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Controllability from Rest to Arbitrary Position of the Nonhomogeneous Timoshenko Beam

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The controllability of a slowly rotating beam clamped to a disc is considered. It is assumed that at the beginning the beam remains at the position of rest and it is supposed to rotate by the given angle to achieve a desired position. The movement is governed by the system of two differential equations with nonhomogeneous coefficients: mass density, rotary inertia, flexural rigidity and shear stiffness. The problem of controllability is reduced to the moment problem that is, in turn, solved with the use of the asymptotics of the spectrum of the operator connected with the movement.

Key words: controllability, Timoshenko beam, minimal system, moment problem, Ullrich theorem.

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1. Introduction

The problem of controllability of a slowly rotating Timoshenko beam was studied by many authors ([7, 11, 15]). In this paper we consider the beam clamped to a disc that is rotated by a motor. The controllability depends on the radius of the disc then, and there are values of the radius (at most countably many of them — as it is shown in Sect. 4) for which controllability is not given. The control of a rotating homogeneous beam was described by G.M. Sklyar and W. Krabs in monograph [7] and in a series of papers (for example, [5, 8]). In [15] F. Woittennek and J. Rudolph considered the homogeneous Timoshenko beam with the load attached to the other (i.e., not clamped to the disc) end of the beam. The most complex analysis of the Timoshenko beam (not clamped to a disc, but directly to a rotating motor) was given by M. Shubov ([12, 10, 11]). In addition, she was the first author to consider the nonhomogeneous case. She studied the control of dumped beam.

It turns out that her study leads to the theory of not self-adjoint operators. This theory has not been well developed yet and only few authors study it. The control of an undumped beam is not a special case of dumped beam control and it is not trivial. It is just a different subject that should be elaborated from the very beginning. The theory developed here deals with selfadjoint operators, so Ullrich's theorem [14] may be used. An analysis of the dumped Timoshenko beam presented by M. Shubov excludes the case $RK = E$. In the case of undumped beam, that equality is naturally included into the theory. G.M. Sklyar, W. Krabs and V.I. Korobov [5] considered the case when $RK = E = 1$ and they solved the problem of exact controllability for that case. The spectral analysis, independent of assumption $RK \neq E$, was given in [13]. In the present paper we solve the problem of controllability of nonhomogeneous Timoshenko beam generalizing results from [5] and [8].

2. Timoshenko Beam Model

We consider the rotational motion of a beam in a horizontal plane. Its left end is clamped to the disk of a driving motor. We denote by r the radius of the disk, and let $\theta = \theta(t)$ be the rotation angle considered as a function of time ($t \geq 0$). Further on, we assign to a (uniform) cross section at x with $0 \leq x \leq 1$ the following: $E(x)$ which is the flexural rigidity, $K(x)$ — shear stiffness, $\varrho(x)$ — linear mass density, i.e., the weight of a cross section, $R(x)$ — rotary inertia. All of the above functions are assumed to be real and bounded by two positive numbers. It is assumed that they vary slowly, so their first and second derivatives are bounded. The length of the beam is assumed to be 1. We denote by $w(x,t)$ the deflection of the center line of beam and by $\xi(x,t)$ — the rotation angle of cross section area at the location x and at time t. Then w and ξ are governed by the following system of differential equations [13, 7]:

$$\begin{aligned} &\varrho(x)\ddot{w}(x,t) - \big(K(x)(w'(x,t) + \xi(x,t))\big)' = -\ddot{\theta}(t)\varrho(x)(x+r) \\ &R(x)\ddot{\xi}(x,t) - (E(x)\xi'(x,t))' + K(x)(w'(x,t) + \xi(x,t)) = \ddot{\theta}(t)R(x). \end{aligned} \tag{1}$$

Here for the given function g of two variables t and x, we adopt the notation $\dot{g} = g_t$, $g' = g_x$ for derivatives. In addition to (1) we impose the following boundary conditions:

$$\begin{aligned} &w(0,t) = \xi(0,t) = 0, \\ &w'(1,t) + \xi(1,t) = 0, \quad \xi'(1,t) = 0 \end{aligned}$$

for $t \geq 0$.

