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On the Berg–Chen–Ismail Theorem and the Nevanlinna–Pick Problem

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In 2002, C. Berg, Y. Chen, and M. Ismail found a nice relation between the determinacy of the Hamburger moment problem and asymptotic behavior of the smallest eigenvalues of the corresponding Hankel matrices. We investigate whether an analog of this statement holds for the Nevanlinna–Pick interpolation problem.

Key words: moment problem, Blaschke product, Carleson measure, Pick matrix.

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The Hamburger moment problem is a problem of finding conditions on a sequence $\{s_j\}$, $j = 0, 1, \dots$, so that there exists a positive Borel measure σ and

$$s_j = \int_{\mathbb{R}} x^j \, d\sigma(x), \qquad j = 0, 1, \dots . \tag{1}$$

With σ one can associate the infinite Hankel matrix and the sequence of its principal submatrices

$$H = \|s_{i+j}\|_{i,j=0}^{\infty}, \qquad H_n = \|s_{i+j}\|_{i,j=0}^{n}, \quad n = 0, 1, 2, \dots . \tag{2}$$

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By the famous result of Hamburger (1) has a solution if and only if $H_n \geq 0$ for all n.

Denote by $\{\lambda_{n,j}\}_{j=0}^{n}$ the eigenvalues of H_n (2) labelled in increasing order. Because of the interlacing property we have

$$0 \leq \lambda_{n+1,0} \leq \lambda_{n,0} \leq \lambda_{n+1,1} \leq \lambda_{n,1} \leq \ldots \leq \lambda_{n+1,n} \leq \lambda_{n,n} \leq \lambda_{n+1,n+1}, \qquad (3)$$

and so for each k $\lambda_{n,k}$ are monotone decreasing, $\lambda_k(H) = \lim_{n\to\infty} \lambda_{n,k}$ exists, and

$$0 \leq \lambda_0(H) \leq \lambda_1(H) \leq \ldots . \qquad (4)$$

Let us call the problem (1) *regular* if $\lambda_0(H) > 0$, and *singular* otherwise. In 2002, C. Berg, Y. Chen, and M. Ismail [2] proved a beautiful result which states that (1) is regular if and only if it has infinitely many solutions (indeterminate).

The Hamburger moment problem is one of the representatives of the so-called classical interpolation problems [1]. In this note we address to another one, specifically, the Nevanlinna–Pick interpolation problem in the Schur class $\mathcal{S}$ of functions contractive and analytic in the unit disk $\mathbb{D}$. This is a problem of finding the solutions of

$$f(z_k) = w_k, \qquad k = 0, 1, 2, \ldots, \qquad (5)$$

where z_k are distinct points in $\mathbb{D}$, w_k complex numbers, and $f \in \mathcal{S}$. The well-known criterion for (5), to have at least one solution, is given in terms of Pick matrices by

$$P_n := \left\| \frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right\|_{i,j=0}^{n} \geq 0$$

for all $n = 0, 1, \ldots$. For the eigenvalues $\{\lambda_{n,j}\}_{j=0}^{n}$ of the matrices P_n labelled in increasing order the above relations (3), (4) hold, and, again, we distinguish between the regular and singular Nevanlinna–Pick problem, i.e., in this paper we say that the problem (5) is regular if $\lambda_0(P) > 0$, and it is singular if $\lambda_0(P) = 0$.

With respect to a number of various questions there is a strong similarity between different classical interpolation problems, that is, if one can prove this or that statement with respect to one of the classical problems, a quite parallel statement holds for another one. In this note we study the question: *Is it true that a Nevanlinna–Pick problem has infinitely many solutions (indeterminate) if and only if* $\lambda_0(P) > 0$? We give a negative answer to this question. More precisely, we construct data $\{z_k, w_k\}_{k=0}^{\infty}$ of an indeterminate Nevanlinna–Pick problem such that $\lambda_0(P) = 0$.

