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Scattering from Sparse Potentials on Graphs

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We study the spectral structure of Schrödinger operators $H = \Delta + V$ for random potentials supported on sparse sets. In the past years examples of such operators whose spectra almost surely satisfy the following properties have been exhibited: Anderson localization holds outside spec(Δ), while the wave operators $\Omega^{\pm}(H, \Delta)$ exist inside this last set. We continue this program by presenting sparseness conditions under which $\Omega^{\pm}(\Delta, H)$ also exist.

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1. Introduction

Since its introduction in 1958, there has been considerable interest in the *Anderson model* [4], which describes potentials that are not completely known, but are affected by a probability distribution. By focusing on almost sure results (and hence by discarding pathological constructions), research on this model has given a new insight into quantum physics. A random potential, V, lies on a lattice $\mathbb{Z}^d$. It is described by the following operator on $l^2(\mathbb{Z}^d)$:

$$V = \sum_{N \in \mathbb{Z}^d} V(N) \langle \delta_N \mid \cdot \rangle \delta_N,$$

where $\delta_N(M)$ is the Kronecker delta and $\{V(N)\}_{N\in\mathbb{Z}^d}$ is a family of i.i.d. random variables of law ν.* The spectral structure of the random Hamiltonian

$$H = \Delta + \lambda V$$

*Explicitly, the probability space is given by $\Omega = \mathbb{R}^{(\mathbb{Z}^d)}$ equipped with its Borel σ-algebra and the probability measure $\mathbb{P} = \prod_{\mathbb{Z}^d} \nu$. The random variable $V(N)$ then maps an element of Ω to its N-th coordinate.

has been investigated—where λ is a positive number (the so-called *disorder*) and Δ is the centered discrete Laplacian. It was proven by L. Pastur that the absolutely continuous, essential, singular continuous and point spectra of H are almost surely constant [20]. Indeed, from the first days Anderson has conjectured that H has the following spectral structure (almost surely): if λ is small, spec(H) is purely absolutely continuous (*delocalization*) except near its edges, where it is pure point with exponentially decaying eigenfunctions (*Anderson localization*); on the other hand, if λ is large, Anderson localization occurs on the whole spec(H). While the structure of the a.c. spectrum of H is still not completely understood, the localization part of the above conjecture was proven by M. Aizenman and S. Molchanov [3, 1]. In their works these authors developed a method for estimating the s^{th}-moment of the resolvent's matrix elements

$$R(M,N,z) = \langle \delta_M \mid (H-z)^{-1}\delta_N \rangle$$

(in absolute value) for suitable λ, s and z approaching the real line. This method, which is used in the present paper, is based on the following *decoupling lemmas* — which apply to a large class of probability measures including Gaussian, Cauchy, and uniform distributions [1–3, 5, 11, 15]:*

Proposition 1. *Suppose there exists an $s \in (0,1)$ such that*

$$k_s = \inf_{\alpha,\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x-\alpha|^s |x-\beta|^{-s}\,\mathrm{d}\nu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\,\mathrm{d}\nu(x)} > 0.$$

Then, for any deterministic function $F(M,N,z)$,

$$\mathbb{E}|V(M) - F(M,N,z)|^s |R(M,N,z)|^s \geqslant k_s \mathbb{E}|R(M,N,z)|^s.$$

Suppose instead there exists an $s \in (0,1)$ such that

$$K_s = \sup_{\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x|^s |x-\beta|^{-s}\,\mathrm{d}\nu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\,\mathrm{d}\nu(x)} < \infty.$$

Then, $\mathbb{E}|V(M)|^s |R(M,N,z)|^s \leqslant K_s \mathbb{E}|R(M,N,z)|^s$.

In addition to the Anderson model, several researchers (M. Krishna *et al.* [13, 14], W. Kirsch *et al.* [6, 12], S. Molchanov *et al.* [15–19]) have investigated various *sparse models*, which describe random potentials lying on a set Γ subject to various geometric constraints, having in common that *the distance between*

*In the sequel we use parentheses with $\mathbb{E}$ in the same way as with $\sum$. For instance, $\mathbb{E}X^s = \mathbb{E}(X^s)$, not $(\mathbb{E}X)^s$.

$N \in \Gamma$ *and its closest neighbor in* Γ *tends to infinity when* $|N| \to \infty$. In the discrete case the following Hamiltonian on $l^2(\mathbb{Z}^d)$ has been investigated,

$$H = \Delta + V, \qquad V = \sum_{n \in \Gamma} V(n) \langle \delta_n \mid \cdot \rangle \delta_n,$$

where $\{V(n)\}_{n \in \Gamma}$ is a family of i.i.d. random variables.

Since such a model is not ergodic, Pastur's theorem fails for the singular continuous and point spectra of H, but still holds for the essential and continuous spectra. Indeed, the essential spectrum of H has been completely characterized by S. Molchanov and B. Vainberg under appropriate sparseness conditions [17, 19]. In addition, the spectral structure of H (for the above model or its continuous analog) has been clarified in different cases. Families of random Hamiltonians with the following, almost sure properties have been exhibited: the spectrum of H is (possibly dense) pure point outside spec(Δ), while the wave operators

$$\Omega_E^{\pm}(H, \Delta) = \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}t\Delta} \mathbf{1}_E(\Delta) \quad \text{(strongly)}$$

exist on the whole $E = \mathrm{spec}(\Delta)$—yielding that $\mathrm{spec}_{\mathrm{ac}}(H) = \mathrm{spec}(\Delta)$.

In order to complete this program we show that under suitable sparseness conditions the above wave operators are almost surely *complete*, *i.e.*, $\Omega_E^{\pm}(\Delta, H)$ also exist. We conclude this work by exhibiting a family of random operators $H = \Delta + V$ with sparse potentials satisfying almost surely the following properties: 1^o the spectrum of H is purely absolutely continuous on spec(Δ), 2^o the wave operators exist and are complete on spec(Δ), 3^o the spectrum of H is (possibly dense) pure point outside spec(Δ).

This work, based on a private communication with V. Jakšić, is an application of a completeness criterion found in [9] — a paper of V. Jakšić and Y. Last dedicated to L. Pastur.

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2. Abstract Results

2.1. The Model

At a higher level of generality the lattice $\mathbb{Z}^d$ is replaced with a countable set X endowed with a graph structure. We assume that this graph consists of finitely many connected components and that the degrees of the vertices are bounded. Let

$\mathrm{d}(M,N)$ be the distance between $M,N \in X$, that is, the length of the shortest path connecting them in X (∞ if M and N lie on two different components). The usual centered Laplacian is then replaced with the adjacency operator of X: for $\varphi \in l^2(X)$,

$$\Delta\varphi(N) = \sum_{\mathrm{d}(M,N)=1} \varphi(M).$$

Notice that Δ is a bounded selfadjoint operator on $l^2(X)$. The Euclidean distance is replaced with a *weight* on the set X, that is, a function $\gamma\colon X \times X \to [0,\infty)$ satisfying all axioms of metric distance, except that $\gamma(M,N) = 0$ does not necessarily imply $M = N$.