We define

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \int_0^1 \varrho(x) y_1(x) \overline{y_2(x)} dx + \int_0^1 R(x) z_1(x) \overline{z_2(x)} dx \tag{2}$$

and consider the space H, whose underlying set is $L^2\big((0,1), \mathbb{C}^2\big)$ and with inner product (2). Due to the hypotheses imposed on ϱ, the norm generated by (2) is equivalent to the standard L^2 norm. Next, we define the linear operator $A : D(A) \to H$ by the formula

$$A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\varrho}\big(K(y'+z)\big)' \\ -\dfrac{1}{R}\big((Ez')' - K(y'+z)\big) \end{pmatrix}, \tag{3}$$

where K, E, ϱ, R, y and z are functions of variable $x \in [0,1]$ and

$$D(A) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H^2\big((0,1), \mathbb{C}^2\big) : \begin{matrix} y(0) = z(0) = 0, \\ y'(1) + z(1) = z'(1) = 0 \end{matrix} \right\} \subset H.$$

It is easy to see that $D(A)$ is dense in H. Using the defined operator A and putting

$$f_1(x,t) = -\ddot{\theta}(t)(r+x), \qquad f_2(x,t) = \ddot{\theta}(t), \tag{4}$$

we can rewrite the equations (1) in the vector form

$$\begin{pmatrix} \ddot{w}(x,t) \\ \ddot{\xi}(x,t) \end{pmatrix} + A \begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}. \tag{5}$$

It follows readily [13] that the operator $A : D(A) \to H$ is positive, symmetric, invertible and selfadjoint. Therefore there exists the unique weak solution to (1) and it is given by

$$\begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \sin \sqrt{\lambda_j}(t-s) ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \tag{6}$$

The inner product used here is defined in (2), the functions f_1 and f_2 are defined in (4) and $\begin{pmatrix} y_j \\ z_j \end{pmatrix}$ for $j \in \mathbb{N}$ is the eigenvector of the operator A that corresponds to eigenvalue λ_j. Also, we notice that the first (time) derivative of the above solution is

$$\begin{pmatrix} \dot{w}(x,t) \\ \dot{\xi}(x,t) \end{pmatrix} = \sum_{j=1}^{\infty} \int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \cos \sqrt{\lambda_j}(t-s) ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \tag{7}$$

As it was proved in [13] the spectrum asymptotically splits into two subsets:

$$\Lambda^{(0)} = \left\{ \left(\int_0^1 \sqrt{\frac{\varrho(t)}{K(t)}}\, dt \right)^{-2} \left(-\frac{\pi}{2} + n\pi + \varepsilon_n^{(0)} \right)^2 : n > N \right\}, \tag{8}$$

$$\Lambda^{(1)} = \left\{ \left(\int_0^1 \sqrt{\frac{R(t)}{E(t)}}\, dt \right)^{-2} \left(-\frac{\pi}{2} + n\pi + \varepsilon_n^{(1)} \right)^2 : n > N \right\} \tag{9}$$

with $\varepsilon_n^{(0)}, \varepsilon_n^{(1)} \to \infty$ as $n \to \infty$. We denote by $\lambda_n^{(k)}$ the elements of the set $\Lambda^{(k)}$, where $k \in \{0, 1\}$, and by $J^{(k)}$ the corresponding integrals, i.e.,

$$J^{(0)} = \int_0^1 \sqrt{\frac{\varrho(t)}{K(t)}}\, dt, \qquad J^{(1)} = \int_0^1 \sqrt{\frac{R(t)}{E(t)}}\, dt.$$

R e m a r k. In [13] we proved that the eigenvalues of the operator A were at most double. It is not enough for exact controllability, where asymptotic singularity of eigenvalues is required. However, for a wide class of models (that includes all homogeneous case patterns) the singularity of eigenvalues is proved. As we do not have the general proof of singularity, we set an assumption that all eigenspaces of the operator A are one-dimensional.

3. Ullrich Theorem and its Modification

Generalizing the classical theorem of R.E.A.C. Paley and N. Wiener (see [9]), D. Ullrich proved the following theorem in [14]. We simplify it to the case that fulfills our requirements.