First of all, we note that the Blaschke condition on the interpolation nodes $Z = \{z_k\}$

$$\sum_{k=0}^{\infty} (1 - |z_k|) < \infty \qquad (6)$$

guarantees that the interpolation problem

$$f(z_k) = 0, \qquad k = 0, 1, 2, \dots, \tag{7}$$

has infinitely many solutions. Indeed, every function of the form

$$f = g(z)B(z),$$

where $g(z) \in \mathcal{S}$, and $B(z)$ is the Blaschke product

$$B(z) = \prod_k \frac{z_k - z}{1 - z\bar{z}_k} \frac{|z_k|}{z_k},$$

solves (7). Note that (6) is necessary and sufficient for the problem (7) to be indeterminate.

Thus, our goal is to construct a Blaschke set Z such that $\lambda_0(P) = 0$ for the sequence of Nevanlinna–Pick matrices of the specific form $P_n = K_n$, where

$$K_n := \left\| \frac{1}{1 - z_i \bar{z}_j} \right\|_{i,j=0}^{n}.$$

In fact, *our main statement here characterizes completely the regularity of such Nevanlinna–Pick problems in the above sense.*

Recall (see, e.g., [4]), that a (finite) Borel measure ν on $\mathbb{D}$ is called a *Carleson measure*, if

$$\int_{\mathbb{D}} |f|^2 \, d\nu \le C \int_{\mathbb{T}} |f|^2 \, dm \tag{8}$$

for all $f \in H^2$. Here dm is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Due to Carleson's theorem [3] such measures are characterized completely by the following property: there exists $C > 0$ such that

$$\nu(Q_\epsilon(\phi)) \le C\epsilon$$

for all $-\pi < \phi \le \pi$, $0 < \epsilon < 1$,

$$Q_\epsilon(\phi) := \{z \in \mathbb{D} : |\arg z - \phi| \le \pi\epsilon, \ 1 - \epsilon \le |z| < 1\}.$$

Theorem 1. *Let Z satisfy (6), $B(z)$ be the corresponding Blaschke product. The Nevanlinna–Pick problem (7) is regular if and only if the measure ν, defined by*

$$\nu(\{z_k\}) = |B'(z_k)|^{-2}, \tag{9}$$

is a Carleson measure in $\mathbb{D}$.

P r o o f. Let ν (9) be a Carleson measure. For arbitrary $c_0, \dots, c_n \in \mathbb{C}$ put

$$h(z) = B(z) \sum_{k=0}^{n} \frac{c_k}{z - z_k} \in H^2$$

with

$$\|h\|^2 = \sum_{i,j=0}^{n} K_{ij} c_i \overline{c_j}, \qquad K = \|K_{ij}\|_{i,j=0}^{\infty} = \left\| \frac{1}{1 - z_i \bar{z}_j} \right\|_{i,j=0}^{\infty}.$$

We have $h(z_j) = c_j B'(z_j)$ for $j = 0, 1, \dots, n$ and $h(z_j) = 0$ for $j \geq n+1$, so

$$\int_{\mathbb{D}} |h|^2 \, d\nu = \sum_{j=0}^{n} |c_j|^2.$$

Hence by (8)

$$\int_{\mathbb{T}} |h|^2 \, dm = \|h\|^2 \geq \frac{1}{C} \sum_{j=0}^{n} |c_j|^2,$$

and so $\lambda_0(K) \geq C^{-1} > 0$, as claimed.

Conversely, assume that the Nevanlinna–Pick problem in question is regular. We use the standard orthogonal decomposition $L^2(\mathbb{T}) = H^2 \bigoplus H^2_-$. Put

$$\mathcal{K}_B = (BH^2)^{\perp} = H^2 \cap BH^2_-, \qquad \phi_k(z) = \frac{B(z)}{z - z_k}, \quad k = 0, 1, \dots .$$

It is easy to see that $\{\phi_k\}$ is complete in $\mathcal{K}_B$. Indeed, $(z - z_k)^{-1} \in H^2_-$, so $\phi_k \in \mathcal{K}_B$. Let $g \in \mathcal{K}_B$, $g \perp \phi_k$ for all $k = 0, 1, \dots$. Then $g = B\overline{\zeta g_1(\zeta)}$, $g_1 \in H^2$, and

$$0 = (g, \phi_k) = \int_{\mathbb{T}} \frac{B(\zeta)}{\zeta - z_k} \bar{B}(\zeta) \zeta g_1(\zeta) \, dm = g_1(z_k),$$

that is, $g_1 \in BH^2$, so $g \in H^2_-$ and $g \equiv 0$, as claimed. Hence the system of functions

$$f(z) = B(z) \sum_{k=0}^{n} \frac{c_k}{z - z_k} + B(z) g(z) = h(z) + B(z) g(z), \quad n = 0, 1, \dots,$$