For a fixed $\Gamma \subset X$, a family $\{V(n)\}_{n\in\Gamma}$ of i.i.d. random variables is given. Their law, ν, is assumed to be absolutely continuous and to satisfy both hypotheses of Prop. 1 for a fixed $s \in (0,1)$. We study the following random Hamiltonian on $l^2(X)$:

$$H = \Delta + V, \qquad V = \sum_{n\in\Gamma} V(n)\langle\delta_n \mid \cdot\rangle\delta_n.$$

N o t a t i o n. In the sequel the connected components of the graph are denoted by X_j. For $0 \leqslant R \leqslant \infty$, the *R-fattening* of Γ is defined as

$$\Gamma_R = \{\underline{N} \in X \; ; \; \mathrm{d}(\underline{N},\Gamma) \leqslant R\},$$

while the projection on $l^2(\Gamma_R)$ is denoted by $\mathbf{1}_R$. For the sake of clarity, we shall use the following fonts: $\underline{n}$ varies in a certain subset of Γ, n varies in Γ, $\underline{N}$ varies in a certain fattening of Γ and N in the whole X.

The abbreviation a.e. & a.s. stands for *almost everywhere and almost surely*, where the former refers to the Lebesgue measure and the latter to the given probability measure $\mathbb{P}$. Here, the underlying probability space is given by $\Omega = \mathbb{R}^{(\mathbb{Z}^d)}$ equipped with its Borel σ-algebra and the probability measure $\mathbb{P} = \prod_{\mathbb{Z}^d} \nu$.

2.2. Preliminaries

Our work is based on the following *Jakšić–Last criterion of completeness* [9], whose conclusion trivially persists for disconnected graphs:*

Proposition 2. *Suppose that the spectrum of H is purely a.c. on a given Borel set $E \subseteq \mathbb{R}$. Suppose also that $\mathbf{1}_1$ is Δ-smooth on E, that is,***

$$\sup_{\substack{0<\varepsilon<1 \\ e\in E}} \|\mathbf{1}_1(\Delta - e - \mathrm{i}\varepsilon)^{-1}\mathbf{1}_1\| < \infty.$$

*This last observation is deduced from elementary properties of the projections, P_j, onto $l^2(X_j)$, namely: $P_jP_k = 0$ if $j \neq k$; $\sum P_j$ is the identity; $P_j\mathbf{1}_R = \mathbf{1}_R P_j$ for any j and R; $f(T)P_j = f(TP_j) = P_j f(T) P_j$ for any bounded Borel function f and $T \in \{\Delta, V, H\}$.

** See [25].

If for all $n \in \Gamma$ and almost all $e \in E$

$$\sum_{\underline{M} \in \Gamma_1} |\mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1} \delta_n \rangle|^2 < \infty,$$

then the wave operators $\Omega_E^{\pm}(\Delta, H)$ exist.

Since in this context the usual wave operators are $\Omega_E^{\pm}(H, \Delta)$, this last criterion asserts their completeness, but without assuming their existence.

In order to prove localization we shall use the following *Simon–Wolff theorem* [27]. It is easily seen that its conclusion is valid for disconnected graphs with finitely many components, except regarding simplicity of the eigenvalues — which follows from a recent theorem of V. Jakšić and Y. Last [10].

Proposition 3. *Let $E \subseteq \mathbb{R}$ be a Borel set. If with probability one*

$$\|(H - e - \mathrm{i}0)^{-1} \delta_n\| < \infty$$

*for all $n \in \Gamma$ and almost all $e \in E$, then the spectrum of H on E is almost surely pure point with simple eigenvalues.**

Suppose in addition that for almost all $V \in \Omega$, almost all $e \in E$, and all $n \in \Gamma$ there exist constants $K, k > 0$ independent of $M \in X$ such that

$$|\langle \delta_n \,|\, (H - e - \mathrm{i}0)^{-1} \delta_M \rangle| \leqslant K \mathrm{e}^{-k\gamma(n,M)}.$$

Then, the eigenfunctions are exponentially bounded in the following sense: for such an eigenfunction $\psi(N)$ and an arbitrarily fixed site N_0, there exists a coefficient Const (depending on V, N_0 and the associated eigenvalue) and a universal exponent $k > 0$ such that

$$|\psi(N)| \leqslant Const\ \mathrm{e}^{-k\gamma(N,N_0)}$$

for all $N \in X$.

Given a selfadjoint operator T on $l^2(X)$, let T_j be its restriction to $l^2(X_j)$. The *essential support* of the a.c. spectrum of T_j is given by

$$\Sigma(T_j) = \{e \in \mathbb{R}\,;\ \sum_{N \in X_j} |\mathrm{Im}\, \langle \delta_N \,|\, (T_j - e - \mathrm{i}0)^{-1} \delta_N \rangle| > 0\} \text{ a.e.}$$

Notice that $\Sigma(T_j)$ is defined up to a set of Lebesgue measure zero; however, its equivalence class is almost surely constant (by a variant of Pastur's theorem). We define

$$\Sigma(T) = \cap_j \Sigma(T_j).$$

The *Jakšić–Last theorem* [8] asserts:

*Recall that the *spectrum of H on E* is defined as $\mathrm{spec}(H\chi_E(H))$, where χ_E is the characteristic function of E; it is *not* equal to $\mathrm{spec}(H) \cap E$ in general. Moreover, the above conclusion includes the trivial case where H has no spectrum on E.

Proposition 4. *Let $E \subseteq \mathbb{R}$ be a Borel set. If with probability one $E \subset \Sigma(H)$ (in the sense that $E \setminus \Sigma(H)$ has Lebesgue measure zero), then the spectrum of H on E is purely a.c., almost surely.*

2.3. Main Results

As mentioned in the previous section we shall determine the spectral structure of H on a given interval $[a,b]$ by using the Jakšić–Last and the Simon–Wolff criteria (depending on the location of $[a,b]$). In both cases the matrix elements of the resolvent of H have to be estimated. This will be done in one step, using the Aizenman–Molchanov method.*

Consider the following quantity,

$$\tau(M,N) = \sup_{z \in \mathcal{S}} |\langle \delta_M \mid (\Delta - z)^{-1} \delta_N \rangle|,$$

where M, $N \in X$ and $\mathcal{S} = \{a \leqslant \operatorname{Re} z \leqslant b,\, 0 < \operatorname{Im} z < 1\}$. In concrete models $\tau(M,N)$ decays when M and N become distant. This motivates our choice in the present abstract setting to make sparseness assumptions on $\tau(M,N)$:

A s s u m p t i o n A. *For all $\varepsilon > 0$ there exists a finite set $\mathcal{F} \subseteq \Gamma$ such that $\sum_{n \in \Gamma \setminus \{\underline{m}\}} \tau(n, \underline{m})^s < \varepsilon$ for all $\underline{m} \in \Gamma \setminus \mathcal{F}$.*

Given an $R \in [0, \infty]$,

A s s u m p t i o n B. $\sup_{n \in \Gamma} \sum_{\underline{M} \in \Gamma_R} \tau(n, \underline{M})^s < \infty$.

Let $\mathfrak{I} = \inf_{n \in \Gamma, z \in \mathcal{S}} |\langle \delta_n \mid (\Delta - z)^{-1} \delta_n \rangle|$. We also assume

A s s u m p t i o n C. $\mathfrak{I} > 0$.

Our chief lemma is:

Lemma 1. *Suppose $0 \leqslant R \leqslant \infty$. Under Assumptions A, B and C, for all $n \in \Gamma$,*

$$\|\mathbf{1}_R (H - e - \mathrm{i}0)^{-1} \delta_n\| < \infty \text{ a.e. \& a.s.}$$

on $[a,b] \times \Omega$.