Theorem 1. *Suppose that for every integer number n, the distinct complex numbers ω_{n0}, ω_{n1} with*

$$\lim_{n \to \pm\infty} |n - \omega_{nk}| = 0 \quad \text{for } k = 0, 1$$

are given. Then for any set of complex numbers $\{c_{nk}\}$ with $n \in \mathbb{Z}$, $k = 0, 1$, the system of integral equations

$$\int_{-2\pi}^{2\pi} f(t) \exp(it\omega_{nk}) dt = c_{nk} \tag{10}$$

has a solution $f \in L^2(-2\pi, 2\pi)$, if and only if

$$\sum_{n=-\infty}^{\infty} \left(|c_{n0}|^2 + \left| \frac{c_{n0} - c_{n1}}{\omega_{n0} - \omega_{n1}} \right|^2 \right) < \infty, \tag{11}$$

for every integer n and $k = 0, 1$. If the solution to (10) exists, then it is unique.

The above theorem should be altered to serve the purpose of controllability. V.I. Korobov, W. Krabs and G.M. Sklyar proved the following modification in [5].

Theorem 2. *The condition (11) remains necessary and sufficient for the solvability of the system of integral equations (10) if the interval of integration* $[-2\pi, 2\pi]$ *is replaced by* $[0, T]$ *with* $T \geq 4\pi$*. The solution is not unique unless* $T = 4\pi$.

A modification of the above theorem was proved by W. Krabs, G.M. Sklyar and J. Woźniak in [8]. We state a little bit altered version of it.

Theorem 3. *Assume* n *is an integer,*

$$\omega_{n0} = \gamma\left(-\frac{\pi}{2} + n\pi\right) + \varepsilon_{n0},$$
$$\omega_{n1} = \left(-\frac{\pi}{2} + n\pi\right) + \varepsilon_{n1},$$

where $1 < \gamma = \frac{p}{q}$, p, q *are relatively prime positive integers and* $\lim\limits_{n\to\pm\infty} \varepsilon_{nk} = 0$ *for* $k = 0, 1$*. Let* $I_\gamma = \left[0, 2\frac{1+\gamma}{\gamma}\right]$.

1) If both p *and* q *are odd, then the system*

$$\int\limits_{I_\gamma} f(t)\exp(-i\omega_{nk}t)dt = c_{nk}, \quad n \in \mathbb{Z}, k \in \{0, 1\}, \tag{12}$$

has a solution $f \in L^2(I_\gamma)$ *if and only if*

$$\sum_{n=-\infty}^{\infty}\left(|c_{n0}|^2 + |c_{n1}|^2 + \left|\frac{c_{((1-q)/2)+qn,0} - c_{((1-p)/2)+pn,1}}{\omega_{((1-q)/2)+qn,0} - \omega_{((1-p)/2)+pn,1}}\right|^2\right) < \infty. \tag{13}$$

2) If exactly one of p, q *is even, then the system (12) has a solution if and only if*

$$\sum_{n=-\infty}^{\infty}\left(|c_{n0}|^2 + |c_{n1}|^2\right) < \infty. \tag{14}$$

In both cases, if the solution exists, it is unique. The interval I_γ *may be replaced by any interval of the length* $T = 2\frac{\gamma+1}{\gamma}$*. If* $T > 2\frac{\gamma+1}{\gamma}$*, then the solution still exists, but it is not unique.*

4. Solution to the Problem of Controllability

We assume that at time $t = 0$ the beam remains at the position of rest, i.e.,

$$w(x,0) = \dot{w}(x,0) = \xi(x,0) = \dot{\xi}(x,0) = \theta(0) = \dot{\theta}(0) = 0$$

for $x \in [0,1]$. At a given time T, we need to achieve the following position:

$$\begin{aligned} w(x,T) &= w_T(x), & \dot{w}(x,T) &= \dot{w}_T(x), \\ \xi(x,T) &= \xi_T(x), & \dot{\xi}(x,T) &= \dot{\xi}_T(x), \end{aligned} \tag{15}$$

where functions w_T, $\dot{w}_T$, ξ_T, $\dot{\xi}_T$ defined on $[0,1]$ are given. The problem of controllability from rest to arbitrary position is:

Problem of Controllability. *Given time $T > 0$, numbers θ_T, $\dot{\theta}_T \in \mathbb{R}$ and position (15), find a function $\theta \in H_0^2(0,T)$ satisfying*

$$\theta(T) = \theta_T, \qquad \dot{\theta}(T) = \dot{\theta}_T \tag{16}$$

and such that the weak solution (6) of (1) satisfies (15).

