-m|x|} \quad \text{for all } x \in \mathbb{Z}^d. \quad (\text{B.2})$$

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