*Compared with the original Aizenman–Molchanov argument complications from two sources arise: since we play with sparseness instead of the disorder, in order to control the norm of a certain operator we remove a finite number of sites and then put them back using the resolvent identity repeatedly; moreover, deletion of these sites never prevents a remaining site to be close to itself, so the diagonal elements have to be treated differently.

We deduce the following result inside spec(Δ):

Theorem 1. *Suppose A, C, and* $\sup_{\underline{N}\in\Gamma_1}\sum_{\underline{M}\in\Gamma_1}\tau(\underline{N},\underline{M})^s<\infty$ *for an interval* $[a,b]\subset\Sigma(\Delta)$. *If* $\Omega^{\pm}_{[a,b]}(H,\Delta)$ *exist a.s., then the spectrum of H on* $[a,b]$ *is purely a.c. and the wave operators are complete there, almost surely.*

In order to derive Anderson localization outside spec(Δ) we make the following assumptions on the weight:

Assumption D. *For any* $k>0$, $\sup_{N\in X}\sum_{M\in X}\mathrm{e}^{-k\gamma(N,M)}<\infty$.

Assumption E. *For each* $L>0$ *there exists a finite set* $\mathcal{E}\subseteq\Gamma$ *such that for all* $\underline{m}\in\Gamma\setminus\mathcal{E}$, $\inf_{n\in\Gamma\setminus\{\underline{m}\}}\gamma(n,\underline{m})\geqslant L$.

Given an $R\in[0,\infty]$,

Assumption F. *There exist* D,β *such that* $\tau(n,\underline{M})^s\leqslant D\mathrm{e}^{-\beta\gamma(n,\underline{M})}$ *for all* $n\in\Gamma$ *and* $\underline{M}\in\Gamma_R$.

Our main lemma is:

Lemma 2. *Suppose* $0\leqslant R\leqslant\infty$. *Under Assumptions C, D, E, and F, there exists a universal constant* $k>0$ *such that the following holds a.e. & a.s. on* $[a,b]\times\Omega$: *for all* $n\in\Gamma$ *there exists a* $K>0$ *such that*

$$|\langle\delta_n\,|\,(H-e-\mathrm{i}0)^{-1}\delta_{\underline{M}}\rangle|\leqslant K\mathrm{e}^{-k\gamma(n,\underline{M})}$$

for all $\underline{M}\in\Gamma_R$.

From Lemmas 1 and 2 we deduce:

Theorem 2. *Suppose C, D, and E. Suppose in addition F holds with* $R=\infty$. *Then, the spectrum of H on* $[a,b]$ *is almost surely pure point with simple eigenvalues and exponentially bounded eigenfunctions (in the sense of Prop. 3).*

2.4. Proof of the First Lemma

In this section Assumption A is used in the following form: there exists a finite set $\mathcal{F}\subset\Gamma$ such that

$$\sup_{\underline{m}\in\Gamma\setminus\mathcal{F}}\sum_{n\in\Gamma\setminus\{\underline{m}\}}\tau(n,\underline{m})^s<\frac{\mathfrak{I}^s k_s}{2K_s}. \tag{1}$$

We also assume B for an arbitrary $R\in[0,\infty]$, and C.

Let $\widehat{H} = \Delta + \sum_{\underline{n}\in\Gamma\setminus\mathcal{F}} V(\underline{n})\langle\delta_{\underline{n}}\,|\,\cdot\rangle\delta_{\underline{n}}$. We use the abbreviations

$$\begin{aligned} R_0(N,M,z) &= \langle\delta_N\,|\,(\Delta - z)^{-1}\delta_M\rangle, \\ R(N,M,z) &= \langle\delta_N\,|\,(H - z)^{-1}\delta_M\rangle, \\ \widehat{R}(N,M,z) &= \langle\delta_N\,|\,(\widehat{H} - z)^{-1}\delta_M\rangle, \end{aligned}$$

where M, $N \in X$ and $z \in \mathcal{S}$. Since the spectral measure of δ_M and δ_N with respect to H is real-valued [9], $R(N,M,z) = R(M,N,z)$ for any $z \in \mathcal{S}$; similar relations hold for R_0 and $\widehat{R}$.

In the sequel we use the Aizenman–Molchanov decoupling lemmas (Prop. 1) in conjunction with the resolvent identity; this latter implies

$$\widehat{R}(N,M,z) = R_0(N,M,z) - \sum_{\underline{p}\in\Gamma\setminus\mathcal{F}} R_0(N,\underline{p},z)V(\underline{p})\widehat{R}(\underline{p},M,z) \tag{2}$$

for all $M,N \in X$. As a first instance, with the convention that $\underline{p}$ varies in $\Gamma\setminus\mathcal{F}$,

Lemma 3. *For all* $\underline{n}$, $\underline{m} \in \Gamma\setminus\mathcal{F}$ *and* $z \in \mathcal{S}$,

$$\mathbb{E}|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant \frac{1}{k_s\mathfrak{I}^s}\tau(\underline{n},\underline{m})^s + \frac{K_s}{k_s\mathfrak{I}^s}\sum_{\underline{p}\neq\underline{n}}\tau(\underline{n},\underline{p})^s\mathbb{E}|\widehat{R}(\underline{p},\underline{m},z)|^s.$$

P r o o f. By the equation (2),

$$\widehat{R}(\underline{n},\underline{m},z)(1 + R_0(\underline{n},\underline{n},z)V(\underline{n})) = R_0(\underline{n},\underline{m},z) - \sum_{\underline{p}\neq\underline{n}} R_0(\underline{n},\underline{p},z)V(\underline{p})\widehat{R}(\underline{p},\underline{m},z).$$

Using the triangle inequality for $|\cdot|^s$, taking the expectation, and then applying the decoupling lemmas give $k_s|R_0(\underline{n},\underline{n},z)|^s\mathbb{E}|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant |R_0(\underline{n},\underline{m},z)|^s + K_s\sum_{\underline{p}\neq\underline{n}}|R_0(\underline{n},\underline{p},z)|^s\mathbb{E}|\widehat{R}(\underline{p},\underline{m},z)|^s$, from which the result follows. ■

Let us fix $\underline{m} \in \Gamma\setminus\mathcal{F}$ and $z \in \mathcal{S}$, $\underline{n} \in \Gamma\setminus\mathcal{F}$ being thought as the only variable. We define the following vectors on $l^\infty(\Gamma\setminus\mathcal{F})$:

$$\begin{aligned} X(\underline{n}) &= \mathbb{E}|\widehat{R}(\underline{n},\underline{m},z)|^s, \\ B(\underline{n}) &= \frac{1}{k_s\mathfrak{I}^s}\tau(\underline{n},\underline{m})^s. \end{aligned}$$

They are well defined, since $\|X\|_\infty \leqslant |\mathrm{Im}\,z|^{-s}$ and $\|B\|_\infty < \infty$, the latter by Assumption B (which also ensures $\|B\|_1 < \infty$). Let us define the operator

$$(A\psi)(\underline{n}) = \frac{K_s}{k_s\mathfrak{I}^s}\sum_{\underline{p}\neq\underline{n}}\tau(\underline{n},\underline{p})^s\psi(\underline{p}),$$

which acts on both $l^\infty(\Gamma \setminus \mathcal{F})$ and $l^1(\Gamma \setminus \mathcal{F})$. By the equation (1),

$$\|A\|_\infty = \|A\|_1 = \frac{K_s}{k_s \mathfrak{I}^s} \sup_{\underline{n}} \sum_{\underline{p} \neq \underline{n}} \tau(\underline{n}, \underline{p})^s < \frac{1}{2}. \tag{3}$$

In addition, the previous lemma gives $(1 - A)X \leqslant B$ (pointwise).