Employing the end conditions (15) to (6), (7) and comparing the coefficients, we obtain

$$\begin{aligned} a_n^{(k)} \int_0^T \ddot{\theta}(t) \sin \sqrt{\lambda_n^{(k)}}(T-t)dt &= \sqrt{\lambda_n^{(k)}} \left\langle \begin{pmatrix} w_T \\ \xi_T \end{pmatrix}, \begin{pmatrix} y_n^{(k)} \\ z_n^{(k)} \end{pmatrix} \right\rangle, \\ a_n^{(k)} \int_0^T \ddot{\theta}(t) \cos \sqrt{\lambda_n^{(k)}}(T-t)dt &= \left\langle \begin{pmatrix} \dot{w}_T \\ \dot{\xi}_T \end{pmatrix}, \begin{pmatrix} y_n^{(k)} \\ z_n^{(k)} \end{pmatrix} \right\rangle \end{aligned} \tag{17}$$

for all $n \in \mathbb{N}$ and $k \in \{0,1\}$, where

$$a_n^{(k)} = \int_0^1 R(x)\overline{z_n^{(k)}(x)}dx - \int_0^1 \varrho(x)(r+x)\overline{y_n^{(k)}(x)}dx.$$

From now on we assume that all the $a_n^{(k)}$'s are different from 0. Actually, we call the value r of the radius *regular* if $a_n^{(k)} \neq 0$ for all positive integers n and $k \in \{0,1\}$. Other values of r are called *singular*. We notice that there are only countably many singular values of r. To see this, we write a instead of $a_n^{(k)}$ and y, z for the corresponding to a coordinates of an eigenvector of A. From the spectral equation of the operator A we gather that

$$\begin{aligned} \varrho(x)y(x) &= -\frac{1}{\lambda}\big(K(x)(y'(x)+z(x))\big)', \\ R(x)z(x) &= -\frac{1}{\lambda}\big(E(x)(z'(x))' - K(x)(y'(x)+z(x))\big). \end{aligned}$$

Therefore, using integration by parts and the fact that $\begin{pmatrix} y \\ z \end{pmatrix} \in D(A)$, we obtain

$$a = -\frac{1}{\lambda}\left(\int_0^1 \overline{\left(E(x)z'(x)\right)'}dx - \int_0^1 K(x)\overline{(y'(x)+z(x))}dx\right.$$
$$\left. - \int_0^1 (r+x)\overline{\left(K(x)(y'(x)+z(x))\right)'}dx\right)$$
$$= -\frac{1}{\lambda}\overline{(E(0)z'(0) - rK(0)y'(0))}.$$

Thus to each $a = 0$ there corresponds at most one value of radius. As $\left\{a_n^{(k)}\right\}_{n,k}$ is a countable set, the set of singular values of r is at most countable.

We define

$$\begin{aligned} c_n^{(k)} &= \frac{\sqrt{\lambda_n^{(k)}}}{a_n^{(k)}}\left\langle \begin{pmatrix} w_T \\ \xi_T \end{pmatrix}, \begin{pmatrix} y_n^{(k)} \\ z_n^{(k)} \end{pmatrix} \right\rangle, \\ \dot{c}_n^{(k)} &= \frac{1}{a_n^{(k)}}\left\langle \begin{pmatrix} \dot{w}_T \\ \dot{\xi}_T \end{pmatrix}, \begin{pmatrix} y_n^{(k)} \\ z_n^{(k)} \end{pmatrix} \right\rangle \end{aligned} \tag{18}$$

and put $u(t) = \ddot{\theta}(T-t)$ for $t \in [0,T]$. Then (17) can be rewritten in the form

$$\int_0^T u(t)\sin\sqrt{\lambda_n^{(k)}}tdt = c_n^{(k)}, \qquad \int_0^T u(t)\cos\sqrt{\lambda_n^{(k)}}tdt = \dot{c}_n^{(k)}. \tag{19}$$

Also we have the end conditions (16) equivalent to

$$\int_0^T u(t)dt = \theta_T \qquad \text{and} \qquad \int_0^T tu(t)dt = \dot{\theta}_T. \tag{20}$$

Thus the problem of controllability from rest to arbitrary position is equivalent to the following moment problem.