$c_0, \dots, c_n \in \mathbb{C}$, and $g \in H^2$, is dense in H^2. We prove (8) for such functions. As above,

$$f(z_j) = c_j B'(z_j), \quad j = 0, 1, \dots, n, \qquad f(z_j) = 0, \quad j \geq n+1,$$

and

$$\int_{\mathbb{D}} |f|^2\, d\nu = \sum_{j=0}^{n} |c_j|^2, \qquad \|f\|^2 = \|h\|^2 + \|g\|^2 \ge \|h\|^2 = \sum_{i,j=0}^{n} K_{ij} c_i \overline{c_j},$$

and so

$$\int_{\mathbb{T}} |f|^2\, dm \ge \sum_{i,j=0}^{n} K_{ij} c_i \overline{c_j} \ge \lambda_0(K) \sum_{j=0}^{n} |c_j|^2 = \lambda_0(K) \int_{\mathbb{D}} |f|^2\, d\nu.$$

The proof is complete. ■

If the Nevanlinna–Pick problem (7) is singular (as in the example below), then so is the general problem (5). Indeed, for $D = \mathrm{diag}(w_0, w_1, \dots, w_n)$

$$P_n = K_n - DK_nD^* \le K_n,$$

and so $\lambda_0(P) \le \lambda_0(K)$.

E x a m p l e. Put $z_k = 1 - k^{-p}$, $p > 1$. Evidently Z is a Blaschke set. On the other hand

$$\begin{aligned}
(1-|z_n|^2)|B'(z_n)| &= \prod_{k=1}^{n-1} \frac{z_n - z_k}{1 - z_n z_k} \prod_{k=n+1}^{\infty} \frac{z_k - z_n}{1 - z_n z_k} \\
&\le \prod_{k=n+1}^{\infty} \frac{k^p - n^p}{n^p + k^p - 1} \le \prod_{k=n+1}^{\infty} \left(1 - \left(\frac{n}{k}\right)^p\right) \\
&\le \exp\left(-\sum_{k=n+1}^{\infty} \left(\frac{n}{k}\right)^p\right) \le \exp\left(-\frac{n+1}{2^p(p-1)}\right),
\end{aligned}$$

so

$$|B'(z_n)|^{-2} \ge \frac{1}{n^{2p}} \exp\left(\frac{n+1}{2^{p-1}(p-1)}\right).$$

Thus the measure ν (9) is infinite and, moreover, it is not of Carleson type.

There is a simple way of manufacturing regular Nevanlinna–Pick problems (7). Recall that $Z = \{z_k\}$ is the Carleson (uniformly separated) sequence if

$$\delta(Z) := \inf_n \left| \prod_{k \ne n} \frac{|z_k|}{z_k} \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0. \tag{10}$$

Assume that Z satisfies (10). By the theorem of H.S. Shapiro and A.L. Shields [5] the system of functions

$$x_k(z) = \frac{(1-|z_k|^2)^{1/2}}{1 - \bar{z}_k z} \quad k = 0, 1, \dots,$$

forms a Riesz basis in $\mathcal{K}_B$. So, for all $c_0, \dots, c_n \in \mathbb{C}$ there is $c > 0$ such that

$$\sum_{i,j=0}^{n} K_{ij}(1-|z_i|^2)^{1/2}\,(1-|z_j|^2)^{1/2}\,c_i\overline{c_j} \ge c \sum_{j=0}^{n} |c_j|^2,$$

or

$$\sum_{i,j=0}^{n} K_{ij}\,d_i\overline{d_j} \ge c \sum_{j=0}^{n} \frac{|d_j|^2}{(1-|z_j|^2)} \ge c \sum_{j=0}^{n} |d_j|^2,$$

as claimed.

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