Lemma 4. $\sup_{z \in \mathcal{S}} \sup_{\underline{m} \in \Gamma \setminus \mathcal{F}} \sum_{\underline{n} \in \Gamma \setminus \mathcal{F}} \mathbb{E}|\widehat{R}(\underline{n}, \underline{m}, z)|^s < \infty$.

P r o o f. The relation (3) implies that $(1 - A)^{-1} = \sum_{j=0}^{\infty} A^j$ is well-defined and satisfies $\|(1 - A)^{-1}\|_1 \leqslant 2$. Observe that, since all matrix elements of A are positive, those of $(1 - A)^{-1}$ are also positive, *i.e.*, $(1 - A)^{-1}$ preserves pointwise positivity. Therefore, by the previous lemma

$$X \leqslant (1 - A)^{-1} B \quad \text{(pointwise)}, \tag{4}$$

so $\|X\|_1 \leqslant 2\|B\|_1$. In other words, $\sum_{\underline{n}} \mathbb{E}|\widehat{R}(\underline{n}, \underline{m}, z)|^s \leqslant \frac{2}{k_s \mathfrak{I}^s} \sum_{\underline{n}} \tau(\underline{n}, \underline{m})^s$. Since $\underline{m}$ and z are arbitrary, Assumption B yields the result. ∎

Lemma 5. *For all M, $N \in X$ and $z \in \mathcal{S}$,*

$$\mathbb{E}|\widehat{R}(N, M, z)|^s \leqslant \tau(N, M)^s + K_s \sum_{\underline{p} \in \Gamma \setminus \mathcal{F}} \tau(N, \underline{p})^s \mathbb{E}|\widehat{R}(\underline{p}, M, z)|^s.$$

P r o o f. The result is obtained by applying the triangle inequality for $|\cdot|^s$ to (2), taking the expectation, and then using the decoupling lemma. ∎

Lemma 6. $\sup_{z \in \mathcal{S}} \sup_{n \in \Gamma} \sum_{\underline{M} \in \Gamma_R} \mathbb{E}|\widehat{R}(n, \underline{M}, z)|^s < \infty$.

P r o o f. Assumption B and Lemma 4 imply that $C = \sup_{n \in \Gamma} \sum_{\underline{M} \in \Gamma_R} \tau(n, \underline{M})^s$ and $D = \sup_{z \in \mathcal{S}} \sup_{\underline{m} \in \Gamma \setminus \mathcal{F}} \sum_{\underline{n} \in \Gamma \setminus \mathcal{F}} \mathbb{E}|\widehat{R}(\underline{n}, \underline{m}, z)|^s$ are finite. By the previous lemma, for all $\underline{N} \in \Gamma_R$, $\underline{m} \in \Gamma \setminus \mathcal{F}$ and $z \in \mathcal{S}$,

$$\mathbb{E}|\widehat{R}(\underline{N}, \underline{m}, z)|^s \leqslant \tau(\underline{N}, \underline{m})^s + K_s \sum_{\underline{p} \in \Gamma \setminus \mathcal{F}} \tau(\underline{N}, \underline{p})^s \mathbb{E}|\widehat{R}(\underline{p}, \underline{m}, z)|^s,$$

and hence $\sup_{z \in \mathcal{S}} \sup_{\underline{m} \in \Gamma \setminus \mathcal{F}} \sum_{\underline{N} \in \Gamma_R} \mathbb{E}|(\widehat{R}(\underline{N}, \underline{m}, z)|^s \leqslant C + K_s CD$. By the same lemma, for all $n \in \Gamma$, $\underline{M} \in \Gamma_R$ and $z \in \mathcal{S}$

$$\mathbb{E}|\widehat{R}(n, \underline{M}, z)|^s \leqslant \tau(n, \underline{M})^s + K_s \sum_{\underline{p} \in \Gamma \setminus \mathcal{F}} \tau(n, \underline{p})^s \mathbb{E}|\widehat{R}(\underline{p}, \underline{M}, z)|^s,$$

and hence $\sum_{\underline{M} \in \Gamma_R} \mathbb{E}|\widehat{R}(n, \underline{M}, z)|^s \leqslant C + K_s C(C + K_s CD)$ uniformly in $n \in \Gamma$ and $z \in \mathcal{S}$, as desired. ∎

We want to deduce information about $\widehat{R}(n, \underline{M}, e + \mathrm{i}0)$ for $n \in \Gamma$, $\underline{M} \in \Gamma_R$ and $e \in [a, b]$; this last limit exists a.e. & a.s. on $[a, b] \times \Omega$ (by classical Analysis and Fubini's theorem).

Lemma 7. *For all* $n \in \Gamma$, $\sum_{\underline{M} \in \Gamma_R} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^2 < \infty$ *a.e. & a.s. on* $[a, b] \times \Omega$.

P r o o f. For a fixed $n \in \Gamma$,

$$\int_a^b \mathbb{E} \sum_{\underline{M} \in \Gamma_R} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \,\mathrm{d}e \leqslant (b - a) \operatorname*{ess\,sup}_{a<e<b} \sum_{\underline{M} \in \Gamma_R} \mathbb{E} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s,$$

where ess sup denotes the essential supremum w.r.t. the Lebesgue measure. Hence, by Fatou's lemma

$$\int_a^b \mathbb{E} \sum_{\underline{M} \in \Gamma_R} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \,\mathrm{d}e \leqslant (b - a) \sup_{z \in \mathcal{S}} \sum_{\underline{M} \in \Gamma_R} \mathbb{E} |\widehat{R}(n, \underline{M}, z)|^s.$$

The result follows from the previous lemma and the triangle inequality for $|\cdot|^{\frac{s}{2}}$. ■

We are now ready to prove Lemma 1. Let $n \in \Gamma$. By the resolvent identity, for all $\underline{M} \in \Gamma_R$

$$R(n, \underline{M}, e + \mathrm{i}0) = \widehat{R}(n, \underline{M}, e + \mathrm{i}0) - \sum_{p \in \mathcal{F}} V(p) \widehat{R}(p, \underline{M}, e + \mathrm{i}0) R(n, p, e + \mathrm{i}0)$$

a.e. & a.s. on $[a, b] \times \Omega$. Consequently, $\sum_{\underline{M} \in \Gamma_R} |R(n, \underline{M}, e + \mathrm{i}0)|^2$ is less than or equal to $A\,(\sum_{\underline{M} \in \Gamma_R} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^2 + M(e) \sum_{p \in \mathcal{F}} |V(p)|^2 |R(n, p, e + \mathrm{i}0)|^2)$ a.e. & a.s., where $M(e) = \max_{p \in \mathcal{F}} \sum_{\underline{M} \in \Gamma_R} |\widehat{R}(p, \underline{M}, e + \mathrm{i}0)|^2$ and A is the number of elements of $\mathcal{F}$ plus one. Then the finiteness of $\mathcal{F}$ and the previous lemma complete the proof.