Moment Problem. *Find $u \in L^2(0,T)$ such that for all $n \in \mathbb{N}$ and $k \in \{0,1\}$ the conditions*

$$\int_0^T u(t)\cos t\sqrt{\lambda_n^{(k)}}dt = \dot{c}_n^{(k)} \qquad \int_0^T u(t)\sin t\sqrt{\lambda_n^{(k)}}dt = c_n^{(k)}$$
$$\int_0^T u(t)dt = \dot{\theta}_T \qquad \int_0^T tu(t)dt = \theta_T.$$

are satisfied.

Once $u(t)$ is found, we also have $\theta(t) = \int_0^t (t-s)u(T-s)ds$.
For to solve the stated problem we divide it into three cases.
Case 1: $J^{(0)} = J^{(1)} = J$.
Using (18), we define

$$c_{nk} = \frac{\pi}{J}\left(\dot{c}_n^{(k)} + i c_n^{(k)}\right). \tag{21}$$

Then we can rewrite (19) in the form

$$\begin{aligned} &\int_0^T u(t)\exp\left(i\sqrt{\lambda_n^{(k)}}t\right) = \frac{J}{\pi}c_{nk} \\ \text{and}\quad &\int_0^T u(t)\exp\left(-i\sqrt{\lambda_n^{(k)}}t\right) = \frac{J}{\pi}\overline{c_{nk}}. \end{aligned} \tag{22}$$

We are going to use Theorem 2. Therefore we replace the last two equations with the one, where n ranges over the integers. We make some changes. Let

$$\omega_{nk} = \frac{J}{\pi}\left(\sqrt{\lambda_n^{(k)}} - \frac{\pi}{2J}\right) \tag{23}$$

and $\omega_{-m+1,k} = -\omega_{mk}+1$ for $n, m \in \mathbb{N}$ and $k \in \{0,1\}$. In addition to equation (21) we define $c_{-m+1,k} = \overline{c_{mk}}$ for $m \in \mathbb{N}$. Thus

$$\lim_{n\to\pm\infty} |\,\omega_{nk} - n| = 0.$$

According to Theorem 2, there exists a unique solution v to the system of integral equations

$$\int_0^{4\pi} v(t)\exp\left(i\omega_{nk}t\right)dt = c_{nk}, \qquad n \in \mathbb{Z},$$

if and only if (11) holds.

Now, let us define $u_1(t) = v\left(\frac{\pi t}{J}\right)\exp\left(-\frac{i\pi t}{2J}\right)$ for $t \in [0, 4J]$. Then the function u_1 is a member of $L^2(0,4J)$ and after standard computation including changing of variable, we get

$$\int_0^{4J} u_1(t)\exp\left(i\sqrt{\lambda_n^{(k)}}t\right)dt = \frac{J}{\pi}c_{nk}$$

and

$$\int_0^{4J} u_1(t) \exp\left(-i\sqrt{\lambda_n^{(k)}}t\right) dt = \frac{J}{\pi}\overline{c_{nk}}.$$

Thus for $T = 4J$ the system (22) has the unique solution u_1 if and only if (11) is satisfied. Let us set $T = 4J$ for a while.

We will show that u_1 is in fact a real function. To achieve this we put first $u_1 = \mathrm{Re}u_1 + i\mathrm{Im}u_1$ and then we notice that

$$\frac{J}{\pi}c_{nk} = \overline{\frac{J}{\pi}\overline{c_{nk}}} = \int_0^{4J} \overline{u_1(t)} \exp\left(i\sqrt{\lambda_n^{(k)}}t\right) dt.$$

Thus it follows from (22) and the above equation that

$$2\frac{J}{\pi}c_{nk} = 2\int_0^{4J} \mathrm{Re}u_1(t) \exp\left(i\sqrt{\lambda_n^{(k)}}t\right) dt.$$

Comparing this with (22), we obtain immediately

$$\int_0^{4J} \mathrm{Im}u_1(t) \exp\left(i\sqrt{\lambda_n^{(k)}}t\right) dt = 0.$$

Similarly, we get

$$\int_0^{4J} \mathrm{Im}u_1(t) \exp\left(-i\sqrt{\lambda_n^{(k)}}t\right) dt = 0$$

for all positive integers n and $k \in \{0,1\}$. Well, we need to show that u_1 itself is a real function. Thus, further on, we put $v_0(t) = \mathrm{Im}u_1\left(\frac{Jt}{\pi}\right)e^{\frac{it}{2}}$, so we have

$$\int_0^{4\pi} v_0(t) \exp(i\omega_{nk}t)dt = 0$$

for all $n \in \mathbb{Z}$ and $k \in \{0,1\}$. But because $v_0 \in L^2(0,4\pi)$, this system has the unique solution (Th. 2). Therefore $v_0(t) = 0$ and $\mathrm{Im}u_1(t) = 0$.

From now on let $T > 4J$. Then the system (22) has a (nonunique this time) solution $u_1 \in L^2(0,T)$. Proceeding like before, we observe that for the integer n and $k \in \{0,1\}$ the system

$$\int_0^{\frac{\pi T}{J}} v(t) \exp\left(i\omega_{nk}t\right) dt = c_{nk}$$

has a solution $v(t) = u_1\left(\frac{Jt}{\pi}\right)\exp\left(\frac{it}{2}\right)$ for $t \in \left[0, \frac{\pi T}{J}\right]$ if and only if (11) is satisfied.

We define

$$u_2(t) = \begin{cases} u_1(t) & \text{for } 0 \le t \le 4J, \\ 0 & \text{for } 4J < t \le T. \end{cases}$$

Provided (11) is satisfied, we have $u_2 \in L^2(0,T)$ and

$$\int_0^T u_2(t)\exp\left(i\sqrt{\lambda_n^{(k)}}t\right)dt = \frac{J}{\pi}c_{nk},$$

and

$$\int_0^T u_2(t)\exp\left(-i\sqrt{\lambda_n^{(k)}}t\right)dt = \frac{J}{\pi}\overline{c_{nk}}$$

for $n \in \mathbb{Z}$ and $k \in \{0,1\}$. The last equation is equivalent to

$$\int_0^T u_2(t)\sin\sqrt{\lambda_n^{(k)}}tdt = c_n^{(k)}$$

and

$$\int_0^T u_2(t)\cos\sqrt{\lambda_n^{(k)}}tdt = \dot{c}_n^{(k)}.$$

In [7] it is shown that the system

$$\left\{t, 1, \cos t\sqrt{\lambda_n^{(k)}}, \sin t\sqrt{\lambda_n^{(k)}} : n \in \mathbb{N}, k \in \{0,1\}\right\} \tag{24}$$

is minimal in $L^2(0,T)$ for $T > 4J$. The minimality implies, in particular, the existence of functions u_3, $u_4 \in L^2(0,T)$ that satisfy

$$\int_0^T tu_3(t)dt = 1, \qquad \int_0^T u_3(t)dt = 0,$$

$$\int_0^T tu_4(t)dt = 0, \qquad \int_0^T u_4(t)dt = 1,$$

$$\int_0^T u_j(t)\sin\sqrt{\lambda_n^{(k)}}tdt = 0, \qquad \int_0^T u_j(t)\cos\sqrt{\lambda_n^{(k)}}tdt = 0 \quad \text{for } j \in \{3,4\}$$

for all positive integers n and $k \in \{0, 1\}$.

Now, let

$$\tilde{\theta}_T = \int_0^T t u_2(t) dt \quad \text{and} \quad \dot{\tilde{\theta}}_T = \int_0^T u_2(t) dt$$

and then we put

$$u(t) = u_2(t) + \left(\theta_T - \tilde{\theta}_T\right) u_3(t) + \left(\dot{\theta}_T - \dot{\tilde{\theta}}_T\right) u_4(t)$$

for $t \in [0, T]$. The defined above function u is a member of $L^2(0, T)$ and it solves the stated Moment Problem.

On the other hand, when we consider the definition of c_{nk} and ω_{nk} ((18) and (23), respectively), we get the equivalence of (11) with

$$\sum_{n=1}^{\infty} \left(|c_{n0}|^2 + \left| \frac{c_{n0} - c_{n1}}{\sqrt{\lambda_n^{(0)}} - \sqrt{\lambda_n^{(1)}}} \right|^2 \right) < \infty. \tag{25}$$

We conclude our study with the following theorem:

Theorem 4. *Provided* $\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}} dx = \int_0^1 \sqrt{\frac{R(x)}{E(x)}} dx$ *and* $T \geq 4 \int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}} dx$*, the problem of controllability from the state of rest to arbitrary position is solvable if and only if the condition (25) is satisfied.*

Case 2: $J^{(1)}/J^{(0)} = \frac{p}{q}$ is a rational number and p and q are relatively prime positive odd integers.