2.5. Proof of the Second Lemma

Now we assume C, D, E, and F. Assumption D extends by induction:

Lemma 8. *For any* k *and* α *such that* $0 < \alpha < k$ *there exists a* $C_{k,\alpha} > 0$ *satisfying*

$$\sum_{P_1, \dots, P_l \in X} \mathrm{e}^{-k(\gamma(N, P_1) + \gamma(P_1, P_2) + \dots + \gamma(P_l, M))} \leqslant C_{k,\alpha}^l \mathrm{e}^{-\alpha \gamma(N, M)} \tag{5}$$

for every $N, M \in X$ *and* $l \in \mathbb{N}$.

P r o o f. There exists an $s \in (0,1)$ such that $\alpha = sk$. By Assumption D, $B_{k'} = \sup_{N\in X} \sum_{M\in X} \mathrm{e}^{-k'\gamma(N,M)} < \infty$ for any $k' > 0$. Let us show that $C_{k,\alpha} = B_{tk}$ then satisfies the desired property, where $t = 1 - s$.

The triangle inequality for γ implies that the left-hand side in (5) is bounded above by $\sum_{P_1,\dots,P_l} \mathrm{e}^{-tk(\gamma(N,P_1)+\dots+\gamma(P_l,M))}\mathrm{e}^{-\alpha\gamma(N,M)}$ for any fixed $l \geqslant 0$. It is thus sufficient to show $\sum_{P_1,\dots,P_l} \mathrm{e}^{-tk(\gamma(N,P_1)+\dots+\gamma(P_l,M))} \leqslant B_{tk}^l$ for any $l \geqslant 0$.

The result is trivial if $l = 0$. Suppose it holds for $l - 1$. Then,

$$\begin{aligned}\sum_{P_1,\dots,P_l} \mathrm{e}^{-tk(\gamma(N,P_1)+\dots+\gamma(P_l,M))} &= \sum_{P_1} \mathrm{e}^{-tk\gamma(N,P_1)} \sum_{P_2,\dots,P_l} \mathrm{e}^{-tk(\gamma(P_1,P_2)+\dots+\gamma(P_l,M))} \\ &\leqslant B_{tk}B_{tk}^{l-1} = B_{tk}^l,\end{aligned}$$

as desired. ■

As a final preliminary remark,

Lemma 9. *All assumptions of the previous section are satisfied.*

P r o o f. Assumption B follows from Assumptions D and F. Assumption A is satisfied, since for any finite $\mathcal{E} \subset \Gamma$ and $\underline{n} \in \Gamma \setminus \mathcal{E}$,

$$\sum_{m\in\Gamma\setminus\{\underline{n}\}} \tau(m,\underline{n})^s \leqslant (D \sup_{p\in\Gamma} \sum_{q\in\Gamma\setminus\{p\}} \mathrm{e}^{-\frac{\beta}{2}\gamma(p,q)}) \sup_{m\in\Gamma\setminus\{\underline{n}\}} \mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{n},m)},$$

where the right-hand side goes to zero as $\mathcal{E} \uparrow X$ (by Assumptions D and E). Finally, Assumption C is satisfied by fiat. ■

We are thus free to use the results and computations of the previous section. Recall that $\mathcal{F} \subset \Gamma$ is a finite set chosen in such a way that the relation (1) holds. From now, by enlarging $\mathcal{F}$ if necessary, we also require*

$$\mathrm{e}^{-\frac{\beta}{2}\widehat{d}} < \frac{\mathfrak{I}^s k_s}{K_s C_{\frac{\beta}{2},\frac{\beta}{3}} D}, \tag{6}$$

where $\widehat{d} = \inf_{\underline{m}\in\Gamma\setminus\mathcal{F}} \inf_{\underline{n}\in(\Gamma\setminus\mathcal{F})\setminus\{\underline{m}\}} \gamma(\underline{n},\underline{m})$; this may be done by Assumption E.

Let $\underline{m} \in \Gamma\setminus\mathcal{F}$ and $z \in \mathcal{S}$ be fixed, $\underline{n} \in \Gamma\setminus\mathcal{F}$ being thought as the only variable. Then, with the notation of the previous section the inequation (4) applies, namely $X \leqslant (1 - A)^{-1}B$ (pointwise). Consequently,

Lemma 10. $X \leqslant$ *Const* $(1 - A)^{-1}\delta_{\underline{m}}$ *(pointwise).*

*Here, β, D, and $C_{\frac{\beta}{2},\frac{\beta}{3}}$ refer to Assumption F and Lem. 8.

P r o o f. Observe that $(A\delta_{\underline{m}})(\underline{n}) = K_s B(\underline{n}) - \frac{K_s}{k_s\mathfrak{I}^s}\tau(\underline{m},\underline{m})^s\delta_{\underline{m}}(\underline{n})$, and hence $B = \frac{1}{K_s}A\delta_{\underline{m}} + \frac{1}{k_s\mathfrak{I}^s}\tau(\underline{m},\underline{m})^s\delta_{\underline{m}}$. By the inequation (4),

$$X \leqslant \left(\frac{1}{K_s} + \frac{\tau(\underline{m},\underline{m})^s}{k_s\mathfrak{I}^s}\right)(1-A)^{-1}\delta_{\underline{m}} \ \text{(pointwise)}.$$

The result follows, with *Const* $= \frac{1}{K_s} + \frac{1}{k_s\mathfrak{I}^s}\sup_{\underline{p}\in\Gamma\setminus\mathcal{F}}\tau(\underline{p},\underline{p})^s$ (which is finite by Assumption B). ■

Lemma 11. *There exist universal constants Const and k such that*

$$\mathbb{E}|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant Const\ \mathrm{e}^{-k\gamma(\underline{n},\underline{m})}$$

for all $\underline{n},\underline{m} \in \Gamma\setminus\mathcal{F}$ *and* $z \in \mathcal{S}$.

P r o o f. By the previous lemma,

$$\mathbb{E}|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant Const \sum_{j=0}^{\infty}\langle\delta_{\underline{n}}\,|\,A^j\delta_{\underline{m}}\rangle. \tag{7}$$

Moreover,

$$A^j(\underline{n},\underline{m}) = \left(\frac{K_s}{k_s\mathfrak{I}^s}\right)^j \sum_{\underline{p}_1,\cdots,\underline{p}_{j-1}\in\Gamma\setminus\mathcal{F}} \mathbf{1}_{\underline{n}\neq\underline{p}_1}\tau(\underline{n},\underline{p}_1)^s \ldots \mathbf{1}_{\underline{p}_{j-1}\neq\underline{m}}\tau(\underline{p}_{j-1},\underline{m})^s,$$

where $\mathbf{1}_{\underline{p}\neq\underline{q}} = 1-\delta_{\underline{p}}(\underline{q})$. By Assumption F, $\mathbf{1}_{\underline{p}\neq\underline{q}}\tau(\underline{p},\underline{q})^s \leqslant D\mathrm{e}^{-\frac{\beta\widehat{d}}{2}}\mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{p},\underline{q})}$ for $\underline{p}$, $\underline{q} \in \Gamma\setminus\mathcal{F}$. Hence, Lem. 8 implies

$$\begin{aligned} A^j(\underline{n},\underline{m}) &\leqslant \left(\frac{K_s D\mathrm{e}^{-\frac{\beta\widehat{d}}{2}}}{k_s\mathfrak{I}^s}\right)^j \sum_{\underline{p}_1,\cdots,\underline{p}_{j-1}} \mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{n},\underline{p}_1)}\ldots\mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{p}_{j-1},\underline{m})} \\ &\leqslant \frac{1}{C_{\frac{\beta}{2},\frac{\beta}{3}}}\left(\frac{K_s C_{\frac{\beta}{2},\frac{\beta}{3}} D\mathrm{e}^{-\frac{\beta\widehat{d}}{2}}}{k_s\mathfrak{I}^s}\right)^j \mathrm{e}^{-\frac{\beta}{3}\gamma(\underline{n},\underline{m})}. \end{aligned}$$

By choice of $\mathcal{F}$ the equation (6) holds, so there exist constants *Const* and k such that $\sum_{j=0}^{\infty}A^j(\underline{n},\underline{m}) \leqslant Const\,\mathrm{e}^{-k\gamma(\underline{n},\underline{m})}$. The equation (7) then completes the proof. ■

Lemma 12. *There exist constants Const and k such that for each* $n \in \Gamma$, $\underline{M} \in \Gamma_R$ *and* $z \in \mathcal{S}$,

$$\mathbb{E}|\widehat{R}(n,\underline{M},z)|^s \leqslant Const\ \mathrm{e}^{-k\gamma(n,\underline{M})}.$$

P r o o f. For $\underline{N} \in \Gamma_R$ and $\underline{m} \in \Gamma \setminus \mathcal{F}$, Lem. 5, Assumption F, and the previous lemma yield

$$\begin{aligned}\mathbb{E}|\widehat{R}(\underline{N},\underline{m},z)|^s &\leqslant \tau(\underline{N},\underline{m})^s + K_s \sum_{\underline{p}\in\Gamma\setminus\mathcal{F}} \tau(\underline{N},\underline{p})^s \mathbb{E}|\widehat{R}(\underline{p},\underline{m},z)|^s \\ &\leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(\underline{N},\underline{m})} + K_s \sum_{\underline{p}\in\Gamma\setminus\mathcal{F}} \mathit{Const}\ \mathrm{e}^{-k\gamma(\underline{N},\underline{p})}\mathrm{e}^{-k\gamma(\underline{p},\underline{m})},\end{aligned}$$

where *Const* and k denote generic constants. It follows from Lem. 8 that $\mathbb{E}|\widehat{R}(\underline{N},\underline{m},z)|^s \leqslant \mathit{Const}\,\mathrm{e}^{-k\gamma(\underline{N},\underline{m})}$. Using this last inequation and Lem. 5 again, a similar computation then gives the result. ■

Lemma 13. *For all $n \in \Gamma$ and almost all $(e,V) \in [a,b] \times \Omega$ there exist constants, Const and k, the latter being universal, satisfying*

$$|\widehat{R}(n,\underline{M},e+\mathrm{i}0)| \leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(n,\underline{M})}$$

for all $\underline{M} \in \Gamma_R$.

P r o o f. Let $n \in \Gamma$ be fixed and $\underline{M} \in \Gamma_R$. Recall that $\widehat{R}(n,\underline{M},e+\mathrm{i}0)$ exists for almost all $(e,V) \in [a,b] \times \Omega$. Thus, the previous result and Fatou's lemma yield

$$\mathbb{E}\int_a^b |\widehat{R}(n,\underline{M},e+\mathrm{i}0)|^s\,\mathrm{d}e \leqslant \mathit{Const}\,\mathrm{e}^{-k\gamma(n,\underline{M})}.$$

Let $A_{\underline{M}} = \{(e,V) \in [a,b]\times\Omega\ ;\ |\widehat{R}(n,\underline{M},e+\mathrm{i}0)| > \mathrm{e}^{-\frac{k}{2s}\gamma(n,\underline{M})}\}$, where k refers to the previous inequality. Then, denoting by d the Lebesgue measure,

$$\begin{aligned}\sum_{\underline{M}\in\Gamma_R} (\,\mathrm{d}\times\,\mathrm{d}\mathbb{P})(A_{\underline{M}}) &\leqslant \sum_{\underline{M}\in\Gamma_R} \mathbb{E}\int_a^b \mathrm{e}^{\frac{k}{2}\gamma(n,\underline{M})}|\widehat{R}(n,\underline{M},e+\mathrm{i}0)|^s\,\mathrm{d}e \\ &\leqslant \mathit{Const} \sum_{\underline{M}\in\Gamma_R} \mathrm{e}^{-\frac{k}{2}\gamma(n,\underline{M})},\end{aligned}$$

which is finite by Assumption D. Hence, by Cantelli's lemma there exists a finite $\mathcal{E} \subseteq \Gamma_R$ such that for all $\underline{M} \in \Gamma_R \setminus \mathcal{E}$

$$|\widehat{R}(n,\underline{M},e+\mathrm{i}0)| \leqslant \mathrm{e}^{-\frac{k}{2s}\gamma(n,\underline{M})} \quad \text{a.e. \& a.s.},$$

where $n \in \Gamma$ is arbitrarily fixed. Since $\mathcal{E}$ is finite, the result follows. ■

Lemma 14. *Let $\mathcal{E} \subset \Gamma$ be finite. For a given $n \in \Gamma$ and almost all $(e, V) \in [a, b] \times \Omega$ there exist constants, K and k, the latter being universal, satisfying*

$$|\widehat{R}(q, \underline{M}, e + \mathrm{i}0)| \leqslant K \mathrm{e}^{-k\gamma(n, \underline{M})}$$

for all $\underline{M} \in \Gamma_R$ and $q \in \mathcal{E}$.

P r o o f. Since $\mathcal{E}$ is finite, the last lemma ensures for almost all (e, V) the existence of constants satisfying $|\widehat{R}(q, \underline{M}, e + \mathrm{i}0)| \leqslant Const\ \mathrm{e}^{-k\gamma(q, \underline{M})}$ for all $\underline{M} \in \Gamma_R$ and $q \in \mathcal{E}$. Since $\mathrm{e}^{-k\gamma(q, \underline{M})} \leqslant \mathrm{e}^{k\gamma(n,q)} \mathrm{e}^{-k\gamma(n, \underline{M})}$, the result follows, with $K = Const\ \sup_{q \in \mathcal{E}} \mathrm{e}^{k\gamma(n,q)}$. ■

We are now ready to prove Lem. 2. By the resolvent identity, for all $n \in \Gamma$, $\underline{M} \in \Gamma_R$, and almost all $(e, V) \in [a, b] \times \Omega$

$$R(n, \underline{M}, e + \mathrm{i}0) = \widehat{R}(n, \underline{M}, e + \mathrm{i}0) - \sum_{p \in \mathcal{F}} R(n, p, e + \mathrm{i}0) V(p) \widehat{R}(p, \underline{M}, e + \mathrm{i}0).$$

In particular, there exists a constant, namely, $L = \sup_{p \in \mathcal{F}} |R(n, p, e + \mathrm{i}0) V(p)|$, which depends on n, e, and V, but not on $\underline{M}$, satisfying

$$|R(n, \underline{M}, e + \mathrm{i}0)| \leqslant |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)| + L \sum_{p \in \mathcal{F}} |\widehat{R}(p, \underline{M}, e + \mathrm{i}0)|.$$

The result follows from the previous lemma applied to $\mathcal{E} = \mathcal{F} \cup \{n\}$.