Without loss of generality, we may assume that $J^{(1)}/J^{(0)} = \gamma > 1$. Let

$$\omega_{n0} = \gamma \left(-\frac{\pi}{2} + n\pi\right) + \varepsilon_{n0} = J^{(1)} \sqrt{\lambda_n^{(0)}}$$

and

$$\omega_{n1} = \left(-\frac{\pi}{2} + n\pi\right) + \varepsilon_{n1} \quad = J^{(1)} \sqrt{\lambda_n^{(1)}},$$

here $\varepsilon_{n0} = (J^{(1)}/J^{(0)})\varepsilon_n^{(0)}$ and $\varepsilon_{n1} = \varepsilon_n^{(1)}$. Still we have ε_{n0}, $\varepsilon_{n1} \to \infty$ as $n \to \infty$. Let $c_{nk} = \dot{c}_n^{(k)} + i c_n^{(k)}$. We define c_{nk} for nonpositive values of n like in Case 1 and let $\omega_{-m+1,k} = -\omega_{mk}$.

According to Theorem 3, there exists a unique solution v to the system

$$\int_{I_\gamma} v(t) \exp(i\omega_{nk} t) dt = c_{nk}, \qquad n \in \mathbb{Z},\ k = 0, 1,$$

if and only if (13) holds. Let $\Gamma = 2\frac{1+\gamma}{\gamma}$, $T = J^{(1)}\Gamma = 2(J^{(0)} + J^{(1)})$ and $u_1(t) = v(t/J^{(1)})$. Then u_1 is the solution to the system of integral equations

$$\int_0^T f(t)\exp(i\sqrt{\lambda_n^{(k)}}t)dt = c_{nk}, \qquad n \in \mathbb{Z},\ k = 0, 1,$$

if and only if (13) is fulfilled. Similarly as in Case 1, we define u_2, u_3, u_4 and u. Also the proof that u is a real function is analogous to the one given in Case 1.

Similarly as in Case 1, we notice that (13) is equivalent to

$$\sum_{n=1}^{\infty}\left(|c_{n0}|^2 + |c_{n1}|^2 + \left|\frac{c_{((1-q)/2)+qn,0} - c_{((1-p)/2)+pn,1}}{\sqrt{\lambda^{(0)}_{((1-q)/2)+qn}} - \sqrt{\lambda^{(1)}_{((1-p)/2)+pn}}}\right|^2\right) < \infty. \tag{26}$$

Ultimately, we obtain the following theorem:

Theorem 5. *If* $\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx \Big/ \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx = \frac{p}{q}$ *with* p*,* q *being relatively prime odd positive integers and*

$$T \geq 2\left(\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx + \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx\right),$$

the problem of controllability from the state of rest to arbitrary position is solvable if and only if the condition (26) is satisfied.

Case 3: $J^{(1)}/J^{(0)} = \frac{p}{q}$ is a rational number, p and q are relatively prime positive integers and exactly one of them is even.

We proceed in the same way as in Case 2 and finally get the following theorem:

Theorem 6. *If* $\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx \Big/ \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx = \frac{p}{q}$ *with* p*,* q *being relatively prime positive integers, from which exactly one is even, and*

$$T \geq 2\left(\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx + \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx\right),$$

the problem of controllability from the state of rest to arbitrary position is solvable if and only if the condition (14) is satisfied.

5. Final Remarks

We excluded the case when the ratio $\gamma = J^{(1))}/J^{(0)}$ was irrational. Well, in this case the exact controllability was not yet solved even for the homogeneous beam. The reason is that there is no regularity in the distribution of numbers $n\gamma - [n\gamma]$ in the interval $[0, 1]$ (in fact, these numbers form a dense subset of $[0, 1]$). Neither Ullrich's theorem nor the Ullrich-type theorems deal with these cases and therefore we do not include them into this paper. A way of dealing with irrational case of γ is presented in [2]. Because the (possible) methods used there are totally different from the ones presented here, we do not include them into this paper.

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