2.6. Proofs of the Theorems

Lemma 15. *Let $0 \leqslant R \leqslant \infty$. If*

$$\sup_{\underline{N} \in \Gamma_R} \sum_{\underline{M} \in \Gamma_R} \tau(\underline{N}, \underline{M})^s < \infty,$$

then $\mathbf{1}_R$ is Δ-smooth on $[a, b]$.

P r o o f. The triangle inequality for $|\cdot|^s$ and the hypothesis yield

$$\sup_{\underline{N} \in \Gamma_R} \sum_{\underline{M} \in \Gamma_R} |\langle \delta_{\underline{N}} \,|\, (\Delta - z)^{-1} \delta_{\underline{M}} \rangle| \leqslant Const$$

uniformly in $z \in \mathcal{S}$. Interpreting $\mathbf{1}_R (\Delta - z)^{-1} \mathbf{1}_R$ as an operator on $l^2(\Gamma_R)$, its l^1 and l^∞ norms are given by the above expression. Therefore, Schur's interpolation theorem implies $\sup_{z \in \mathcal{S}} \|\mathbf{1}_R (\Delta - z)^{-1} \mathbf{1}_R\| < \infty$, as desired. ■

P r o o f o f t h e f i r s t t h e o r e m. Since $[a,b] \subset \Sigma(\Delta)$, we also have $[a,b] \subset \Sigma(H)$ for all V such that $\Omega^{\pm}_{[a,b]}(H,\Delta)$ exist, *i.e.*, almost surely. Hence, by Prop. 4 the spectrum of H is purely a.c. on $[a,b]$. Moreover, the previous lemma (with $R = 1$) and the assumption of the theorem imply that $\mathbf{1}_1$ is Δ-smooth. Lemma 1 (with $R = 1$) and Prop. 2 thus complete the proof.

P r o o f o f t h e s e c o n d t h e o r e m. Lemma 9 and the assumption of the theorem imply Lems. 1 and 2 (both with $R = \infty$). The result then follows from Prop. 3.

3. Models on $\mathbb{Z}^d$

We now turn our attention to the case where $X = \mathbb{Z}^d$ ($d \geqslant 2$), and the graph distance, $\mathrm{d}(M,N)$, is translational invariant. The graph $(\mathbb{Z}^d, \mathrm{d})$ is then determined by $\mathcal{V} = \{N \in \mathbb{Z}^d\,;\ \mathrm{d}(N,0) = 1\}$. We set $\gamma(M,N) = |N - M|$.

Recall that the *Fourier transform* of $\psi \in l^2(\mathbb{Z}^d)$ is defined as

$$\widehat{\psi}(x) = (\mathcal{F}\psi)(x) = (2\pi)^{-\frac{d}{2}} \sum_{N\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}N\cdot x}\psi(N),$$

where $x \in \mathbb{T}^d$. The *symbol* of Δ is $\widehat{\Delta} = \mathcal{F}\Delta\mathcal{F}^{-1}$. Thus, given a $\mathcal{V} \subset \mathbb{Z}^d$, the symbol of the Laplacian associated with $\mathcal{V}$ is the multiplication by

$$\Phi(x) = \sum_{V\in\mathcal{V}} \mathrm{e}^{\mathrm{i}V\cdot x} = \sum_{V\in\mathcal{V}} \cos\,(V \cdot x).$$

It follows from a change of variables that the spectrum of Δ is purely a.c. and equal to $[\min\Phi, \#\mathcal{V}]$, where $\#\mathcal{V}$ denotes the cardinality of $\mathcal{V}$.

The *Green function* of Δ is defined as $G(M,N,z) = \langle\delta_M \,|\, (\Delta - z)^{-1}\delta_N\rangle$ for $M,N \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$. Since $(\mathbb{Z}^d, \mathrm{d})$ is translational invariant, $G(M,N,z) = G(0,N-M,z)$; this last is abbreviated by $G(N-M,z)$. Hence, for any $N \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$,

$$\begin{aligned} G(N,z) &= \langle\widehat{\delta_0}(x) \,|\, (\widehat{\Delta} - z)^{-1}\widehat{\delta_N}(x)\rangle_2 \\ &= (2\pi)^{-d} \int\limits_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}N\cdot x}}{\Phi(x) - z}\,\mathrm{d}x. \end{aligned} \tag{8}$$

Recall that our sparseness assumptions are formulated in terms of

$$\tau(M,N) = \sup_{z\in\mathcal{S}} |G(N-M,z)|,$$

where $[a,b]$ is a given interval and $\mathcal{S} = \{a \leqslant \mathrm{Re}\, z \leqslant b, 0 < \mathrm{Im}\, z < 1\}$. Hence the decay of $G(N,z)$ when $N \to \infty$ has to be *a priori* known. It is clear that $G(N,z)$

decays exponentially when $[a,b]$ is outside the spectrum of Δ. Moreover, the case where $[a,b]\subset \mathrm{spec}(\Delta)$ has been studied in [24, 23] using material from [26, 28, 29]:

Proposition 5. *Given a real-valued, analytic and periodic function $\Phi(x)$ on $\mathbb{T}^d$, let $\Gamma(e)=\{x\in\mathbb{T}^d\ ;\ \Phi(x)=e\}$ and let $G(N,z)$ be defined by (8). Assume, for $(a',b')\subset \mathrm{Ran}\,\Phi$ and $\mathcal{S}'=\bigcup_{e\in(a',b')}\Gamma(e)$:*

- *$\nabla\Phi(x)\neq 0$ for all $x\in\mathcal{S}'$;*
- *for all $e\in(a',b')$, $\Gamma(e)$ admits at least κ nonvanishing principal curvatures at any point, where $\kappa\geqslant 1$ is a fixed integer.*

Then, for $N=|N|\omega$ and $[a,b]\subset(a',b')$, $\lim_{z\to e,\, z\in\mathbb{C}_+}G(N,z)$ exists and is $O(|N|^{-\frac{\kappa}{2}})$ uniformly in $(e,\omega)\in[a,b]\times \mathrm{S}^{d-1}$. More generally,*

$$G(N,z)=O(|N|^{-\frac{\kappa}{2}}\log|N|)$$

uniformly in $(z,\omega)\in\overline{\mathcal{S}}\times\mathrm{S}^{d-1}$, where $\mathcal{S}=\{e+\mathrm{i}y\ ;\ a\leqslant e\leqslant b,\ 0<y<1\}$.

For example, in the case of the *centered Laplacian*, which is specified by

$$\mathcal{V}=\{(\pm1,0,\ldots,0),(0,\pm1,\ldots,0),\ldots,(0,0,\ldots,\pm1)\}$$

and whose spectrum is equal to $[-2d,2d]$, $\Gamma(e)$ defines a regular surface for $e\notin\{-2d,-2d+4,\ldots,2d-4,2d\}$, exempt of planarity if in addition $e\neq 0$. Hence, letting $E=\{-2d,-2d+4,\ldots,2d-4,2d\}\cup\{0\}$, $G(N,e+\mathrm{i}0)=O(|N|^{-\frac{1}{2}})$ uniformly on compact subsets of $[-2d,2d]\setminus E$. As an alternative, in order to avoid convexity problems, S. Molchanov and B. Vainberg [17] have suggested to base the discretization of the Laplacian on the diagonal neighbors

$$\mathcal{V}=\{(v^{(1)},\ldots,v^{(d)})\ ;\ v^{(j)}\in\{1,-1\}\ \text{ for } j=1,\ldots,d\}.$$

The resulting graph consists of 2^{d-1} connected components, and the spectrum of its Laplacian is equal to $\left[-2^d,2^d\right]$. Remarkably, $\Gamma(e)$ defines a regular, *strictly convex* surface for $e\notin\{-2^d,0,2^d\}$, as shown in [22]; hence, with $E=\{-2^d,0,2^d\}$, $G(N,e+\mathrm{i}0)=O(|N|^{-\frac{d-1}{2}})$ uniformly on compact subsets of $[-2d,2d]\setminus E$.

Let us translate our abstract results to the present concrete models using the previous proposition. Assumption A and the strengthened version of B assumed in Th. 1 easily reduce to the following sparseness assumption:

A s s u m p t i o n G. *There exists an $\epsilon>0$ such that $\sum_{m\in\Gamma\setminus\{n\}}|n-m|^{-\frac{\kappa s}{2}+\epsilon}<\infty$ for all $n\in\Gamma$, and*

$$\lim_{\substack{|n|\to\infty\\ n\in\Gamma}}\sum_{m\in\Gamma\setminus\{n\}}|n-m|^{-\frac{\kappa s}{2}+\epsilon}=0.$$

*Without constraints on the approach.

First consider the case where $[a,b] \subset (a',b') \subset \mathrm{spec}(\Delta)$ for a given (a',b') satisfying the hypotheses of the previous proposition. Since $(\mathbb{Z}^d, \mathrm{d})$ is translational invariant,

$$\mathfrak{I} = \inf_{z\in\mathcal{S}} |\langle \delta_0 \,|\, (\Delta - z)^{-1}\delta_0\rangle| = \inf_{z\in\mathcal{S}} |G(0,z)|.$$

Moreover, by Th. 6.1 in [24]

$$\lim_{\substack{z\to e\\ z\in\mathbb{C}_+}} \mathrm{Im}\, G(0,z) = \pi \int\limits_{\Gamma(e)} \|\nabla_x \Phi(x)\|^{-1} \mathrm{ds}(x) > 0. \tag{9}$$

Since in addition $\mathrm{Im}\, G(0,z) > 0$ on $\mathcal{S}$, the above implies C.

Let $\Delta_j = P_j \Delta P_j$, where P_j denotes the projection onto $l^2(X_j)$. Observe that for any $z \notin \mathbb{R}$

$$\langle \delta_N \,|\, (\Delta_j - z)^{-1}\delta_N\rangle = \begin{cases} G(0,z) & \text{if } N \in X_j \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the equation (9) implies $[a,b] \subset \Sigma(\Delta)$.

Consider now the case where $[a,b]$ is at a positive distance of $\mathrm{spec}(\Delta)$. Then, $\mathfrak{I}$ is clearly positive, *i.e.*, C holds. Assumption D is satisfied for $\gamma(M,N) = |M-N|$. Moreover, Assumption F holds, since $\sup_{z\in\mathcal{S}} |G(N,z)|$ is exponentially decaying. Finally, Assumption E yields our sparseness condition in this case, namely

Assumption H. $\lim\limits_{\substack{|n|\to\infty\\ n\in\Gamma}} \inf\limits_{m\in\Gamma\setminus\{n\}} |n-m| = \infty.$

Let Θ be a reunion of intervals (a',b') like above. We have proven:

Theorem 3. *Suppose* Γ *satisfies G. If the wave operators* $\Omega^{\pm}_{\Theta}(H,\Delta)$ *exist a.e., then they are complete (and the spectrum of* H *is purely a.c.) on* Θ*, almost surely. Suppose instead* Γ *satisfies the weaker assumption H. Then, the spectrum of* H *outside* $\mathrm{spec}(\Delta)$ *is almost surely pure point with simple eigenvalues and exponentially decaying eigenfunctions.*

Remarks.

1. In particular, the previous theorem holds for the standard Laplacian (with $\kappa = 1$) and the Molchanov–Vainberg Laplacian (with $\kappa = d-1$) on $\Theta = \mathrm{spec}(\Delta) \setminus E$, where in both cases E is a finite, deterministic set (described after Prop. 5). By Proposition 4 (for instance), such an E does not contain eigenvalues of H, almost surely. In both cases completeness (a.s.) of the wave operators on the whole $\mathrm{spec}(\Delta)$ follows.

2. Additional conditions may be imposed on the geometry of Γ in order to assure the existence of the wave operators, including additional sparseness conditions [19].

3. As mentioned in the introduction, by Pastur's theorem the essential spectrum of H is almost surely equal to a deterministic set, which was characterized by S. Molchanov and B. Vainberg [17, 19].* Using their result, one may construct examples in which $\mathrm{spec}_{\mathrm{ess}}(H) = \mathbb{R}$. This is the case for instance when the random potential at each site has a Cauchy or a normal distribution. Then, the spectrum of H is *dense pure point* in $\mathbb{R} \setminus \mathrm{spec}(\Delta)$.

4. Our study includes another approach, based on Fredholm analytic theory and valid for bounded, deterministic potentials [23]. Under suitable sparseness conditions both existence and completeness of the wave operators are derived on $\mathrm{spec}(\Delta)$ minus a set of Lebesgue measure zero — which disappears in the random frame.

Example. Consider $H = \Delta + V$, where Δ is the standard (or the Molchanov–Vainberg) Laplacian. Suppose $\{V(n)\}_{n\in\Gamma}$ is a family of i.i.d. random variables lying on $\Gamma = \{(j^4, 0, \ldots, 0) \in \mathbb{Z}^d\ ;\ j \in \mathbb{Z}\}$, whose common distribution is Cauchy (alternatively, normal). Then, Γ is sparse in the sense of Th. 3 (with s sufficiently close to 1). Moreover, since Γ is included in the hyperplane $\mathbb{Z}^{d-1} \subset \mathbb{Z}^d$, the existence of $\Omega^{\pm}(H, \Delta)$ follows from a deterministic result of V. Jakšić and Y. Last [7].** Hence, by Th. 3 (and the first remark following it), $\mathrm{spec}(H)$ is purely a.c. on $\mathrm{spec}(\Delta)$ and the wave operators are complete there (almost surely). Moreover, by the same theorem (and the third remark following it), the spectrum of H on $\mathbb{R}\setminus\mathrm{spec}(\Delta)$ is dense pure point with simple eigenvalues and exponentially decaying eigenfunctions, almost surely.

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*S. Molchanov and B. Vainberg considered the random operator $H = \Delta + V$, where Δ is the standard Laplacian. However, their proof may easily be adapted in order to include Laplacians coming from translational invariant graphs on $\mathbb{Z}^d$; in particular, the spectrum of Δ does not have to be centered.

** V. Jakšić and Y. Last considered the half-space model (in which the Laplacian does not come from a translational invariant graph) with a random potential at the boundary; however, their argument may be slightly modified in order to include the above situation.

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